Weighted limits and colimits in the category of left $S$-posets

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Abstract

Weighted limits and colimits are defined in categories that are enriched over a symmetric monoidal closed category. Since the category $\mathcal{S}\text{Pos}$ of left $S$-posets over a pomonoid $S$ is enriched over the category $\mathcal{Pos}$ of posets (with order-preserving mappings as morphisms) we can speak about weighted limits and colimits in $\mathcal{S}\text{Pos}$.

1 Introduction

By $\mathbf{1} = \{ \ast \}$ we shall denote the one-element ($S$-)poset and by $\mathbf{2} = \{0, 1\}$ the two-element chain with $0 < 1$. We assume the existence of an empty $S$-poset. Recall that morphisms in $\mathcal{S}\text{Pos}$ are order and action preserving mappings and isomorphisms are surjective mappings that preserve and reflect order.

The category $\mathcal{Pos}$ of posets and order-preserving mappings is a symmetric monoidal closed category (see Def. 6.1.1–6.1.3 of [2]) with the cartesian product as a tensor product and $I = \mathbf{1}$.

The category $\mathcal{S}\text{Pos}$ of left $S$-posets (or $\mathcal{Pos}_S$ of right $S$-posets) is a $\mathcal{Pos}$-category (or poset enriched category or a category enriched over $\mathcal{Pos}$) (see Def. 6.2.1 of [2]), where the morphism sets $\mathcal{S}\text{Pos}(A, B), sA, sB \in \mathcal{S}\text{Pos}$ are posets with respect to pointwise order.

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{Pos}$-categories then a $\mathcal{Pos}$-functor $F : \mathcal{A} \to \mathcal{B}$ has to preserve (in addition to composition and identity morphisms) the order of morphism posets. We shall call such functors pofunctors.

$\mathcal{Pos}$-natural transformations (see Def. 6.2.4 of [2]) between pofunctors are just the ordinary natural transformations. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{Pos}$-categories and $\mathcal{A}$ is small then by Proposition 6.3.1 of [2] the category of pofunctors $\mathcal{A} \to \mathcal{B}$ and natural transformations between them can be provided with the structure of a $\mathcal{Pos}$-category, written $\mathcal{Pos}[\mathcal{A}, \mathcal{B}]$. Namely, given two pofunctors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$, the set

$$\text{Nat}(F, G) = \{ (\alpha_A : F(A) \to G(A))_{A \in \mathcal{A}} \mid G(f)\alpha_{A'} = \alpha_A F(f) \text{ for every } f : A' \to A'' \text{ in } \mathcal{A} \}$$

of natural transformations from $F$ to $G$ is a poset with respect to the order

$$(\alpha_A)_{A \in \mathcal{A}} \leq (\beta_A)_{A \in \mathcal{A}} \iff \alpha_A \leq \beta_A \text{ for every } A \in \mathcal{A} \text{ in the poset } \mathcal{B}(F(A), G(A)).$$
2 Weighted limits in $S\text{Pos}$

2.1 Definition

Definition 1 (Cf. Def. 6.6.3 of [2]) Given a pomonoid $S$, small $\text{Pos}$-category $A$, and pofunctors $F : A \to S\text{Pos}$, $G : A \to \text{Pos}$, a $\text{Pos}$-limit of $F$ weighted by $G$ is a pair $(S\text{L}, (\lambda_P)_{P \in S\text{Pos}})$ where $S\text{L}$ is a left $S$-poset and $\lambda = (\lambda_P)_{P \in S\text{Pos}} : S\text{Pos}(-, L) \Rightarrow \text{Nat}(G, S\text{Pos}(-, F(-)))$ is a natural isomorphism, that is, for every $sP \in S\text{Pos}$,

$$\lambda_P : S\text{Pos}(P, L) \to \text{Nat}(G, S\text{Pos}(P, F(-))),$$

are poset isomorphisms that are natural in $P$. We write $\lim_G F$ for a $\text{Pos}$-limit of $F$ weighted by $G$.

\[\begin{array}{c}
A \\
\downarrow F \\
S\text{Pos} \\
\downarrow G \\
\text{Pos} \\
\end{array}\]

Remark 1 For every $sP \in S\text{Pos}$, $S\text{Pos}(P, F(-)) = S\text{Pos}(P, -) \circ F : A \to \text{Pos}$ is a pofunctor and the set $\text{Nat}(G, S\text{Pos}(P, F(-)))$ is a poset with respect to componentwise order of natural transformations. Therefore, there is a contravariant functor

$$\text{Nat}(G, S\text{Pos}(-, F(-))) : S\text{Pos} \to \text{Pos}$$

given by the assignment

\[\begin{array}{c}
sP \\
\downarrow p \\
S\text{Q} \\
\end{array}\]

\[\begin{array}{c}
\text{Nat}(G, S\text{Pos}(P, F(-))) \\
\downarrow (-p)\circ- \\
\text{Nat}(G, S\text{Pos}(Q, F(-))) \\
\end{array}\]

where the mapping $(- \circ p) \circ -$ is defined by

$$((- \circ p) \circ -)(\mu) := ((- \circ p) \circ \mu_A)_{A \in A} : G \Rightarrow S\text{Pos}(P, F(-))$$

for every natural transformation $\mu : G \Rightarrow S\text{Pos}(Q, F(-))$ and $- \circ p : S\text{Pos}(Q, F(A)) \to S\text{Pos}(P, F(A))$. The fact that $\lambda = (\lambda_P)_{P \in S\text{Pos}} : S\text{Pos}(-, L) \Rightarrow \text{Nat}(G, S\text{Pos}(-, F(-)))$ is a natural transformation meant that

$$\lambda_P(\psi \circ p) = ((- \circ p) \circ \lambda_Q(\psi))_{A \in A} ;$$

or

$$\lambda_P(\psi \circ p)_A = (- \circ p) \circ \lambda_Q(\psi)_A ;$$

or

$$\lambda_P(\psi \circ p)_A(x) = \lambda_Q(\psi)_A(x) \circ p \quad (1)$$

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for every $A \in \mathcal{A}$, $x \in G(A)$, $sP, sQ \in s\text{Pos}$, $p \in s\text{Pos}(P, Q)$, $\psi \in s\text{Pos}(Q, L)$.

$$s\text{Pos}(P, L) \xymatrix{\ar[r]^-{\lambda_p} & \text{Nat}(G, s\text{Pos}(P, F(-)))}$$

$$s\text{Pos}(Q, L) \xymatrix{\ar[r]^-{\lambda_Q} & \text{Nat}(G, s\text{Pos}(Q, F(-)))}$$

$$G(A) \xymatrix{\ar[r]^-{\lambda_Q(\psi)_A} & s\text{Pos}(Q, F(A))}$$

$$\xymatrix{\ar[r]^-{\lambda_p(\psi)p_A} & s\text{Pos}(P, F(A))}$$

$$\xymatrix{G(A) \ar[r]^-{\lambda_Q(\psi)_A} & s\text{Pos}(Q, F(A))}$$

$$\xymatrix{\ar[r]^-{\lambda_p(\psi)p_A} & s\text{Pos}(P, F(A))}$$

2.2 Existence of weighted limits in $s\text{Pos}$

Here we give a characterization of a weighted limit in more usual terms of so-called projections of a limit and a universal property. We shall use the notation of Definition 1.

**Theorem 1** There is one-to-one correspondence between $\text{Pos}$-limits of $F$ weighted by $G$ and pairs $(sL, (l^x_A)_{A \in A})$, where $sL$ is a left $S$-poset and $(l^x_A)_{A \in A}$ is a family of left $S$-poset morphisms $l^x_A : sL \to sF(A)$ such that

1. (a) for all $A \in \mathcal{A}$ and $x, x' \in G(A)$
   $$x \leq x' \implies l^x_A \leq l^{x'}_A;$$
   
   (b) for all $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_0)$,
   $$F(a) \circ l^x_A = l^{G(a)(x)}_{A_1};$$

2. for all $sP \in s\text{Pos}$ and $\varphi, \psi \in s\text{Pos}(P, L)$,
   $$((\forall A \in \mathcal{A})(\forall x \in G(A))(l^x_A \circ \varphi \leq l^x_A \circ \psi)) \implies \varphi \leq \psi;$$

3. for every $sP \in s\text{Pos}$ and family $(p^x_A)_{A \in A}$ of left $S$-poset morphisms $p^x_A : sP \to sF(A)$ with properties 1, there is a left $S$-poset morphism $\varphi : sP \to sL$ such that
   $$l^x_A \circ \varphi = p^x_A$$
   for every $A \in \mathcal{A}$ and $x \in G(A).$
**Proof.** Suppose that there is \( sL \in sPos \) and for every \( sP \in sPos \) poset isomorphisms

\[
\lambda_P : sPos(P, L) \to \text{Nat}(G, sPos(P, F(-)))
\]

which are natural in \( P \). For every \( A \in \mathcal{A} \), \( x \in G(A) \) we set

\[
l^x_A := \lambda_L(1_L)_A(x) : sL \to sF(A).
\]

1(a) holds because \( \lambda_L(1_L)_A : G(A) \to sPos(L, F(A)) \) is order preserving for every \( A \in \mathcal{A} \).

1(b). For every \( a : A_0 \to A_1 \) in \( \mathcal{A} \) and \( x \in G(A_0) \),

\[
F(a) \circ l^x_{A_0} = F(a) \circ \lambda_L(1_L)_{A_0}(x) = \lambda_L(1_L)_{A_1}(G(a)(x)) = l^{G(a)(x)}_{A_1},
\]

because \( \lambda_L(1_L) \) is a natural transformation.

\[
\begin{array}{ccc}
G(A_0) & \xrightarrow{\lambda_L(1_L)_{A_0}} & sPos(L, F(A_0)) \\
\downarrow G(a) & & \downarrow F(a) \circ - \\
G(A_1) & \xrightarrow{\lambda_L(1_L)_{A_1}} & sPos(L, F(A_1))
\end{array}
\]

2. Suppose that \( \varphi, \psi \in sPos(P, L) \) are such that \( l^x_A \circ \varphi \leq l^x_A \circ \psi \) for every \( A \in \mathcal{A} \) and \( x \in G(A) \). Since \( \lambda_P \) is natural in \( P \) (see (1)), we obtain

\[
\lambda_P(\varphi)_A(x) = \lambda_P(1_L \circ \varphi)_A(x) = \lambda_L(1_L)_A(x) \circ \varphi = l^x_A \circ \varphi \\
\leq l^x_A \circ \psi = \lambda_L(1_L)_A(x) \circ \psi = \lambda_P(1_L \circ \psi)_A(x) = \lambda_P(\psi)_A(x)
\]

for every \( A \in \mathcal{A} \), \( x \in G(A) \). Hence \( \lambda_P(\varphi) \leq \lambda_P(\psi) \), and so \( \varphi \leq \psi \), because \( \lambda_P \) reflects order.

3. If \( (p^x_A)_{A \in \mathcal{A}} \) is a family of left \( S \)-poset morphisms \( p^x_A : sP \to sF(A) \) that satisfies condition 1, then \( \mu = (\mu_A)_{A \in \mathcal{A}} \), where \( \mu_A : G(A) \to sPos(P, F(A)) \) is defined by

\[
\mu_A(x) := p^x_A,
\]

\( x \in G(A) \), is a natural transformation \( G \Rightarrow sPos(sP, F(-)) \). By the surjectivity of \( \lambda_P \), there exists \( \varphi \in sPos(P, L) \) such that \( \lambda_P(\varphi) = \mu \), and hence, by (1),

\[
l^x_A \circ \varphi = \lambda_L(1_L)_A(x) \circ \varphi = \lambda_P(\varphi)_A(x) = \mu_A(x) = p^x_A
\]

for every \( A \in \mathcal{A} \) and \( x \in G(A) \).

Conversely, let a pair \( (sL, (l^x_A)_{A \in \mathcal{A}}) \) satisfy conditions 1–3. For every \( sP \in sPos \) we define a mapping

\[
\lambda_P : sPos(P, L) \to \text{Nat}(G, sPos(P, F(-)))
\]

by

\[
\lambda_P(\varphi)_A(x) := l^x_A \circ \varphi : P \to F(A),
\]

(3) \( \varphi \in sPos(P, L), A \in \mathcal{A} \) and \( x \in G(A) \).

1. As a composite of two \( S \)-poset morphisms, \( \lambda_P(\varphi)_A(x) \) is an \( S \)-poset morphism.
2. Because of 1(a), \( \lambda_P(\varphi)_A : G(A) \to s\text{Pos}(P, F(A)) \) preserves order.
3. \( \lambda_P(\varphi) : G \Rightarrow s\text{Pos}(P, F(-)) \) is a natural transformation, because
\[
(F(a) \circ -) \circ \lambda_P(\varphi)_{A_0}(x) = F(a) \circ \lambda_P(\varphi)_{A_0}(x) = F(a) \circ l^x_{A_0} \circ \varphi
= l^{G(a)}_{A_1} \circ \varphi = \lambda_P(\varphi)_{A_1}(G(a)(x))
= (\lambda_P(\varphi)_{A_1} \circ G(a))(x)
\]
for every \( a : A_0 \to A_1 \) in \( A \) and \( x \in G(A_0) \).
4. \( \lambda_P \) is order preserving. Indeed, if \( \varphi \leq \psi \) in \( s\text{Pos}(P, L) \) then
\[
\lambda_P(\varphi)_A(x) = l^x_A \circ \varphi \leq l^x_A \circ \psi = \lambda_P(\psi)_A(x)
\]
for every \( A \in A \) and \( x \in G(A) \), thus \( \lambda_P(\varphi) \leq \lambda_P(\psi) \).
5. \( \lambda_P \) is order reflecting, because, assuming that \( \lambda_P(\varphi) \leq \lambda_P(\psi) \), \( \varphi, \psi \in s\text{Pos}(P, L) \), i.e. \( l^x_A \circ \varphi \leq l^x_A \circ \psi \) for every \( A \in A \) and \( x \in G(A) \), we conclude \( \varphi \leq \psi \) by 2.
6. \( \lambda_P \) is surjective. To prove this, consider a natural transformation \( \mu : G \Rightarrow s\text{Pos}(P, F(-)) \). For every \( A \in A \) and \( x \in G(A) \) set
\[
p^x_A := \mu_A(x) : sP \to sF(A).
\]
Since \( \mu_A \) is order preserving, the family \( \{p^x_A\}_{A \in A} \) satisfies 1(a). Since \( \mu \) is a natural transformation,
\[
F(a) \circ p^x_{A_0} = ((F(a) \circ -) \circ \mu_{A_0})(x) = (\mu_{A_1} \circ G(a))(x)
= \mu_{A_1}(G(a)(x)) = p^{G(a)}_{A_1}(x)
\]
for every \( a : A_0 \to A_1 \) in \( A \). Hence 1(b) is also satisfied.
\[
\begin{array}{ccc}
G(A_0) & \xrightarrow{\mu_{A_0}} & s\text{Pos}(sP, F(A_0)) \\
G(a) \downarrow & & \downarrow (F(a) \circ -) \\
G(A_1) & \xrightarrow{\mu_{A_1}} & s\text{Pos}(sP, F(A_1))
\end{array}
\]
By 3, there is an \( S \)-poset morphism \( \varphi : sP \to sL \) such that \( l^x_A \circ \varphi = p^x_A \) for every \( A \in A \) and \( x \in G(A) \). So
\[
\lambda_P(\varphi)_A(x) = l^x_A \circ \varphi = p^x_A = \mu_A(x)
\]
for every \( A \in A \) and \( x \in G(A) \). Hence \( \lambda_P(\varphi) = \mu \) and \( \lambda_P \) is surjective.
7. \( \lambda_P \) is natural in \( P \) by (1), because
\[
\lambda_P(\psi \circ p)_A(x) = l^x_A \circ (\psi \circ p) = (l^x_A \circ \psi) \circ p = \lambda_Q(\psi)_A(x) \circ p
\]
for every \( \psi \in s\text{Pos}(Q, L) \), \( p \in s\text{Pos}(P, Q) \), \( A \in A \) and \( x \in G(A) \).
Now, if \( \left(sL, (l^x_A)_{A \in A}\right) \) is a Pos-lim of \( F \) weighted by \( G \), if we define mappings \( l^x_A \) by (2) and a natural transformation \( \lambda' \) by \( \lambda'_P(\varphi)_A(x) := l^x_A \circ \varphi \), \( sP \in s\text{Pos}, \varphi \in s\text{Pos}(P, L), A \in A, x \in G(A) \), then by (1)
\[
\lambda'_P(\varphi)_A(x) = l^x_A \circ \varphi = \lambda_L(1_L)_A(x) \circ \varphi = \lambda_P(1_L \circ \varphi)_A(x) = \lambda_P(\varphi)_A(x),
\]
Theorem 2

The pair 

\[(sL, (l^x_A)_{A \in \mathcal{A}})\]

satisfies conditions 1–3, we define a natural transformation \(\lambda\) by (3) and thereafter mappings \(k^x_A\) by \(k^x_A := \lambda L(1_L)_A(x), A \in \mathcal{A}, x \in G(A)\), then

\[k^x_A = \lambda L(1_L)_A(x) = l^x_A \circ 1_L = l^x_A.\]

Hence the correspondence is indeed one-to-one. ■

Remark 2 We always can assume that \(\varphi\) in condition 3 of Theorem 1 is unique. Indeed, if also \(\psi : sP \to sL\) is such that \(l^x_A \circ \psi = p^x_A\) for every \(A \in \mathcal{A}\) and \(x \in G(A)\), then \(l^x_A \circ \psi \leq l^x_A \circ \varphi\) and \(l^x_A \circ \varphi \leq l^x_A \circ \psi\) for every \(A \in \mathcal{A}\) and \(x \in G(A)\), which by condition 2 of Theorem 1 implies \(\varphi = \psi\).

Remark 3 Having Theorem 1 in mind, we shall also call the pairs \((sL, (l^x_A)_{A \in \mathcal{A}})\), satisfying conditions 1–3 of Theorem 1, limits of \(F\) weighted by \(G\) and \(l^x_A\) their projections.

2.3 Canonical construction of weighted limits in \(s\mathcal{P}\mathcal{O}\mathcal{s}\)

We shall show that weighted limits always exist in the category \(s\mathcal{P}\mathcal{O}\mathcal{s}\) and give a canonical construction for such limits.

It is easy to see that the poset \(\mathcal{N}\mathcal{a}\mathcal{t}(G, U \circ F)\), where \(U : s\mathcal{P}\mathcal{O}\mathcal{s} \to \mathcal{P}\mathcal{O}\) is the forgetful functor, is an \(S\)-poset if the left \(S\)-action is given by

\[s \cdot f := (s \cdot f)_A : A \in \mathcal{A},\]

where \(s \in S, f = (f_A)_{A \in \mathcal{A}} \in \mathcal{N}\mathcal{a}\mathcal{t}(G, U \circ F)\), and the mapping \(s \cdot f_A : G(A) \to F(A)\) is defined by

\[(s \cdot f_A)(x) := s \cdot f_A(x),\]

\(x \in G(A)\). For every \(A \in \mathcal{A}\) and \(x \in G(A)\) we define a mapping \(l^x_A : \mathcal{N}\mathcal{a}\mathcal{t}(G, U \circ F) \to F(A)\) by

\[l^x_A(f) := f_A(x),\]

\(f = (f_A)_{A \in \mathcal{A}} \in \mathcal{N}\mathcal{a}\mathcal{t}(G, U \circ F)\).

Theorem 2 The pair \((\mathcal{N}\mathcal{a}\mathcal{t}(G, U \circ F), (l^x_A)_{A \in \mathcal{A}})\) is a \(\mathcal{P}\mathcal{O}\mathcal{s}\)-limit of \(F\) weighted by \(G\).

Proof. Since

\[l^x_A(s \cdot f) = l^x_A((s \cdot f)_A) = (s \cdot f_A)(x) = s \cdot f_A(x) = s \cdot l^x_A(f)\]

for every \(A \in \mathcal{A}, x \in G(A)\), \(f = (f_A)_{A \in \mathcal{A}} \in L, s \in S,\) and since \(l^x_A\) are obviously order preserving, they are left \(S\)-poset morphisms. We shall show that they satisfy the conditions of Theorem 1.

1(a). If \(x \leq x', x, x' \in G(A)\), then \(f_A(x) \leq f_A(x')\) for every \(f \in \mathcal{N}\mathcal{a}\mathcal{t}(G, U \circ F)\). Hence \(l^x_A \leq l^{x'}_A\).

1(b). For every \(a : A_0 \to A_1\) in \(\mathcal{A}\), \(x \in G(A_0)\) and \(f \in \mathcal{N}\mathcal{a}\mathcal{t}(G, U \circ F)\),

\[(F(a) \circ l^x_{A_0})(f) = F(a)(f_{A_0}(x)) = f_{A_1}(G(a)(x)) = l^{G(a)(x)}_{A_1}(f).\]
2. Suppose that $sP \in s\text{Pos}$, $\varphi, \psi \in s\text{Pos}(P, \text{Nat}(G, U \circ F))$ are such that $l^x_A \circ \varphi \leq l^x_A \circ \psi$ for every $A \in \mathcal{A}$ and $x \in G(A)$. Then $\varphi(z)_A(x) = l^x_A(\varphi(z)) \leq l^x_A(\psi(z)) = \psi(z)_A(x)$ in $F(A)$ for every $A \in \mathcal{A}$, $x \in G(A)$ and $z \in P$. Since the order in $\text{Pos}(G(A), F(A))$ is pointwise and the order in $\text{Nat}(G, U \circ F)$ is componentwise, $\varphi(z) = (\varphi(z))_{A \in \mathcal{A}} \leq (\psi(z))_{A \in \mathcal{A}} = \psi(z)$ for every $z \in P$, and thus $\varphi \leq \psi$.

3. Let $sP \in s\text{Pos}$ and let $(p^x_A)_{A \in \mathcal{A}}$ be a family of left $S$-poset morphisms $p^x_A : sP \to sF(A)$ such that (a) $x \leq x'$ implies $p^x_A \leq p^{x'}_A$ for all $A \in \mathcal{A}$, $x, x' \in G(A)$, and (b) $F(a) \circ p^x_{A_0} = p^{G(a)(x)}_{A_1}$ for all $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_0)$. We define a mapping $\varphi : P \to \text{Nat}(G, U \circ F)$ by

$$\varphi(z)_A(x) := p^x_A(z),$$

$A \in \mathcal{A}$, $x \in G(A)$, $z \in P$. By (a), $\varphi(z)_A : G(A) \to F(A)$ is order preserving. By (b),

$$(F(a) \circ \varphi(z)_{A_0})(x) = F(a)(p^x_{A_0}(z)) = p^{G(a)(x)}_{A_1}(z) = (\varphi(z))_{A_1}(G(a)(x)) = (\varphi(z))_{A_0} \circ G(a)(x)$$

for every $a : A_0 \to A_1$ in $\mathcal{A}$, $x \in G(A_0)$ and $z \in P$. Hence $\varphi(z) \in L$. Further, $\varphi$ is order preserving, because all mappings $p^x_A$ are. Also

$$\varphi(s \cdot z)_A(x) = p^x_A(s \cdot z) = s \cdot p^x_A(z) = s \cdot \varphi(z)_A(x) = (s \cdot \varphi(z))_A(x)$$

for every $A \in \mathcal{A}$, $x \in G(A)$, $z \in P$ and $s \in S$, which implies $\varphi(s \cdot z) = s \cdot \varphi(z)$, and hence $\varphi$ is a left $S$-poset morphism. Finally,

$$(l^x_A \circ \varphi)(z) = l^x_A(\varphi(z)) = \varphi(z)_A(x) = p^x_A(z)$$

for every $A \in \mathcal{A}$, $x \in G(A)$, $z \in P$, and hence $l^x_A \circ \varphi = p^x_A$. □

**Remark** 4 That weighted limits can be constructed as in Theorem 2 may also follow from (3.2) or (2.1) of [6], but we have preferred to give a direct proof here.

### 2.4 Another existence theorem for weighted limits

Here we show that condition 2 in Theorem 1 is actually redundant.

**Theorem** 3 A pair $\left( S L, (l^x_A)_{A \in \mathcal{A}} \right)$, where $l^x_A : sL \to sF(A)$ are left $S$-poset morphisms, is a limit of $F$ weighted by $G$ if and only if

1. (a) for all $A \in \mathcal{A}$ and $x, x' \in G(A)$

$$x \leq x' \implies l^x_A \leq l^{x'}_A;$$
If \( \exists \) conical limits
3 Some special weighted limits

Remark 6 If I correctly understand a remark on p. 306 of [6] then the redundance of a pair

\[ \psi \] which follows immediately from Theorem 1 and Remark 2.

Proof. Necessity follows immediately from Theorem 1 and Remark 2.

Sufficiency. Suppose that \( M \) with \( I \), \( A \in A \), \( x \in G(A) \), satisfies conditions 1 and 2. Then there exists a unique morphism \( \mu : M \to L \) such that \( I \circ \mu = m_A \) for every \( A \in A \) and \( x \in G(A) \), and a unique morphism \( \nu : L \to M \) such that \( m_A \circ \nu = l_A \) for every \( A \in A \) and \( x \in G(A) \). Hence \( I \circ \mu \circ \nu = l_A \circ \psi \) for every \( A \in A \) and \( x \in G(A) \), which implies \( \mu \circ \nu = 1_L \) by the uniqueness of the comparison morphism \( 1_L : L \to L \).

(b) for all \( a : A_0 \to A_1 \) in \( A \) and \( x \in G(A_0) \),

\[ F(a) \circ I^{G(A)}_{A_0} = I^{G(A)}_{A_1}; \]

2. for every \( S \in \mathcal{S} \) and family \((p_A)_{A \in A}\) of left \( S \)-poset morphisms \( p_A : S \to SF(A) \) with property 1, there is a unique left \( S \)-poset morphism \( \varphi : S \to SL \) such that \( I^A \circ \varphi = p_A \) for every \( A \in A \), \( x \in G(A) \).

Thus conical limits are just the ordinary limits, e.g. products, equalizers, pullbacks.
3.2 Inserters

Consider parallel morphisms \( r, q : sR \rightrightarrows sQ \) in \( s\text{Pos} \). Let the category \( \mathcal{A} \) and its images under \( F \) and \( G \) be

\[
A_0 \xrightarrow{a} A_1 \quad sR \xrightarrow{r \atop q} sQ \quad 1 \xrightarrow{c_1 \atop c_0} 2
\]

where \( a, a' \) are incomparable and \( c_1, c_0 \) map \( * \) to 1 and 0, respectively. Then the limit of \( F \) weighted by \( G \) is called the inserter of \( q \) and \( r \) (see [6], p. 307) and it can be constructed as

\[
\text{Nat}(G, U \circ F) = \{ (f_{A_0}, f_{A_1}) \mid f_{A_0} : 1 \to R, f_{A_1} : 2 \to Q, f_{A_1}(0) \leq f_{A_1}(1), r \circ f_{A_0} = f_{A_1} \circ c_1, q \circ f_{A_0} = f_{A_1} \circ c_0 \}
\]

\[
= \{ (f_{A_0}, f_{A_1}) \mid f_{A_0} : 1 \to R, f_{A_1} : 2 \to Q, f_{A_1}(0) \leq f_{A_1}(1), r(f_{A_0}(*)) = f_{A_1}(1), q(f_{A_0}(*)) = f_{A_1}(0) \}
\]

\[
\cong \{ f_{A_0} \mid f_{A_0} : 1 \to R, q \circ f_{A_0} \leq r \circ f_{A_0} \}
\]

\[
\cong \{ z \in R \mid q(z) \leq r(z) \} =: \text{Ins}(q, r),
\]

where the order and \( S \)-action of \( \text{Ins}(q, r) \) are inherited from \( sR \), and there is an isomorphism

\[
\alpha : s\text{Nat}(G, U \circ F) \to s\text{Ins}(q, r), \quad (f_{A_0}, f_{A_1}) \mapsto f_{A_0}(*)
\]

in \( s\text{Pos} \).

**Lemma 1** There is one-to-one correspondence between inserters of \( q \) and \( r \) and pairs \((sE, e)\), where \( sE \) is a left \( S \)-poset and \( e : sE \to sR \) a morphism such that

1. \( q \circ e \leq r \circ e \),

2. if \( e' : sE' \to sR \) is such that \( q \circ e' \leq r \circ e' \) then there exists unique \( \varphi : sE' \to sE \) in \( s\text{Pos} \) such that \( e \circ \varphi = e' \).

**Proof.** Assume that the pair \((sL, (l_{A_0}^*, l_{A_1}^*, l_{A_1}^{(0)}))\) satisfies conditions 1 and 2 of Theorem 3. We write \((sE, e) = (sL, l_{A_0}^*) = \alpha (sL, (l_{A_0}^*, l_{A_1}^*, l_{A_1}^{(0)})) \). Then

\[
q \circ e = F(a) \circ l_{A_0}^* = l_{A_1}^{G(a)(*)} = l_{A_1}^{(0)} \leq l_{A_1}^* = l_{A_1}^{G(a)(*)} = F(a) \circ l_{A_0}^* = r \circ e.
\]

To prove 2, let \( e' : sE' \to sR \) be such that \( q \circ e' \leq r \circ e' \). Then for \( p_{A_0}^* = e', p_{A_1}^* = q \circ e' \) and \( p_{A_1}^* = r \circ e' \) we have \( p_{A_1}^* \leq p_{A_1}^* \), \( F(a') \circ p_{A_0}^* = q \circ e' = p_{A_1}^* = p_{A_1}^{G(a')(*)} \), and, similarly, \( F(a) \circ p_{A_0}^* = p_{A_1}^{G(a)(*)} \). By the assumption, there is a unique morphism \( \varphi : sE' \to sE \) such that \( e' = e \circ \varphi \).

Conversely, if a pair \((sE, e)\) satisfies 1 and 2, we consider the pair \((sE', (e, r \circ e, q \circ e)) = \beta(sE, e) \). It is easy to see that conditions 1 and 2 of Theorem 3 are satisfied.

Finally,

\[
\beta \left( \alpha \left( sL, (l_{A_0}^*, l_{A_1}^*, l_{A_1}^{(0)}) \right) \right) = \beta \left( sL, (l_{A_0}^*, q \circ l_{A_0}^* r \circ l_{A_0}^*) \right) = \left( sL, (l_{A_0}^*, l_{A_1}^*, l_{A_1}^{(0)}) \right)
\]

for every inserter \((sL, (l_{A_0}^*, l_{A_1}^*, l_{A_1}^{(0)}))\) of \( q \) and \( r \) and

\[
\alpha(\beta(sE, e)) = \alpha(sE, (e, r \circ e, q \circ e)) = (sE, e)
\]

for every pair \((sE, e)\) that satisfies 1 and 2.

**Remark 7** It is easy to check that the pair \((\text{Ins}(q, r), \iota)\), where \( \iota : \text{Ins}(q, r) \to R \) is the inclusion, satisfies conditions 1 and 2 of Lemma 1. We call \((\text{Ins}(q, r), \iota)\) the canonical inserter of \( q \) and \( r \).
3.3 Equifiers

Consider parallel morphisms \( r, q : sR \rightarrow sQ \) with \( q \leq r \) in \( s\text{Pos} \). Let the category \( A \) and its images under \( F \) and \( G \) be

\[
\begin{array}{cccccc}
A_0 & \xrightarrow{a} & A_1 & \xleftarrow{a'} & A' & \xrightarrow{r} & sR & \xrightarrow{q} & sQ & \xleftarrow{c_1} & 1 & \xleftarrow{c_0} & 2
\end{array}
\]

where \( a' \leq a \) and \( c_1, c_0 \) map \( * \) to 1 and 0, respectively. Then the limit of \( F \) weighted by \( G \) is called the **equifier** of \( q \) and \( r \) (see [6], p. 309) and it can be constructed as

\[
\text{Nat}(G, U \circ F) = \{(f_{A_0}, f_{A_1}) \mid f_{A_0} : 1 \rightarrow R, f_{A_1} : 2 \rightarrow Q, f_{A_1}(0) \leq f_{A_1}(1), r \circ f_{A_0} = f_{A_1} \circ c_1, q \circ f_{A_0} = F_{A_0} \circ c_0 \}
\]

\[
= \{(f_{A_0}, f_{A_1}) \mid f_{A_0} : 1 \rightarrow R, f_{A_1} : 2 \rightarrow Q, f_{A_1}(0) \leq f_{A_1}(1), r(f_{A_0}(*)) = f_{A_1}(1), q(f_{A_0}(*)) = f_{A_1}(0) \}
\]

\[
\approx \{f_{A_0} : 1 \rightarrow R, q \circ f_{A_0} \leq r \circ f_{A_0} \}
\]

\[
\approx \{z \in R \mid q(z) \leq r(z) \} = R.
\]

So the equifier of \( (q, r) \) with \( q \leq r \) is just the pair \( (R, 1_R) \) and the universal property is trivially satisfied. Clearly every pofunctor preserves equifiers.

3.4 Comma objects

Consider morphisms \( r : sR \rightarrow sQ \) and \( r' : sR' \rightarrow sQ \) in \( s\text{Pos} \). If the category \( A \) and its images under \( F \) and \( G \) are

\[
\begin{array}{cccccc}
A & \xrightarrow{a} & A_1 & \xleftarrow{a'} & A' & \xrightarrow{r} & sR & \xrightarrow{r'} & sQ & \xleftarrow{1} & 1 & \xleftarrow{1} & 1
\end{array}
\]

then the limit of \( F \) weighted by \( G \) is called the **comma-object** of \( r' \) and \( r \) (see [6], p. 308). Analogously to Lemma 1 one can prove the following result.

**Lemma 2** There is one-to-one correspondence between comma-objects of \( r' \) and \( r \) and triples \( (\text{Co}(r', r), z', z) \), where \( z : \text{Co}(r', r) \rightarrow R, z' : \text{Co}(r', r) \rightarrow R' \) are such that

1. \( r \circ z \leq r' \circ z' \);

2. if \( w : W \rightarrow R \) and \( w' : W \rightarrow R' \) in \( s\text{Pos} \) are such that \( r \circ w \leq r' \circ w' \) then there exists a unique morphism \( \varphi : W \rightarrow \text{Co}(r', r) \) in \( s\text{Pos} \) such that \( z \circ \varphi = w \) and \( z' \circ \varphi = w' \).

Canonically, one can take

\[
\text{Co}(r', r) := \{(x', x) \in R' \times R \mid r'(x') \leq r(x) \}
\]

and \( z', z \) the restrictions of the projections of \( R' \times R \).

Note that inserters and comma objects in \( s\text{Pos} \) were termed sub-equalizers and sub-pullbacks, respectively, in [3].
3.5 Lax limit and op-lax limit of a morphism

Consider a morphism \( r : sR \rightarrow sQ \) in \( s\text{Pos} \). Let the category \( \mathcal{A} \) and its images under \( F \) and \( G \) be

\[
\begin{array}{ccc}
A_0 & \xrightarrow{a} & A_1 \\
\downarrow{sR} & & \downarrow{sQ} \\
1 & \xrightarrow{c_0} & 2.
\end{array}
\]

Then the limit of \( G \) and \( F \) is called the **lax limit** of the morphism \( r \) (replacing \( c_0 \) by \( c_1 \) we obtain the **op-lax limit** of the morphism \( r \)); see [6], p. 308 and it can be canonically constructed as

\[
\text{Nat}(G, U \circ F) = \left\{ (f_{A_0}, f_{A_1}) \mid f_{A_0} : 1 \rightarrow R, f_{A_1} : 2 \rightarrow Q, f_{A_1}(0) \leq f_{A_1}(1), r \circ f_{A_0} = f_{A_1} \circ c_0 \right\}
\]

\[
= \left\{ (f_{A_0}, f_{A_1}) \mid f_{A_0} : 1 \rightarrow R, f_{A_1} : 2 \rightarrow Q, f_{A_1}(0) \leq f_{A_1}(1), r(f_{A_0}(*)) = f_{A_1}(0) \right\}
\]

\[
\cong \left\{ (x, y) \in R \times Q \mid r(x) \leq y \right\} =: \text{Lax}(r),
\]

where the order and left \( S \)-action on \( \text{Lax}(r) \) are componentwise. In more detail, if \( (f_{A_0}, f_{A_1}) \in \text{Nat}(G, U \circ F) \) then \( r(f_{A_0}(*)) \leq f_{A_1}(1) \), and hence we may define a mapping \( \alpha : \text{Nat}(G, U \circ F) \rightarrow \text{Lax}(r) \) by

\[
\alpha(f_{A_0}, f_{A_1}) := (f_{A_0}(*), f_{A_1}(1)).
\]

Obviously, \( \alpha \) is order preserving and, for every \( s \in S \),

\[
\alpha(s \cdot (f_{A_0}, f_{A_1})) = \alpha(s \cdot f_{A_0}, s \cdot f_{A_1}) = ((s \cdot f_{A_0})(*), (s \cdot f_{A_1})(1)) = (s \cdot f_{A_0}(*), f_{A_1}(1)) = s \cdot \alpha(f_{A_0}, f_{A_1}).
\]

Suppose that also \( (g_{A_0}, g_{A_1}) \in \text{Nat}(G, U \circ F) \) and \( (f_{A_0}(*), f_{A_1}(1)) \leq (g_{A_0}(*), g_{A_1}(1)) \). Then \( f_{A_0}(*), f_{A_1}(1) \leq g_{A_1}(1) \), and \( f_{A_1}(0) = r(f_{A_0}(*)) \leq r(g_{A_0}(*)) = g_{A_1}(0) \). Hence \( (f_{A_0}, f_{A_1}) \leq (g_{A_0}, g_{A_1}) \), and \( \alpha \) is order reflecting. Finally, if \( (x, y) \in R \times Q \) and \( r(x) \leq y \) then defining \( f_{A_0}(*):=x, f_{A_1}(1):=y \) and \( f_{A_1}(0):=r(x) \) we have \( (f_{A_0}, f_{A_1}) \in \text{Nat}(G, U \circ F) \) and \( \alpha(f_{A_0}, f_{A_1}) = (x, y) \). Thus we have proved that \( \alpha \) is an isomorphism. Consequently, the pair \( (\text{Lax}(r), (l_{A_0}^r \circ \alpha^{-1}, l_{A_1}^r \circ \alpha^{-1}, l_{A_1}^1 \circ \alpha^{-1})) \) is a lax limit of \( r \).

**Lemma 3** There is one-to-one correspondence between lax limits of a morphism \( r : sR \rightarrow sQ \) and pairs \( (L, (l_R, l_Q)) \) with \( l_R : sL \rightarrow sR, l_Q : sL \rightarrow sQ \) such that

1. \( r \circ l_R \leq l_Q \);

2. if \( l_R' : sL' \rightarrow sR \) and \( l_Q' : sL' \rightarrow sQ \) are such that \( r \circ l_R' \leq l_Q' \) then there exists a unique morphism \( \varphi : sL' \rightarrow sL \) such that \( l_R \circ \varphi = l_R' \) and \( l_Q \circ \varphi = l_Q' \).

**Proof.** Let \( (sL, (l_{A_0}^r, l_{A_1}^r, l_{A_1}^1)) \) be a lax limit of a morphism \( r : sR \rightarrow sQ \), that is, it satisfies conditions 1 and 2 of Theorem 3. We write \( (L, (l_R, l_Q)) = (sL, (l_{A_0}^r, l_{A_1}^r)) = \alpha(sL, (l_{A_0}^r, l_{A_1}^r, l_{A_1}^1)) \).
Then
\[ r \circ l_R = F(a) \circ l_{A_0}^* = l_{A_1}^{0^*} = l_{A_1} = l_Q. \]

Suppose that \( l'_{R} : sL' \to sR \) and \( l'_{Q} : sL' \to sQ \) are such that \( r \circ l'_{R} \leq l'_{Q} \). Taking \( p_{A_0}^* := l'_{R}, p_{A_1}^0 := r \circ l'_{R} \) and \( p_{A_1}^1 := l'_{Q} \) we see that the pair \( \left( sL', (p_{A_0}^*, p_{A_1}^0, p_{A_1}^1) \right) \) satisfies condition 1 of Theorem 3. Hence there exists a left S-poset morphism \( \varphi : sL' \to sL \) such that \( l_R \circ \varphi = l_{A_0}^* \circ \varphi = p_{A_0}^* = l'_{R} \) and \( l_Q \circ \varphi = l_{A_1}^* \circ \varphi = p_{A_1}^1 = l'_{Q} \). If \( \psi : sL' \to sL \) is another morphism such that \( l_R \circ \psi = l'_{R} \) and \( l_Q \circ \psi = l'_{Q} \) then \( l_{A_1}^* \circ \psi = r \circ l_R \circ \psi = r \circ l_R = p_{A_1}^0 \) and hence \( \varphi = \psi \) by the uniqueness of \( \varphi \) in condition 2 of Theorem 3.

Conversely, if a pair \( (sL, (l_{R}, l_{R}, l_Q)) \) satisfies 1 and 2, we consider the pair \( (sL, (l_{R}, r \circ l_{R}, l_Q)) = \beta(sL, (l_{R}, l_Q)) \). It is easy to see that \( (sL, (l_{R}, r \circ l_{R}, l_Q)) \) satisfies conditions 1 and 2 of Theorem 3 and hence is a lax limit of \( r \).

Now,
\[ \beta \left( (sL, (l_{A_0}^*, l_{A_1}^0, l_{A_1}^1)) \right) = \beta \left( (sL, (l_{A_0}^0, l_{A_1}^1)) \right) = (sL, (l_{A_0}, l_{A_1}^1)) = (sL, (l_{A_0}^0, l_{A_1}^1)) \]
for every lax limit \( (sL, (l_{A_0}^*, l_{A_1}^0, l_{A_1}^1)) \) of \( r \), and
\[ \alpha(\beta(sL, (l_{R}, l_Q))) = \alpha(sL, (l_{R}, r \circ l_{R}, l_Q)) = (sL, (l_{R}, l_Q)) \]
for every pair \( (sL, (l_{R}, l_Q)) \) that satisfies conditions 1 and 2.

Having Lemma 3 in mind, we shall call the pairs \( (sL, (l_{R}, l_Q)) \) satisfying conditions 1 and 2 of that lemma the lax limits of \( r \). In particular, we say that the canonical lax limit of \( r \) is the pair \( (\text{Lax}(r), (p_{R}, p_Q)) \), where \( p_{R} := l_{A_0}^* \circ \alpha^{-1} : \text{Lax}(r) \to R \) and \( p_{Q} := l_{A_1}^1 \circ \alpha^{-1} : \text{Lax}(r) \to Q \) are given by
\[ p_{R}(x, y) = l_{A_0}^*(\alpha^{-1}(x, y)) = \alpha^{-1}(x, y)_{A_0}(\star) = x, \]
\[ p_{Q}(x, y) = l_{A_1}(\alpha^{-1}(x, y)) = \alpha^{-1}(x, y)_{A_1}(1) = y, \]
\((x, y) \in \text{Lax}(r)).\]

One can check that a canonical op-lax limit of a morphism \( r : sR \to sQ \) in \( \text{SPos} \) can be constructed as a pair \( (\text{Oplax}(r), (p_{R}, p_{Q})) \), where
\[ \text{Oplax}(r) = \{(x, y) \in R \times Q \mid y \leq r(x)\}, \]
\[ p_{R}(x, y) = x, p_{Q}(x, y) = y \]
for all \((x, y) \in \text{Oplax}(r)) \). Op-lax limits of morphisms together with pullbacks give a possibility to define downwards closed S-subposets of an S-poset in categorical terms.

### 3.6 Cotensor products

If \( \mathcal{A} \) is the the discrete category with a single object \( \star \) then \( F \) and \( G \) can be identified with objects \( F(\star) \) and \( G(\star) \) of \( \text{SPos} \) and of \( \text{Pos} \), respectively. By Theorem 3, \( (sL, (l^x_{sS} \star \in G(\star))) \), where \( l^x : sL \to sF(\star) \), is a limit of \( F \) weighted by \( G \) if and only if

1. for all \( x, x' \in G(\star) \),
\[ x \leq x' \implies l^x \leq l^{x'}; \]
2. for every \( sP \in \text{SPos} \) and family \( (p^x)_{\star \in G(\star)} \) of left \( S \)-poset morphisms \( p^x : sP \to sF(\star) \) with property 1, there is a unique left \( S \)-poset morphism \( \varphi : sP \to sL \) such that \( l^x \circ \varphi = p^x \) for every \( x \in G(\star) \).
Such weighted limit is called a cotensor product of $F$ and $G$ (or of $F(\star)$ and $G(\star)$; see [6], p. 305). By Theorem 2, one such cotensor product is $(s\text{Pos}(G(\star), F(\star)), (lx)_{x \in G(\star)})$, where $lx : s\text{Pos}(G(\star), F(\star)) \to F(\star)$ is the evaluation map at $x \in G(\star)$, i.e. $lx(f) = f(x)$ for every $f \in s\text{Pos}(G(\star), F(\star))$.

### 3.7 Pie limits

For a functor $G : D \to \text{Pos}$ we can consider its category of elements (or Grothendieck category). The objects of this category $\text{el}(G)$ are pairs $(x, i)$, where $i \in I = D_0$ and $x \in G(i)$. A morphism $(x, i) \to (y, j)$ is a morphism $d \in D(i, j)$ such that $G(d)(x) = y$.

**Definition 2 ([7])** A pofunctor $G : D \to \text{Pos}$ is called a pie weight if each component of $\text{el}(G)$ has an initial object.

Since equifiers in $s\text{Pos}$ are trivial, from Proposition 2.1 of [7] we have the following corollary, which we present with a proof.

**Proposition 1** If $G : D \to \text{Pos}$ is a pie weight and $F : D \to s\text{Pos}$ is a pofunctor then $\lim_G F$ can be constructed using products and inserters.

**Proof.** Let $U$ be the set of connected components of $\text{el}(G)$. For every connected component $u \in U$, let $(z_u, j_u)$ be the initial object of $u$. If $(x, i) \in \text{el}(G)_0$, then we write $(x, i) \in U$ for the connected component of $(x, i)$ and $!_{(x, i)}$ for the unique morphism $j_{(x, i)} \to i$ such that $G(!_{(x, i)})(z_{(x, i)}) = x$. Take

$$S := \{(x, y, i) \mid i \in I, x, y \in G(i), x \leq y\}$$

and consider products

$$\left(\prod_{u \in U} F(j_u), (\pi_u)_{u \in U}\right) \quad \text{and} \quad \left(\prod_{(x, y, i) \in S} F(i), (p(x, y, i))_{(x, y, i) \in S}\right).$$

Then there exist unique morphisms $f_0, f_1 : \prod_{u \in U} F(j_u) \to \prod_{(x, y, i) \in S} F(i)$ such that

$$p(x, y, i) \circ f_0 = \pi_{(x, i)} \circ F(!_{(x, i)}) \quad \text{and} \quad p(x, y, i) \circ f_1 = \pi_{(y, i)} \circ F(!_{(y, i)})$$

for every $(x, y, i) \in S$. Let $(E, e)$ be the inserter of $(f_0, f_1)$. In particular, $f_0 \circ e \leq f_1 \circ e$. We claim that

$$\left(E, (l_x)_{x \in G(i)}\right) \approx \lim_G F$$

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where \( l^x_i := F(l^x_{(x,i)}) \circ \pi_{(x,i)} \circ e : E \to F(i) \). If \( d : i_0 \to i_1 \) in \( D \) and \( x \in G(i_0) \) then \( d : (x, i_0) \to (G(d)(x), i_1) \) in \( el(G) \) and \( (x, i_0) = (G(d)(x), i_1) \). Hence \( !_{(G(d)(x), i_1)} = d \circ !_{(x, i_0)} \) and

\[
\begin{align*}
l^{G(d)(x)}_{i_1} &= F \left( l^{(G(d)(x), i_1)} \right) \circ \pi_{(G(d)(x), i_1)} \circ e = F(d) \circ F \left( l^{(x, i_0)} \right) \circ \pi_{(x, i_0)} \circ e \\
&= F(d) \circ l^x_i.
\end{align*}
\]

If \( x, y \in G(i) \) are such that \( x \leq y \) then

\[
l^x_i = F(l^{(x,i)}) \circ \pi_{(x,i)} \circ e = p_{(x,y,i)} \circ f_0 \circ e \leq p_{(x,y,i)} \circ f_1 \circ e = F(l^{(y,i)}) \circ \pi_{(y,i)} \circ e = l^y_i.
\]

To verify the universal property, let \( \left( P, (p^x_i)_{i \in I} \right) \) be such that \( F(d) \circ p^x_{i_0} = p^{G(d)(x)}_{i_1} \) for every \( d : i_0 \to i_1 \) in \( D \) and \( p^x_i \leq p^y_i \) whenever \( x \leq y \) in \( G(i) \). Then there exists a unique morphism \( g : P \to \prod_{u \in U} F(j_u) \) such that \( \pi_u \circ g = p^z_u \) for every \( u \in U \). Now, for every \( (x, y, i) \in S \),

\[
p_{(x,y,i)} \circ f_0 \circ g = F(l^{(x,i)}) \circ \pi_{(x,i)} \circ g = F(l^{(y,i)}) \circ p_{(y,i)}^{z_u} = p^x_i
\]

\[
\leq p^y_i = F(l^{(y,i)}) \circ p_{(y,i)}^{z_u} = F(l^{(y,i)}) \circ \pi_{(y,i)} \circ g = p_{(x,y,i)} \circ f_1 \circ g.
\]

Since products are weighted limits, they satisfy condition 2 of Theorem 1, and hence \( f_0 \circ g \leq f_1 \circ g \). Consequently, there exists a unique morphism \( \varphi : P \to E \) such that \( e \circ \varphi = g \). Then

\[
l^x_i \circ \varphi = F(l^{(x,i)}) \circ \pi_{(x,i)} \circ e \circ \varphi = F(l^{(x,i)}) \circ \pi_{(x,i)} \circ g = F(l^{(y,i)}) \circ p_{(x,i)}^{z_u} = p^x_i.
\]

Finally, suppose that \( \psi : P \to E \) is such that \( l^x_i \circ \psi = p^x_i \) for each \( x \in G(i), i \in I \). Note that \( (z_u, j_u) = u \) and \( l^{(z_u,j_u)}_{(z_u,j_u)} = 1_{j_u} \). Hence \( l^{z_u} = F(1_{j_u}) \circ \pi_u \circ e = \pi_u \circ e \) for every \( u \in U \). Now \( l^x_i \circ \varphi = l^x_i \circ \psi \) implies

\[
\pi_u \circ e \circ \varphi = l^{z_u}_{j_u} \circ \varphi \leq l^{z_u}_{j_u} \circ \psi = \pi_u \circ e \circ \psi
\]

for every \( u \in U \). Applying again condition 2 of Theorem 1, first for product and then for inserter, we obtain \( \varphi \leq \psi \). Symmetrically we get \( \psi \leq \varphi \), and thus \( \varphi = \psi \). ■
4 Weighted colimits in $\mathbb{S}\text{Pos}$

4.1 Definition

Definition 3 (Cf. Def. 6.6.4 of [2]) Given a pomonoid $S$, small Pos-category $\mathcal{A}$, and pofunctors $F : \mathcal{A} \to \mathbb{S}\text{Pos}$, $G : \mathcal{A}^{\text{op}} \to \text{Pos}$ (covariant and contravariant on $\mathcal{A}$, respectively), the Pos-colimit of $F$ weighted by $G$ is a pair $(sL, (\lambda_P)_{P \in \mathbb{S}\text{Pos}})$ where $sL$ is a left $S$-poset and $\lambda = (\lambda_P)_{P \in \mathbb{S}\text{Pos}} : \mathbb{S}\text{Pos}(L, -) \Rightarrow \text{Nat}(G, \mathbb{S}\text{Pos}(F(-), -))$ is a natural isomorphism, that is, for every $sP \in \mathbb{S}\text{Pos}$,

$$\lambda_P : \mathbb{S}\text{Pos}(L, P) \to \text{Nat}(G, \mathbb{S}\text{Pos}(F(-), P)),$$

are poset isomorphisms that are natural in $sP$. We write $\text{colim}_G F$ for a Pos-colimit of $F$ weighted by $G$.

$$\begin{tikzcd}
\mathcal{A}^{\text{op}} \cong \mathcal{A} \arrow[r, F] \arrow[d, G, \text{Pos}] & \mathbb{S}\text{Pos} \\
\text{Pos} \arrow[u, \text{Pos}(\cdot, P)]
\end{tikzcd}$$

Dually to Theorem 1, one can prove the following result.

Theorem 4 There is one-to-one correspondence between Pos-colimits of $F$ weighted by $G$ and pairs $\left(sL, (l_A^x)_{A \in \mathcal{A}}, (l^x_A)_{x \in G(A)}\right)$, where $sL$ is a left $S$-poset and $(l^x_A)_{x \in G(A)}$ is a family of left $S$-poset morphisms $l^x_A : sF(A) \to sL$ such that

1. (a) for all $A \in \mathcal{A}$ and $x, x' \in G(A)$

$$x \leq x' \implies l^x_A \leq l^{x'}_A;$$

(b) for all $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_1)$,

$$l^x_{A_1} \circ F(a) = l^{G(a)(x)}_{A_0};$$

2. for all $sP \in \mathbb{S}\text{Pos}$ and $\varphi, \psi \in \mathbb{S}\text{Pos}(L, P)$,

$$((\forall A \in \mathcal{A})(\forall x \in G(A)) (\varphi \circ l^x_A \leq \psi \circ l^x_A)) \implies \varphi \leq \psi;$$

3. for every $sP \in \mathbb{S}\text{Pos}$ and family $(p^x_A)_{A \in \mathcal{A}}$ of left $S$-poset morphisms $p^x_A : sF(A) \to sP$ with properties 1, there is a left $S$-poset morphism $\varphi : sL \to sP$ such that $\varphi \circ l^x_A = p^x_A$ for every $A \in \mathcal{A}$ and $x \in G(A)$.
4.2 Canonical construction of weighted colimits in $\mathbf{S} \text{Pos}$

We shall show that the $\mathbf{Pos}$-category $\mathbf{S} \text{Pos}$ is $\mathbf{Pos}$-cocomplete by giving an explicit construction of a colimit $\text{colim}_G F$ of $F$ weighted by $G$.

We define a relation $\tau$ on the disjoint union $\bigsqcup_{A \in \mathcal{A}} G(A) \times F(A)$ by

$$(x_A, y_A) \tau (x_{A'}, y_{A'})$$

for some morphisms

$$A \xrightarrow{f_i} A'_i \xleftarrow{g_i} A_1 \xrightarrow{f_2} A'_2 \xleftarrow{g_2} A_2 \xrightarrow{f_3} \cdots A'_{n-1} \xrightarrow{f_n} A'_n \xleftarrow{g_n} A'$$

in $\mathcal{A}$ and elements $x_i \in G(A'_i)$, $i = 1, \ldots, n$, $y_j \in F(A_j)$, $j = 1, \ldots, n - 1$.

**Lemma 4** The relation $\tau$ is reflexive and transitive.

**Proof.** Reflexivity of $\tau$ follows from inequalities

$$x_A \leq G(1_A)(x_A) \quad G(1_A)(x_A) \leq x_A \quad F(1_A)(y_A) \leq F(1_A)(y_A).$$

To prove transitivity, suppose that $(x_A, y_A) \tau (x_{A'}, y_{A'})$ and $(x_{A'}, y_{A'}) \tau (x_{A''}, y_{A''})$, where $x_A \in G(A)$, $x_{A'} \in G(A')$, $x_{A''} \in G(A'')$, $y_A \in F(A)$, $y_{A'} \in F(A')$ and $y_{A''} \in F(A'')$. Then, in addition to inequalities (6), we have inequalities

$$x_{A'} \leq G(h_1)(z_1) \quad G(k_1)(z_1) \leq G(h_2)(z_2) \quad F(h_1)(y_{A'}) \leq F(k_1)(w_1) \quad G(k_2)(z_2) \leq G(h_3)(z_3) \quad F(h_2)(w_1) \leq F(k_2)(w_2) \quad \cdots \quad G(k_m)(z_m) \leq x_{A''} \quad F(h_m)(w_{m-1}) \leq F(k_m)(y_{A''})$$

for some morphisms

$$A' \xrightarrow{h_1} B'_1 \xleftarrow{k_1} B_1 \xrightarrow{h_2} B'_2 \xleftarrow{k_2} B_2 \xrightarrow{h_3} B'_3 \cdots B'_{m-1} \xleftarrow{k_m} B'_{m} \xrightarrow{k_m} A''$$

in $\mathcal{A}$. Hence we have inequalities

$$x_A \leq G(f_1)(x_1) \quad G(g_1)(x_1) \leq G(f_2)(x_2) \quad F(f_1)(y_A) \leq F(g_1)(y_1) \quad G(g_2)(x_2) \leq G(f_3)(x_3) \quad F(f_2)(y_1) \leq F(g_2)(y_2) \quad \cdots \quad G(g_n)(x_n) \leq G(h_1)(z_1) \quad F(f_n)(y_{n-1}) \leq F(g_n)(y_{A''}) \quad G(k_1)(z_1) \leq G(h_2)(z_2) \quad F(h_1)(y_{A'}) \leq F(k_1)(w_1) \quad G(k_2)(z_2) \leq G(h_3)(z_3) \quad F(h_2)(w_1) \leq F(k_2)(w_2) \quad \cdots \quad G(k_m)(z_m) \leq x_{A''} \quad F(h_m)(w_{m-1}) \leq F(k_m)(y_{A''}),$$

i.e. $(x_A, y_A) \tau (x_{A''}, y_{A''})$. ■
Lemma 5 Let $\tau$ be reflexive and transitive binary relation on a set $M$. Define a binary relation $\sigma$ on $M$ by

\[ m \sigma n \iff m \tau n \land n \tau m. \]

Then $\sigma$ is an equivalence relation, and by defining

\[ [m] \leq [n] \iff m \sigma n \]

we obtain a well-defined partial order on the quotient set $M/\sigma = \{[m] \mid m \in M\}$.

By Lemma 5, the relation $\sigma$, defined by

\[ (x_A, y_A) \sigma (x_{A'}, y_{A'}) \iff (x_A, y_A) \tau (x_{A'}, y_{A'}) \land (x_A, y_A) \tau (x_A, y_A) \]

is an equivalence relation on the set $\bigsqcup_{A \in A} G(A) \times F(A)$, and the definition

\[ [x_A, y_A] \leq [x_{A'}, y_{A'}] \iff (x_A, y_A) \tau (x_{A'}, y_{A'}) \]

gives a partial order on the quotient set

\[ L := \bigsqcup_{A \in A} G(A) \times F(A)/\sigma = \{[x_A, y_A] \mid A \in A, x_A \in G(A), y_A \in F(A)\}. \]

We define a left $S$-action on $L$ by

\[ s \cdot [x_A, y_A] := [x_A, s \cdot y_A]. \]

Lemma 6 This way, $L$ becomes a left $S$-poset.

Proof. Since $F(A)$ is a left $S$-act for every $A \in A$, so is $L$.

Suppose that $s \leq t$, $s, t \in S$, $x_A \in G(A)$, $y_A \in F(A)$, $A \in A$. Since $F(A)$ is a left $S$-poset, $s \cdot y_A \leq t \cdot y_A$. From

\[ x_A \leq G(1_A)(x_A) \]
\[ G(1_A)(x_A) \leq x_A \]
\[ F(1_A)(s \cdot y_A) \leq F(1_A)(t \cdot y_A) \]

we see that $(x_A, s \cdot y_A) \tau (x_A, t \cdot y_A)$, i.e. $[x_A, s \cdot y_A] \leq [x_A, t \cdot y_A]$.

Suppose that $[x_A, y_A] \leq [x_{A'}, y_{A'}]$ and $s \in S$. Then we have inequalities (6). Using that the elements in the right-hand column belong to left $S$-posets and all $F(f_i), F(g_i)$ are left $S$-poset morphisms, we obtain

\[ x_A \leq G(f_1)(x_1) \]
\[ G(g_1)(x_1) \leq G(f_2)(x_2) \]
\[ G(g_2)(x_2) \leq G(f_3)(x_3) \]
\[ \vdots \]
\[ G(g_n)(x_n) \leq x_{A'} \]
\[ F(f_n)(s \cdot y_{n-1}) \leq F(g_n)(s \cdot y_A). \]

Hence

\[ s \cdot [x_A, y_A] = [x_A, s \cdot y_A] \leq [x_{A'}, s \cdot y_{A'}] = s \cdot [x_{A'}, y_{A'}]. \]

(Note that the condition, we have just verified, implies that the $S$-action is well-defined.)

\[ \blacksquare \]
Lemma 7  The poset $L$ satisfies conditions

1. $(\forall x \in G(A))(\forall y, y' \in F(A))(y \leq y' \Rightarrow [x, y] \leq [x, y'])$,
2. $(\forall x, x' \in G(A))(\forall y \in F(A))(x \leq x' \Rightarrow [x, y] \leq [x', y])$,
3. $(\forall x \in G(A))(\forall y' \in F(A'))(\forall f : A' \to A \text{ in } A)([x, F(f)(y')] = [G(f)(x), y'])$.

**Proof.** The proof follows from the existence of the following inequalities:

\[
\begin{align*}
x & \leq G(1_A)(x) \\
G(1_A)(x) & \leq x \\
F(1_A)(y) & \leq F(1_A)(y'),
\end{align*}
\]

\[
\begin{align*}
x & \leq G(1_A)(x') \\
G(1_A)(x') & \leq x' \\
F(1_A)(y) & \leq F(1_A)(y),
\end{align*}
\]

\[
\begin{align*}
x & \leq G(1_A)(x) \\
G(f)(x) & \leq G(f)(x) \\
F(1_A)(F(f)(y)) & \leq F(f)(y'),
\end{align*}
\]

and

\[
\begin{align*}
x & \leq G(1_A)(x) \\
G(1_A)(x) & \leq x \\
F(f)(y') & \leq F(1_A)(F(f)(y')).
\end{align*}
\]

Theorem 5  The left $S$-poset $sL$ is a Pos-colimit of $F$ weighted by $G$.

**Proof.** We define a mapping $l^x_A : F(A) \to L$, $A \in \mathcal{A}$, $x \in G(A)$, by

\[l^x_A(y) := [x, y],\]

$y \in F(A)$. By Lemma 7(1), $l^x_A$ is order preserving. Since it obviously preserves $S$-action, it is a left $S$-poset morphism. We shall check that the pair $\left(\mathcal{sL}, \{l^x_A\}_{A \in \mathcal{A}}\right)$ satisfies conditions 1–3 of Theorem 4.

1(a) follows from Lemma 7(2).

1(b) For every $a : A_0 \to A_1$ in $\mathcal{A}$, $x \in G(A_1)$ and $y \in F(A_0)$ we have

\[(l^x_{A_1} \circ F(a))(y) = l^x_{A_1}(F(a)(y)) = [x, F(a)(y)] = [G(a)(x), y] = l^{G(a)(x)}_{A_0}(y)\]

by Lemma 7(3).

2. Suppose that $sP \in \text{sPos}$, $\varphi, \psi \in \text{sPos}(L, P)$ and $\varphi \circ l^x_A \leq \psi \circ l^x_A$ for all $A \in \mathcal{A}$ and $x \in G(A)$. Then, for every $A \in \mathcal{A}$, $x \in G(A)$ and $y \in F(A)$,

\[\varphi([x, y]) = (\varphi \circ l^x_A)(y) \leq (\psi \circ l^x_A)(y) = \psi([x, y]),\]

and hence $\varphi \leq \psi$.

3. Suppose that the morphisms $p^x_A : sF(A) \to sP$ satisfy condition 1. We define a mapping $\varphi : L \to P$ by

\[\varphi([x, y]) := p^x_A(y)\]
for every $A \in \mathcal{A}$, $x \in G(A)$ and $y \in F(A)$. Since $p^x_A$ are left $S$-act morphisms, so is $\varphi$. Suppose that $[x_A, y_A] \leq [x_{A'}, y_{A'}]$ in $L$, i.e. we have inequalities (6). Then

$$p^x_A(y_A) \leq p^x_{A'}(y_A) = (p^x_{A_1} \circ F(f_1)) (y_A) \leq \left( p^x_{A_1} \circ F(g_1) \right)(y_1) = p^{G(g_1)(x_1)}_{A_1}(y_1)$$

$$\leq p^{G(f_2)(x_2)}_{A_1}(y_1) \leq \ldots \leq p^{G(f_{n-1})(x_{n-1})}_{A_1}(y_{n-1}) = \left( p^x_{A_{n-1}} \circ F(f_{n-1}) \right)(y_{n-1})$$

$$\leq \left( p^x_{A_n} \circ F(g_n) \right)(y_{A'}) = p^{G(g_n)(x_n)}_{A'}(y_{A'}) \leq p^x_{A'}(y_{A'}).$$

This proves that $\varphi$ is well defined and order preserving. Finally, $(\varphi \circ l^x_A)(y) = \varphi([x, y]) = p^x_A(y)$ for every $A \in \mathcal{A}$, $x \in G(A)$ and $y \in F(A)$. $\blacksquare$

Dually to Theorem 3, one can prove the following result.

**Theorem 6** There is one-to-one correspondence between $\text{Pos}-\text{colimits}$ of $F$ weighted by $G$ and pairs $(sL, (l^x_A)_{A \in \mathcal{A}})$, where $sL$ is a left $S$-poset and $(l^x_A)_{A \in \mathcal{A}}$ is a family of left $S$-poset morphisms $l^x_A : sF(A) \to sL$ such that

1. (a) for all $A \in \mathcal{A}$ and $x, x' \in G(A)$
$$x \leq x' \implies l^x_A \leq l^{x'}_A;$$

(b) for all $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_1)$,
$$l^x_{A_1} \circ F(a) = l^{G(a)(x)}_{A_0};$$

2. for every $sP \in s\text{Pos}$ and family $(p^x_A)_{A \in \mathcal{A}}$ of left $S$-poset morphisms $p^x_A : sF(A) \to sP$ with properties 1, there is a unique left $S$-poset morphism $\varphi : sL \to sP$ such that $\varphi \circ l^x_A = p^x_A$ for every $A \in \mathcal{A}$ and $x \in G(A)$.

## 5 Some special weighted colimits

### 5.1 Conical colimits

$\text{Pos}$-colimits of a functor $F$ weighted by the constant functor $G = \Delta 1$ are called **conical colimits**. These turn out to be ordinary colimits.

### 5.2 Coinserters

Consider parallel morphisms $r, q : sR \rightleftharpoons sQ$ in $s\text{Pos}$. Let the category $\mathcal{A}$ and its images under $F$ and $G$ be

$$A_0 \xrightarrow{a} A_1 \quad S \xrightarrow{\begin{array}{c} r \\phantom{\rightarrow} \nearrow \\phantom{\rightarrow} \searrow \phantom{\rightarrow} \end{array}} \phantom{S} \quad 2 \xrightarrow{\begin{array}{c} c_1 \\phantom{\rightarrow} \nearrow \\phantom{\rightarrow} \searrow \phantom{\rightarrow} \end{array}} 1.$$

Then the colimit of $F$ weighted by $G$ is called the **coinserter** of $q$ and $r$.

**Lemma 8** There is one-to-one correspondence between coinserters of $q$ and $r$ and pairs $(sL, l)$, where $sL$ is a left $S$-poset and $l : sQ \to sL$ a morphism such that
1. \( l \circ q \leq l \circ r \),

2. if \( l' : sQ \to sL' \) is such that \( l' \circ q \leq l' \circ r \) then there exists unique \( \varphi : sL \to sL' \) in \( s\text{Pos} \) such that \( \varphi \circ l = l' \).

By Lemma 4.2 of [3] (where coiners were called subcoequalizers), one such pair is \((s\text{Coins}(q, r), \pi)\), where \( s\text{Coins}(q, r) = Q/\nu(H) \) is the quotient \( S\)-poset of \( sQ \) by the congruence \( \nu(H) \) induced by the set \( H = \{(q(x), r(x)) \mid x \in R \} \subseteq Q^2 \) and \( \pi : Q \to Q/\nu(H) \) is the natural surjection. We call \((s\text{Coins}(q, r), \pi)\) the canonical coinserter of \( q \) and \( r \).

### 5.3 Co-comma-objects

Consider morphisms \( r : sR \to sQ \) and \( r' : sR' \to sQ \) in \( s\text{Pos} \). If the category \( A \) and its images under \( F \) and \( G \) are

\[
\begin{array}{c}
A \xleftarrow{a} A_1 \xrightarrow{a'} A' \\
S_0 \xleftarrow{r} \text{ } S_1 \xrightarrow{r'} \text{ } S_2 \\
1 \xleftarrow{c_1} 2 \xrightarrow{c_0} 1
\end{array}
\]

then the colimit of \( F \) weighted by \( G \) is called the co-comma-object of \( r' \) and \( r \).

**Lemma 9** There is one-to-one correspondence between co-comma-objects of \( r' \) and \( r \) and triples \((sL, l', l)\), where \( l : sR \to sL \), \( l' : sR' \to sL \) are such that

1. \( l \circ r \leq l' \circ r' \);

2. if \( p : sR \to sL' \) and \( p' : sR' \to sL' \) in \( s\text{Pos} \) are such that \( p \circ r \leq p' \circ r' \) then there exists a unique morphism \( \varphi : L \to L' \) in \( s\text{Pos} \) such that \( \varphi \circ l = p \) and \( \varphi \circ l' = p' \).

By Section 2.1 of [4] (where co-comma-objects were called subpushouts), one such triple is \((s\text{Coco}(r', r), l', l)\), where \( s\text{Coco}(r', r) \) is the quotient \( S\)-poset of the coproduct \( sR \sqcup sR' = (1 \times R) \cup (\{2\} \times R') \) by the congruence \( \nu(H) \) induced by the set \( H = \{(2, r'(x)), (1, r(x)) \mid x \in Q \} \subseteq (R \sqcup R')^2 \) and the mapping \( l : R \to s\text{Coco}(r', r) \) \((l' : R' \to s\text{Coco}(r', r))\) is defined by \( l(y) := [1, y] \) \((l'(y') := [2, y'])\). We call \((s\text{Coco}(r', r), l', l)\) the canonical co-comma-object of \( r' \) and \( r \).
5.4 Lax colimit of a morphism

(WARNING: The results of this section may be incorrect!)

Consider a morphism \( h : \mathcal{S}R \to \mathcal{S}Q \) in \( \mathcal{S}\text{Pos} \). Let the category \( \mathcal{A} \) and its images under (covariant) \( F \) and (contravariant) \( G \) be

\[
\begin{array}{cccc}
A_0 \xrightarrow{a} A_1 & \mathcal{S}R \xrightarrow{h} \mathcal{S}Q & 1 \xrightarrow{c_0} 2.
\end{array}
\]

Then the colimit of \( F \) weighted by \( G \) is called the **lax colimit** of the morphism \( h \) (replacing \( c_0 \) by \( c_1 \) we obtain the **op-lax colimit** of the morphism \( h \)).

By the canonical construction of weighted colimits we know that

\[
\text{Laxco}(h) = (1 \times Q \sqcup 2 \times R)/\sigma \cong (Q \sqcup 2 \times R)/\sigma,
\]

where

\[(x_A, y_A)\sigma(x_{A'}, y_{A'}) \iff (x_A, y_A)\tau(x_{A'}, y_{A'}) \land (x_{A'}, y_{A'}) \tau(x_A, y_A)\]

and \( \tau \) is defined as in Section 4.2.

Let us examine the relation \( \tau \). Suppose that \((x_A, y_A)\tau(x_{A'}, y_{A'})\). Then we have a scheme (6), assume that it has a minimal length. Note that if \( f_i = g_i = 1_A \), or \( g_i = f_i+1 = 1_A \) for some \( i \) then the scheme could be shortened. Otherwise, consider the following cases.

1. Zigzag (7) contains \( 1_{A_0} \). Then

\[(x_A, y_A) \leq (x_{A'}, y_{A'}), \tag{9}\]

because otherwise either the morphism preceeding \( 1_{A_0} \) or the morphism following it would also be \( 1_{A_0} \).

2. Zigzag (7) contains no \( 1_{A_0} \). We have two subcases.

   2.1. \( f_1 = 1_A \). Then \( g_1 = a \). If \( n > 1 \) then we must have \( f_2 = a \), hence \( x_A = * = x_1 = x_2 \),

   \[
   x_A \leq G(f_1)(x_1) \quad G(g_2)(x_2) \leq G(f_3)(x_3) \quad F(f_1)(y_A) \leq F(g_1)(y_1) = F(f_2)(y_1) \leq F(g_2)(y_2),
   \]

   contradicting the minimality of \( n \). Hence \( n = 1 \) and

   \[
   * = x_A \leq x_1 = * \quad c_0(\ast) = 0 \leq x_{A'} \quad y_A \leq h(y_{A'}),
   \]

   i.e.

   \[
   (x_A, y_A) \in 1 \times Q, \quad (x_{A'}, y_{A'}) \in 2 \times R, \quad y_A \leq h(y_{A'}). \tag{10}
   \]

2.2. \( f_1 = a \). If \( g_1 = 1_A \) then also \( f_2 = 1_A \), contradicting our assumption. Hence \( g_1 = a \). If \( n > 1 \) then \( f_2 = a \) (because \( \text{dom} f_2 = A_0 \)), but then the sequence can be shortened. Hence \( n = 1 \) and

   \[
   x_A \leq c_0(x_1) \quad c_0(x_1) \leq x_{A'} \quad h(y_A) \leq h(y_{A'}),
   \]

   i.e.

   \[
   (x_A, y_A), (x_{A'}, y_{A'}) \in 2 \times R, \quad x_A = 0, \quad h(y_A) \leq h(y_{A'}). \tag{11}
   \]

So it seems that the cases when \((x_A, y_A)\tau(x_{A'}, y_{A'})\) can only be (9), (10) and (11). Also, it seems that \((x_A, y_A)\sigma(x_{A'}, y_{A'})\) if and only if

1) \((x_A, y_A) = (x_{A'}, y_{A'}), \) or

2) \( x_A = x_A' = 0 \in 2, \ y_A, y_{A'} \in R \) and \( h(y_A) = h(y_{A'}). \)
5.4.1 An application: coconvexity

Let \( sR \) be a \( S \)-subposet of \( QS \) and \( r : RS \to QS \) the inclusion mapping. Then the left \( S \)-poset \( \text{Oplax}(r) = \{(x, y) \in R \times Q \mid y \leq r(x)\} \) together with the restrictions \( l_R \) and \( l_Q \) of projections is an op-lax limit (see [6] for the definition) of the morphism \( r \).

It is easy to see that the \( S \)-subposet \( RS \) is down-closed if and only if the projection \( \pi_2 : \text{Pb}(r, l_Q) \to \text{Oplax}(r) \) of the canonical pullback \( (\text{Pb}(r, l_Q), \pi_1, \pi_2) \) of \( r \) and \( l_Q \) is an epimorphism (i.e. a surjective morphism). Note that

\[
\text{Pb}(r, l_Q) = \{(x_1, x_2, y) \in R \times R \times Q \mid x_1 = y \leq x_2\}.
\]

Using pullbacks and lax limits of morphisms one can categorically define up-closedness.

Convex \( S \)-subposets are precisely the intersections of up-closed and down-closed \( S \)-subposets.

We say that a (regular?) epimorphism \( h : R \to Q \) in \( S \text{Pos} \) is down-coclosed if the injection \( u_2 : \text{Oplaxco}(h) \to \text{Po}(h, n_R) \) of the pushout

\[
\begin{array}{c}
R \\
\downarrow h \\
Q \\
\end{array} \xrightarrow{\text{Oplaxco}(h)} \begin{array}{c}
\text{Oplaxco}(h) \\
\downarrow u_2 \\
\text{Po}(h, n_R) \\
\end{array}
\]

is a (regular?) monomorphism, where \((\text{Oplaxco}(h), n_R, n_Q)\) is the op-lax colimit of the morphism \( h \).

Using pushouts and lax colimits we define up-coclosedness. We say that a factor \( S \)-poset is \textit{coconvex} if it is a cointersection (!) of a down-coclosed and an up-coclosed factor \( S \)-poset. (I have no idea, what are the cointersections, but they must exist!)
5.5 Weighted tensor product

If $\mathcal{A}$ is the discrete category with a single object $\star$ then we call a colimit of $F$ weighted by $G$ a **weighted tensor product** of $F$ and $G$ (to distinguish it from the tensor product that is used in the study of flatness properties of $S$-posets). The weighted tensor product, constructed as in Theorem 2 is just the direct product $G(\star) \times F(\star)$, where the order is componentwise and the $S$-action is defined by

$$s \cdot (x, y) := (x, s \cdot y),$$

[together with left $S$-poset morphisms $l^x : F(\star) \to G(\star) \times F(\star)$, $x \in G(\star)$, defined by $l^x(y) := (x, y)$, $y \in F(\star)$.]

By Theorem 6, weighted tensor products of $F$ and $G$ are pairs $(sL, (l^x)_{x \in G(\star)})$, where $l^x : sF(\star) \to sL$ are morphisms such that

1. for all $x, x' \in G(\star)$, $x \leq x'$ implies $l^x \leq l^{x'}$;

2. for every $sP \in s\text{Pos}$ and family $(p^x)_{x \in G(\star)}$ of left $S$-poset morphisms $p^x : sF(\star) \to sP$ such that $x \leq x'$ implies $p^x \leq p^{x'}$ for all $x, x' \in G(\star)$ then there is a unique left $S$-poset morphism $\varphi : sL \to sP$ such that $\varphi \circ l^x = p^x$ for every $x \in G(\star)$.

In the case when $F(\star) = sS$, the weighted tensor product of $F$ and $G$ is the free $S$-poset on $G(\star)$ (see Theorem 10 of [4]).

Since $\text{Pos}$ also admits weighted tensor products ($=$direct products) of $P$ and $2$, for every poset $P$, the two-dimensional universal property of any limit follows from the one-dimensional one (see p. 306 of [6], or Theorem 4.85 of [5]). **WHAT DOES THIS MEAN?**

References


