Weighted limits and colimits in the category of left S-posets

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Abstract

Weighted limits and colimits are defined in categories that are enriched over a symmetric monoidal closed category. Since the category ${}_S\mathsf{Pos}$ of left S-posets over a pomonoid S is enriched over the category Pos of posets (with order-preserving mappings as morphisms) we can speak about weighted limits and colimits in ${}_S\mathsf{Pos}$.

1 Introduction

By $\mathbf{1} = \{*\}$ we shall denote the one-element (S-)poset and by $\mathbf{2} = \{0, 1\}$ the two-element chain with $0 < 1$. We assume the existence of an empty S-poset. Recall that morphisms in ${}_S\mathsf{Pos}$ are order and action preserving mappings and isomorphisms are surjective mappings that preserve and reflect order.

The category Pos of posets and order-preserving mappings is a symmetric monoidal closed category (see Def. 6.1.1–6.1.3 of [2]) with the cartesian product as a tensor product and $I = \mathbf{1}$.

The category ${}_S\mathsf{Pos}$ of left S-posets (or Pos_S of right S-posets) is a Pos-category (or poset enriched category or a category enriched over Pos) (see Def. 6.2.1 of [2]), where the morphism sets ${}_S\mathsf{Pos}(A, B)$, ${}_SA, {}_SB \in {}_S\mathsf{Pos}$ are posets with respect to pointwise order.

If $\mathcal{A}$ and $\mathcal{B}$ are Pos-categories then a Pos-functor $F : \mathcal{A} \to \mathcal{B}$ has to preserve (in addition to composition and identity morphisms) the order of morphism posets. We shall call such functors **pofunctors**.

Pos-natural transformations (see Def. 6.2.4 of [2]) between pofunctors are just the ordinary natural transformations. If $\mathcal{A}$ and $\mathcal{B}$ are Pos-categories and $\mathcal{A}$ is small then by Proposition 6.3.1 of [2] the category of pofunctors $\mathcal{A} \to \mathcal{B}$ and natural transformations between them can be provided with the structure of a Pos-category, written $\mathsf{Pos}[\mathcal{A}, \mathcal{B}]$. Namely, given two pofunctors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$, the set

$$\mathsf{Nat}(F, G) = \{(\alpha_A : F(A) \to G(A))_{A \in \mathcal{A}} \mid G(f)\alpha_{A'} = \alpha_{A''}F(f) \text{ for every } f\!: A' \to A'' \text{ in } \mathcal{A}\}$$

of natural transformations from F to G is a poset with respect to the order

$$(\alpha_A)_{A \in \mathcal{A}} \le (\beta_A)_{A \in \mathcal{A}} \iff \alpha_A \le \beta_A \text{ for every } A \in \mathcal{A} \text{ in the poset } \mathcal{B}(F(A), G(A)).$$

2 Weighted limits in ${}_S\mathsf{Pos}$

2.1 Definition

Definition 1 (Cf. Def. 6.6.3 of [2]) Given a pomonoid S, small Pos-category $\mathcal{A}$, and pofunctors $F : \mathcal{A} \to {}_S\mathsf{Pos}$, $G : \mathcal{A} \to \mathsf{Pos}$, a **$\mathsf{Pos}$-limit of F weighted by G** is a pair $({}_SL, (\lambda_P)_{P \in {}_S\mathsf{Pos}})$ where ${}_SL$ is a left S-poset and $\lambda = (\lambda_P)_{P \in {}_S\mathsf{Pos}} : {}_S\mathsf{Pos}(-, L) \Rightarrow \mathsf{Nat}(G, {}_S\mathsf{Pos}(-, F(-)))$ is a natural isomorphism, that is, for every ${}_SP \in {}_S\mathsf{Pos}$,

$$\lambda_P : {}_S\mathsf{Pos}(P, L) \longrightarrow \mathsf{Nat}(G, {}_S\mathsf{Pos}(P, F(-))),$$

are poset isomorphisms that are natural in P. We write $\lim_G F$ for a Pos-limit of F weighted by G.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & {}_S\mathsf{Pos} \\ & {\scriptstyle G}\searrow & \downarrow {\scriptstyle {}_S\mathsf{Pos}(P,-)} \\ & & \mathsf{Pos} \end{array}$$

Remark 1 For every ${}_SP \in {}_S\mathsf{Pos}$, ${}_S\mathsf{Pos}(P, F(-)) = {}_S\mathsf{Pos}(P, -) \circ F : \mathcal{A} \to \mathsf{Pos}$ is a pofunctor and the set $\mathsf{Nat}(G, {}_S\mathsf{Pos}(P, F(-)))$ is a poset with respect to componentwise order of natural transformations. Therefore, there is a contravariant functor

$$\mathsf{Nat}(G, {}_S\mathsf{Pos}(-, F(-))) : {}_S\mathsf{Pos} \to \mathsf{Pos}$$

given by the assignment

$$\begin{array}{ccc} {}_SP & \longmapsto & \mathsf{Nat}(G, {}_S\mathsf{Pos}(P, F(-))) \\ {\scriptstyle p}\downarrow & & \uparrow {\scriptstyle (-\circ p)\circ -} \\ {}_SQ & \longmapsto & \mathsf{Nat}(G, {}_S\mathsf{Pos}(Q, F(-))) \end{array}$$

where the mapping $(- \circ p) \circ -$ is defined by

$$((- \circ p) \circ -)(\mu) := ((- \circ p) \circ \mu_A)_{A \in \mathcal{A}} : G \Rightarrow {}_S\mathsf{Pos}(P, F(-))$$

for every natural transformation $\mu : G \Rightarrow {}_S\mathsf{Pos}(Q, F(-))$ and $- \circ p : {}_S\mathsf{Pos}(Q, F(A)) \to {}_S\mathsf{Pos}(P, F(A))$. The fact that $\lambda = (\lambda_P)_{P \in {}_S\mathsf{Pos}} : {}_S\mathsf{Pos}(-, L) \Rightarrow \mathsf{Nat}(G, {}_S\mathsf{Pos}(-, F(-)))$ is a natural transformation meand that

$$\lambda_P(\psi \circ p) = ((- \circ p) \circ \lambda_Q(\psi)_A)_{A \in \mathcal{A}},$$

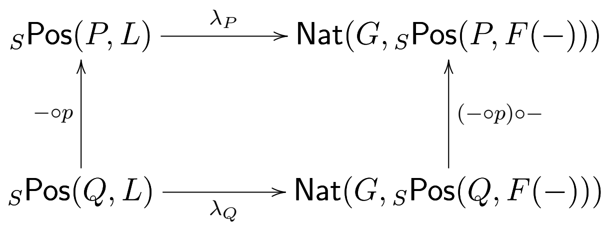
or

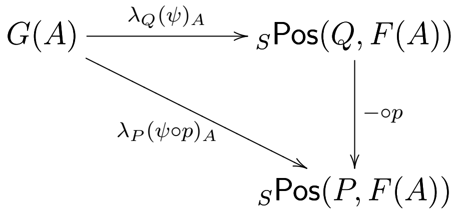
$$\lambda_P(\psi \circ p)_A = (- \circ p) \circ \lambda_Q(\psi)_A,$$

or

$$\lambda_P(\psi \circ p)_A(x) = \lambda_Q(\psi)_A(x) \circ p \tag{1}$$

for every $A \in \mathcal{A}$, $x \in G(A)$, ${}_SP, {}_SQ \in {}_S\mathsf{Pos}$, $p \in {}_S\mathsf{Pos}(P,Q)$, $\psi \in {}_S\mathsf{Pos}(Q,L)$.





2.2 Existence of weighted limits in ${}_S\mathsf{Pos}$

Here we give a characterization of a weighted limit in more usual terms of so-called projections of a limit and a universal property. We shall use the notation of Definition 1.

Theorem 1 *There is one-to-one correspondence between* Pos*-limits of* F *weighted by* G *and pairs* $\left({}_SL, (l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}\right)$*, where* ${}_SL$ *is a left* S*-poset and* $(l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}$ *is a family of left* S*-poset morphisms* $l_A^x : {}_SL \to {}_SF(A)$ *such that*

1. *(a) for all* $A \in \mathcal{A}$ *and* $x, x' \in G(A)$

$$x \leq x' \Longrightarrow l_A^x \leq l_A^{x'};$$

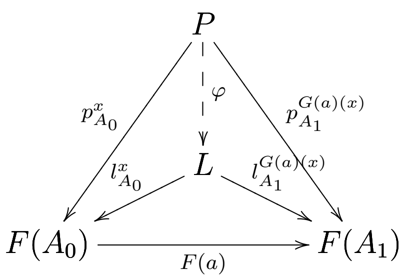
   (b) for all $a : A_0 \to A_1$ *in* $\mathcal{A}$ *and* $x \in G(A_0)$,

$$F(a) \circ l_{A_0}^x = l_{A_1}^{G(a)(x)};$$

2. *for all* ${}_SP \in {}_S\mathsf{Pos}$ *and* $\varphi, \psi \in {}_S\mathsf{Pos}(P,L)$,

$$((\forall A \in \mathcal{A})(\forall x \in G(A))(l_A^x \circ \varphi \leq l_A^x \circ \psi)) \Longrightarrow \varphi \leq \psi;$$

3. *for every* ${}_SP \in {}_S\mathsf{Pos}$ *and family* $(p_A^x)_{A\in\mathcal{A}}^{x\in G(A)}$ *of left* S*-poset morphisms* $p_A^x : {}_SP \to {}_SF(A)$ *with properties 1, there is a left* S*-poset morphism* $\varphi : {}_SP \to {}_SL$ *such that* $l_A^x \circ \varphi = p_A^x$ *for every* $A \in \mathcal{A}$ *and* $x \in G(A)$.



Proof. Suppose that there is ${}_SL \in {}_S\mathsf{Pos}$ and for every ${}_SP \in {}_S\mathsf{Pos}$ poset isomorphisms

$$\lambda_P : {}_S\mathsf{Pos}(P, L) \longrightarrow \mathsf{Nat}(G, {}_S\mathsf{Pos}(P, F(-)))$$

which are natural in P. For every $A \in \mathcal{A}$, $x \in G(A)$ we set

$$l_A^x := \lambda_L(1_L)_A(x) : {}_SL \to {}_SF(A). \tag{2}$$

1(a) holds because $\lambda_L(1_L)_A : G(A) \to {}_S\mathsf{Pos}(L, F(A))$ is order preserving for every $A \in \mathcal{A}$.

1(b). For every $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_0)$,

$$F(a) \circ l_{A_0}^x = F(a) \circ \lambda_L(1_L)_{A_0}(x) = \lambda_L(1_L)_{A_1}(G(a)(x)) = l_{A_1}^{G(a)(x)},$$

because $\lambda_L(1_L)$ is a natural transformation.

$$\begin{array}{ccc} G(A_0) & \xrightarrow{\lambda_L(1_L)_{A_0}} & {}_S\mathsf{Pos}(L, F(A_0)) \\ {\scriptstyle G(a)}\downarrow & & \downarrow{\scriptstyle F(a)\circ -} \\ G(A_1) & \xrightarrow[\lambda_L(1_L)_{A_1}]{} & {}_S\mathsf{Pos}(L, F(A_1)) \end{array}$$

2. Suppose that $\varphi, \psi \in {}_S\mathsf{Pos}(P, L)$ are such that $l_A^x \circ \varphi \leq l_A^x \circ \psi$ for every $A \in \mathcal{A}$ and $x \in G(A)$. Since λ_P is natural in P (see (1)), we obtain

$$\begin{aligned} \lambda_P(\varphi)_A(x) &= \lambda_P(1_L \circ \varphi)_A(x) = \lambda_L(1_L)_A(x) \circ \varphi = l_A^x \circ \varphi \\ &\leq l_A^x \circ \psi = \lambda_L(1_L)_A(x) \circ \psi = \lambda_P(1_L \circ \psi)_A(x) = \lambda_P(\psi)_A(x) \end{aligned}$$

for every $A \in \mathcal{A}$, $x \in G(A)$. Hence $\lambda_P(\varphi) \leq \lambda_P(\psi)$, and so $\varphi \leq \psi$, because λ_P reflects order.

3. If $(p_A^x)_{A \in \mathcal{A}}^{x \in G(A)}$ is a family of left S-poset morphisms $p_A^x : {}_SP \to {}_SF(A)$ that satisfies condition 1, then $\mu = (\mu_A)_{A \in \mathcal{A}}$, where $\mu_A : G(A) \to {}_S\mathsf{Pos}(P, F(A))$ is defined by

$$\mu_A(x) := p_A^x,$$

$x \in G(A)$, is a natural transformation $G \Rightarrow {}_S\mathsf{Pos}({}_SP, F(-))$. By the surjectivity of λ_P, there exists $\varphi \in {}_S\mathsf{Pos}(P, L)$ such that $\lambda_P(\varphi) = \mu$, and hence, by (1),

$$l_A^x \circ \varphi = \lambda_L(1_L)_A(x) \circ \varphi = \lambda_P(\varphi)_A(x) = \mu_A(x) = p_A^x$$

for every $A \in \mathcal{A}$ and $x \in G(A)$.

Conversely, let a pair $\left({}_SL, (l_A^x)_{A \in \mathcal{A}}^{x \in G(A)}\right)$ satisfy conditions 1–3. For every ${}_SP \in {}_S\mathsf{Pos}$ we define a mapping

$$\lambda_P : {}_S\mathsf{Pos}(P, L) \longrightarrow \mathsf{Nat}(G, {}_S\mathsf{Pos}(P, F(-)))$$

by

$$\lambda_P(\varphi)_A(x) := l_A^x \circ \varphi : P \to F(A), \tag{3}$$

$\varphi \in {}_S\mathsf{Pos}(P, L)$, $A \in \mathcal{A}$ and $x \in G(A)$.

1. As a composite of two S-poset morphisms, $\lambda_P(\varphi)_A(x)$ is an S-poset morphism.

2. Because of 1(a), $\lambda_P(\varphi)_A : G(A) \to {}_S\mathsf{Pos}(P, F(A))$ preserves order.

3. $\lambda_P(\varphi) : G \Rightarrow {}_S\mathsf{Pos}(P, F(-))$ is a natural transformation, because

$$\begin{aligned} ((F(a) \circ -) \circ \lambda_P(\varphi)_{A_0})(x) &= F(a) \circ \lambda_P(\varphi)_{A_0}(x) = F(a) \circ l^x_{A_0} \circ \varphi \\ &= l^{G(a)(x)}_{A_1} \circ \varphi = \lambda_P(\varphi)_{A_1}(G(a)(x)) \\ &= (\lambda_P(\varphi)_{A_1} \circ G(a))(x) \end{aligned}$$

for every $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_0)$.

4. λ_P is order preserving. Indeed, if $\varphi \le \psi$ in ${}_S\mathsf{Pos}(P, L)$ then

$$\lambda_P(\varphi)_A(x) = l^x_A \circ \varphi \le l^x_A \circ \psi = \lambda_P(\psi)_A(x)$$

for every $A \in \mathcal{A}$ and $x \in G(A)$, thus $\lambda_P(\varphi) \le \lambda_P(\psi)$.

5. λ_P is order reflecting, because, assuming that $\lambda_P(\varphi) \le \lambda_P(\psi)$, $\varphi, \psi \in {}_S\mathsf{Pos}(P, L)$, i.e. $l^x_A \circ \varphi \le l^x_A \circ \psi$ for every $A \in \mathcal{A}$ and $x \in G(A)$, we conclude $\varphi \le \psi$ by 2.

6. λ_P is surjective. To prove this, consider a natural transformation $\mu : G \Rightarrow {}_S\mathsf{Pos}(P, F(-))$. For every $A \in \mathcal{A}$ and $x \in G(A)$ set

$$p^x_A := \mu_A(x) : {}_SP \to {}_SF(A).$$

Since μ_A is order preserving, the family $(p^x_A)^{x \in G(A)}_{A \in \mathcal{A}}$ satisfies 1(a). Since μ is a natural transformation,

$$\begin{aligned} F(a) \circ p^x_{A_0} &= ((F(a) \circ -) \circ \mu_{A_0})(x) = (\mu_{A_1} \circ G(a))(x) \\ &= \mu_{A_1}(G(a)(x)) = p^{G(a)(x)}_{A_1} \end{aligned}$$

for every $a : A_0 \to A_1$ in $\mathcal{A}$. Hence 1(b) is also satisfied.

$$\begin{array}{ccc} G(A_0) & \xrightarrow{\mu_{A_0}} & {}_S\mathsf{Pos}({}_SP, F(A_0)) \\ \big\downarrow{\scriptstyle G(a)} & & \big\downarrow{\scriptstyle F(a)\circ -} \\ G(A_1) & \xrightarrow[\mu_{A_1}]{} & {}_S\mathsf{Pos}({}_SP, F(A_1)) \end{array}$$

By 3, there is an S-poset morphism $\varphi : {}_SP \to {}_SL$ such that $l^x_A \circ \varphi = p^x_A$ for every $A \in \mathcal{A}$ and $x \in G(A)$. So

$$\lambda_P(\varphi)_A(x) = l^x_A \circ \varphi = p^x_A = \mu_A(x)$$

for every $A \in \mathcal{A}$ and $x \in G(A)$. Hence $\lambda_P(\varphi) = \mu$ and λ_P is surjective.

7. λ_P is natural in P by (1), because

$$\lambda_P(\psi \circ p)_A(x) = l^x_A \circ (\psi \circ p) = (l^x_A \circ \psi) \circ p = \lambda_Q(\psi)_A(x) \circ p$$

for every $\psi \in {}_S\mathsf{Pos}(Q, L)$, $p \in {}_S\mathsf{Pos}(P, Q)$, $A \in \mathcal{A}$ and $x \in G(A)$.

Now, if $\left({}_SL, (l^x_A)^{x \in G(A)}_{A \in \mathcal{A}}\right)$ is a Pos-limit of F weighted by G, if we define mappings l^x_A by (2) and a natural transformation λ' by $\lambda'_P(\varphi)_A(x) := l^x_A \circ \varphi$, ${}_SP \in {}_S\mathsf{Pos}$, $\varphi \in {}_S\mathsf{Pos}(P, L)$, $A \in \mathcal{A}$, $x \in G(A)$, then by (1)

$$\lambda'_P(\varphi)_A(x) = l^x_A \circ \varphi = \lambda_L(1_L)_A(x) \circ \varphi = \lambda_P(1_L \circ \varphi)_A(x) = \lambda_P(\varphi)_A(x),$$

and so $\lambda = \lambda'$. Also, if $\left({}_S L, (l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}\right)$ satisfies conditions 1–3, we define a natural transformation λ by (3) and thereafter mappings k_A^x by $k_A^x := \lambda_L(1_L)_A(x)$, $A \in \mathcal{A}$, $x \in G(A)$, then

$$k_A^x = \lambda_L(1_L)_A(x) = l_A^x \circ 1_L = l_A^x.$$

Hence the correspondence is indeed one-to-one. ■

Remark 2 We always can assume that φ in condition 3 of Theorem 1 is unique. Indeed, if also $\psi : {}_S P \to {}_S L$ is such that $l_A^x \circ \psi = p_A^x$ for every $A \in \mathcal{A}$ and $x \in G(A)$, then $l_A^x \circ \psi \leq l_A^x \circ \varphi$ and $l_A^x \circ \varphi \leq l_A^x \circ \psi$ for every $A \in \mathcal{A}$ and $x \in G(A)$, which by condition 2 of Theorem 1 implies $\varphi = \psi$.

Remark 3 Having Theorem 1 in mind, we shall also call the pairs $\left({}_S L, (l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}\right)$, satisfying conditions 1–3 of Theorem 1, **limits of F weighted by G** and l_A^x their **projections**.

2.3 Canonical construction of weighted limits in ${}_S\mathsf{Pos}$

We shall show that weighted limits always exist in the category ${}_S\mathsf{Pos}$ and give a canonical construction for such limits.

It is easy to see that the poset $\mathsf{Nat}(G, U \circ F)$, where $U : {}_S\mathsf{Pos} \to \mathsf{Pos}$ is the forgetful functor, is an S-poset if the left S-action is given by

$$s \cdot f := (s \cdot f_A)_{A\in\mathcal{A}},$$

where $s \in S$, $f = (f_A)_{A\in\mathcal{A}} \in \mathsf{Nat}(G, U \circ F)$, and the mapping $s \cdot f_A : G(A) \to F(A)$ is defined by

$$(s \cdot f_A)(x) := s \cdot f_A(x),$$

$x \in G(A)$. For every $A \in \mathcal{A}$ and $x \in G(A)$ we define a mapping $l_A^x : \mathsf{Nat}(G, U\circ F) \to F(A)$ by

$$l_A^x(f) := f_A(x), \tag{4}$$

$f = (f_A)_{A\in\mathcal{A}} \in \mathsf{Nat}(G, U \circ F)$.

Theorem 2 *The pair* $\left(\mathsf{Nat}(G, U \circ F), (l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}\right)$ *is a* Pos*-limit of* F *weighted by* G.

Proof. Since

$$l_A^x(s \cdot f) = l_A^x((s \cdot f_A)_{A\in\mathcal{A}}) = (s \cdot f_A)(x) = s \cdot f_A(x) = s \cdot l_A^x(f)$$

for every $A \in \mathcal{A}$, $x \in G(A)$, $f = (f_A)_{A\in\mathcal{A}} \in L$, $s \in S$, and since l_A^x are obviously order preserving, they are left S-poset morphisms. We shall show that they satisfy the conditions of Theorem 1.

1(a). If $x \leq x'$, $x, x' \in G(A)$, then $f_A(x) \leq f_A(x')$ for every $f \in \mathsf{Nat}(G, U \circ F)$. Hence $l_A^x \leq l_A^{x'}$.

1(b). For every $a : A_0 \to A_1$ in $\mathcal{A}$, $x \in G(A_0)$ and $f \in \mathsf{Nat}(G, U \circ F)$,

$$(F(a) \circ l_{A_0}^x)(f) = F(a)(f_{A_0}(x)) = f_{A_1}(G(a)(x)) = l_{A_1}^{G(a)(x)}(f).$$

$$\begin{array}{ccc} G(A_0) & \xrightarrow{f_{A_0}} & U(F(A_0)) \\ {\scriptstyle G(a)}\downarrow & & \downarrow{\scriptstyle F(a)} \\ G(A_1) & \xrightarrow[f_{A_1}]{} & U(F(A_1)) \end{array}$$

2. Suppose that ${}_SP \in {}_S\mathsf{Pos}$, $\varphi, \psi \in {}_S\mathsf{Pos}(P, \mathsf{Nat}(G, U \circ F))$ are such that $l_A^x \circ \varphi \leq l_A^x \circ \psi$ for every $A \in \mathcal{A}$ and $x \in G(A)$. Then $\varphi(z)_A(x) = l_A^x(\varphi(z)) \leq l_A^x(\psi(z)) = \psi(z)_A(x)$ in $F(A)$ for every $A \in \mathcal{A}$, $x \in G(A)$ and $z \in P$. Since the order in $\mathsf{Pos}(G(A), F(A))$ is pointwise and the order in $\mathsf{Nat}(G, U \circ F)$ is componentwise, $\varphi(z) = (\varphi(z)_A)_{A \in \mathcal{A}} \leq (\psi(z)_A)_{A \in \mathcal{A}} = \psi(z)$ for every $z \in P$, and thus $\varphi \leq \psi$.

3. Let ${}_SP \in {}_S\mathsf{Pos}$ and let $(p_A^x)_{A \in \mathcal{A}}^{x \in G(A)}$ be a family of left S-poset morphisms $p_A^x : {}_SP \to {}_SF(A)$ such that (a) $x \leq x'$ implies $p_A^x \leq p_A^{x'}$ for all $A \in \mathcal{A}$, $x, x' \in G(A)$, and (b) $F(a) \circ p_{A_0}^x = p_{A_1}^{G(a)(x)}$ for all $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_0)$. We define a mapping $\varphi : P \to \mathsf{Nat}(G, U \circ F)$ by

$$\varphi(z)_A(x) := p_A^x(z),$$

$A \in \mathcal{A}$, $x \in G(A)$, $z \in P$. By (a), $\varphi(z)_A : G(A) \to F(A)$ is order preserving. By (b),

$$(F(a) \circ \varphi(z)_{A_0})(x) = F(a)(p_{A_0}^x(z)) = p_{A_1}^{G(a)(x)}(z) = \varphi(z)_{A_1}(G(a)(x)) = (\varphi(z)_{A_1} \circ G(a))(x)$$

for every $a : A_0 \to A_1$ in $\mathcal{A}$, $x \in G(A_0)$ and $z \in P$. Hence $\varphi(z) \in L$. Further, φ is order preserving, because all mappings p_A^x are. Also

$$\begin{aligned} \varphi(s \cdot z)_A(x) &= p_A^x(s \cdot z) = s \cdot p_A^x(z) = s \cdot \varphi(z)_A(x) \\ &= (s \cdot \varphi(z)_A)(x) = (s \cdot \varphi(z))_A(x) \end{aligned}$$

for every $A \in \mathcal{A}$, $x \in G(A)$, $z \in P$ and $s \in S$, which implies $\varphi(s \cdot z) = s \cdot \varphi(z)$, and hence φ is a left S-poset morphism. Finally,

$$(l_A^x \circ \varphi)(z) = l_A^x(\varphi(z)) = \varphi(z)_A(x) = p_A^x(z)$$

for every $A \in \mathcal{A}$, $x \in G(A)$, $z \in P$, and hence $l_A^x \circ \varphi = p_A^x$. ■

Remark 4 That weighted limits can be constructed as in Theorem 2 may also follow from (3.2) or (2.1) of [6], but we have preferred to give a direct proof here.

2.4 Another existence theorem for weighted limits

Here we show that condition 2 in Theorem 1 is actually redundant.

Theorem 3 *A pair* $\left({}_SL, (l_A^x)_{A \in \mathcal{A}}^{x \in G(A)}\right)$, *where* $l_A^x : {}_SL \to {}_SF(A)$ *are left* S*-poset morphisms, is a limit of* F *weighted by* G *if and only if*

1. *(a) for all* $A \in \mathcal{A}$ *and* $x, x' \in G(A)$

$$x \leq x' \Longrightarrow l_A^x \leq l_A^{x'};$$

(b) for all $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_0)$,

$$F(a) \circ l^x_{A_0} = l^{G(a)(x)}_{A_1};$$

2. *for every ${}_SP \in {}_S\mathsf{Pos}$ and family $(p^x_A)^{x\in G(A)}_{A\in\mathcal{A}}$ of left S-poset morphisms $p^x_A : {}_SP \to {}_SF(A)$ with properties 1, there is a unique left S-poset morphism $\varphi : {}_SP \to {}_SL$ such that $l^x_A \circ \varphi = p^x_A$ for every $A \in \mathcal{A}$ and $x \in G(A)$.*

Proof. Necessity follows immediately from Theorem 1 and Remark 2.

Sufficiency. Suppose that ${}_SL$ with l^x_A, $A \in \mathcal{A}$, $x \in G(A)$, satisfies conditions 1 and 2. Let ${}_SM$ together with left S-poset morphisms $m^x_A : {}_SM \to {}_SF(A)$ that satisfy conditions 1–3 of Theorem 1 be a limit of F weighted by G (by Theorem 2 we know that at least one such ${}_SM$ exists). Then there exists a unique morphism $\mu : {}_SM \to {}_SL$ such that $l^x_A \circ \mu = m^x_A$ for every $A \in \mathcal{A}$ and $x \in G(A)$, and a unique morphism $\nu : {}_SL \to {}_SM$ such that $m^x_A \circ \nu = l^x_A$ for every $A \in \mathcal{A}$ and $x \in G(A)$. Hence $l^x_A \circ (\mu \circ \nu) = l^x_A = l^x_A \circ 1_L$ for every $A \in \mathcal{A}$ and $x \in G(A)$, which implies $\mu \circ \nu = 1_L$ by the uniqueness of the comparison morphism $1_L : {}_SL \to {}_SL$.

Suppose now that $\varphi, \psi \in {}_S\mathsf{Pos}(P, L)$ and $l^x_A \circ \varphi \le l^x_A \circ \psi$ for every $A \in \mathcal{A}$ and $x \in G(A)$. Then

$$m^x_A \circ (\nu \circ \varphi) = l^x_A \circ \varphi \le l^x_A \circ \psi = m^x_A \circ (\nu \circ \psi)$$

for every $A \in \mathcal{A}$ and $x \in G(A)$. Since, for the limit ${}_SM$, condition 2 of Theorem 1 is satisfied, we have $\nu \circ \varphi \le \nu \circ \psi$, which yields $\varphi \le \psi$ by multiplying by μ on the left. ■

Remark 5 If I correctly understand a remark on p. 306 of [6] then the redundance of condition 2 in Theorem 1 should somehow follow from the existence of a tensor product (= direct product, $\neq$ the "homological tensor product", see Section 5) of **2** and ${}_SP$ for every left S-poset ${}_SP$. HOW?

Remark 6 In view of Theorem 3, in what follows, by a limit of F weighted by G we mean a pair $\left({}_SL, (l^x_A)^{x\in G(A)}_{A\in\mathcal{A}}\right)$ that satisfies conditions 1 and 2 of Theorem 3.

3 Some special weighted limits

3.1 Conical limits

If $G = \Delta\mathbf{1}$ is the constant functor at the one-element poset **1** then the limit of F weighted by G is called a **conical limit** (see [6], p. 305). By Theorem 3, $({}_SL, (l_A)_{A\in\mathcal{A}})$ is such a limit if and only if

1. for all $a : A_0 \to A_1$ in $\mathcal{A}$, $F(a) \circ l_{A_0} = l_{A_1}$;

2. for every ${}_SP \in {}_S\mathsf{Pos}$ and family $(p_A)_{A\in\mathcal{A}}$ of left S-poset morphisms $p_A : {}_SP \to {}_SF(A)$ with property 1, there is a unique left S-poset morphism $\varphi : {}_SP \to {}_SL$ such that $l_A \circ \varphi = p_A$ for every $A \in \mathcal{A}$.

Thus conical limits are just the ordinary limits, e.g. products, equalizers, pullbacks.

3.2 Inserters

Consider parallel morphisms $r, q : {}_SR \rightrightarrows {}_SQ$ in ${}_S\mathsf{Pos}$. Let the category $\mathcal{A}$ and its images under F and G be

$$A_0 \underset{a'}{\overset{a}{\rightrightarrows}} A_1 \qquad {}_SR \underset{q}{\overset{r}{\rightrightarrows}} {}_SQ \qquad \mathbf{1} \underset{c_0}{\overset{c_1}{\rightrightarrows}} \mathbf{2}$$

where a, a' are incomparable and c_1, c_0 map $*$ to 1 and 0, respectively. Then the limit of F weighted by G is called the **inserter** of q and r (see [6], p. 307) and it can be constructed as

$$\begin{aligned}
\mathsf{Nat}(G, U\circ F) &= \{(f_{A_0}, f_{A_1}) \mid f_{A_0} : \mathbf{1} \to R, f_{A_1} : \mathbf{2} \to Q, f_{A_1}(0) \le f_{A_1}(1), \\
&\quad r \circ f_{A_0} = f_{A_1} \circ c_1, q \circ f_{A_0} = f_{A_1} \circ c_0\} \\
&= \{(f_{A_0}, f_{A_1}) \mid f_{A_0} : \mathbf{1} \to R, f_{A_1} : \mathbf{2} \to Q, f_{A_1}(0) \le f_{A_1}(1), \\
&\quad r(f_{A_0}(*)) = f_{A_1}(1), q(f_{A_0}(*)) = f_{A_1}(0)\} \\
&\cong \{f_{A_0} \mid f_{A_0} : \mathbf{1} \to R, q \circ f_{A_0} \le r \circ f_{A_0}\} \\
&\cong \{z \in R \mid q(z) \le r(z)\} =: \mathsf{Ins}(q, r),
\end{aligned}$$

where the order and S-action of $\mathsf{Ins}(q,r)$ are inherited from ${}_SR$, and there is an isomorphism

$$\alpha : {}_S\mathsf{Nat}(G, U\circ F) \to {}_S\mathsf{Ins}(q, r), \quad (f_{A_0}, f_{A_1}) \mapsto f_{A_0}(*)$$

in ${}_S\mathsf{Pos}$.

Lemma 1 *There is one-to-one correspondence between inserters of q and r and pairs $({}_SE, e)$, where ${}_SE$ is a left S-poset and $e : {}_SE \to {}_SR$ a morphism such that*

1. *$q \circ e \le r \circ e$,*
2. *if $e' : {}_SE' \to {}_SR$ is such that $q \circ e' \le r \circ e'$ then there exists unique $\varphi : {}_SE' \to {}_SE$ in ${}_S\mathsf{Pos}$ such that $e \circ \varphi = e'$.*

Proof. Assume that the pair $({}_SL, (l^*_{A_0}, l^1_{A_1}, l^0_{A_1}))$ satisfies conditions 1 and 2 of Theorem 3. We write $({}_SE, e) = ({}_SL, l^*_{A_0}) = \alpha\left({}_SL, (l^*_{A_0}, l^1_{A_1}, l^0_{A_1})\right)$. Then

$$q \circ e = F(a') \circ l^*_{A_0} = l^{G(a')(*)}_{A_1} = l^0_{A_1} \le l^1_{A_1} = l^{G(a)(*)}_{A_1} = F(a) \circ l^*_{A_0} = r \circ e.$$

To prove 2, let $e' : {}_SE' \to {}_SR$ be such that $q \circ e' \le r \circ e'$. Then for $p^*_{A_0} = e'$, $p^0_{A_1} = q \circ e'$ and $p^1_{A_1} = r \circ e'$ we have $p^0_{A_1} \le p^1_{A_1}$, $F(a') \circ p^*_{A_0} = q \circ e' = p^0_{A_1} = p^{G(a')(*)}_{A_1}$, and, similarly, $F(a) \circ p^*_{A_0} = p^{G(a)(*)}_{A_1}$. By the assumption, there is a unique morphism $\varphi : {}_SE' \to {}_SE$ such that $e' = e \circ \varphi$.

Conversely, if a pair $({}_SE, e)$ satisfies 1 and 2, we consider the pair $({}_SE, (e, r\circ e, q \circ e)) = \beta({}_SE, e)$. It is easy to see that conditions 1 and 2 of Theorem 3 are satisfied.

Finally,

$$\beta\left(\alpha\left({}_SL, (l^*_{A_0}, l^1_{A_1}, l^0_{A_1})\right)\right) = \beta\left({}_SL, l^*_{A_0}\right) = \left({}_SL, (l^*_{A_0}, r \circ l^*_{A_0}, q \circ l^*_{A_0})\right) = \left({}_SL, (l^*_{A_0}, l^1_{A_1}, l^0_{A_1})\right)$$

for every inserter $\left({}_SL, (l^*_{A_0}, l^1_{A_1}, l^0_{A_1})\right)$ of q and r and

$$\alpha(\beta({}_SE, e)) = \alpha\left({}_SE, (e, r \circ e, q \circ e)\right) = ({}_SE, e))$$

for every pair $({}_SE, e))$ that satisfies 1 and 2. ■

Remark 7 It is easy to check that the pair $(\mathsf{Ins}(q,r), \iota)$, where $\iota : \mathsf{Ins}(q,r) \to R$ is the inclusion, satisfies conditions 1 and 2 of Lemma 1. We call $(\mathsf{Ins}(q,r), \iota)$ the **canonical inserter** of q and r.

3.3 Equifiers

Consider parallel morphisms $r, q : {}_S R \rightrightarrows {}_S Q$ with $q \leq r$ in ${}_S\mathsf{Pos}$. Let the category $\mathcal{A}$ and its images under F and G be

$$A_0 \underset{a'}{\overset{a}{\rightrightarrows}} A_1 \qquad {}_S R \underset{q}{\overset{r}{\rightrightarrows}} {}_S Q \qquad \mathbf{1} \underset{c_0}{\overset{c_1}{\rightrightarrows}} \mathbf{2}$$

where $a' \leq a$ and c_1, c_0 map $*$ to 1 and 0, respectively. Then the limit of F weighted by G is called the **equifier** of q and r (see [6], p. 309) and it can be constructed as

$$\begin{aligned}
\mathsf{Nat}(G, U \circ F) &= \{(f_{A_0}, f_{A_1}) \mid f_{A_0} : \mathbf{1} \to R, f_{A_1} : \mathbf{2} \to Q, f_{A_1}(0) \leq f_{A_1}(1), \\
&\quad r \circ f_{A_0} = f_{A_1} \circ c_1, q \circ f_{A_0} = f_{A_1} \circ c_0\} \\
&= \{(f_{A_0}, f_{A_1}) \mid f_{A_0} : \mathbf{1} \to R, f_{A_1} : \mathbf{2} \to Q, f_{A_1}(0) \leq f_{A_1}(1), \\
&\quad r(f_{A_0}(*)) = f_{A_1}(1), q(f_{A_0}(*)) = f_{A_1}(0)\} \\
&\cong \{f_{A_0} \mid f_{A_0} : \mathbf{1} \to R, q \circ f_{A_0} \leq r \circ f_{A_0}\} \\
&\cong \{z \in R \mid q(z) \leq r(z)\} = R.
\end{aligned}$$

So the equifier of (q, r) with $q \leq r$ is just the pair $(R, 1_R)$ and the universal property is trivially satisfied. Clearly every pofunctor preserves equifiers.

3.4 Comma objects

Consider morphisms $r : {}_S R \to {}_S Q$ and $r' : {}_S R' \to {}_S Q$ in ${}_S\mathsf{Pos}$. If the category $\mathcal{A}$ and its images under F and G are

$$A \xrightarrow{a} A_1 \xleftarrow{a'} A' \qquad {}_S R \xrightarrow{r} {}_S Q \xleftarrow{r'} R' \qquad \mathbf{1} \xrightarrow{c_1} \mathbf{2} \xleftarrow{c_0} \mathbf{1}$$

then the limit of F weighted by G is called the **comma-object** of r' and r (see [6], p. 308). Analogously to Lemma 1 one can prove the following result.

Lemma 2 *There is one-to-one correspondence between comma-objects of r' and r and triples $(\mathsf{Co}(r', r), z', z)$, where $z : \mathsf{Co}(r', r) \to R$, $z' : \mathsf{Co}(r', r) \to R'$ are such that*

1. $r \circ z \leq r' \circ z'$;
2. *if $w : W \to R$ and $w' : W \to R'$ in ${}_S\mathsf{Pos}$ are such that $r \circ w \leq r' \circ w'$ then there exists a unique morphism $\varphi : W \to \mathsf{Co}(r', r)$ in ${}_S\mathsf{Pos}$ such that $z \circ \varphi = w$ and $z' \circ \varphi = w'$.*

Canonically, one can take

$$\mathsf{Co}(r', r) := \{(x', x) \in R' \times R \mid r'(x') \leq r(x)\}$$

and z', z the restrictions of the projections of $R' \times R$.

Note that inserters and comma objects in ${}_S\mathsf{Pos}$ were termed sub-equalizers and sub-pullbacks, respectively, in [3].

3.5 Lax limit and op-lax limit of a morphism

Consider a morphism $r : {}_SR \longrightarrow {}_SQ$ in ${}_S\mathsf{Pos}$. Let the category $\mathcal{A}$ and its images under F and G be

$$A_0 \xrightarrow{a} A_1 \qquad {}_SR \xrightarrow{r} {}_SQ \qquad \mathbf{1} \xrightarrow{c_0} \mathbf{2}. \tag{5}$$

Then the limit of F weighted by G is called the **lax limit** of the morphism r (replacing c_0 by c_1 we obtain the **op-lax limit** of the morphism r; see [6], p. 308) and it can be canonically constructed as

$$\begin{aligned}
& \mathsf{Nat}(G, U \circ F) \\
= \; & \{(f_{A_0}, f_{A_1}) \mid f_{A_0} : \mathbf{1} \to R, f_{A_1} : \mathbf{2} \to Q, f_{A_1}(0) \leq f_{A_1}(1), r \circ f_{A_0} = f_{A_1} \circ c_0\} \\
= \; & \{(f_{A_0}, f_{A_1}) \mid f_{A_0} : \mathbf{1} \to R, f_{A_1} : \mathbf{2} \to Q, f_{A_1}(0) \leq f_{A_1}(1), r(f_{A_0}(*)) = f_{A_1}(0)\} \\
\cong \; & \{(x, y) \in R \times Q \mid r(x) \leq y\} =: \mathsf{Lax}(r),
\end{aligned}$$

where the order and left S-action on $\mathsf{Lax}(r)$ are componentwise. In more detail, if $(f_{A_0}, f_{A_1}) \in \mathsf{Nat}(G, U \circ F)$ then $r(f_{A_0}(*)) \leq f_{A_1}(1)$, and hence we may define a mapping $\alpha : \mathsf{Nat}(G, U \circ F) \to \mathsf{Lax}(r)$ by

$$\alpha(f_{A_0}, f_{A_1}) := (f_{A_0}(*), f_{A_1}(1)).$$

Obviously, α is order preserving and, for every $s \in S$,

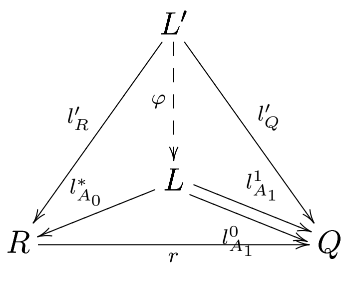
$$\begin{aligned}
\alpha(s \cdot (f_{A_0}, f_{A_1})) &= \alpha(s \cdot f_{A_0}, s \cdot f_{A_1}) = ((s \cdot f_{A_0})(*), (s \cdot f_{A_1})(1)) \\
&= (s \cdot f_{A_0}(*), s \cdot f_{A_1}(1)) = s \cdot (f_{A_0}(*), f_{A_1}(1)) = s \cdot \alpha(f_{A_0}, f_{A_1}).
\end{aligned}$$

Suppose that also $(g_{A_0}, g_{A_1}) \in \mathsf{Nat}(G, U \circ F)$ and $(f_{A_0}(*), f_{A_1}(1)) \leq (g_{A_0}(*), g_{A_1}(1))$. Then $f_{A_0}(*) \leq g_{A_0}(*)$, $f_{A_1}(1) \leq g_{A_1}(1)$, and $f_{A_1}(0) = r(f_{A_0}(*)) \leq r(g_{A_0}(*)) = g_{A_1}(0)$. Hence $(f_{A_0}, f_{A_1}) \leq (g_{A_0}, g_{A_1})$, and α is order reflecting. Finally, if $(x, y) \in R \times Q$ and $r(x) \leq y$ then defining $f_{A_0}(*) := x$, $f_{A_1}(1) := y$ and $f_{A_1}(0) := r(x)$ we have $(f_{A_0}, f_{A_1}) \in \mathsf{Nat}(G, U \circ F)$ and $\alpha(f_{A_0}, f_{A_1}) = (x, y)$. Thus we have proved that α is an isomorphism. Consequently, the pair $\left(\mathsf{Lax}(r), \left(l^*_{A_0} \circ \alpha^{-1}, l^0_{A_1} \circ \alpha^{-1}, l^1_{A_1} \circ \alpha^{-1}\right)\right)$ is a lax limit of r.

Lemma 3 *There is one-to-one correspondence between lax limits of a morphism $r : {}_SR \to {}_SQ$ and pairs $(L, (l_R, l_Q))$ with $l_R : {}_SL \to {}_SR$, $l_Q : {}_SL \to {}_SQ$ such that*

1. $r \circ l_R \leq l_Q$;
2. *if $l'_R : {}_SL' \to {}_SR$ and $l'_Q : {}_SL' \to {}_SQ$ are such that $r \circ l'_R \leq l'_Q$ then there exists a unique morphism $\varphi : {}_SL' \to {}_SL$ such that $l_R \circ \varphi = l'_R$ and $l_Q \circ \varphi = l'_Q$.*

Proof. Let $\left({}_SL, \left(l^*_{A_0}, l^0_{A_1}, l^1_{A_1}\right)\right)$ be a lax limit of a morphism $r : {}_SR \to {}_SQ$, that is, it satisfies conditions 1 and 2 of Theorem 3. We write $(L, (l_R, l_Q)) = \left({}_SL, \left(l^*_{A_0}, l^1_{A_1}\right)\right) = \alpha\left({}_SL, \left(l^*_{A_0}, l^0_{A_1}, l^1_{A_1}\right)\right)$.



Then

$$r \circ l_R = F(a) \circ l^*_{A_0} = l^{G(a)(*)}_{A_1} = l^0_{A_1} \le l^1_{A_1} = l_Q.$$

Suppose that $l'_R : {}_SL' \to {}_SR$ and $l'_Q : {}_SL' \to {}_SQ$ are such that $r \circ l'_R \le l'_Q$. Taking $p^*_{A_0} := l'_R$, $p^0_{A_1} := r \circ l'_R$ and $p^1_{A_1} := l'_Q$ we see that the pair $\left({}_SL', \left(p^*_{A_0}, p^0_{A_1}, p^1_{A_1}\right)\right)$ satisfies condition 1 of Theorem 3. Hence there exists a left S-poset morphism $\varphi : {}_SL' \to {}_SL$ such that $l_R \circ \varphi = l^*_{A_0} \circ \varphi = p^*_{A_0} = l'_R$ and $l_Q \circ \varphi = l^1_{A_1} \circ \varphi = p^1_{A_1} = l'_Q$. If $\psi : {}_SL' \to {}_SL$ is another morphism such that $l_R \circ \psi = l'_R$ and $l_Q \circ \psi = l'_Q$ then $l^0_{A_1} \circ \psi = r \circ l_R \circ \psi = r \circ l'_R = p^0_{A_1}$ and hence $\varphi = \psi$ by the uniqueness of φ in condition 2 of Theorem 3.

Conversely, if a pair $({}_SL, (l_R, l_Q))$ satisfies 1 and 2, we consider the pair $({}_SL, (l_R, r \circ l_R, l_Q)) = \beta({}_SL, (l_R, l_Q))$. It is easy to see that $({}_SL, (l_R, r \circ l_R, l_Q))$ satisfies conditions 1 and 2 of Theorem 3 and hence is a lax limit of r.

Now,

$$\beta\left(\alpha\left({}_SL, (l^*_{A_0}, l^0_{A_1}, l^1_{A_1})\right)\right) = \beta\left({}_SL, (l^*_{A_0}, l^1_{A_1})\right) = \left({}_SL, (l^*_{A_0}, r \circ l^*_{A_0}, l^1_{A_1})\right) = \left({}_SL, (l^*_{A_0}, l^0_{A_1}, l^1_{A_1})\right)$$

for every lax limit $\left({}_SL, (l^*_{A_0}, l^0_{A_1}, l^1_{A_1})\right)$ of r, and

$$\alpha(\beta({}_SL, (l_R, l_Q))) = \alpha\left({}_SL, (l_R, r \circ l_R, l_Q)\right) = ({}_SL, (l_R, l_Q)))$$

for every pair $({}_SL, (l_R, L_Q)))$ that satisfies conditions 1 and 2. ■

Having Lemma 3 in mind, we shall call the pairs $({}_SL, (l_R, l_Q))$ satisfying conditions 1 and 2 of that lemma the **lax limits** of r. In particular, we say that the **canonical lax limit** of r is the pair $(\mathsf{Lax}(r), (p_R, p_Q))$, where $p_R := l^*_{A_0} \circ \alpha^{-1} : \mathsf{Lax}(r) \to R$ and $p_Q := l^1_{A_1} \circ \alpha^{-1} : \mathsf{Lax}(r) \to Q$ are given by

$$\begin{aligned} p_R(x, y) &= l^*_{A_0}(\alpha^{-1}(x, y)) = \alpha^{-1}(x, y)_{A_0}(*) = x, \\ p_Q(x, y) &= l^1_{A_1}(\alpha^{-1}(x, y)) = \alpha^{-1}(x, y)_{A_1}(1) = y, \end{aligned}$$

$(x, y) \in \mathsf{Lax}(r)$.

One can check that a canonical op-lax limit of a morphism $r : {}_SR \to {}_SQ$ in ${}_S\mathsf{Pos}$ can be constructed as a pair $(\mathsf{Oplax}(r), (p_R, p_Q))$, where

$$\mathsf{Oplax}(r) = \{(x, y) \in R \times Q \mid y \le r(x)\},$$

$p_R(x, y) = x$, $p_Q(x, y) = y$ for all $(x, y) \in \mathsf{Oplax}(r)$. Op-lax limits of morphisms together with pullbacks give a possibility to define downwards closed S-subposets of an S-poset in categorical terms.

3.6 Cotensor products

If $\mathcal{A}$ is the the discrete category with a single object $\star$ then F and G can be identified with objects $F(\star)$ and $G(\star)$ of ${}_S\mathsf{Pos}$ and of Pos, respectively. By Theorem 3, $({}_SL, (l^x)^{x \in G(\star)})$, where $l^x : {}_SL \to {}_SF(\star)$, is a limit of F weighted by G if and only if

1. for all $x, x' \in G(\star)$,
$$x \le x' \Longrightarrow l^x \le l^{x'};$$
2. for every ${}_SP \in {}_S\mathsf{Pos}$ and family $(p^x)^{x \in G(\star)}$ of left S-poset morphisms $p^x : {}_SP \to {}_SF(\star)$ with property 1, there is a unique left S-poset morphism $\varphi : {}_SP \to {}_SL$ such that $l^x \circ \varphi = p^x$ for every $x \in G(\star)$.

Such weighted limit is called a **cotensor product** of F and G (or of $F(\star)$ and $G(\star)$; see [6], p. 305). By Theorem 2, one such cotensor product is $\big({}_S\mathsf{Pos}(G(\star), F(\star)), (l^x)^{x \in G(\star)}\big)$, where $l^x : {}_S\mathsf{Pos}(G(\star), F(\star)) \to F(\star)$ is the evaluation map at $x \in G(\star)$, i.e. $l^x(f) = f(x)$ for every $f \in {}_S\mathsf{Pos}(G(\star), F(\star))$.

3.7 Pie limits

For a functor $G : \mathcal{D} \to \mathsf{Pos}$ we can consider its category of elements (or Grothendieck category). The objects of this category $\mathsf{el}(G)$ are pairs (x, i), where $i \in I = \mathcal{D}_0$ and $x \in G(i)$. A morphism $(x, i) \to (y, j)$ is a morphism $d \in \mathcal{D}(i, j)$ such that $G(d)(x) = y$.

Definition 2 ([7]) A pofunctor $G : \mathcal{D} \to \mathsf{Pos}$ is called a **pie weight** if each component of $\mathsf{el}(G)$ has an initial object.

Since equifiers in ${}_S\mathsf{Pos}$ are trivial, from Proposition 2.1 of [7] we have the following corollary, which we present with a proof.

Proposition 1 *If $G : \mathcal{D} \to \mathsf{Pos}$ is a pie weight and $F : \mathcal{D} \to {}_S\mathsf{Pos}$ is a pofunctor then $\lim_G F$ can be constructed using products and inserters.*

Proof. Let U be the set of connected components of $\mathsf{el}(G)$. For every connected component $u \in U$, let (z_u, j_u) be the initial object of u. If $(x, i) \in \mathsf{el}(G)_0$, then we write $\overline{(x, i)} \in U$ for the connected component of (x, i) and $!_{\overline{(x,i)}}$ for the unique morphism $j_{\overline{(x,i)}} \to i$ such that $G(!_{\overline{(x,i)}})(z_{\overline{(x,i)}}) = x$. Take

$$S := \{(x, y, i) \mid i \in I, x, y \in G(i), x \leq y\}$$

and consider products

$$\left(\prod_{u \in U} F(j_u), (\pi_u)_{u \in U}\right) \quad \text{and} \quad \left(\prod_{(x,y,i) \in S} F(i), (p_{(x,y,i)})_{(x,y,i) \in S}\right).$$

$$\begin{array}{ccccc}
F(j_{\overline{(x,i)}}) & \xleftarrow{\pi_{\overline{(x,i)}}} & \prod_{u \in U} F(j_u) & \xrightarrow{\pi_{\overline{(y,i)}}} & F(j_{\overline{(y,i)}}) \\
\downarrow F(!_{\overline{(x,i)}}) & & f_0 \downarrow\downarrow f_1 & & \downarrow F(!_{\overline{(y,i)}}) \\
F(i) & \xleftarrow[p_{(x,y,i)}]{} & \prod_{(x,y,i) \in S} F(i) & \xrightarrow[p_{(x,y,i)}]{} & F(i)
\end{array}$$

Then there exist unique morphisms $f_0, f_1 : \prod_{u \in U} F(j_u) \to \prod_{(x,y,i) \in S} F(i)$ such that

$$p_{(x,y,i)} \circ f_0 = \pi_{\overline{(x,i)}} \circ F(!_{\overline{(x,i)}}) \quad \text{and} \quad p_{(x,y,i)} \circ f_1 = \pi_{\overline{(y,i)}} \circ F(!_{\overline{(y,i)}})$$

for every $(x, y, i) \in S$. Let (E, e) be the inserter of (f_0, f_1). In particular, $f_0 \circ e \leq f_1 \circ e$. We claim that

$$\left(E, (l_i^x)_{i \in I}^{x \in G(i)}\right) \approx \lim_G F$$

where $l_i^x := F(!_{\overline{(x,i)}}) \circ \pi_{\overline{(x,i)}} \circ e : E \to F(i)$. If $d : i_0 \to i_1$ in $\mathcal{D}$ and $x \in G(i_0)$ then $d : (x, i_0) \to (G(d)(x), i_1)$ in $\mathsf{el}(G)$ and $\overline{(x, i_0)} = \overline{(G(d)(x), i_1)}$. Hence $!_{\overline{(G(d)(x),i_1)}} = d \circ !_{\overline{(x,i_0)}}$ and

$$\begin{aligned} l_{i_1}^{G(d)(x)} &= F\left(!_{\overline{(G(d)(x),i_1)}}\right) \circ \pi_{\overline{(G(d)(x),i_1)}} \circ e = F(d) \circ F(!_{\overline{(x,i_0)}}) \circ \pi_{\overline{(x,i_0)}} \circ e \\ &= F(d) \circ l_i^x. \end{aligned}$$

If $x, y \in G(i)$ are such that $x \leq y$ then

$$l_i^x = F(!_{\overline{(x,i)}}) \circ \pi_{\overline{(x,i)}} \circ e = p_{(x,y,i)} \circ f_0 \circ e \leq p_{(x,y,i)} \circ f_1 \circ e = F(!_{\overline{(y,i)}}) \circ \pi_{\overline{(y,i)}} \circ e = l_i^y.$$

To verify the universla property, let $\left(P, (p_i^x)_{i \in I}^{x \in G(i)}\right)$ be such that $F(d) \circ p_{i_0}^x = p_{i_1}^{G(d)(x)}$ for every $d : i_0 \to i_1$ in $\mathcal{D}$ and $p_i^x \leq p_i^y$ whenever $x \leq y$ in $G(i)$. Then there exists a unique morphism $g : P \to \prod_{u \in U} F(j_u)$ such that $\pi_u \circ g = p_{j_u}^{z_u}$ for every $u \in U$. Now, for every $(x, y, i) \in S$,

$$\begin{aligned} p_{(x,y,i)} \circ f_0 \circ g &= F(!_{\overline{(x,i)}}) \circ \pi_{\overline{(x,i)}} \circ g = F(!_{\overline{(x,i)}}) \circ p_{j_{\overline{(x,i)}}}^{z_{\overline{(x,i)}}} = p_i^x \\ &\leq p_i^y = F(!_{\overline{(y,i)}}) \circ p_{j_{\overline{(y,i)}}}^{z_{\overline{(y,i)}}} = F(!_{\overline{(y,i)}}) \circ \pi_{\overline{(y,i)}} \circ g = p_{(x,y,i)} \circ f_1 \circ g. \end{aligned}$$

Since products are weighted limits, they satisfy condition 2 of Theorem 1, and hence $f_0 \circ g \leq f_1 \circ g$. Consequently, there exists a unique morphism $\varphi : P \to E$ such that $e \circ \varphi = g$. Then

$$l_i^x \circ \varphi = F(!_{\overline{(x,i)}}) \circ \pi_{\overline{(x,i)}} \circ e \circ \varphi = F(!_{\overline{(x,i)}}) \circ \pi_{\overline{(x,i)}} \circ g = F(!_{\overline{(x,i)}}) \circ p_{j_{\overline{(x,i)}}}^{z_{\overline{(x,i)}}} = p_i^x.$$

Finally, suppose that $\psi : P \to E$ is sucht that $l_i^x \circ \psi = p_i^x$ for each $x \in G(i)$, $i \in I$. Note that $\overline{(z_u, j_u)} = u$ and $!_{\overline{(z_u,j_u)}} = 1_{j_u}$. Hence $l_{j_u}^{z_u} = F(1_{j_u}) \circ \pi_u \circ e = \pi_u \circ e$ for every $u \in U$. Now $l_i^x \circ \varphi = p_i^x = l_i^x \circ \psi$ implies

$$\pi_u \circ e \circ \varphi = l_{j_u}^{z_u} \circ \varphi \leq l_{j_u}^{z_u} \circ \psi = \pi_u \circ e \circ \psi$$

for every $u \in U$. Applying again condition 2 of Theorem 1, first for product and then for inserter, we obtain $\varphi \leq \psi$. Symmetrically we get $\psi \leq \varphi$, and thus $\varphi = \psi$. ■

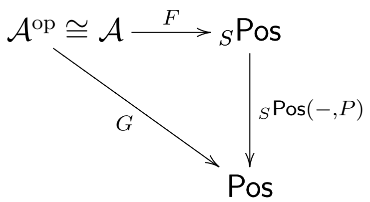
4 Weighted colimits in ${}_S\mathsf{Pos}$

4.1 Definition

Definition 3 (Cf. Def. 6.6.4 of [2]) Given a pomonoid S, small Pos-category $\mathcal{A}$, and pofunctors $F : \mathcal{A} \to {}_S\mathsf{Pos}$, $G : \mathcal{A}^{\mathrm{op}} \to \mathsf{Pos}$ (covariant and contravariant on $\mathcal{A}$, respectively), the Pos**-colimit of** F **weighted by** G is a pair $({}_SL, (\lambda_P)_{P\in {}_S\mathsf{Pos}})$ where ${}_SL$ is a left S-poset and $\lambda = (\lambda_P)_{P\in {}_S\mathsf{Pos}} : {}_S\mathsf{Pos}(L, -) \Rightarrow \mathsf{Nat}(G, {}_S\mathsf{Pos}(F(-), -))$ is a natural isomorphism, that is, for every ${}_SP \in {}_S\mathsf{Pos}$,

$$\lambda_P : {}_S\mathsf{Pos}(L, P) \longrightarrow \mathsf{Nat}(G, {}_S\mathsf{Pos}(F(-), P)),$$

are poset isomorphisms that are natural in ${}_SP$. We write $\mathrm{colim}_G F$ for a Pos-colimit of F weighted by G.



Dually to Theorem 1, one can prove the following result.

Theorem 4 *There is one-to-one correspondence between* Pos*-colimits of* F *weighted by* G *and pairs* $\left({}_SL, (l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}\right)$*, where* ${}_SL$ *is a left* S*-poset and* $(l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}$ *is a family of left* S*-poset morphisms* $l_A^x : {}_SF(A) \to {}_SL$ *such that*

1. *(a) for all* $A \in \mathcal{A}$ *and* $x, x' \in G(A)$

$$x \le x' \Longrightarrow l_A^x \le l_A^{x'};$$

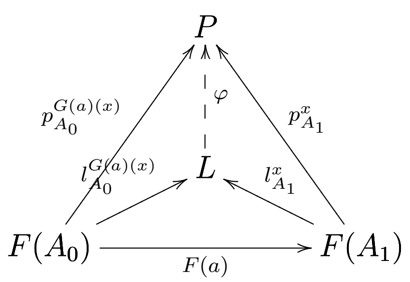
   (b) for all $a : A_0 \to A_1$ *in* $\mathcal{A}$ *and* $x \in G(A_1)$,

$$l_{A_1}^x \circ F(a) = l_{A_0}^{G(a)(x)};$$

2. *for all* ${}_SP \in {}_S\mathsf{Pos}$ *and* $\varphi, \psi \in {}_S\mathsf{Pos}(L, P)$,

$$((\forall A \in \mathcal{A})(\forall x \in G(A))(\varphi \circ l_A^x \le \psi \circ l_A^x)) \Longrightarrow \varphi \le \psi;$$

3. *for every* ${}_SP \in {}_S\mathsf{Pos}$ *and family* $(p_A^x)_{A\in\mathcal{A}}^{x\in G(A)}$ *of left* S*-poset morphisms* $p_A^x : {}_SF(A) \to {}_SP$ *with properties 1, there is a left* S*-poset morphism* $\varphi : {}_SL \to {}_SP$ *such that* $\varphi \circ l_A^x = p_A^x$ *for every* $A \in \mathcal{A}$ *and* $x \in G(A)$.



4.2 Canonical construction of weighted colimits in ${}_S\mathsf{Pos}$

We shall show that the Pos-category ${}_S\mathsf{Pos}$ is Pos-cocomplete by giving an explicit construction of a colimit ${}_SL \cong \mathrm{colim}_G F$ of F weighted by G.

We define a relation τ on the disjoint union $\bigsqcup_{A\in\mathcal{A}} G(A)\times F(A)$ by

$$(x_A, y_A)\tau(x_{A'}, y_{A'})$$

$x_A \in G(A)$, $y_A \in F(A)$, $x_{A'} \in G(A')$, $y_{A'} \in F(A')$, if and only if either $(x_A, y_A) \le (x_{A'}, y_{A'})$ or

$$\begin{array}{rclrcl} x_A & \le & G(f_1)(x_1) & & & \\ G(g_1)(x_1) & \le & G(f_2)(x_2) & F(f_1)(y_A) & \le & F(g_1)(y_1) \\ G(g_2)(x_2) & \le & G(f_3)(x_3) & F(f_2)(y_1) & \le & F(g_2)(y_2) \\ & \dots & & & \dots & \\ G(g_n)(x_n) & \le & x_{A'} & F(f_n)(y_{n-1}) & \le & F(g_n)(y_{A'}) \end{array} \tag{6}$$

for some morphisms

$$A \xrightarrow{f_1} A_1' \xleftarrow{g_1} A_1 \xrightarrow{f_2} A_2' \xleftarrow{g_2} A_2 \xrightarrow{f_3} A_3' \dots A_{n-1} \xrightarrow{f_n} A_n' \xleftarrow{g_n} A' \tag{7}$$

in $\mathcal{A}$ and elements $x_i \in G(A_i')$, $i = 1, \dots, n$, $y_j \in F(A_j)$, $j = 1, \dots, n-1$.

Lemma 4 *The relation τ is reflexive and transitive.*

Proof. Reflexivity of τ follows from inequalities

$$\begin{array}{rclrcl} x_A & \le & G(1_A)(x_A) & & & \\ G(1_A)(x_A) & \le & x_A & F(1_A)(y_A) & \le & F(1_A)(y_A). \end{array}$$

To prove transitivity, suppose that $(x_A, y_A)\tau(x_{A'}, y_{A'})$ and $(x_{A'}, y_{A'})\tau(x_{A''}, y_{A''})$, where $x_A \in G(A)$, $x_{A'} \in G(A')$, $x_{A''} \in G(A'')$, $y_A \in F(A)$, $y_{A'} \in F(A')$ and $y_{A''} \in F(A'')$. Then, in addition to inequalities (6), we have inequalities

$$\begin{array}{rclrcl} x_{A'} & \le & G(h_1)(z_1) & & & \\ G(k_1)(z_1) & \le & G(h_2)(z_2) & F(h_1)(y_{A'}) & \le & F(k_1)(w_1) \\ G(k_2)(z_2) & \le & G(h_3)(z_3) & F(h_2)(w_1) & \le & F(k_2)(w_2) \\ & \dots & & & \dots & \\ G(k_m)(z_m) & \le & x_{A''} & F(h_m)(w_{m-1}) & \le & F(k_m)(y_{A''}) \end{array}$$

for some morphisms

$$A' \xrightarrow{h_1} B_1' \xleftarrow{k_1} B_1 \xrightarrow{h_2} B_2' \xleftarrow{k_2} B_2 \xrightarrow{h_3} B_3' \dots B_{m-1} \xrightarrow{h_m} B_m' \xleftarrow{k_m} A''$$

in $\mathcal{A}$. Hence we have inequalities

$$\begin{array}{rclrcl} x_A & \le & G(f_1)(x_1) & & & \\ G(g_1)(x_1) & \le & G(f_2)(x_2) & F(f_1)(y_A) & \le & F(g_1)(y_1) \\ G(g_2)(x_2) & \le & G(f_3)(x_3) & F(f_2)(y_1) & \le & F(g_2)(y_2) \\ & \dots & & & \dots & \\ G(g_n)(x_n) & \le & G(h_1)(z_1) & F(f_n)(y_{n-1}) & \le & F(g_n)(y_{A'}) \\ G(k_1)(z_1) & \le & G(h_2)(z_2) & F(h_1)(y_{A'}) & \le & F(k_1)(w_1) \\ G(k_2)(z_2) & \le & G(h_3)(z_3) & F(h_2)(w_1) & \le & F(k_2)(w_2) \\ & \dots & & & \dots & \\ G(k_m)(z_m) & \le & x_{A''} & F(h_m)(w_{m-1}) & \le & F(k_m)(y_{A''}), \end{array}$$

i.e. $(x_A, y_A)\tau(x_{A''}, y_{A''})$. ■

Lemma 5 *Let τ be reflexive and transitive binary relation on a set M. Define a binary relation σ on M by*

$$m\sigma n \iff m\tau n \wedge n\tau m.$$

Then σ is an equivalence relation, and by defining

$$[m] \leq [n] \iff m\tau n$$

we obtain a well-defined partial order on the quotient set $M/\sigma = \{[m] \mid m \in M\}$.

By Lemma 5, the relation σ, defined by

$$(x_A, y_A)\sigma(x_{A'}, y_{A'}) \iff (x_A, y_A)\tau(x_{A'}, y_{A'}) \wedge (x_{A'}, y_{A'})\tau(x_A, y_A)$$

is an equivalence relation on the set $\bigsqcup_{A\in\mathcal{A}} G(A) \times F(A)$, and the definition

$$[x_A, y_A] \leq [x_{A'}, y_{A'}] \iff (x_A, y_A)\tau(x_{A'}, y_{A'})$$

gives a partial order on the quotient set

$$L := \bigsqcup_{A\in\mathcal{A}} G(A) \times F(A)/\sigma = \{[x_A, y_A] \mid A \in \mathcal{A}, x_A \in G(A), y_A \in F(A)\}.$$

We define a left S-action on L by

$$s \cdot [x_A, y_A] := [x_A, s \cdot y_A].$$

Lemma 6 *This way, L becomes a left S-poset.*

Proof. Since $F(A)$ is a left S-act for every $A \in \mathcal{A}$, so is L.

Suppose that $s \leq t$, $s, t \in S$, $x_A \in G(A)$, $y_A \in F(A)$, $A \in \mathcal{A}$. Since $F(A)$ is a left S-poset, $s \cdot y_A \leq t \cdot y_A$. From

$$\begin{array}{rclcrcl} x_A & \leq & G(1_A)(x_A) & & & & \\ G(1_A)(x_A) & \leq & x_A & & F(1_A)(s \cdot y_A) & \leq & F(1_A)(t \cdot y_A) \end{array}$$

we see that $(x_A, s \cdot y_A)\tau(x_A, t \cdot y_A)$, i.e. $[x_A, s \cdot y_A] \leq [x_A, t \cdot y_A]$.

Suppose that $[x_A, y_A] \leq [x_{A'}, y_{A'}]$ and $s \in S$. Then we have inequalities (6). Using that the elements in the right-hand column belong to left S-posets and all $F(f_i), F(g_i)$ are left S-poset morphisms, we obtain

$$\begin{array}{rclcrcl} x_A & \leq & G(f_1)(x_1) & & & & \\ G(g_1)(x_1) & \leq & G(f_2)(x_2) & & F(f_1)(s \cdot y_A) & \leq & F(g_1)(s \cdot y_1) \\ G(g_2)(x_2) & \leq & G(f_3)(x_3) & & F(f_2)(s \cdot y_1) & \leq & F(g_2)(s \cdot y_2) \\ & \cdots & & & & \cdots & \\ G(g_n)(x_n) & \leq & x_{A'} & & F(f_n)(s \cdot y_{n-1}) & \leq & F(g_n)(s \cdot y_{A'}). \end{array}$$

Hence

$$s \cdot [x_A, y_A] = [x_A, s \cdot y_A] \leq [x_{A'}, s \cdot y_{A'}] = s \cdot [x_{A'}, y_{A'}].$$

(Note that the condition, we have just verified, implies that the S-action is well-defined.)

■

Lemma 7 *The poset L satisfies conditions*

1. $(\forall x \in G(A))(\forall y, y' \in F(A))(y \leq y' \Rightarrow [x,y] \leq [x,y'])$,
2. $(\forall x, x' \in G(A))(\forall y \in F(A))(x \leq x' \Rightarrow [x,y] \leq [x',y])$,
3. $(\forall x \in G(A))(\forall y' \in F(A'))(\forall f : A' \to A \text{ in } \mathcal{A})([x, F(f)(y')] = [G(f)(x), y'])$.

Proof. The proof follows from the existence of the following inequalities:

$$\begin{array}{rcl} x & \leq & G(1_A)(x) \\ G(1_A)(x) & \leq & x \end{array} \qquad F(1_A)(y) \leq F(1_A)(y'),$$

$$\begin{array}{rcl} x & \leq & G(1_A)(x') \\ G(1_A)(x') & \leq & x' \end{array} \qquad F(1_A)(y) \leq F(1_A)(y),$$

$$\begin{array}{rcl} x & \leq & G(1_A)(x) \\ G(f)(x) & \leq & G(f)(x) \end{array} \qquad F(1_A)(F(f)(y')) \leq F(f)(y')$$

and

$$\begin{array}{rcl} G(f)(x) & \leq & G(f)(x) \\ G(1_A)(x) & \leq & x \end{array} \qquad F(f)(y') \leq F(1_A)(F(f)(y')).$$

■

Theorem 5 *The left S-poset ${}_SL$ is a* Pos*-colimit of F weighted by G.*

Proof. We define a mapping $l_A^x : F(A) \to L$, $A \in \mathcal{A}$, $x \in G(A)$, by

$$l_A^x(y) := [x,y],$$

$y \in F(A)$. By Lemma 7(1), l_A^x is order preserving. Since it obviously preserves S-action, it is a left S-poset morphism. We shall check that the pair $\left({}_SL, (l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}\right)$ satisfies conditions 1–3 of Theorem 4.

1(a) follows from Lemma 7(2).

1(b) For every $a : A_0 \to A_1$ in $\mathcal{A}$, $x \in G(A_1)$ and $y \in F(A_0)$ we have

$$\left(l_{A_1}^x \circ F(a)\right)(y) = l_{A_1}^x(F(a)(y)) = [x, F(a)(y)] = [G(a)(x), y] = l_{A_0}^{G(a)(x)}(y)$$

by Lemma 7(3).

2. Suppose that ${}_SP \in {}_S\mathsf{Pos}$, $\varphi, \psi \in {}_S\mathsf{Pos}(L,P)$ and $\varphi \circ l_A^x \leq \psi \circ l_A^x$ for all $A \in \mathcal{A}$ and $x \in G(A)$. Then, for every $A \in \mathcal{A}$, $x \in G(A)$ and $y \in F(A)$,

$$\varphi([x,y]) = (\varphi \circ l_A^x)(y) \leq (\psi \circ l_A^x)(y) = \psi([x,y]),$$

and hence $\varphi \leq \psi$.

3. Suppose that the morphisms $p_A^x : {}_SF(A) \to {}_SP$ satisfy condition 1. We define a mapping $\varphi : L \to P$ by

$$\varphi([x,y]) := p_A^x(y)$$

for every $A \in \mathcal{A}$, $x \in G(A)$ and $y \in F(A)$. Since p_A^x are left S-act morphisms, so is φ. Suppose that $[x_A, y_A] \leq [x_{A'}, y_{A'}]$ in L, i.e. we have inequalities (6). Then

$$\begin{aligned} p_A^{x_A}(y_A) &\leq p_A^{G(f_1)(x_1)}(y_A) = \left(p_{A_1'}^{x_1} \circ F(f_1)\right)(y_A) \leq \left(p_{A_1'}^{x_1} \circ F(g_1)\right)(y_1) = p_{A_1}^{G(g_1)(x_1)}(y_1) \\ &\leq p_{A_1}^{G(f_2)(x_2)}(y_1) \leq \ldots \leq p_{A_{n-1}}^{G(f_n)(x_n)}(y_{n-1}) = \left(p_{A_n'}^{x_n} \circ F(f_n)\right)(y_{n-1}) \\ &\leq \left(p_{A_n'}^{x_n} \circ F(g_n)\right)(y_{A'}) = p_{A'}^{G(g_n)(x_n)}(y_{A'}) \leq p_{A'}^{x_{A'}}(y_{A'}). \end{aligned}$$

This proves that φ is well defined and order preserving. Finally, $(\varphi \circ l_A^x)(y) = \varphi([x,y]) = p_A^x(y)$ for every $A \in \mathcal{A}$, $x \in G(A)$ and $y \in F(A)$. ■

Dually to Theorem 3, one can prove the following result.

Theorem 6 *There is one-to-one correspondence between* Pos*-colimits of F weighted by G and pairs $\left({}_SL, (l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}\right)$, where ${}_SL$ is a left S-poset and $(l_A^x)_{A\in\mathcal{A}}^{x\in G(A)}$ is a family of left S-poset morphisms $l_A^x : {}_SF(A) \to {}_SL$ such that*

1. *(a) for all $A \in \mathcal{A}$ and $x, x' \in G(A)$*

$$x \leq x' \Longrightarrow l_A^x \leq l_A^{x'};$$

   (b) for all $a : A_0 \to A_1$ in $\mathcal{A}$ and $x \in G(A_1)$,

$$l_{A_1}^x \circ F(a) = l_{A_0}^{G(a)(x)};$$

2. *for every ${}_SP \in {}_S$Pos and family $(p_A^x)_{A\in\mathcal{A}}^{x\in G(A)}$ of left S-poset morphisms $p_A^x : {}_SF(A) \to {}_SP$ with properties 1, there is a unique left S-poset morphism $\varphi : {}_SL \to {}_SP$ such that $\varphi \circ l_A^x = p_A^x$ for every $A \in \mathcal{A}$ and $x \in G(A)$.*

5 Some special weighted colimits

5.1 Conical colimits

Pos-colimits of a functor F weighted by the constant functor $G = \Delta\mathbf{1}$ are called **conical colimits**. These turn out to be ordinary colimits.

5.2 Coinserters

Consider parallel morphisms $r, q : {}_SR \rightrightarrows {}_SQ$ in ${}_S$Pos. Let the category $\mathcal{A}$ and its images under F and G be

$$A_0 \overset{a}{\underset{a'}{\rightrightarrows}} A_1 \qquad {}_SR \overset{r}{\underset{q}{\rightrightarrows}} {}_SQ \qquad \mathbf{2} \overset{c_1}{\underset{c_0}{\leftleftarrows}} \mathbf{1}.$$

Then the colimit of F weighted by G is called the **coinserter** of q and r.

Lemma 8 *There is one-to-one correspondence between coinserters of q and r and pairs $({}_SL, l)$, where ${}_SL$ is a left S-poset and $l : {}_SQ \to {}_SL$ a morphism such that*

1. $l \circ q \leq l \circ r$,

2. *if* $l' : {}_S Q \to {}_S L'$ *is such that* $l' \circ q \leq l' \circ r$ *then there exists unique* $\varphi : {}_S L \to {}_S L'$ *in* ${}_S\mathsf{Pos}$ *such that* $\varphi \circ l = l'$.

By Lemma 4.2 of [3] (where coinserters were called subcoequalizers), one such pair is $({}_S\mathsf{Coins}(q, r), \pi)$, where ${}_S\mathsf{Coins}(q, r) = Q/\nu(H)$ is the quotient S-poset of ${}_S Q$ by the congruence $\nu(H)$ induced by the set $H = \{(q(x), r(x)) \mid x \in R\} \subseteq Q^2$ and $\pi : Q \to Q/\nu(H)$ is the natural surjection. We call $({}_S\mathsf{Coins}(q, r), \pi)$ the **canonical coinserter** of q and r.

5.3 Co-comma-objects

Consider morphisms $r : {}_S R \to {}_S Q$ and $r' : {}_S R' \to {}_S Q$ in ${}_S\mathsf{Pos}$. If the category $\mathcal{A}$ and its images under F and G are

$$A \xleftarrow{a} A_1 \xrightarrow{a'} A' \qquad\qquad {}_S R \xleftarrow{r} {}_S Q \xrightarrow{r'} R' \qquad\qquad \mathbf{1} \xrightarrow{c_1} \mathbf{2} \xleftarrow{c_0} \mathbf{1}$$

then the colimit of F weighted by G is called the **co-comma-object** of r' and r.

Lemma 9 *There is one-to-one correspondence between co-comma-objects of* r' *and* r *and triples* $({}_S L, l', l)$, *where* $l : {}_S R \to {}_S L$, $l' : {}_S R' \to {}_S L$ *are such that*

1. $l \circ r \leq l' \circ r'$;

2. *if* $p : {}_S R \to {}_S L'$ *and* $p' : {}_S R' \to {}_S L'$ *in* ${}_S\mathsf{Pos}$ *are such that* $p \circ r \leq p' \circ r'$ *then there exists a unique morphism* $\varphi : L \to L'$ *in* ${}_S\mathsf{Pos}$ *such that* $\varphi \circ l = p$ *and* $\varphi \circ l' = p'$.

By Section 2.1 of [4] (where co-comma-objects were called subpushouts), one such triple is $({}_S\mathsf{Coco}(r', r), l', l)$, where ${}_S\mathsf{Coco}(r', r)$ is the quotient S-poset of the coproduct ${}_S R \sqcup {}_S R' = (1 \times R) \cup (\{2\} \times R')$ by the congruence $\nu(H)$ induced by the set $H = \{((2, r'(x)), (1, r(x))) \mid x \in Q\} \subseteq (R \sqcup R')^2$ and the mapping $l : R \to {}_S\mathsf{Coco}(r', r)$ ($l' : R' \to {}_S\mathsf{Coco}(r', r)$) is defined by $l(y) := [1, y]$ ($l'(y') := [2, y']$). We call $({}_S\mathsf{Coco}(r', r), l', l)$ the **canonical co-comma-object** of r' and r.

5.4 Lax colimit of a morphism

(WARNING: The results of this section may be incorrect!)

Consider a morphism $h : {}_SR \longrightarrow {}_SQ$ in ${}_S\mathsf{Pos}$. Let the category $\mathcal{A}$ and its images under (covariant) F and (contravariant) G be

$$A_0 \xrightarrow{a} A_1 \qquad {}_SR \xrightarrow{h} {}_SQ \qquad \mathbf{1} \xrightarrow{c_0} \mathbf{2}. \tag{8}$$

Then the colimit of F weighted by G is called the **lax colimit** of the morphism h (replacing c_0 by c_1 we obtain the **op-lax colimit** of the morphism h).

By the canonical construction of weighted colimits we know that

$$\mathsf{Laxco}(h) = (\mathbf{1} \times Q \sqcup \mathbf{2} \times R)/\sigma \cong (Q \sqcup \mathbf{2} \times R)/\sigma,$$

where

$$(x_A, y_A)\sigma(x_{A'}, y_{A'}) \iff (x_A, y_A)\tau(x_{A'}, y_{A'}) \wedge (x_{A'}, y_{A'})\tau(x_A, y_A)$$

and τ is defined as in Section 4.2.

Let us examine the relation τ. Suppose that $(x_A, y_A)\tau(x_{A'}, y_{A'})$. Then we have a scheme (6), assume that it has a minimal length. Note that if $f_i = g_i = 1_{A_i}$ or $g_i = f_{i+1} = 1_{A_i}$ for some i then the scheme could be shortened. Otherwise, consider the following cases.

1. Zigzag (7) contains 1_{A_0}. Then

$$(x_A, y_A) \leq (x_{A'}, y_{A'}), \tag{9}$$

because otherwise either the morpism preceeding 1_{A_0} or the morphism following it would also be 1_{A_0}.

2. Zigzag (7) contains no 1_{A_0}. We have two subcases.

2.1. $f_1 = 1_{A_1}$. Then $g_1 = a$. If $n > 1$ then we must have $f_2 = a$, hence $x_A = * = x_1 = x_2$,

$$\begin{array}{rcl} x_A & \leq & G(f_1)(x_1) \\ G(g_2)(x_2) & \leq & G(f_3)(x_3) \end{array} \qquad F(f_1)(y_A) \;\leq\; F(g_1)(y_1) = F(f_2)(y_1) \leq F(g_2)(y_2),$$

contradicting the minimality of n. Hence $n = 1$ and

$$\begin{array}{rcl} * = x_A & \leq & x_1 = * \\ c_0(*) = 0 & \leq & x_{A'} \end{array} \qquad y_A \;\leq\; h(y_{A'}),$$

i.e.

$$(x_A, y_A) \in \mathbf{1} \times Q, \;\; (x_{A'}, y_{A'}) \in \mathbf{2} \times R, \;\; y_A \leq h(y_{A'}). \tag{10}$$

2.2. $f_1 = a$. If $g_1 = 1_{A_1}$ then also $f_2 = 1_{A_1}$, contradicting our assumption. Hence $g_1 = a$. If $n > 1$ then $f_2 = a$ (because $dom f_2 = A_0$), but then the sequence can be shortened. Hence $n = 1$ and

$$\begin{array}{rcl} x_A & \leq & c_0(x_1) \\ c_0(x_1) & \leq & x_{A'} \end{array} \qquad h(y_A) \;\leq\; h(y_{A'}),$$

i.e.

$$(x_A, y_A), (x_{A'}, y_{A'}) \in \mathbf{2} \times R, \;\; x_A = 0, \;\; h(y_A) \leq h(y_{A'}). \tag{11}$$

So it seems that the cases when $(x_A, y_A)\tau(x_{A'}, y_{A'})$ can only be (9), (10) and (11). Also, it seems that $(x_A, y_A)\sigma(x_{A'}, y_{A'})$ if and only if

1) $(x_A, y_A) = (x_{A'}, y_{A'})$, or

2) $x_A = x_{A'} = 0 \in \mathbf{2}$, $y_A, y_{A'} \in R$ and $h(y_A) = h(y_{A'})$.

5.4.1 An application: coconvexity

Let ${}_SR$ be a S-subposet of Q_S and $r : R_S \to Q_S$ the inclusion mapping. Then the left S-poset $\mathsf{Oplax}(r) = \{(x, y) \in R \times Q \mid y \leq r(x)\}$ together with the restrictions l_R and l_Q of projections is an op-lax limit (see [6] for the definition) of the morphism r. It is easy to see that the S-subposet R_S is down-closed if and only if the projection $\pi_2 : \mathsf{Pb}(r, l_Q) \to \mathsf{Oplax}(r)$ of the canonical pullback $(\mathsf{Pb}(r, l_Q), \pi_1, \pi_2)$ of r and l_Q is an epimorphism (i.e. a surjective morphism). Note that

$$\mathsf{Pb}(r, l_Q) = \{(x_1, x_2, y) \in R \times R \times Q \mid x_1 = y \leq x_2\}.$$

$$\begin{array}{ccc} \mathsf{Pb}(r, l_Q) & \xrightarrow{\pi_2} & \mathsf{Oplax}(r) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle l_Q} \\ R & \xrightarrow[r]{} & Q \end{array}$$

Using pullbacks and lax limits of morphisms one can categorically define up-closedness.

Convex S-subposets are precisely the intersections of up-closed and down-closed S-subposets.

We say that a (regular?) epimorphism $h : R \to Q$ in ${}_S\mathsf{Pos}$ is down-coclosed if the injection $u_2 : \mathsf{Oplaxco}(h) \to \mathsf{Po}(h, n_R)$ of the pushout

$$\begin{array}{ccc} R & \xrightarrow{n_R} & \mathsf{Oplaxco}(h) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle u_2} \\ Q & \xrightarrow[u_1]{} & \mathsf{Po}(h, n_R) \end{array}$$

is a (regular?) monomorphism, where $(\mathsf{Oplaxco}(h), n_R, n_Q)$ is the op-lax colimit of the morphism h.

Using pushouts and lax colimits we define up-coclosedness. We say that a factor S-poset is *coconvex* if it is a cointersection (!) of a down-coclosed and an up-coclosed factor S-poset. (I have no idea, what are the cointersections, but they must exist!)

5.5 Weighted tensor product

If $\mathcal{A}$ is the the discrete category with a single object $\star$ then we call a colimit of F weighted by G a **weighted tensor product** of F and G (to distinguish it from the tensor product that is used in the study of flatness properites of S-posets). The weighted tensor product, constructed as in Theorem 2 is just the direct product $G(\star) \times F(\star)$, where the order is componentwise and the S-action is defined by

$$s \cdot (x, y) := (x, s \cdot y),$$

together with left S-poset morphisms $l^x : F(\star) \to G(\star) \times F(\star)$, $x \in G(\star)$, defined by $l^x(y) := (x, y)$, $y \in F(\star)$.

By Theorem 6, weighted tensor products of F and G are pairs $\big({}_S L, (l^x)^{x \in G(\star)}\big)$, where $l^x : {}_S F(\star) \to {}_S L$ are morphisms such that

1. for all $x, x' \in G(\star)$, $x \leq x'$ implies $l^x \leq l^{x'}$;
2. for every ${}_S P \in {}_S\mathsf{Pos}$ and family $(p^x)^{x \in G(\star)}$ of left S-poset morphisms $p^x : {}_S F(\star) \to {}_S P$ such that $x \leq x'$ implies $p^x \leq p^{x'}$ for all $x, x' \in G(\star)$ then there is a unique left S-poset morphism $\varphi : {}_S L \to {}_S P$ such that $\varphi \circ l^x = p^x$ for every $x \in G(\star)$.

In the case when $F(\star) = {}_S S$, the weighted tensor product of F and G is the free S-poset on $G(\star)$ (see Theorem 10 of [4]).

Since Pos also admits weighted tensor products (=direct products) of P and $\mathbf{2}$, for every poset P, the two-dimensional universal property of any limit follows from the one-dimensional one (see p. 306 of [6], or Theorem 4.85 of [5]). WHAT DOES THIS MEAN?

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