# Generators in the category of $S$-posets

Valdis Laan

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**Def. 1** A monoid $S$ together with a partial order relation $\leq$ on it is called a **partially ordered monoid** (shortly **pomonoid**) if

$$s \leq t \Longrightarrow us \leq ut \quad \text{and} \quad su \leq tu$$

for all $s, t, u \in S$.

**Def. 2** Let $S$ be a pomonoid. A poset $(A, \leq)$ is called a **right $S$-poset** if there is a mapping $A \times S \to A$, $(a, s) \mapsto a \cdot s$, such that

1. $(a \cdot s) \cdot t = a \cdot st$,
2. $a \cdot 1 = a$,
3. $a \leq b$ implies $a \cdot s \leq b \cdot s$,
4. $s \leq t$ implies $a \cdot s \leq a \cdot t$,

for all $a, b \in A$, $s, t \in S$. We write $A_S$. Similarly, for a pomonoid $T$, **left $T$-posets** and $(T, S)$-**biposets** can be defined.

**Def. 3 Right $S$-poset morphisms** $A_S \to B_S$ are mappings $f : A \to B$ such that

1. $f(a \cdot s) = f(a) \cdot s$,

2. $a \leq b$ implies $f(a) \leq f(b)$,

for all $a, b \in A$, $s \in S$. We write $\mathsf{Pos}_S(A, B)$ for the set of all $S$-poset morphisms from $A_S$ to $B_S$.

Right $S$-posets and their morphisms form a category $\mathsf{Pos}_S$, which is enriched over the category $\mathsf{Pos}$ of posets, i.e. the morphism sets $\mathsf{Pos}_S(A, B)$ are posets with respect to the pointwise order

$$f \leq g \iff f(a) \leq g(a) \quad \text{for every } a \in A.$$

Regular monomorphisms in the category $\mathsf{Pos}_S$ are mappings $f : A_S \to B_S$ with

$$a \leq a' \iff f(a) \leq f(a') \quad \text{for every } a, a' \in A.$$

A right $S$-poset $A_S$ is a **retract** of $B_S$ if there exist morphisms $A \underset{\gamma}{\overset{\pi}{\rightleftarrows}} B$ in $\mathsf{Pos}_S$ such that $\gamma \circ \pi = 1_A$.

**Proposition 1** *Cyclic projectives in* $\mathsf{Pos}_S$ *are precisely retracts of* $S_S$.

## Generators

An object $A_S$ in the category $\mathsf{Pos}_S$ is a **generator** if the mapping $\mathsf{Pos}_S(A, -)$ restricted to every morphism set is injective. In other words, for every $X_S, Y_S \in \mathsf{Pos}_S$ and $f, g \in \mathsf{Pos}_S(X, Y)$,

$$\mathsf{Pos}_S(A, f) = \mathsf{Pos}_S(A, g) \Longrightarrow f = g,$$

or

$$((\forall k \in \mathsf{Pos}_S(A, X))(f \circ k = g \circ k)) \Longrightarrow f = g.$$

**Theorem 1** *The following assertions are equivalent for a right $S$-poset $A_S$:*

1. *$A_S$ is a generator;*
2. *for every $X_S \in \mathsf{Pos}_S$ there exists a set $I$ and an epimorphism $h : \sqcup_I A \to X$ in $\mathsf{Pos}_S$;*
3. *there exists an epimorphism $\pi : A \to S$ in $\mathsf{Pos}_S$;*
4. *$S_S$ is a retract of $A_S$.*

For fixed elements $s \in S$, $t \in T$ and ${}_TA_S \in {}_T\mathsf{Pos}_S$, the mappings $\rho_s : A \to A$, $a \mapsto a \cdot s$, and $\lambda_t : A \to A$, $a \mapsto t \cdot a$, are morphisms in ${}_T\mathsf{Pos}$ and $\mathsf{Pos}_S$, respectively.

**Lemma 1** *For every ${}_TA_S \in {}_T\mathsf{Pos}_S$*

1. *if ${}_TB \in {}_T\mathsf{Pos}$ then ${}_T\mathsf{Pos}(B, A) \in \mathsf{Pos}_S$ with*

$$f \cdot s := \rho_s \circ f, \tag{1}$$

   *$f \in {}_T\mathsf{Pos}(B, A), s \in S$;*

2. *the assignment ${}_TB \mapsto {}_T\mathsf{Pos}(B, A)$ defines a contravariant pofunctor ${}_T\mathsf{Pos}(-, A) : {}_T\mathsf{Pos} \to \mathsf{Pos}_S$;*

3. *${}_T\mathsf{Pos}(T, A) \cong A_S$ in $\mathsf{Pos}_S$;*

4. *${}_T\mathsf{Pos}(A, A) \in {}_S\mathsf{Pos}_S$ with (1) and*

$$s \cdot f := f \circ \rho_s,$$

   *$f \in {}_T\mathsf{Pos}(A, A)$, $s \in S$;*

5. *the mapping $\rho : S \to {}_T\mathsf{Pos}(A, A)$, $s \mapsto \rho_s$, is a morphism in ${}_S\mathsf{Pos}_S$.*

For every $A_S \in \mathsf{Pos}_S$ we consider the set $\mathsf{End}(A_S) = \mathsf{Pos}_S(A, A)$ as a pomonoid with respect to composition and pointwise order. For every ${}_TA \in {}_T\mathsf{Pos}$ we consider the set $\mathsf{End}({}_TA) = {}_T\mathsf{Pos}(A, A)$ as a pomonoid with multiplication

$$f \bullet g := g \circ f,$$

$f, g \in \mathsf{End}({}_TA)$.

**Lemma 2** *For every* $A_S \in \mathsf{Pos}_S$, ${}_{\mathsf{End}(A_S)}A_S \in {}_{\mathsf{End}(A_S)}\mathsf{Pos}_S$ *with the left* $\mathsf{End}(A_S)$*-action defined by*

$$f \cdot a := f(a), \tag{2}$$

$a \in A$, $f \in \mathsf{End}(A_S)$.

**Proposition 2** *For every* ${}_TA_S \in {}_T\mathsf{Pos}_S$*, the mappings*

$$\lambda : T \to \mathsf{End}(A_S), \quad t \mapsto \lambda_t,$$
$$\rho : S \to \mathsf{End}({}_TA), \quad s \mapsto \rho_s,$$

*are pomonoid homomorphisms.*

**Def. 4** We call a biposet ${}_TA_S$ **faithful** (**regularly faithful, balanced, faithfully balanced**) if the pomonoid homomorphisms $\lambda : T \to \mathsf{End}(A_S)$ and $\rho : S \to \mathsf{End}({}_TA)$ are monomorphisms (resp. regular monomorphisms, epimorphisms, isomorphisms).

We say that a right $S$-poset $A_S$ is **faithful** (**regularly faithful, balanced, faithfully balanced**) if the biposet ${}_{\mathsf{End}(A_S)}A_S$ is faithful (resp. regularly faithful, balanced, faithfully balanced).

Thus ${}_TA_S$ (or $A_S$) is faithfully balanced if and only if it is regularly faithful and balanced.

Note that $\lambda : \mathsf{End}(A_S) \to \mathsf{End}(A_S)$ is the identity mapping, because $\lambda_f(a) = f \cdot a = f(a)$ for all $a \in A$, $f \in \mathsf{End}(A_S)$. Hence $A_S$ is faithful (regularly faithful, balanced, faithfully balanced) if and only if the mapping

$$\rho : S \to \mathsf{End}\left({}_{\mathsf{End}(A_S)}A\right)$$

is a monomorphism (resp. regular monomorphism, epimorphism, isomorphism). Moreover, the set $\mathsf{End}\left({}_{\mathsf{End}(A_S)}A\right) = {}_{\mathsf{End}(A_S)}\mathsf{Pos}(A, A)$ consists of order preserving mappings $f : A \to A$ (i.e. morphisms in $\mathsf{Pos}$) such that $f \circ k = k \circ f$ for all $k \in \mathsf{End}(A_S)$. That is, $\mathsf{End}\left({}_{\mathsf{End}(A_S)}A\right)$ is the centralizer of the submonoid $\mathsf{Pos}_S(A, A)$ in the monoid $\mathsf{Pos}(A, A)$. Thus $A_S$ is balanced if and only if only right translations by some element of $S$ commute with all endomorphisms of $A_S$, i.e.

$$C_{\mathsf{Pos}(A,A)}\left(\mathsf{Pos}_S(A, A)\right) = \{\rho_s : A \to A \mid s \in S\},$$

and $A_S$ is faithfully balanced if, in addition, the last pomonoid is isomorphic to $S$.

**Proposition 3** *Let ${}_TA_S \in {}_T\mathsf{Pos}_S$ be a faithfully balanced biposet. Then $A_S$ is a generator if and only if ${}_TA$ is a cyclic projective.*

**Proof.** If ${}_TA_S$ is faithfully balanced, by Lemma 1, the mappings $\lambda : T \to \mathsf{Pos}_S(A, A)$, $\rho : S \to {}_T\mathsf{Pos}(A, A)$ are isomorphisms in ${}_T\mathsf{Pos}$ and $\mathsf{Pos}_S$, respectively.

**Necessity.** If $A_S$ is a generator, by Theorem 1 there exist morphisms $A \underset{\gamma}{\overset{\pi}{\rightleftarrows}} S$ in $\mathsf{Pos}_S$ such that $\pi \circ \gamma = 1_S$. Applying the functor $\mathsf{Pos}_S(-, A)$ and Lemma 1, we obtain morphisms and isomorphisms

$$ {}_TA \cong \mathsf{Pos}_S(S, A) \underset{\mathsf{Pos}_S(\gamma,A)}{\overset{\mathsf{Pos}_S(\pi,A)}{\rightleftarrows}} \mathsf{Pos}_S(A, A) \cong {}_TT $$

in ${}_T\mathsf{Pos}$ such that $\mathsf{Pos}_S(\gamma, A) \circ \mathsf{Pos}_S(\pi, A) = \mathsf{Pos}_S(\pi \circ \gamma, A) = 1_{\mathsf{Pos}_S(S,A)}$. Hence ${}_TA$ is a retract of ${}_TT$, consequently a cyclic projective by Proposition 1.

**Sufficiency.** If ${}_TA$ is cyclic projective then, by Proposition 1, there are morphisms $T \underset{\gamma}{\overset{\pi}{\rightleftarrows}} A$ in ${}_T\mathsf{Pos}$ such that $\pi \circ \gamma = 1_A$. Applying the functor ${}_T\mathsf{Pos}(-, A)$ and Lemma 1, we obtain the diagram

$$ S_S \cong {}_T\mathsf{Pos}(A, A) \underset{{}_T\mathsf{Pos}(\gamma,A)}{\overset{{}_T\mathsf{Pos}(\pi,A)}{\rightleftarrows}} {}_T\mathsf{Pos}(T, A) \cong A_S $$

in $\mathsf{Pos}_S$ with ${}_T\mathsf{Pos}(\gamma, A) \circ {}_T\mathsf{Pos}(\pi, A) = 1_{{}_T\mathsf{Pos}(A,A)}$. So $S_S$ is a retract of $A_S$ and hence $A_S$ is a generator by Theorem 1. ■

**Theorem 2** *If a right $S$-poset $A_S$ is a generator then*

1. *$A_S$ is regularly faithful,*

2. *${}_{\mathrm{End}(A_S)}A$ is a cyclic projective.*

## Cyclic projective generators

**Proposition 4** *If $A_S$ is a cyclic projective then ${}_{\mathsf{End}(A_S)}A$ is a generator.*

**Theorem 3** *A right $S$-poset $A_S$ is a cyclic projective generator if and only if*

1. *$A_S$ is faithfully balanced,*
2. *${}_{\mathsf{End}(A_S)}A$ is a cyclic projective generator.*

**Proposition 5** *An $S$-poset $A_S$ is a cyclic projective generator in $\mathsf{Pos}_S$ if and only if $A_S \cong eS_S$ for an idempotent $e \in S$ with $e\mathcal{J}1$ (that is, $k'ek = 1$ for some $k, k' \in S$).*

## References

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