

Transposed Poisson Algebras and Superalgebras

Recent developments and connections with 3-Lie structures

Viktor Abramov

University of Tartu
Institute of Mathematics and Statistics

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- 1 Familiar ground: Poisson algebra and phase space
- 2 Transposition: transposed Poisson compatibility
- 3 Super and ternary extensions
- 4 Our construction: even derivation $\Rightarrow$ 3-Lie superalgebra
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Poisson $\longrightarrow$ transposed Poisson $\longrightarrow$ transposed Poisson superalgebra $\longrightarrow$ 3-Lie superalgebra

Poisson algebra

A Poisson algebra is a vector space P with two compatible binary operations:

- $(P, \cdot)$ is a commutative associative algebra;
- $(P, \{, \})$ is a Lie algebra;
- the operations satisfy the Leibniz rule.

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$$

For fixed x ,

$$ad_x(y) = \{x, y\}$$

is a derivation of the associative product.

Canonical example: phase space

For canonical coordinates (p_i, q_i) on phase space,

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

The space $C^\infty(M)$ has simultaneously:

- the Poisson Lie bracket;
- the pointwise commutative associative product.

This is the prototype we are going to transpose.

Transposed Poisson algebra

Keep the same two binary structures, but transpose the compatibility condition:

$$2z \cdot \{x, y\} = \{z \cdot x, y\} + \{x, z \cdot y\}$$

For fixed z , the multiplication operator

$$L_z(x) = z \cdot x$$

behaves like a derivation of the Lie bracket, up to the factor 2.

The derivation role moves from the bracket to multiplication.

A basic construction by a derivation

Let $(A, \cdot)$ be a commutative associative algebra and $D : A \rightarrow A$ a derivation:

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y).$$

Define

$$\boxed{[x, y] = x \cdot D(y) - y \cdot D(x).}$$

Then $(A, \cdot, [,])$ is a transposed Poisson algebra, since

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y].$$

Let $L = L_0 \oplus L_1$ be a $\mathbb{Z}_2$ -graded vector space.

$$[L_i, L_j] \subseteq L_{i+j}, \quad [x, y] = -(-1)^{|x||y|}[y, x].$$

The graded Jacobi identity is

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

3-Lie algebra

A vector space L equipped with a trilinear operation

$$[\ , \ , \] : L \times L \times L \longrightarrow L$$

is called a **3-Lie algebra** if

- the bracket is totally skew-symmetric:

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = \operatorname{sgn}(\sigma) [x_1, x_2, x_3], \quad \sigma \in S_3;$$

- the **Filippov–Jacobi identity** holds:

$$[x_1, x_2, [y_1, y_2, y_3]] = [[x_1, x_2, y_1], y_2, y_3] + [y_1, [x_1, x_2, y_2], y_3] + [y_1, y_2, [x_1, x_2, y_3]]$$

3-Lie superalgebra

A 3-Lie superalgebra has a trilinear bracket

$$[, ,] : L^{\otimes 3} \rightarrow L, \quad |[x, y, z]| = |x| + |y| + |z|.$$

It is graded skew-symmetric, e.g.

$$[y, x, z] = -(-1)^{|x||y|}[x, y, z],$$

$$[x, z, y] = -(-1)^{|y||z|}[x, y, z],$$

and satisfies the super Filippov–Jacobi identity

$$\begin{aligned} [x_1, x_2, [y_1, y_2, y_3]] &= [[x_1, x_2, y_1], y_2, y_3] \\ &\quad + (-1)^{(|x_1|+|x_2|)|y_1|}[y_1, [x_1, x_2, y_2], y_3] \\ &\quad + (-1)^{(|x_1|+|x_2|)(|y_1|+|y_2|)}[y_1, y_2, [x_1, x_2, y_3]]. \end{aligned}$$

Transposed Poisson superalgebra

Let $P = P_0 \oplus P_1$.

- $(P, \cdot)$ is a commutative associative superalgebra;
- $(P, [\cdot, \cdot])$ is a Lie superalgebra;
- the compatibility condition is

$$2z \cdot [x, y] = [z \cdot x, y] + (-1)^{|x||z|} [x, z \cdot y]$$

If D is even, then

$$[x, y] = x \cdot D(y) - (-1)^{|x||y|} y \cdot D(x)$$

gives the basic derivation construction.

Our construction: even derivation $\Rightarrow$ 3-Lie superalgebra

Let $(P, \cdot, [,])$ be a transposed Poisson superalgebra and D an even derivation of P . Define

$$\begin{aligned} [x, y, z]_D = & D(x) \cdot [y, z] + (-1)^{|x|(|y|+|z|)} D(y) \cdot [z, x] \\ & + (-1)^{(|x|+|y|)|z|} D(z) \cdot [x, y]. \end{aligned}$$

Theorem

$(P, [, ,]_D)$ is a 3-Lie superalgebra.

Why does the construction work?

Our proof of super Filippov-Jacobi identity is based on

- ① Transposed compatibility condition (graded version).
- ② Proved identities.
- ③ We expand the super Filippov–Jacobi identity, use the graded Leibniz rule for the even derivation D and identities.

The remaining terms cancel in structured cyclic families.

Even derivation D

$$|D(x)| = |x|$$

$\implies$ 3-Lie superalgebra

Odd derivation δ

$$|\delta(x)| = |x| + 1$$

The naive bracket has the wrong parity.
The paper instead introduces a
TP-compatible module structure with a
Jordan bracket.

Odd differentials are natural in graded differential geometry and BRST-type constructions.

For a commutative associative algebra $(A, \cdot)$ with a 3-Lie bracket $[\cdot, \cdot, \cdot]$, transposed compatibility is

$$3u \cdot [x, y, z] = [u \cdot x, y, z] + [x, u \cdot y, z] + [x, y, u \cdot z].$$

Equivalently, every left multiplication $L_u : x \mapsto u \cdot x$ is a $\frac{1}{3}$ -derivation:

$$\varphi[x, y, z] = \frac{1}{3}([\varphi x, y, z] + [x, \varphi y, z] + [x, y, \varphi z]).$$

$\frac{1}{3}$ -derivations $\longrightarrow$ compatible commutative products $\longrightarrow$ classification

Jiang–Kang–Lü: a 3-dimensional classification

The 3-Lie algebra studied is

$$A_3 : \quad [e_1, e_2, e_3] = e_1.$$

Under the additional assumption

$$L_{e_1} = 0,$$

the authors obtain ten pairwise non-isomorphic families

$$T_1, \dots, T_{10}.$$

For example,

$$T_1 : \quad e_2^2 = \alpha e_2, \quad e_2 e_3 = -\alpha e_3, \quad e_3^2 = -3\alpha e_2.$$

The theorem is a classification under $L_{e_1} = 0$, rather than a classification of all 3-dimensional transposed Poisson 3-Lie algebras.

Transposed Poisson 3-Lie Superalgebra

A triple $(P, \cdot, [\cdot, \cdot, \cdot])$ is called a **transposed Poisson 3-Lie superalgebra** if $(P, \cdot)$ is a commutative superalgebra, $(P, [\cdot, \cdot, \cdot])$ is a 3-Lie superalgebra, and for any elements $x, y, z, u \in P$ the following compatibility condition holds:

$$3u \cdot [x, y, z] = [u \cdot x, y, z] + (-1)^{|x||u|}[x, u \cdot y, z] + (-1)^{|u|(|x|+|y|)}[x, y, u \cdot z].$$

$\frac{1}{3}$ -derivations in the super case

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a 3-Lie superalgebra.

A homogeneous linear map $D : L \rightarrow L$ is a δ -**derivation** if

$$D[x, y, z] = \delta \left([D(x), y, z] + (-1)^{|D||x|} [x, D(y), z] + (-1)^{|D|(|x|+|y|)} [x, y, D(z)] \right).$$

For $\delta = \frac{1}{3}$:

$$D[x, y, z] = \frac{1}{3} \left([D(x), y, z] + (-1)^{|D||x|} [x, D(y), z] + (-1)^{|D|(|x|+|y|)} [x, y, D(z)] \right).$$

For a transposed Poisson 3-Lie superalgebra,

$$D = L_u, \quad L_u(x) = u \cdot x, \quad |L_u| = |u|,$$

is a $\frac{1}{3}$ -derivation.

Known classification of low-dimensional 3-Lie superalgebras

Let

$$L = L_{\bar{0}} \oplus L_{\bar{1}}, \quad \dim L = m|n.$$

In our previous work we classified low-dimensional 3-Lie superalgebras and determined their commutation relations.

For a homogeneous basis

$$\{e_i\} \subset L_{\bar{0}}, \quad \{f_\alpha\} \subset L_{\bar{1}},$$

the ternary bracket has four possible types:

$$[e_i, e_j, e_k] \in L_{\bar{0}},$$

$$[e_i, e_j, f_\alpha] \in L_{\bar{1}},$$

$$[e_i, f_\alpha, f_\beta] \in L_{\bar{0}},$$

$$[f_\alpha, f_\beta, f_\gamma] \in L_{\bar{1}}.$$

The graded Filippov identity imposes quadratic constraints on the structure constants.

The $3|1$ case

Consider

$$\dim L = 3|1, \quad L_{\bar{0}} = \langle e_1, e_2, e_3 \rangle, \quad L_{\bar{1}} = \langle f_1 \rangle.$$

The possible non-zero brackets are of the form

$$[e_i, e_j, e_k] \in L_{\bar{0}}, \quad [e_i, e_j, f_1] \in L_{\bar{1}}, \quad [e_i, f_1, f_1] \in L_{\bar{0}}.$$

Because of graded skew-symmetry,

$$[f_1, f_1, f_1] = 0.$$

The $3|1$ 3-Lie superalgebras are already classified.

The classification problem for us is therefore:

Which commutative associative products are compatible with these brackets?

$\frac{1}{3}$ -derivation of a 3-Lie superalgebra

Let

$$L = L_{\bar{0}} \oplus L_{\bar{1}}$$

be a 3-Lie superalgebra.

A homogeneous linear map

$$D : L \longrightarrow L$$

is called a $\frac{1}{3}$ -**derivation** if

$$D[x, y, z] = \frac{1}{3} \left([D(x), y, z] + (-1)^{|D||x|} [x, D(y), z] + (-1)^{|D|(|x|+|y|)} [x, y, D(z)] \right).$$

For a transposed Poisson 3-Lie superalgebra, define

$$L_u(x) = u \cdot x, \quad |L_u| = |u|.$$

Then

$$L_u \text{ is a } \frac{1}{3}\text{-derivation.}$$

A possible classification problem

Known: classification of low-dimensional 3-Lie superalgebras



Known: transposed Poisson 3-Lie algebras in dimension 3



Open problem: transposed Poisson 3-Lie superalgebras of dimension 3|1

For each classified 3-Lie superalgebra L :

classify compatible commutative associative superalgebra products $\cdot$

up to isomorphism.

The main tool:

$$u \mapsto L_u \quad \text{and} \quad L_u \in \text{Der}_{1/3}(L).$$

From Jiang–Kang–Lü to the super case

Jiang–Kang–Lü

$$3u[x, y, z] = [ux, y, z] + [x, uy, z] + [x, y, uz]$$

$$L_u \in \text{Der}_{1/3}(L)$$



Classification of compatible commutative products

Super extension

$$\begin{aligned} 3u[x, y, z] &= [ux, y, z] + (-1)^{|u||x|}[x, uy, z] \\ &\quad + (-1)^{|u|(|x|+|y|)}[x, y, uz] \end{aligned}$$

$$L_u \in \text{Der}_{1/3}^{\text{gr}}(L)$$



Classification in superdimension $3|1$

Same philosophy, new graded classification problem

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