

Ternary Structures

Ternary Lie Algebras at Cube Root of Unity

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The Navier–Stokes Millennium Problem

For an incompressible viscous fluid in three dimensions:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + f, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad x \in \mathbb{R}^3, \quad \nu > 0.$$

Existence and smoothness

Do smooth, divergence-free initial data always generate a globally smooth solution, or can a singularity develop in finite time?

One of the seven Millennium Prize Problems posed by the Clay Mathematics Institute in 2000.

A Claimed Breakthrough: Finite-Time Blow-Up

On September 8, 2026, OpenAI announced an analytical construction of smooth data for which a Navier–Stokes solution develops a singularity in finite time.

- The fluid starts from rest: $u_0 = 0$.
- The external force f is smooth and rapidly decaying.
- The total kinetic energy remains finite.
- Nevertheless, the velocity becomes unbounded in finite time.
- The proof is accompanied by a formal verification in Lean.

Current status

The result claims to establish the “breakdown” alternatives (C) and (D) in the official Clay formulation. Independent evaluation is still under way.

Sources: OpenAI, *On the Navier–Stokes Millennium Prize Problem* (8 September 2026); C. Fefferman, official Clay problem description.

From Commutative Algebras to Spaces

Theorem (Gelfand–Naimark)

Every unital commutative C^* -algebra A is isometrically $*$ -isomorphic to

$$C(X),$$

the algebra of continuous complex-valued functions on a compact Hausdorff space X .

The space X is recovered as the spectrum

$$X = \operatorname{Spec}(A) = \{\chi : A \rightarrow \mathbb{C} \mid \chi \neq 0, \chi \text{ is a } * \text{-homomorphism}\}.$$

Thus, the algebraic structure of $C(X)$ completely determines the underlying topological space X .

The Idea of Noncommutative Geometry

Classical geometry may be encoded by the commutative algebra

$$X \longleftrightarrow C(X).$$

Noncommutative geometry replaces $C(X)$ by a noncommutative algebra:

$$\text{"noncommutative space"} \longleftrightarrow A, \quad ab \neq ba.$$

The algebra A is interpreted as an algebra of functions on a virtual space whose ordinary points may no longer exist.

The deformation principle

The replacement is not arbitrary: A should preserve or deform essential structural properties of the original function algebra. The classical geometry must be recovered in an appropriate limit.

Two Models of Noncommutative Geometry

The Manin plane

Its coordinate algebra is generated by x and y with relation

$$xy = qyx, \quad q \in \mathbb{C}^\times.$$

For $q = 1$, the coordinates commute and the classical polynomial algebra $\mathbb{C}[x, y]$ is recovered.

Connes' spectral approach

A noncommutative geometry is encoded by a spectral triple

$$(A, \mathcal{H}, D),$$

where A is represented by operators on a Hilbert space $\mathcal{H}$, and D is a Dirac-type operator. The spectrum and commutators $[D, a]$ encode geometric data such as dimension, differential structure, distance and integration.

The Wess–Zumino Quantum Plane

J. Wess and B. Zumino, *Covariant Differential Calculus on the Quantum Hyperplane*, Nucl. Phys. B Proc. Suppl. **18B** (1991), 302–312; doi:10.1016/0920-5632(91)90143-3.

The quantum plane is described by the associative coordinate algebra

$$\mathcal{A}_q = \mathbb{C}\langle x, y \rangle / \langle xy - qyx \rangle, \quad q \in \mathbb{C}^\times.$$

Thus, its coordinates satisfy

$$xy = qyx.$$

- For $q = 1$, one recovers the classical polynomial algebra $\mathbb{C}[x, y]$.
- For $q \neq 1$, the order of coordinates matters.
- The “space” is encoded by $\mathcal{A}_q$, rather than by a set of ordinary points.

The algebra is covariant under the corresponding quantum group, in particular under $GL_q(2)$.

Covariant Differential Calculus

The coordinate algebra is extended by one-forms dx, dy and a differential

$$d = dx \partial_x + dy \partial_y, \quad d^2 = 0, \quad d(fg) = (df)g + f dg.$$

Compatibility with $xy = qyx$ leads to deformed commutation rules, for example

$$\begin{aligned} x dx &= q^2 dx x, & y dx &= q dx y, \\ x dy &= q dy x + (q^2 - 1) dx y, & y dy &= q^2 dy y, \end{aligned}$$

together with

$$dx dy = -q dy dx, \quad (dx)^2 = (dy)^2 = 0.$$

The central idea

Coordinates, differential forms and derivatives constitute one consistent algebraic system, covariant under quantum-group transformations.

Two Types of Noncommutative Deformation

In the usual deformation approach, a commutative function algebra is replaced by a family of noncommutative algebras:

$$\mathcal{A}_1 = C(X) \longrightarrow \mathcal{A}_q, \quad q \in \mathbb{C}^\times.$$

The parameter q varies continuously and measures the departure from classical commutative geometry:

$$q \longrightarrow 1 \implies \mathcal{A}_q \longrightarrow \mathcal{A}_1.$$

Deformation at a root of unity

A different algebraic regime appears when

$$q^N = 1, \quad q^k \neq 1 \quad (1 \leq k < N).$$

The admissible values of q are now discrete, and

$$[N]_q = 1 + q + \cdots + q^{N-1} = 0.$$

This setting naturally involves q -numbers, q -factorials and Gaussian binomial coefficients.

The Cubic Root of Unity as a Structural Principle

For the ternary case, the fundamental parameter is

$$j = e^{2\pi i/3}, \quad j^3 = 1, \quad 1 + j + j^2 = 0.$$

It enters directly into the defining relations:

Grassmann: $\theta^A \theta^B \theta^C = j \theta^B \theta^C \theta^A,$

Clifford: $Q_k Q_l Q_m = \rho_{klm} Q_l Q_m Q_k, \quad \rho_{klm} \in \{j, j^2\},$

Lie: $\{X, Y, Z\} = j \{Y, Z, X\} = j^2 \{Z, X, Y\}.$

Thus, ternarity is governed by the nontrivial one-dimensional representations of the cyclic group

$$\mathbb{Z}_3 = \{1, j, j^2\}.$$

V. Abramov, R. Kerner and B. Le Roy, J. Math. Phys. **38** (1997), 1650–1669.

The Grassmann Algebra

The Grassmann algebra Λ_N is the unital associative algebra generated by $\theta^1, \dots, \theta^N$ with relations

$$\theta^A \theta^B = -\theta^B \theta^A, \quad (\theta^A)^2 = 0.$$

Every element has a unique expansion

$$F = \sum_{k=0}^N \frac{1}{k!} F_{A_1 \dots A_k} \theta^{A_1} \dots \theta^{A_k},$$

where the coefficients $F_{A_1 \dots A_k}$ are skew-symmetric. Thus,

$$\dim \Lambda_N = 2^N.$$

The left Grassmann derivatives are defined by

$$\partial_A(\theta^B) = \delta_A^B, \quad \partial_A(FG) = (\partial_A F)G + (-1)^{|F|} F(\partial_A G),$$

for every homogeneous element F .

A $\mathbb{Z}_3$ -Analogue of the Grassmann Algebra

V. Abramov, R. Kerner and B. Le Roy, *Hypersymmetry: A $\mathbb{Z}_3$ -Graded Generalization of Supersymmetry*, J. Math. Phys. **38** (1997), 1650–1669; doi:10.1063/1.531821.

Let

$$j = e^{2\pi i/3}, \quad j^3 = 1, \quad 1 + j + j^2 = 0.$$

The algebra $\Lambda_N^{(3)}$ is generated by $\theta^1, \dots, \theta^N$ and defined by the ternary relations

$$\theta^A \theta^B \theta^C = j \theta^B \theta^C \theta^A.$$

Consequently,

$$(\theta^A)^3 = 0, \quad \theta^A \theta^B \theta^C \theta^D = 0.$$

Its homogeneous components have dimensions

$$N, \quad N^2, \quad 1 + \frac{N^3 - N}{3},$$

and therefore

$$\dim \Lambda_N^{(3)} = 1 + \frac{N(N+1)(N+2)}{3}.$$

A Ternary Analogue of the Lie Bracket

Let A be an associative complex algebra. Define the ternary commutator by

$$\begin{aligned}\{X, Y, Z\} = & XYZ + jYZX + j^2ZXY \\ & + ZYX + j^2YXZ + jXZY.\end{aligned}$$

It satisfies the cyclic covariance relations

$$\{X, Y, Z\} = j\{Y, Z, X\} = j^2\{Z, X, Y\}.$$

In particular,

$$\{X, Y, Z\} + \{Y, Z, X\} + \{Z, X, Y\} = 0$$

and

$$\{X, X, X\} = 0.$$

This construction replaces binary skew-symmetry by a $\mathbb{Z}_3$ -symmetry of a trilinear operation.

Recovering the Ordinary Lie Algebra

Suppose that the associative algebra A has a central unit 1 . Then

$$\{X, 1, Y\} = XY - YX = [X, Y].$$

Thus, every ternary algebra of this type contains the ordinary commutator algebra as a binary sector.

The usual Jacobi identity is recovered by inserting the unit:

$$\begin{aligned} \{\{X, 1, Y\}, 1, Z\} + \{\{Y, 1, Z\}, 1, X\} \\ + \{\{Z, 1, X\}, 1, Y\} = 0. \end{aligned}$$

An essential difference

Replacing the two occurrences of 1 by arbitrary elements does not produce a Jacobi identity. The genuinely ternary bracket satisfies more elaborate identities instead.

A Systematic Approach to Ternary Structures

A.M. Aader, V. Abramov and O. Liivapuu, *Associative Ternary Algebras and Ternary Lie Algebras at Cube Roots of Unity*, *Mathematics* **13** (2025), 3894; doi:10.3390/math13243894.

This work further develops the ternary structures introduced by Abramov, Kerner and Le Roy in 1997.

- We provide a more systematic and rigorous axiomatic framework.
- The starting point is a complex vector space equipped with a ternary algebraic operation

$$\mu: V \times V \times V \longrightarrow V.$$

- We relate two fundamental notions:

ternary associativity $\longleftrightarrow$ a ternary Jacobi-type identity.

- The underlying cyclic symmetry is governed by

$$\omega = e^{2\pi i/3}, \quad 1 + \omega + \omega^2 = 0.$$

Ternary ω -Lie Algebra

Definition

A complex vector space L equipped with a ternary bracket

$$[-, -, -]: L \times L \times L \longrightarrow L$$

is called a *ternary Lie algebra at cube roots of unity*, or a *ternary ω -Lie algebra*, if:

- ① the bracket is either $\mathbb{R}$ -linear or $\mathbb{C}$ -linear;
- ② it satisfies the ω -symmetry condition

$$[u, v, w] = \omega[v, w, u] = \omega^2[w, u, v];$$

- ③ it satisfies the $GA(1, 5)$ -identity

$$\circlearrowleft ([[u, v, w], x, y] + [[u, x, v], y, w] + [[u, y, x], w, v] + [[u, w, y], v, x]) = 0.$$

Here $\circlearrowleft$ denotes the sum over the five cyclic permutations of u, v, w, x, y .