

Transposed Poisson 3-Lie Superalgebras

Nikolai Sovetnikov

University of Tartu
Institute of Mathematics and Statistics

Geometry and Topology Seminar
08.10.2026

Lie algebras

All vector spaces are considered over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition

A **Lie algebra** is a vector space L with a bilinear map $[\cdot, \cdot]: L \times L \rightarrow L$ that satisfies, for $x, y, z \in L$,

$$[x, y] = -[y, x], \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

The bilinear map $[\cdot, \cdot]$ is called the **Lie bracket**.

In a basis $e_1, \dots, e_n$, the structure constants are defined by

$$[e_i, e_j] = c_{ij}^k e_k, \quad c_{ij}^k = -c_{ji}^k, \quad c_{ij}^m c_{mk}^l + c_{jk}^m c_{mi}^l + c_{ki}^m c_{mj}^l = 0.$$

Example: the cross product on $\mathbb{R}^3$

The Euclidean space $\mathbb{R}^3$ with the cross product is a Lie algebra. For an orthonormal basis $\{e_1, e_2, e_3\}$,

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

The remaining non-zero brackets follow from skew-symmetry. This example will later motivate the ternary construction on $\mathbb{R}^4$.

Matrix Lie groups

Definition

A **matrix Lie group** is any subgroup G of $GL(n; \mathbb{K})$ such that, if a sequence $A_m \in G$ converges to a matrix $A \in GL(n; \mathbb{K})$, then $A \in G$.

Examples are

$$SL(n; \mathbb{K}) = \{A \in GL(n; \mathbb{K}) : \det A = 1\},$$

$$O(n; \mathbb{K}) = \{A \in GL(n; \mathbb{K}) : A^t = A^{-1}\},$$

$$U(n) = \{A \in GL(n; \mathbb{C}) : A^\dagger = A^{-1}\}.$$

For a matrix Lie group G , its **matrix Lie algebra** consists of the matrices A for which $e^{tA} \in G$ for every $t \in \mathbb{R}$, with bracket $[A, B] = AB - BA$.

Lie algebras of matrix Lie groups

The corresponding Lie algebras are:

$$\mathrm{GL}(n; \mathbb{K}) \longmapsto \mathfrak{gl}(n; \mathbb{K}) = M(n; \mathbb{K})$$

$$\mathrm{SL}(n; \mathbb{K}) \longmapsto \mathfrak{sl}(n; \mathbb{K}) = \{A \in M(n; \mathbb{K}) : \mathrm{Tr}(A) = 0\}$$

$$\mathrm{O}(n; \mathbb{K}), \mathrm{SO}(n; \mathbb{K}) \longmapsto \mathfrak{o}(n; \mathbb{K}) = \mathfrak{so}(n; \mathbb{K}) = \{A \in M(n; \mathbb{K}) : A^t = -A\}$$

$$\mathrm{U}(n) \longmapsto \mathfrak{u}(n) = \{A \in M(n; \mathbb{C}) : A^\dagger = -A\}$$

$$\mathrm{SU}(n) \longmapsto \mathfrak{su}(n) = \{A \in M(n; \mathbb{C}) : A^\dagger = -A, \mathrm{Tr}(A) = 0\}$$

In every case, the Lie bracket is the matrix commutator

$$[A, B] = AB - BA.$$

The commutator

Proposition

Let $(A, \cdot)$ be an algebra. Define

$$[x, y] := x \cdot y - y \cdot x.$$

Then $(A, [,])$ is a Lie algebra.

Proof. The bracket is bilinear and skew-symmetric. Moreover,

$$[x, [y, z]] = xyz - xzy - yzx + zyx,$$

$$[y, [z, x]] = yzx - yxz - zxy + xzy, \quad [z, [x, y]] = zxy - zyx - xyz + yxz.$$

Hence

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Derivations of algebras

Definition

A derivation of an algebra A is a linear map $D: A \rightarrow A$ such that, for all $x, y \in A$,

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y).$$

This condition is called the **Leibniz rule**.

Proposition

If $D_1, D_2 \in \text{Der}(A)$, then

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1 \in \text{Der}(A).$$

Hence $(\text{Der}(A), [,]) is a Lie algebra.$

A Lie bracket with derivation

Let $(A, \cdot)$ be an algebra and D a derivation of it. Define

$$[x, y] = D(x) \cdot y - x \cdot D(y).$$

Proposition

If $(A, \cdot)$ is a commutative algebra, then $(A, [,])$ is a Lie algebra.

Thus a single derivation on a commutative algebra produces a Lie bracket.

Derivations of Lie algebras

Definition

Let $(L, [\cdot, \cdot])$ be a Lie algebra. A derivation of L is a linear map $D: L \rightarrow L$ such that

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

For every $x \in L$, the map

$$\operatorname{ad}_x: L \rightarrow L, \quad \operatorname{ad}_x(y) := [x, y],$$

is a derivation. Such derivations are called *inner*.

3-Lie algebras

Definition

A **3-Lie algebra** is a vector space L with a trilinear map $[\cdot, \cdot, \cdot]: L \times L \times L \rightarrow L$ such that, for $x, y, z, u, v \in L$,

$$[y, x, z] = -[x, y, z] = [x, z, y],$$

$$[[x, y, z], u, v] = [[x, u, v], y, z] + [x, [y, u, v], z] + [x, y, [z, u, v]].$$

The trilinear map is called the **ternary Lie bracket**; the second identity is the **Filippov–Jacobi identity**.

V. T. Filippov, “ n -Lie algebras” *Siberian Mathematical Journal* **26**(6) (1985), pp. 879–891.

Example: a 3-Lie algebra on $\mathbb{R}^4$ (Filippov, 1985)

Let $\mathbb{R}^4$ have an orthonormal basis $\{e_1, e_2, e_3, e_4\}$. For $x = x^i e_i$, $y = y^i e_i$, and $z = z^i e_i$, define

$$[x, y, z] = \begin{vmatrix} x^1 & x^2 & x^3 & x^4 \\ y^1 & y^2 & y^3 & y^4 \\ z^1 & z^2 & z^3 & z^4 \\ e_1 & e_2 & e_3 & e_4 \end{vmatrix}.$$

This ternary operation is the four-dimensional analogue of the vector product on $\mathbb{R}^3$. It defines a 3-Lie algebra structure on $\mathbb{R}^4$.

Example: commuting derivations (Filippov, 1985)

Let $C^\infty(\mathbb{R}^n)$ be the algebra of smooth functions and let D_1, D_2, D_3 be pairwise commuting derivations. Define

$$\{f, g, h\} = \begin{vmatrix} D_1(f) & D_1(g) & D_1(h) \\ D_2(f) & D_2(g) & D_2(h) \\ D_3(f) & D_3(g) & D_3(h) \end{vmatrix}.$$

Then $(C^\infty(\mathbb{R}^n), \{, , \})$ is a 3-Lie algebra.

Poisson algebras

Definition

A vector space P with two bilinear maps $\cdot: P \times P \rightarrow P$ and $\{, \}: P \times P \rightarrow P$ is a **Poisson algebra** if $(P, \cdot)$ is a commutative algebra, $(P, \{, \})$ is a Lie algebra, and

$$\{z, x \cdot y\} = \{z, x\} \cdot y + x \cdot \{z, y\}$$

holds for every $x, y, z \in P$. The operation $\{, \}$ is called the **Poisson bracket**.

A derivation of a Poisson algebra is a linear map $D: P \rightarrow P$ that is simultaneously a derivation of $(P, \cdot)$ and $(P, \{, \})$:

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y),$$

$$D(\{x, y\}) = \{D(x), y\} + \{x, D(y)\}.$$

The Poisson bracket

On $C^\infty(\mathbb{R}^{2n})$, with coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$, define

$$\{f, g\} = \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right).$$

This bracket is bilinear, skew-symmetric, satisfies the Jacobi identity, and obeys

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Therefore $C^\infty(\mathbb{R}^{2n})$ is a Poisson algebra.

Transposed Poisson algebras

Definition

A vector space T with two bilinear maps $\cdot: T \times T \rightarrow T$ and $[\cdot, \cdot]: T \times T \rightarrow T$ is called a **transposed Poisson algebra** if $(T, \cdot)$ is a commutative algebra, $(T, [\cdot, \cdot])$ is a Lie algebra, and

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y]$$

holds for all $x, y, z \in T$. The compatibility condition is called the **transposed Leibniz rule**.

C. Bai et al., “Transposed Poisson algebras, Novikov–Poisson algebras and 3-Lie algebras”
Journal of Algebra **632** (2023), pp. 535–566.

A construction of transposed Poisson algebras

Theorem

Let $(A, \cdot)$ be a commutative algebra and $D: A \rightarrow A$ its derivation. Define

$$[x, y] = D(x) \cdot y - x \cdot D(y).$$

Then $(A, \cdot, [,])$ is a transposed Poisson algebra.

The coefficient 2 follows directly when the right-hand side is expanded using the Leibniz rule.

When both compatibilities hold

Proposition

Let $(L, \cdot, [\cdot, \cdot])$ be such that $(L, \cdot)$ is a commutative algebra and $(L, [\cdot, \cdot])$ is a Lie algebra. The algebra $(L, \cdot, [\cdot, \cdot])$ is simultaneously a Poisson algebra and a transposed Poisson algebra if and only if

$$x \cdot [y, z] = [x \cdot y, z] = 0$$

for all $x, y, z \in L$.

Transposed Poisson 3-Lie algebras

Definition

Let $(L, \cdot)$ be a commutative algebra and $(L, [\cdot, \cdot])$ a 3-Lie algebra. The algebra $(L, \cdot, [\cdot, \cdot])$ is called a transposed Poisson 3-Lie algebra if, for all $x, y, z, u \in L$,

$$3u \cdot [x, y, z] = [u \cdot x, y, z] + [x, u \cdot y, z] + [x, y, u \cdot z].$$

Theorem

Let $(L, \cdot, [\cdot, \cdot])$ be a transposed Poisson algebra and $D: L \rightarrow L$ its derivation. Define

$$[x, y, z] = D(x)[y, z] + D(y)[z, x] + D(z)[x, y].$$

Then $(L, \cdot, [\cdot, \cdot])$ is a transposed Poisson 3-Lie algebra.

Super vector spaces and superalgebras

A **super vector space** is a vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ ($\bar{0}, \bar{1} \in \mathbb{Z}_2$) with a **parity** defined on homogeneous elements:

$$\bar{x} = \begin{cases} \bar{0}, & x \in V_{\bar{0}}, \\ \bar{1}, & x \in V_{\bar{1}}. \end{cases}$$

Definition

An algebra $(A, \cdot)$ is called a **superalgebra** if $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a super vector space and

$$\overline{x \cdot y} = \bar{x} + \bar{y}.$$

Commutative superalgebras

One of the important tools widely used in computations in superalgebras is the **Koszul sign rule**: *if in an algebraic expression two neighboring elements with parities a and b are interchanged, then this interchange is accompanied by multiplication of the expression by the factor $(-1)^{ab}$.*

Definition

A superalgebra $(A, \cdot)$ is called a **commutative superalgebra** if

$$x \cdot y = (-1)^{\bar{x}\bar{y}} y \cdot x.$$

Derivations of superalgebras

A linear map $D: A \rightarrow A$ is called **even** if $\overline{D(x)} = \bar{x}$ and **odd** if $\overline{D(x)} = \bar{x} + \bar{1}$.

Definition

A linear map $D: A \rightarrow A$ is a derivation of the superalgebra $(A, \cdot)$ if

$$D(x \cdot y) = D(x) \cdot y + (-1)^{\bar{x}\bar{D}} x \cdot D(y)$$

for all $x, y \in A$.

Definition

A super vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with a bilinear map $[\cdot, \cdot]: L \times L \rightarrow L$ is a Lie superalgebra if, for all $x, y, z \in L$,

$$\overline{[x, y]} = \bar{x} + \bar{y},$$

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x],$$

$$(-1)^{\bar{x}\bar{z}}[[x, y], z] + (-1)^{\bar{y}\bar{x}}[[y, z], x] + (-1)^{\bar{z}\bar{y}}[[z, x], y] = 0.$$

3-Lie superalgebras

Definition

A super vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with a trilinear bracket $[\cdot, \cdot, \cdot]: L^3 \rightarrow L$ is a 3-Lie superalgebra if, for all $x, y, z, u, v \in L$,

$$\overline{[x, y, z]} = \bar{x} + \bar{y} + \bar{z},$$

$$-(-1)^{\bar{x}\bar{y}}[y, x, z] = [x, y, z] = -(-1)^{\bar{y}\bar{z}}[x, z, y],$$

$$\begin{aligned} [[x, y, z], u, v] &= (-1)^{(\bar{y}+\bar{z})(\bar{u}+\bar{v})} [[x, u, v], y, z] \\ &\quad + (-1)^{\bar{x}(\bar{y}+\bar{z})+(\bar{x}+\bar{z})(\bar{u}+\bar{v})} [[y, u, v], z, x] \\ &\quad + (-1)^{(\bar{x}+\bar{y})(\bar{z}+\bar{u}+\bar{v})} [[z, u, v], x, y]. \end{aligned}$$

Transposed Poisson superalgebras

Definition

A triple $(T, \cdot, [\cdot, \cdot])$ is called a transposed Poisson superalgebra if $(T, \cdot)$ is a commutative superalgebra, $(T, [\cdot, \cdot])$ is a Lie superalgebra, and

$$2z \cdot [x, y] = [z \cdot x, y] + (-1)^{\bar{x}\bar{z}}[x, z \cdot y]$$

for any $x, y, z \in T$.

This superalgebraic generalization was first introduced and studied in

V. Abramov and O. Liivapuu, "Transposed Poisson superalgebra," *Proceedings of the Estonian Academy of Sciences* **73**(1) (2024), pp. 50–59.

A construction with an even derivation

Theorem

Let $(A, \cdot)$ be a commutative superalgebra and $D: A \rightarrow A$ its even derivation. Define

$$[x, y] = D(x) \cdot y - (-1)^{\bar{x}\bar{y}} x \cdot D(y).$$

Then $(A, \cdot, [,])$ is a transposed Poisson superalgebra.

Identities in a transposed Poisson superalgebra

Theorem

Let $(P, \cdot, [\cdot, \cdot])$ be a transposed Poisson superalgebra. The following identities hold:

$$(-1)^{\bar{x}\bar{z}}x[y, z] + (-1)^{\bar{x}\bar{y}}y[z, x] + (-1)^{\bar{y}\bar{z}}z[x, y] = 0,$$

$$(-1)^{\bar{x}\bar{z}}[h \cdot [x, y], z] + (-1)^{\bar{x}\bar{y}}[h \cdot [y, z], x] + (-1)^{\bar{y}\bar{z}}[h \cdot [z, x], y] = 0,$$

$$(-1)^{\bar{x}\bar{z}}[h \cdot x, [y, z]] + (-1)^{\bar{x}\bar{y}}[h \cdot y, [z, x]] + (-1)^{\bar{y}\bar{z}}[h \cdot z, [x, y]] = 0,$$

$$(-1)^{\bar{x}\bar{z}}[h, x][y, z] + (-1)^{\bar{x}\bar{y}}[h, y][z, x] + (-1)^{\bar{y}\bar{z}}[h, z][x, y] = 0,$$

$$2u \cdot v \cdot [x, y] = (-1)^{\bar{x}\bar{v}}[u \cdot x, v \cdot y] + (-1)^{\bar{u}(\bar{x}+\bar{v})}[v \cdot x, u \cdot y],$$

$$(-1)^{\bar{u}(\bar{y}+\bar{v})}x \cdot [u, y \cdot v] + (-1)^{(\bar{x}+\bar{y})\bar{v}}v \cdot [x \cdot y, u] + (-1)^{\bar{x}(\bar{y}+\bar{v})}y \cdot [v, x] \cdot u = 0.$$

Transposed Poisson 3-Lie superalgebras

Definition

A triple $(T, \cdot, [\cdot, \cdot])$ is called a transposed Poisson 3-Lie superalgebra if $(T, \cdot)$ is a commutative superalgebra, $(T, [\cdot, \cdot])$ is a 3-Lie superalgebra, and, for all $x, y, z, u \in T$,

$$3u[x, y, z] = [ux, y, z] + (-1)^{\bar{x}\bar{u}}[x, uy, z] + (-1)^{\bar{u}(\bar{x}+\bar{y})}[x, y, uz].$$

Theorem (Abramov, Sovetnikov, 2025)

Let $(T, \cdot, [\cdot, \cdot])$ be a transposed Poisson superalgebra and let $D : T \rightarrow T$ be its even derivation. Define

$$[x, y, z] := D(x) \cdot [y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})} D(y) \cdot [z, x] + (-1)^{(\bar{x}+\bar{y})\bar{z}} D(z) \cdot [x, y].$$

Then $(T, [\cdot, \cdot, \cdot])$ is a 3-Lie superalgebra.

Transposed Poisson 3-Lie superalgebras

Theorem

Let $(T, \cdot, [\cdot, \cdot])$ be a transposed Poisson superalgebra and let $D: T \rightarrow T$ be its even derivation. Then the ternary bracket

$$[x, y, z] := D(x) \cdot [y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})} D(y) \cdot [z, x] + (-1)^{(\bar{x}+\bar{y})\bar{z}} D(z) \cdot [x, y]$$

satisfies the compatibility condition

$$3u[x, y, z] = [ux, y, z] + (-1)^{\bar{x}\bar{u}}[x, uy, z] + (-1)^{\bar{u}(\bar{x}+\bar{y})}[x, y, uz],$$

and $(T, \cdot, [\cdot, \cdot, \cdot])$ is a transposed Poisson 3-Lie superalgebra.

A broad class of transposed Poisson 3-Lie superalgebras

Corollary

Let $(A, \cdot)$ be a commutative superalgebra and let D_1, D_2 be its even commuting derivations. Define

$$\begin{aligned} [x, y, z] := & D_2(x) \left(yD_1(z) - (-1)^{\bar{y}\bar{z}} zD_1(y) \right) \\ & + (-1)^{\bar{x}(\bar{y}+\bar{z})} D_2(y) \left(zD_1(x) - (-1)^{\bar{x}\bar{z}} xD_1(z) \right) \\ & + (-1)^{(\bar{x}+\bar{y})\bar{z}} D_2(z) \left(xD_1(y) - (-1)^{\bar{x}\bar{y}} yD_1(x) \right). \end{aligned}$$

Then $(A, \cdot, [, ,]) is a transposed Poisson 3-Lie superalgebra.$

References

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