Superpositional graphs

Binary graph is directed acyclic weakly connected graph, which has root node and two terminal nodes - 0 and 1. Every intermediate node \( v \) has two successors: \( \text{low}(v) \) and \( \text{high}(v) \). Therefore, edge \( a \to b \) is 0-edge (1-edge) if \( \text{low}(a) = b \) (\( \text{high}(a) = b \)).

**Definition 1.** A superposition of \( E \) into \( G \) instead of \( v \) (\( G_{v \leftarrow E} \)) is a graph, which we receive by deleting \( v \) from \( G \) and redirecting all edges, pointing to \( v \), to the root of \( E \), all edges of \( E \) pointing to terminal 1 to the node \( \text{high}(v) \) and all edges pointing to the terminal 0 to the node \( \text{low}(v) \).

We define binary graphs \( A, C \) and \( D \), whose descriptions are in Fig. 1. We define inductively set of superpositional graphs (SPG):

**Definition 2.**

1° Graph \( A \in \text{SPG} \).

2° If \( G \in \text{SPG} \) and \( v \in V(G) \), then \( G_{v \leftarrow C} \in \text{SPG} \) and \( G_{v \leftarrow D} \in \text{SPG} \).

Let the Hamiltonian path of \( n \)-node SPG \( G \) consists of nodes \( v_1, \ldots, v_n \); and \( v_{n+1} \) represents terminal 0 and also terminal 1. The meaning of the propositional variables \( x_{i,j} \) and \( y_{i,j} \) is respectively existence of 0-edge and 1-edge from node \( v_i \) to node \( v_j \):

\[
  x_{i,j} = \begin{cases} 
    1, & \text{if there exists 0-edge } v_i \to v_j; \\
    0, & \text{otherwise.}
  \end{cases}
\]

\[
  y_{i,j} = \begin{cases} 
    1, & \text{if there exists 1-edge } v_i \to v_j; \\
    0, & \text{otherwise.}
  \end{cases}
\]

Therefore propositional formulae for class \( \text{SPG} \) is defined on the set

\[
  X = \{ x_{i,j}, y_{i,j} : 1 \leq i < j \leq n + 1 \} \]
Description of SPG by means of propositional formulae

The formula is

\[ F = P_1 \& P_2 \& P_3 \& P_6 \& P_7 \& P_8 \& P_9 \]

where

\[ P_1 = \bigwedge_{1 \leq i \leq (n-1)} \text{xor}(x_{i,i+1}, y_{i,i+1}) \]

\[ P_2 = \bigwedge_{1 \leq i \leq n-1} \text{exactlyone}(x_{i,j} : i < j \leq n + 1) \]

\[ P_3 = \bigwedge_{1 \leq i \leq n-1} \text{exactlyone}(y_{i,j} : i < j \leq n + 1) \]

\[ P_6 = \bigwedge_{1 \leq k < l < p < r \leq (n+1)} (x_{k,p} \& x_{l,r}) \]

\[ P_7 = \bigwedge_{1 \leq k < l < p < r \leq (n+1)} (y_{k,p} \& y_{l,r}) \]

\[ P_8 = \bigwedge_{1 \leq k < s < l < p < r \leq (n+1)} (x_{k,p} \& y_{l,r} \& y_{s,t}) \]

\[ P_9 = \bigwedge_{1 \leq k < s < l < p < r \leq (n+1)} (y_{k,p} \& x_{l,r} \& x_{s,t}) \]