Monotone Boolean Functions

Function $f: \{0, 1\}^n \to \{0, 1\}$ is a Boolean function of $n$ variables. Let $\alpha = \alpha_1 \ldots \alpha_n \in \{0, 1\}^n$ and $\beta = \beta_1 \ldots \beta_n \in \{0, 1\}^n$. Let

$$\alpha \prec \beta \equiv \exists i [\alpha_i < \beta_i \& \forall j ((j \neq i) \to (\alpha_j = \beta_j))].$$

By $\prec^+$ we denote the transitive closure of the relation $\prec$.

**Definition 1.** Function $f: \{0, 1\}^n \to \{0, 1\}$ is monotone, if $f(\alpha_1, \ldots, \alpha_n) \leq f(\beta_1, \ldots, \beta_n)$ whenever $(\alpha_1, \ldots, \alpha_n) \prec^+ (\beta_1, \ldots, \beta_n)$

Our goal is to describe the notion of monotonicity using family of propositional formulae. Every Boolean function of $n$ variables can be represented by its truth-table, so we use $2^n$ propositional variables $x_0, \ldots, x_{2^n-1}$; one variable for each entry of the truth-table. For convenience we suppose that indexes of the variables are binary numbers. The correspondence of the Boolean function $f: \{0, 1\}^n \to \{0, 1\}$ to the bitstring $f_0, \ldots, f_{2^n-1}$ is defined by $f(\alpha) = f_\alpha$ for every $\alpha \in \{0, 1\}^n$. Using this encoding, we can consider propositional formulae with variables $x_0, \ldots, x_{2^n-1}$ as descriptions of the properties of $n$-variable Boolean functions.

**Theorem 1.** Function $f: \{0, 1\}^n \to \{0, 1\}$ with truth-table $f_0, \ldots, f_{2^n-1}$ is monotone if and only if truth assignment $f_0, \ldots, f_{2^n-1}$ satisfies propositional formula

$$M_n(x_0, \ldots, x_{2^n-1}) \equiv \bigwedge_{\alpha, \beta \in \{0, 1\}^n: \alpha \prec \beta} (\overline{x_\alpha} \lor x_\beta) \quad (1)$$