A Product Quasi-Interpolation Method for Weakly Singular Volterra Integral Equations

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Abstract. For a weakly singular Volterra integral equation, we propose a method of Nyström type of accuracy $O(h^m)$ based on the smoothing change of variables and on the product quasi-interpolation by smooth splines of degree $m-1$ on the uniform grid.

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1. INTRODUCTION

Different methods of Nyström type for weakly singular Volterra and Fredholm integral equations have been constructed in [1], [2], [4]. In the present paper, we propose for a weakly singular Volterra integral equation a method of Nyström type of accuracy $O(h^m)$ based on the smoothing change of variables and on the product quasi-interpolation by smooth splines of degree $m-1$ on the uniform grid. Similar method for weakly singular Fredholm equations has been developed in [5].

2. THE PROBLEM

Consider the weakly singular Volterra integral equation

$$u(x) = \int_0^x \left(a(x,y)(x-y)^{-\nu} + b(x,y)\right) u(y)dy + f(x), \quad 0 \le x \le 1, \tag{1}$$

where $0 < \nu < 1$, a and b are defined and C^m-smooth for $0 \le x \le 1$, $0 < y \le x+\delta$, $\delta > 0$, $m \in \mathbf{N}$, and satisfy there for $k+l \le m$ the inequalities

$$|\,\partial_x^k \partial_y^l a(x,y)\,| \le cy^{-\lambda-l}, \quad |\,\partial_x^k \partial_y^l b(x,y)\,| \le cy^{-\mu-l}, \quad \nu+\lambda < 1, \quad \mu < 1. \tag{2}$$

With the change of variables

$$x = t^r, \quad y = s^r, \quad 0 \le t \le 1, \quad 0 \le s \le t+\delta_r, \quad r \in \mathbf{N}, \quad (1+\delta_r)^r = 1+\delta, \tag{3}$$

equation (1) takes with respect to $v(t) = u(t^r)$ the form

$$v(t) = \int_0^t \left(\mathscr{A}(t,s)(t-s)^{-\nu} + \mathscr{B}(t,s)\right) v(s)ds + g(t), \quad 0 \le t \le 1, \tag{4}$$

which is similar to (1). Here

$$g(t) = f(t^r), \quad \mathscr{A}(t,s) = ra(t^r,s^r)\Phi(t,s)^{-\nu}s^{r-1}, \quad \mathscr{B}(t,s) = rb(t^r,s^r)s^{r-1},$$

$$\Phi(t,s) = \left\{ \begin{array}{ll} \frac{t^r-s^r}{t-s}, & t \ne s \\ rt^{r-1}, & t = s \end{array} \right\} = \sum_{k=0}^{r-1} t^{r-1-k}s^k, \quad 0 \le t \le 1, \quad 0 < s \le t+\delta_r.$$

We assume that the smoothing parameter $r \in \mathbf{N}$ satisfies the inequalities

$$r > (1-\nu)/(1-\nu-\lambda), \quad r > 1/(1-\mu). \tag{5}$$

Then $\mathscr{A}(t,s) \to 0$, $\mathscr{B}(t,s) \to 0$ as $s \to 0$, $0 \le t \le 1$. Extending $\mathscr{A}(t,s)$ and $\mathscr{B}(t,s)$ by the zero value for $s \le 0$, the extended $\mathscr{A}(t,s)$ and $\mathscr{B}(t,s)$ are continuous for $0 \le t \le 1$, $-\infty < s \le t+\delta_r$.

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3. OPERATOR FORM OF THE METHOD

Let $h = 1/n$, $n \in \mathbf{N}$, $n \geq (m-1)/\delta_r$. We call attention to a product quasi-interpolation method which we first present in the operator form:

$$v_n(t) = \int_0^t [(t-s)^{-\nu} Q'_{h,m}(\mathscr{A}(t,s)v_n(s)) + Q'_{h,m}(\mathscr{B}(t,s)v_n(s))]ds + g(t), \quad 0 < t < 1, \tag{6}$$

$$v_n(t) = (\Lambda_m v_n)(t) \quad \text{for } 1 \leq t \leq 1 + (m-1)h. \tag{7}$$

Here $\Lambda_m v$ is the Lagrange interpolant of v by polynomials of degree $m-1$ constructed using, in case of even m, the knots $1 - jh$, $j = 0, ..., m-1$, and in case of odd m, the knots $1 - (j + \frac{1}{2})h$, $j = 0, ..., m-1$, whereas $Q'_{h,m} w$ is the quasi-interpolant of w by polynomial splines of degree $m-1 \geq 2$, defect 1, with spline knots jh, $j \geq -m+1$ constructed in [3]. Namely, for a function $w(s)$, $s \in [-(m-1)h, (\lceil nt \rceil + (m-1))h]$, depending on t, $0 < t \leq 1$, as a parameter, the quasi-interpolant $Q'_{h,m} w$ is defined for $s \in [0,t]$ by the formula

$$(Q'_{h,m} w)(s) = \sum_{j=-m+1}^{\lceil nt \rceil - 1} \left(\sum_{|p| \leq m_1 - 1} \alpha'_{p,m} w((j - p + \frac{m}{2})h) \right) B_m(ns - j),$$

where $\lceil nt \rceil$ is the smallest integer $\geq nt$,

$$m_1 = \left\{ \begin{array}{ll} \frac{m}{2} + 1, & m \text{ even} \\ \frac{m+1}{2}, & m \text{ odd} \end{array} \right\} = m - m_0, \quad m_0 = \left\{ \begin{array}{ll} \frac{m}{2} - 1, & m \text{ even} \\ \frac{m-1}{2}, & m \text{ odd} \end{array} \right\},$$

$$B_m(x) = \frac{1}{(m-1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x-i)_+^{m-1}, \quad x \in \mathbf{R}, \text{ is the father B-spline},$$

$$\alpha'_{p,m} = \sum_{q=|p|}^{m_1 - 1} (-1)^{k+q} \binom{2q}{k+q} \gamma_{q,m}, \quad |\, p \,| \leq m_1 - 1,$$

$$\gamma_{0,m} = 1, \quad \gamma_{q,m} = \sum_{l=1}^{m_0} \frac{(1 + z_{l,m}) z_{l,m}^{m_0 + q - 1}}{(1 - z_{l,m})^{2q+1} P'_m(z_{l,m})}, \quad q \geq 1,$$

$z_{l,m} \in (-1, 0)$, $l = 1, ..., m_0$, are roots of the characteristic polynomial $P_m(z) = \sum_{|k| \leq m_0} B_m(k + \frac{m}{2}) z^{k+m_0}$ (they are simple; $1/z_{l,m} \in (-\infty, -1)$, $l = 1, ..., m_0$, are the other m_0 roots of $P_m \in \mathscr{P}_{2m_0}$).

4. MATRIX FORM OF THE METHOD

Note that $v_n(0) = g(0) = f(0)$. The solution v_n of problem (6)–(7) is uniquely determined on $[0,1]$ by the knot values $v_n((i + \frac{m}{2})h)$ for $0 < (i + \frac{m}{2})h \leq 1$. Collocating (6) at these points, the matrix form of method (6)–(7) follows. For even m, we obtain with respect to $v_{i,n} := v_n(ih)$, $i = 1, ..., n+m$, the system of linear equations

$$v_{i,n} = \sum_{k=1}^{i+m-1} \tau_{i,k} v_{k,n} + g(ih), \quad i = 1, ..., n, \quad v_{i,n} = \sum_{j=0}^{m-1} \sigma_{i,j} v_{n-j,n}, \quad i = n+1, ..., n+m-1, \tag{8}$$

where

$$\sigma_{i,j} = \prod_{j \neq j'=0}^{m-1} \frac{j' + (i-n)}{j' - j}, \quad i = n+1, ..., n+m-1, \quad j = 0, ..., m-1,$$

$$\tau_{i,k} = a_{i,k} \sum_{j=k-m}^{\min\{k,i-1\}} \beta_{i,j} \alpha'_{j-k+m/2,m} + b_{i,k} \sum_{j=k-m}^{\min\{k,i-1\}} \beta^0_{i,j} \alpha'_{j-k+m/2,m}, \quad i = 1, ..., n, \quad k = 1, ..., n+m-1,$$

$$a_{i,k} = \mathscr{A}(ih, kh), \quad b_{i,k} = \mathscr{B}(ih, kh), \quad , i = 1, ..., n, \quad k = 1, ..., n+m-1,$$

$$\beta_{i,j} = \int_0^{ih} (ih - s)^{-\nu} B_m(ns - j) ds, \quad \beta^0_{i,j} = \int_0^{ih} B_m(ns - j) ds, \quad i = 1, ..., n, \quad j = -m+1, ..., i-1. \tag{9}$$

The unknowns $v_{i,n}$, $i = n+1, ..., n+m$, can be eliminated from system (8).

5. FORMULAE FOR QUADRATURE COEFFICIENTS (9)

Again for even $m \geq 3$,

$$\beta_{i,j}^0 = \frac{h}{m!} \triangle^m \gamma_{i,j}^0, \quad \beta_{i,j} = h^{1-\nu} \triangle^m \gamma_{i,j}, \quad i = 1, ..., n, \quad j = -m+1, ..., i-1,$$

where $\triangle^m$ is the forward difference of order m, $\Delta\gamma_{i,j} = \gamma_{i,j+1} - \gamma_{i,j}$,

$$\gamma_{i,j}^0 = (j-i)^m - j^m, \quad \gamma_{i,j} = \sum_{k=0}^{m-1} \frac{(-1)^{m-k} i^{m-\nu-k}}{k!(1-\nu)...(m-k-\nu)} j^k \quad \text{for } j = -m+1, ..., -1,$$

$$\gamma_{i,j}^0 = (j-i)^m, \quad \gamma_{i,j} = \frac{(-1)^m}{(1-\nu)...(m-\nu)} (i-j)^{m-\nu} \quad \text{for } 0 \leq j \leq i-1, \quad \gamma_{i,j}^0 = \gamma_{i,j} = 0 \quad \text{for } j \geq i.$$

There are some symmetries for $\beta_{i,j}$ and $\beta_{i,j}^0$; it holds $\beta_{i,j}^0 = h$ for $0 \leq j \leq i-m$.

6. CONVERGENCE AND ERROR ESTIMATES

Having solved system (8) we can use the Nyström extension to compute the solution $v_n(t)$ of problem (6)–(7) for all $t \in [0,1]$; a cheaper extension $\tilde{v}_n(t)$ can be constructed quasi-interpolating by splines of degree $m-1$ the solution of system (8) completed by $v_{i,n} = f(0)$ for $i = -m+1, ..., -1$. Introduce the space

$$C_\star^m(0,1] = \{f \in C[0,1] \cap C^m(0,1] : | f^{(k)}(x) | \leq c_f x^{-k}, \quad 0 < x \leq 1, \quad k = 0, ..., m\};$$

the smallest constant c_f defines the norm $\| f \|_{C_\star^m(0,1]}$.

Theorem 1.

(i) *If* $f \in C[0,1]$, *the functions* a, b *are continuous and satisfy* (2) *for* $k = l = 0$, *and* $r \in \mathbf{N}$ *satisfies* (5), *then* $\max_{0 \leq t \leq 1} | v(t) - v_n(t) | \to 0$ *as* $n \to \infty$ *where* v *and* v_n *are the solutions of* (4) *and* (6)–(7), *respectively.*

(ii) *If* $f \in C_\star^m(0,1]$, *the functions* a, b *are* C^m*-smooth for* $0 \leq x \leq 1$, $0 < y \leq x + \delta$ *and satisfy* (2) *for* $k + l \leq m$, *and* $r \in \mathbf{N}$ *satisfies the inequalities* $r > (m+\nu)/(1-\lambda)$, $r > m/(1-\mu)$, *then*

$$\delta_{m,n,r} := \max_{0 \leq t \leq 1} t^{(r-1)\nu} \mid v(t) - v_n(t) \mid \leq c_{a,b,m,\nu,\lambda,\mu,r} h^m \| f \|_{C_\star^m(0,1)} .$$

(iii) *Under the same conditions on* f, a, b *as in* (ii) *but* $r > m/(1-\nu-\lambda)$, $r > m/(1-\mu)$, *it holds*

$$\varepsilon_{m,n,r} := \max_{0 \leq t \leq 1} \mid v(t) - v_n(t) \mid \leq c_{a,b,m,\nu,\lambda,\mu,r} h^m \| f \|_{C_\star^m(0,1)} .$$

Proof. The proof is based on the compact convergence of operators and on the error estimates of quasi-interpolation established in [3].

Remark 1. Claim (i) is true also for $\tilde{v}_n$; error estimates like in (i) and (ii) hold for $\tilde{v}_n$ under a slightly strengthened condition on $f \in C[0,1] \cap C^m(0,1]$.

Remark 2. If $f(0) = 0$, the first condition on r in (ii) and (iii) can be relaxed.

7. SOME EXTENSIONS OF THE CONSIDERATIONS

The results of Sections 2– 6 have been extended in the the following directions:

– in cases $m = 1$ and $m = 2$, the algorithms have a special treatment;
– in the case of odd $m \geq 3$, the algorithms are similar to those in Sections 4– 5;
– equations with logarithmic diagonal singularity of the kernel are treated;
– the case of a and b in (1) given only for $0 \leq s \leq t \leq 1$ is treated.

8. NUMERICAL TESTING

Method (6)–(7) and its modifications were tested numerically on the equation (1) with $\nu = 1/2$, $a \equiv 1$, $b \equiv 0$, $f(x) = 1 - x^{1/2} - \frac{\pi}{2}x$; the exact solution is then $u(x) = 1 + x^{1/2}$. About numerical results in the case of Fredholm equation, see [5].

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