“Bestimmung von Strahlungskorrekturen zur Physik schwerer Hadronen im Rahmen verschiedener effektiver Theorien”

“Calculation of radiative corrections within heavy hadron physics for different effective theories”

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Dr. Stefan Groote
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Introduction

At the end of several decades of a successful search for the constituents of matter a successful picture describing the smallest structures of nature has emerged. This picture is better known as the Standard Model of Elementary Particle Physics. The Higgs boson is the last missing particle of the Standard Model but there are hopes that it can be found in this decade. At this point, proposals for so-called "new physics" are of interest. However, before taking a step into the direction of "new physics", it is worthwhile to prove the solidness of the foundation of the Standard Model. The theory of Quantumchromodynamics (QCD) was developed as quantum field theory in the 70's in close analogy to Quantumelectrodynamics (QED) to describe the strong interaction of particles. QCD is one important ingredient of the Standard Model.

While perturbation theory has proven its usefulness in many applications, the ramifications of QCD are not fully known. At present there is no closed form solution of QCD available which is valid for all regions of phase space. Instead, perturbative descriptions, for instance at low and at high energy regions have to be interpolated into regions where perturbation theory is not applicable and have to be adjusted to nonperturbative parameters and models. The present thesis cannot completely fill this gap. But by considering various effective theories derived from QCD one has the appropiate tools at hand to make QCD applicable in different phase space regions. In this way at least a silhouette of full QCD becomes visible.

What characterizes effective theories?

Besides rather well-known effective theories such as Chiral Perturbation Theory (ChPT) for light particles or the Heavy Quark Effective Theory (HQET) for very massive particles, there are a countless number of (not explicitly named) other effective theories available for other applications. In this thesis, therefore, the author does not only deal with the well-known effective theories but also with effective theories which contain elements such as effective scales and effective parameters. In this sense, improvement techniques for the parametrization of QCD on the lattice are included as well as resummation techniques.

The main characteristic of effective theories become obvious in resummation techniques. In resumming perturbation theory, a part of the perturbation theory calculation is done before-hand, either to finite or infinite order, to give rise to effective couplings, masses and scales which are used instead of the original (not resummed) ones. With this technique, one obtains a faster convergence of the perturbation theory series. In the same manner, radiative QCD corrections are subsumed in the effective vertices of HQET as well as in Symanzik improved quark and gluon actions within lattice QCD. Therefore, it makes sense to present several effective theories in a common thesis. Parallel concepts in the different approaches become obvious, and a technique used in one approach can be used to improve techniques used in other approaches.

How is the thesis organized?

The thesis presented here covers work done by the author and his collaborators during the last five years. The author hopes to be able to give the reader an insight into a variety of questions dealt with during this time. Most of the material presented here constitutes that part of the work which the author has contributed to the collaborative projects and which have been published in over thirty articles in a variety of journals. In addition to details which can also be found in the articles cited in the text, the author provides further insights into the underlying ideas and techniques. Calculations which are of technical nature are left to the appendices. In doing so, the author was able to arrange the projects in a somewhat continuous manner, such that this thesis may be of help as a handbook for future research.

The thesis is organized as follows: In Chap. 1, the author starts by considering heavy quarks and extends previous calculations of first-order radiative corrections for polarization and correlation observables. Necessary ingredients such as the elements of the electroweak coupling matrix and the decay rate terms are found in Appendices A to C. While running coupling and masses are already used in this chapter, in Chap. 2 the emphasis is directed to renormalization and resummation techniques which give rise to quantities that "run" because of the necessary renormalization of perturbation theory. Different types of effective couplings are introduced. Using moments, the spectral density of e^+e^- annihilation and τ decay processes are analyzed using perturbation theory methods.

The two-point function and its spectral density are main ingredients of the QCD sum rule analysis. For a special topology of two-point diagrams, the sunrise-type diagrams, it is possible to obtain results in case of arbitrary mass parameters and an arbitrary number of propagators for general space-time dimensions. The calculations described in Chap. 3 are done within configuration space. The properties of Bessel functions of different types used in this chapter are collected in Appendix D, together with other functions which are needed in the following. In Appendix E, polylogarithms are dealt with, and in Appendix F shuffle methods for solving nested integrals are introduced as a technical tool for the calculation of the complex class of nested integrals.

The correlator of finite mass baryon currents is the subject of Chap. 4. Because of their complexity, correlators of baryonic currents have not been treated widely in the literature. The work presented in this chapter fills the gap and provides results which can be used for a QCD sum rule analysis. While the results in Chap. 4 are obtained for the finite mass case, Chap. 5 deals with the heavy quark mass limit of HQET. Vertices are matched, a threshold mass is defined, and first steps for the calculations of radiative corrections for the Isgur–Wise function are taken. Older calculations which are of help in other respects are collected in Appendices G and H. Further details of calculations of this chapter are found in Appendices I and J.

Chap. 6 is related to a subject which at first sight seems to be totally disjunct to the previous subject, because lattice QCD does not appear to be an effective theory at first sight. However, the Symanzik improvement of anisotropic quark actions leads again to effective vertices and propagators, according to the same principles as in HQET. Pole masses and wave function renormalization constants can be determined, as well as a special feature of anisotropic lattices, the so-called “speed-of-light coefficient”. The improvement is done for Wilson quarks and for staggered quarks. The values obtained by numerical integration are of importance for simulations on the lattice.

While the representation of QCD on a lattice is one of the most promising attempts to approach QCD nonperturbatively, QCD sum rules are another approach in this direction. Following a first QCD sum rule analysis for the determination of the electromagnetic coupling at the Z-pole in Chap. 7, ideas about the relation between perturbative and nonperturbative contributions are discussed. This is done by using different moments as well as considerations about the consequence of low-energy contributions. The charm quark mass is determined by means of finite energy sum rules. Details of the calculations are found in Appendix K.

In some parts, the work presented here is not yet finished. Like a tree in spring, there are not only blooms but also buds to be developed in the future. Writing this thesis, therefore, was a step of leaning back for a moment in the different projects, getting an overview and collecting material worked out up to this point, in order to become prepared for further steps. In this sense, the thesis can also be seen as a project description for possible extensions and completions.

Whom do I want to thank?

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———— S. D. Gl. ————

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Contents

Chapter 1

Polarization and heavy quarks

The present thesis mainly deals with heavy quarks and hadrons, especially baryons which contain a heavy quark. Quark mass effects, therefore, are one of the main concerns in this and the following chapters. Recently there has been renewed interest in the role of quark mass effects in the production of quarks and gluons in e^+e^- annihilations. Jet definition schemes, event shape variables, heavy flavour momentum correlations and the polarization of the gluon [1, 2] are affected by the presence of quark masses for charm and botton quarks even when they are produced at the scale of the Z-mass [3, 4, 5]. A careful investigation of quark mass effects in e^+e^- annihilations may even lead to a alternative determination of the quark mass values [3, 4, 5, 6].

Consequently, in this chapter the considerations are started with the heaviest quark which have been found up to now and which according to the *Standard Model (SM) of elementary particle physics* is the heaviest fermion that occurs. The top quark was discovered in 1995 at the Tevatron. The planned high energy linear e^+e^- colliders are copious sources of top quark pairs in the future. Therefore, many different aspects of top quarks are of interest, for instance polarization observables like the longitudinal and transverse polarization of the heavy quarks and their decay products and the correlation of spins which are considered in this chapter. Quark mass effects are important in the $m \to 0$ calculation of radiative corrections to quark polarization variables because residual mass effects change the naive no-flip pattern of the $m = 0$ polarization results [7, 8, 9, 10]. If the linear colliders planned at SLAC and DESY come into operation, it is necessary to have detailed radiative corrections to the production and decay of top quark pairs available, taking into account the aforementioned special aspects of polarization.

As concerns the production of heavy quarks, there are a number of *unpolarized and single spin polarized structure functions* that describe the e^+e^- production process of massive top quark pairs. In the unpolarized case one has the three structure functions H_U (transverse), H_L (longitudinal) and H_F (forward-backward) which determine the polar angle orientation of the top pair relative to the beam axis. Partial results on the full $O(\alpha_s)$ radiative corrections to the unpolarized structure functions H_U, H_L and H_F have been written down in Refs. [6, 11] starting with the early work on the $O(\alpha)$ QED radiative corrections to the vector current γe^+e^- vertex function [12]. Complete results on the $O(\alpha_s)$ unpolarized structure functions have been first given in Ref. [13].

All of the unpolarized $O(\alpha_s)$ structure functions were recalculated in the course of computing the top quark's $O(\alpha_s)$ polarization asymmetries where the unpolarized structure functions were needed to normalize the polarization asymmetries [7, 8, 14, 15, 16]. The numerators of the polarization asymmetries are expressed in terms of polarized structure functions. In the case of the longitudinal polarization of the top one has the three structure functions H_U^ℓ, H_L^ℓ and H_F^ℓ for which the full $O(\alpha_s)$ radiative corrections were given in Refs. [7, 8, 15]. In the case of a top quark polarized transverse or normal to the event plane, one has two structure functions in each case, $H_I^\perp$ and $H_A^\perp$ resp. H_I^N and H_A^N (see Ref. [16]).

When doing the full $O(\alpha_s)$ radiative corrections one integrates over the full (hard and soft) gluon phase space. For some applications it is also interesting to consider radiative corrections where one integrates over the gluon phase space up to a given gluon energy cut E_c. Such radiative corrections may be dictated by experimental considerations when soft gluons accompanying the top quark pair cannot be resolved by the detector. Alternatively one could attempt to measure the cross section for top–antitop–gluon production with a given gluon energy cut E_c and compare the energy cut dependence of the cross section with the predictions of QCD. The results for this analysis are given in this chapter. Finally, the correlation between the longitudinal and transverse top and antitop spins are worthwile to be considered. The longitudinal spin correlation has been considered already in Refs. [17, 18] without taking into account the polar angle dependence. The results with polar angle dependence will be given in this chapter as well.

The decay of the top quark can be described by structure functions H_U, H_L and H_F similar to the case of top quark pair production. When the top quark decays into the W^+ boson and the b quark, the polarization of the top quark is transferred to the boson. Therefore, a further analysis is in order here, namely the decay of polarized top quarks into W^+ and b where the W^+ polarization is considered. The CDF collaboration has already presented some results on the measurement of the longitudinal component of the W^+ based on the limited RUN I data [19]. The measurement has confirmed the expected dominance of the longitudinal mode. Theory results for one-loop radiative corrections with polar angle dependence are presented in the third main part of this chapter.

1.1 Gluon cut for top quark pair production

In this section analytical results for the $O(\alpha_s)$ radiative corrections to the three unpolarized structure functions H_U, H_L and H_F as well as for the seven polarized structure functions H_U^ℓ, H_L^ℓ, H_F^ℓ, $H_I^{\perp,N}$ and $H_A^{\perp,N}$ for polarized top quarks are provided where the integration runs over the gluon energy phase space up to a given energy cut E_c. Partial results on radiative corrections with an energy cut have been obtained before in the unpolarized case [20, 21]. A practical tool for the calculation of tree graph contributions up to a given gluon energy cut is the *soft gluon approximation*. The soft gluon approximation consists in the factorization of the tree graph contribution into the Born term contribution and a universal soft gluon piece which can be easily integrated. An $O(\alpha_s)$ calculation of polarized top pair production using the soft gluon approximation has been done in Refs. [8, 22]. One of the aims of the present investigation is to find out to which extent new coupling structure is generated in $O(\alpha_s)$ top pair production by an exact treatment of the gluon emission. This is done by comparing the results of an exact calculation with the results of the soft gluon approximation for some given small gluon energy cuts [23].

1.1.1 Polarized and unpolarized structure functions

In order to get acquainted with the notation, this section is used to outline the main structure of the cross section calculation and to indicate the point where the structure functions come into play. To start with, for the three body process $(\gamma_V, Z) \to q(p_1) + \bar{q}(p_2) + G(p_3)$ one defines a polarized *hadron tensor* according to

$$H_{\mu\nu}(q, p_1, p_2, s) = \sum_{\bar{q}, G \text{ spins}} \langle \bar{q}qG | j_\mu | 0 \rangle \langle 0 | j_\nu^\dagger | \bar{q}qG \rangle \tag{1.1}$$

where p_1, p_2 and p_3 are the four-momenta of the quark, antiquark and gluon, resp. and $q = p_1 + p_2 + p_3$ is the four-momentum of the boson. s is the spin vector of the quark. This definition also includes the Born case in which the gluon is left out. Note though that in this case the normalization is different while the same normalization is obtained if one integrates the three particle contribution over the (full or partial) phase space. The hadron tensor defined in Eq. (1.1) depends on the vector (V: γ_μ) and axial-vector (A: $\gamma_\mu\gamma_5$) composition of the currents j_μ and j_ν. One therefore has four independent components $H^i_{\mu\nu}$ $(i = 1, 2, 3, 4)$ which are defined according to

$$\begin{aligned} H^1_{\mu\nu} &= \frac{1}{2}(H^{VV}_{\mu\nu} + H^{AA}_{\mu\nu}), \qquad & H^2_{\mu\nu} &= \frac{1}{2}(H^{VV}_{\mu\nu} - H^{AA}_{\mu\nu}), \\ H^3_{\mu\nu} &= \frac{i}{2}(H^{VA}_{\mu\nu} - H^{AV}_{\mu\nu}), \qquad & H^4_{\mu\nu} &= \frac{1}{2}(H^{VA}_{\mu\nu} + H^{AV}_{\mu\nu}). \end{aligned} \tag{1.2}$$

In this notation the arguments for the hadron tensor components are avoided altogether. In the following arguments of the functions are used only when they are necessary, like the spin vector argument as one defines unpolarized and polarized structure functions $H^i_{\mu\nu}$ and $H^{im}_{\mu\nu}$ $(i = 1, 2, 3, 4,\ m = \ell, \perp, N)$ according to

$$H^i_{\mu\nu} = H^i_{\mu\nu}(s^m) + H^i_{\mu\nu}(-s^m), \qquad H^{im}_{\mu\nu} = H^i_{\mu\nu}(s^m) - H^i_{\mu\nu}(-s^m) \tag{1.3}$$

where s^m is the spin vector corresponding to longitudinal $(m = \ell)$, transversal $(m = \perp)$, or normal polarization of the top quark $(m = N)$. The explicit representation of the spin vectors will be given later. For the hadron tensor components one introduces the compact notation $H^{i\{m\}}_{\mu\nu}$ where the curly braces indicate that in the unpolarized case the index m is suppressed. This compact notation is used to show the general features common for the unpolarized part as well as the polarized parts.

If one looks at the process $e^+e^- \to \bar{q}q(G)$, the cross section is given by a modular structure consisting of the hadron tensor, the *lepton tensor* which is constructed according to the same principles, coefficients g_{ij} connecting both parts and known as *coupling coefficients of the electroweak theory* (they are listed in Appendix A), and a *phase space factor* dPS,

$$d\sigma^{\{m\}} = 2\pi \frac{e^4}{q^4} \sum_{i,j=1}^{4} g_{ij} L^{i\,\mu\nu} H^{j\{m\}}_{\mu\nu} dPS \tag{1.4}$$

where the contraction of the indices μ and ν is understood. The process $e^+e^- \to \bar{q}q(G)$ looked at can be described in two equivalent frames, each spanned by three linearly dependent vectors, namely the lepton or *beam plane* spanned by the electron and positron beam and the outgoing quark and the hadron or *event plane* spanned again by the quark, the antiquark and the gluon. In the Born case where no gluon is emitted, both planes coincide by convention. The relative angle between the quark momentum and the electron momentum is the *polar angle* θ, and the event plane is turned against the beam plane about this common axis by the *azimuthal angle* χ.

The natural frame for describing the hadron tensor is the event plane. In this plane obviously no angle relative to the beam axis occurs, except for the transverse and normal spin vectors which are represented in the beam plane because this is the laboratory system where the polarizations are to be measured. The beam particles are described most naturally in the beam plane. While $L^{3\mu\nu}$ vanishes identically and $L^{2\mu\nu}$ vanishes if the lepton mass is set to zero, the other two components have the simple form

$$L^1=\frac{q^2}{2}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\qquad L^4=\frac{q^2}{2}\begin{pmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{pmatrix}. \tag{1.5}$$

In order to calculate the contraction of lepton tensor and hadron tensor, one has to turn the lepton tensor into the even plane. In doing so a variety of angular dependences occurs. Actually one can decompose the lepton tensors according to

$$\begin{aligned}L^1&=\frac{q^2}{2}\left\{\frac{1}{2}(1+\cos^2\theta)\mathit{\Pi}_U+\sin^2\theta\,\mathit{\Pi}_L+\sin\theta\cos\theta\,\mathit{\Pi}_I\right\},\\ L^4&=\frac{q^2}{2}\left\{\cos\theta\,\mathit{\Pi}_F+\sin\theta\,\mathit{\Pi}_A\right\}.\end{aligned} \tag{1.6}$$

Note that $\mathit{\Pi}_I$ and $\mathit{\Pi}_A$ depend linearly on $\sin\chi$ and $\cos\chi$. The matrices $\mathit{\Pi}_U$, $\mathit{\Pi}_L$, $\mathit{\Pi}_I$, $\mathit{\Pi}_F$ and $\mathit{\Pi}_A$ are called *projectors* because in contracting the lepton tensor with the hadron tensor they project out the contribution of the hadron tensor to the different angle dependences. The decomposition in Eq. (1.6) covers all possible angle dependences which occur in the process with a single polarized quark. This decomposition gives rise to the decomposition of the differential cross section according to

$$\frac{d\sigma^{\{m\}}}{d\cos\theta}=\frac{3}{8}(1+\cos^2\theta)\sigma_U^{\{m\}}+\frac{3}{4}\sin^2\theta\,\sigma_L^{\{m\}}+\frac{3}{4}\cos\theta\,\sigma_F^{\{m\}}+\frac{3}{4}\sin\theta\cos\theta\,\sigma_I^{\{m\}}+\frac{3}{4}\sin\theta\,\sigma_A^{\{m\}} \tag{1.7}$$

where

$$\begin{aligned}\sigma_a^{\{m\}}&=\frac{(4\pi\alpha)^2}{3q^4}\sum_{j=1}^4 g_{1j}\int H_a^{j\{m\}}\frac{dPS}{d\cos\theta},\quad H_a^{j\{m\}}=\mathit{\Pi}_a^{\mu\nu}H_{\mu\nu}^{j\{m\}}\quad\text{for } a=U,L,I\\ \sigma_a^{\{m\}}&=\frac{(4\pi\alpha)^2}{3q^4}\sum_{j=1}^4 g_{4j}\int H_a^{j\{m\}}\frac{dPS}{d\cos\theta},\quad H_a^{j\{m\}}=\mathit{\Pi}_a^{\mu\nu}H_{\mu\nu}^{j\{m\}}\quad\text{for } a=F,A.\end{aligned} \tag{1.8}$$

The integration sign in Eqs. (1.8) means that one has to integrate over the measure which remain after division by $d\cos\theta$. For the Born contribution as well as for the loop contribution with the two particle final state the phase space

$$dPS_2=\frac{v}{8(2\pi)^2}\,d\cos\theta\,d\chi\to\frac{v}{16\pi}\,d\cos\theta \tag{1.9}$$

is obtained where $v=\sqrt{1-4m^2/q^2}$ is the velocity of the outgoing quark. The transition to the last expression in Eq. (1.9) stands for the fact that the azimuthal integration (over

χ) is always implied here. If one performs the integration over the azimuth angle χ, the transverse and normal spin dependence in the components H_a^j drop out for $a = U, L, F$ while they are retained for $a = I, A$. Just the opposite happens to the spin independent parts. Therefore, not all structure functions are populated. It turns out that they are alternatively populated by either spin dependent or spin independent parts. For the two particle final state there is no integration measure left, therefore one obtains

$$\begin{aligned} \sigma_a^{\{m\}}(Born, loop) &= \frac{\pi\alpha^2 v}{3q^4}\sum_{j=1}^{4} g_{1j} H_a^{j\{m\}}(Born, loop) \quad \text{for } a = U, L, I \\ \sigma_a^{\{m\}}(Born, loop) &= \frac{\pi\alpha^2 v}{3q^4}\sum_{j=1}^{4} g_{4j} H_a^{j\{m\}}(Born, loop) \quad \text{for } a = F, A. \end{aligned} \tag{1.10}$$

For the three particle final state the phase space

$$dPS_3 = \frac{v}{8(2\pi)^2}\, d\cos\theta\, d\chi\, \frac{q^2}{16\pi^2 v}\, dy\, dz \to \frac{v}{16\pi}\, d\cos\theta\, \frac{q^2}{16\pi^2 v}\, dy\, dz \tag{1.11}$$

is obtained where the phase space variables $y = 1 - 2p_1 \cdot q/q^2$ and $z = 1 - 2p_2 \cdot q/q^2$ are introduced. The $O(\alpha_s)$ tree contributions to $\sigma_a^{\{m\}}$ are therefore given by

$$\begin{aligned} \sigma_a^{\{m\}}(tree) &= \frac{\pi\alpha^2 v}{3q^4}\left(\frac{q^2}{16\pi^2 v}\sum_{j=1}^{4} g_{1j}\int H_a^{j\{m\}}(y,z)dy\, dz\right) \quad \text{for } a = U, L, I \\ \sigma_a^{\{m\}}(tree) &= \frac{\pi\alpha^2 v}{3q^4}\left(\frac{q^2}{16\pi^2 v}\sum_{j=1}^{4} g_{4j}\int H_a^{j\{m\}}(y,z)dy\, dz\right) \quad \text{for } a = F, A. \end{aligned} \tag{1.12}$$

$H_a^{j\{m\}}(Born)$ and $H_a^{j\{m\}}(\alpha_s) = H_a^{j\{m\}}(tree) + H_a^{j\{m\}}(loop)$ with

$$H_a^{j\{m\}}(tree) = \frac{q^2}{16\pi^2 v}\int H_a^{j\{m\}}(y,z)dy\, dz \tag{1.13}$$

are called the *polarized* and *unpolarized structure functions* to leading and next-to-leading order, respecticely, of which results for the case of an explicit gluon energy cut will be presented.

1.1.2 Covariant expressions for the projectors

The projectors Π_a can be represented in a covariant way, as will be seen in the following. As a consequence of this they do not depend explicitly on y and z and can therefore be taken out of the integral. For the derivation one first goes to the rest frame of the boson with $q = (\sqrt{q^2}; 0, 0, 0)$ and takes the z-axis to be the quark momentum direction. Starting with the quark momentum in this frame,

$$p_1 = \frac{1}{2}\sqrt{q^2}\left(1 - y; 0, 0, \sqrt{(1-y)^2 - \xi}\right) \tag{1.14}$$

($y = 0$ for the two-body decay) with $\xi = 1 - v^2 = 4m^2/q^2$, one constructs a four-transversal quark momentum and a four-transverse metric tensor,

$$\hat{g}_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \qquad \hat{p}_{1\mu} = \hat{g}_{\mu\nu} p_1^\nu = p_{1\mu} - \frac{p_1 \cdot q}{q^2} q_\mu \tag{1.15}$$

and uses q and $\hat{p}_1$ to build up two elements of a coordinate basis,

$$\begin{aligned} e_0 &= (e_0^\mu) = \left(q^\mu/\sqrt{q^2}\right) = (1;0,0,0), \\ e_3 &= (e_3^\mu) = \left(\hat{p}_1^\mu/\sqrt{(p_1\cdot q)^2/q^2-p_1^2}\right) = (0;0,0,1). \end{aligned} \tag{1.16}$$

In the case of longitudinal polarization of the quark the spin vector

$$s^\ell = \frac{1}{\sqrt{\xi}}\left(\sqrt{(1-y)^2-\xi};0,0,1-y\right) \tag{1.17}$$

is the third possible quantity for the construction. However, it is a linear combination of e_0 and e_3 and therefore of no help for the construction. The projectors one can construct with e_0 and e_3 are therefore limited to

$$\begin{aligned} \mathit{\Pi}_U^{\mu\nu} &= -\hat{g}^{\mu\nu} - e_3^\mu e_3^\mu, \\ \mathit{\Pi}_L^{\mu\nu} &= e_3^\mu e_3^\nu, \\ \mathit{\Pi}_F^{\mu\nu} &= i\varepsilon_{\mu\nu\rho\sigma}e_3^\rho e_0^\sigma \end{aligned} \tag{1.18}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the total antisymmetric tensor. For the transverse and normal polarizations as measured in the beam plane, the spin vectors expressed in the event plane are given by

$$s^\perp = (0;\cos\chi,-\sin\chi,0), \qquad s^N = (0;\sin\chi,\cos\chi,0). \tag{1.19}$$

These two vectors, therefore, can help to span the beam plane, respectively, a plane perpendicular to it in the event system. With these new elements the construction of additional projectors is possible,

$$\begin{aligned} \mathit{\Pi}_I^{\mu\nu}(s) = s^\mu e_3^\nu + e_3^\mu s^\nu, \qquad \mathit{\Pi}_I^{\mu\nu\prime}(s) = -\left(\varepsilon_{\mu\rho\sigma\tau}e_3^\nu + \varepsilon_{\nu\rho\sigma\tau}e_3^\mu\right)e_0^\rho e_3^\sigma s^\tau, \\ \mathit{\Pi}_A^{\mu\nu}(s) = i\varepsilon_{\mu\nu\rho\sigma}e_0^\rho s^\sigma, \qquad \mathit{\Pi}_A^{\mu\nu\prime}(s) = i(s^\mu e_3^\nu - e_3^\mu s^\nu). \end{aligned} \tag{1.20}$$

Note that $\mathit{\Pi}_I(s^N) = \mathit{\Pi}_I'(s^\perp)$, $\mathit{\Pi}_I(s^\perp) = \mathit{\Pi}_I'(s^N)$, $\mathit{\Pi}_A(s^N) = \mathit{\Pi}_A'(s^\perp)$, $\mathit{\Pi}_A(s^\perp) = \mathit{\Pi}_A'(s^N)$, therefore the primed projectors are redundant. One now can use this set of covariant and unique projectors to calculate the different contributions of the hadron tensor.

In the following the Born term and loop contributions calculated in Refs. [8, 15, 16] are presented. The non-vanishing unpolarized Born term contributions are given by

$$\begin{aligned} H_U^1(\mathit{Born}) = 2N_cq^2(1+v^2), \qquad H_L^1(\mathit{Born}) = N_cq^2(1-v^2) = H_L^2(\mathit{Born}), \\ H_U^2(\mathit{Born}) = 2N_cq^2(1-v^2), \qquad H_F^4(\mathit{Born}) = 4N_cq^2v. \end{aligned} \tag{1.21}$$

The longitudinally polarized contributions read

$$\begin{aligned} H_U^{4\ell}(\mathit{Born}) = 4N_cq^2v, \qquad H_F^{1\ell}(\mathit{Born}) = 2N_cq^2(1+v^2), \\ H_L^{4\ell}(\mathit{Born}) = 0, \qquad H_F^{2\ell}(\mathit{Born}) = 2N_cq^2(1-v^2). \end{aligned} \tag{1.22}$$

For the transverse and normal polarization one finally obtains

$$\begin{aligned} H_I^{4\perp}(\mathit{Born}) = -2N_cq^2v\sqrt{\xi}, \qquad H_A^{1\perp}(\mathit{Born}) = -2N_cq^2\sqrt{\xi} = H_A^{2\perp}(\mathit{Born}), \\ H_A^{3N}(\mathit{Born}) = 2N_cq^2\sqrt{\xi}. \end{aligned} \tag{1.23}$$

The non-vanishing loop contributions are given by

$$\begin{aligned}
H_U^1(loop) &= 4N_cq^2\left((1+v^2)\operatorname{Re}A-2v^2\operatorname{Re}B\right),\\
H_U^2(loop) &= 4N_cq^2\left((1-v^2)\operatorname{Re}A+2v^2\operatorname{Re}B\right),\\
H_L^1(loop) &= 2N_cq^2\left((1-v^2)\operatorname{Re}A+v^2\operatorname{Re}B\right)\ =\ H_L^2(loop),\\
H_F^4(loop) &= 8N_cq^2v\left(\operatorname{Re}A-\operatorname{Re}B\right),\\[1ex]
H_U^{4\ell}(loop) &= 8N_cq^2v\left(\operatorname{Re}A-\operatorname{Re}B\right),\\
H_L^{4\ell}(loop) &= 0,\\
H_F^{1\ell}(loop) &= 4N_cq^2\left((1+v^2)\operatorname{Re}A-2v^2\operatorname{Re}B\right),\\
H_F^{2\ell}(loop) &= 4N_cq^2\left((1-v^2)\operatorname{Re}A+2v^2\operatorname{Re}B\right),\\[1ex]
H_I^{3\perp}(loop) &= -2N_cq^2v\sqrt{\xi}(1+\xi)\operatorname{Im}B/\xi,\\
H_I^{4\perp}(loop) &= -2N_cq^2v\sqrt{\xi}\left(2\operatorname{Re}A+(1-3\xi)\operatorname{Re}B/\xi\right),\\
H_A^{1\perp}(loop) &= -2N_cq^2\sqrt{\xi}\left(\operatorname{Re}A-v^2\operatorname{Re}B/\xi\right)\ =\ H_A^{2\perp}(loop),\\[1ex]
H_I^{1N}(loop) &= -2N_cq^2v\sqrt{\xi}(1-\xi)\operatorname{Im}B/\xi\ =\ H_I^{2N}(loop),\\
H_A^{3N}(loop) &= 2N_cq^2v\sqrt{\xi}\left(2\operatorname{Re}A+(1-3\xi)\operatorname{Re}B/\xi\right),\\
H_A^{4N}(loop) &= 2N_cq^2v\sqrt{\xi}(1+\xi)\operatorname{Im}B/\xi.
\end{aligned}\tag{1.24}$$

in terms of the *vector form factors* A and B *of the vertex correction* with

$$\begin{aligned}
\operatorname{Re}A &= -\frac{\alpha_sC_F}{4\pi}\Bigg\{\left(2+\frac{1+v^2}{v}\ln\left(\frac{1-v}{1+v}\right)\right)\ln\left(\frac{\Lambda q^2}{m^2}\right)+3v\ln\left(\frac{1-v}{1+v}\right)+4\\
&\qquad +\frac{1+v^2}{v}\left(\operatorname{Li}_2\left(\frac{2v}{1+v}\right)+\frac{1}{4}\ln^2\left(\frac{1-v}{1+v}\right)-\frac{\pi^2}{2}\right)\Bigg\},\\
\operatorname{Re}B &= \frac{\alpha_sC_F}{4\pi}\,\frac{1-v^2}{v}\ln\left(\frac{1-v}{1+v}\right),\qquad \operatorname{Im}B\ =\ \frac{\alpha_sC_F}{4\pi}\,\frac{1-v^2}{v}\pi
\end{aligned}\tag{1.25}$$

were the *axial-vector form factor* C is expressed by A according to $C=A-2B$, while the second axial-vector form factor D does not occur. $m_G^2=\Lambda q^2$ is the squared mass of the gluon which is used for the regularization of the first order tree contributions. A connection to the parameter $\varepsilon=(4-D)/2$ of the dimensional regularization is given by the correspondence $1/\varepsilon-\gamma_E+\ln(4\pi\mu^2/m^2)\to\ln(\Lambda q^2/m^2)$.

In order to be able to compare with these results afterwards, the total $O(\alpha_s)$ results $H_a^{j\{m\}}(\alpha_s)=H_a^{j\{m\}}(tree)+H_a^{j\{m\}}(loop)$ are listed here (see e.g. Refs. [8, 15, 16]) in terms of the decay rate terms t_1 to t_{12} (see Appendix B),

$$\begin{aligned}
H_U^1(\alpha_s) &= N\Big\{2(2+7\xi)v+(48-48\xi+7\xi^2)t_3+2\sqrt{\xi}\left(2-7\xi+\sqrt{\xi}(2+3\xi)\right)t_4\ +\\
&\qquad -2\xi(2+3\xi)t_5-4(2-\xi)\left((2-\xi)(t_8-t_9)+2v(t_{10}+2t_{12})\right)\Big\},\\
H_U^2(\alpha_s) &= \xi N\Big\{12v+2(6-\xi)t_3+2\sqrt{\xi}(1-\sqrt{\xi})t_4\ +
\end{aligned}$$

$$
\begin{aligned}
& +2\xi t_5 - 4\left((2-\xi)(t_8-t_9)+2v(t_{10}+2t_{12})\right)\Big\}, \\
H_L^1(\alpha_s) &= N\Big\{\frac{1}{2}(16-46\xi+3\xi^2)v+\frac{\xi}{4}(88-32\xi+3\xi^2)t_3 \; + \\
& -2\sqrt{\xi}\left(2-7\xi+\sqrt{\xi}(2+3\xi)\right)t_4+2\xi(2+3\xi)t_5 \; + \\
& -2\xi\left((2-\xi)(t_8-t_9)+2v(t_{10}+2t_{12})\right)\Big\}, \\
H_L^2(\alpha_s) &= \xi N\Big\{\frac{3}{2}(10-\xi)v+\frac{1}{4}(24-16\xi-3\xi^2)t_3 \; + \\
& -2\sqrt{\xi}(1-\sqrt{\xi})t_4-2\xi t_5-2\left((2-\xi)(t_8-t_9)+2v(t_{10}+2t_{12})\right)\Big\}, \\
H_F^3(\alpha_s) &= -8\xi N v\pi, \\
H_F^4(\alpha_s) &= N\Big\{-16\sqrt{\xi}(1-\sqrt{\xi})-16(t_1-t_2)+8(2-3\xi)vt_3 \; + \\
& -4(4-5\xi)t_6-8v\left((2-\xi)(t_8-t_7)+2v(t_{10}+t_{11})\right)\Big\}, \qquad (1.26)
\end{aligned}
$$

$$
\begin{aligned}
H_U^{3\ell}(\alpha_s) &= -8\xi N v\pi, \\
H_U^{4\ell}(\alpha_s) &= N\Big\{-2(2+35\xi)+2\sqrt{\xi}(8+29\xi)-\frac{1}{2}(32-60\xi+17\xi^2)(t_1-t_2) \; + \\
& +4(4+9\xi)vt_3-2(8+2\xi+3\xi^2)t_6-8v\left((2-\xi)(t_8-t_7)+2v(t_{10}+t_{11})\right)\Big\}, \\
H_L^{3\ell}(\alpha_s) &= 0, \\
H_L^{4\ell}(\alpha_s) &= N\Big\{4(2+19\xi)-4\sqrt{\xi}(8+13\xi)-\xi(24-7\xi)(t_1-t_2) \; + \\
& -52\xi vt_3+2\xi(10+3\xi)t_6\Big\}, \\
H_F^{1\ell}(\alpha_s) &= N\Big\{-4(2+3\xi)v+2(24-12\xi+\xi^2)t_3+2\sqrt{\xi}\left(8+\xi-\sqrt{\xi}(10-\xi)\right)t_4 \; + \\
& +2\xi(10-\xi)t_5-4(2-\xi)\left((2-\xi)(t_8-t_9)+2v(t_{10}+2t_{12})\right)\Big\}, \\
H_F^{2\ell}(\alpha_s) &= \xi N\Big\{12v+2(6-\xi)t_3+2\sqrt{\xi}(1-\sqrt{\xi})t_4 \; + \\
& +2\xi t_5-4(2-\xi)\left((t_8-t_9)+2v(t_{10}+2t_{12})\right)\Big\}, \qquad (1.27)
\end{aligned}
$$

$$
\begin{aligned}
H_I^{3\perp}(\alpha_s) &= -2\sqrt{\xi}N(1+\xi)v\pi, \\
H_I^{4\perp}(\alpha_s) &= \sqrt{\xi}N\Big\{48+17\xi-\sqrt{\xi}(62+3\xi)-\frac{1}{4}(4-\xi)(10+3\xi)(t_1-t_2) \; + \\
& -2(21+2\xi)vt_3+(16+7\xi)t_6+4v\left((2-\xi)(t_8-t_7)+2v(t_{10}+t_{11})\right)\Big\}, \\
H_A^{1\perp}(\alpha_s) &= \sqrt{\xi}N\Big\{(8-3\xi)v-\frac{1}{2}(72-38\xi+3\xi^2)t_3+\left(8+\xi-\sqrt{\xi}(10-\xi)\right)t_4 \; + \\
& -(8+\xi)t_5+4\left((2-\xi)(t_8-t_9)+2v(t_{10}+2t_{12})\right)\Big\}, \\
H_A^{2\perp}(\alpha_s) &= \sqrt{\xi}N\Big\{-(20-3\xi)v-\frac{1}{2}(32-14\xi-3\xi^2)t_3+\xi(1-\sqrt{\xi})t_4 \; + \\
& -\xi t_5+4\left((2-\xi)(t_8-t_9)+2v(t_{10}+2t_{12})\right)\Big\}, \qquad (1.28)
\end{aligned}
$$

$$
H_I^{1N}(\alpha_s) = -2\sqrt{\xi}Nv^2\pi = H_I^{2N}(\alpha_s),
$$

$$
\begin{aligned}
H_A^{3N}(\alpha_s) &= \sqrt{\xi}N\Big\{20+9\xi-\sqrt{\xi}(26+3\xi)-\frac{1}{4}(24-2\xi-3\xi^2)(t_1-t_2)+ \\
&\quad +2(1-6\xi)vt_3-(8-13\xi)t_6-4v\left((2-\xi)(t_8-t_7)+2v(t_{10}+t_{11})\right)\Big\}, \\
H_A^{4N}(\alpha_s) &= 2\sqrt{\xi}N(1+\xi)v\pi \qquad (1.29)
\end{aligned}
$$

where $N=\alpha_sN_cC_Fq^2/4\pi v$. The results in this section are given in terms of the projections H_U, H_L, H_F, H_I, and H_A. Because of practical reasons the new results depending on the exact gluon energy cut shown in the following section are presented not for H_U but for

$$H_{U+L}=H_U+H_L. \qquad (1.30)$$

It is left to the reader or the program which uses these expressions to calculate the difference $H_{U+L}-H_L$.

1.1.3 Exact calculation up to the gluon energy cut

Starting with this section, the calculations for the $O(\alpha_s)$ correction with an exact gluon energy cut are presented [23]. The gluon energy cut is given by the constraint $E_G/\sqrt{q^2}=p_3\cdot q/q^2\le\lambda$ which reads $y+z\le 2\lambda$ in terms of the phase space variables y and z introduced earlier. The phase space looked at is shown in Fig. 1.1 for the parameters $\xi=0.1$ and $\lambda=0.3$. The cut is shown as off-diagonal straight line which cuts the boundary of the phase space at the two symmetric points $(y,z)=(y_1,y_2)$ and (y_2,y_1). The boundary curves of the full phase space are given by

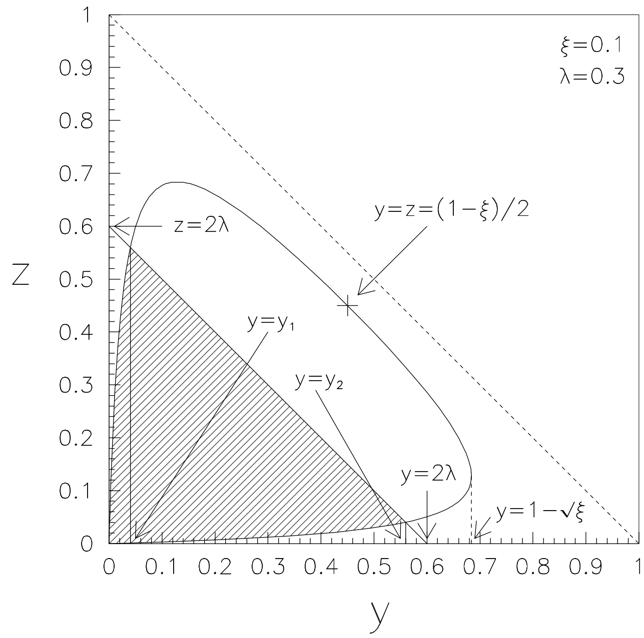


Figure 1.1: Phase space plot with gluon cut

$$z_\pm=\frac{2y-2y^2-\xi y\pm 2y\sqrt{(1-y)^2-\xi}}{4y+\xi}=\frac{\mathcal{A}\pm\mathcal{B}}{\mathcal{C}}. \qquad (1.31)$$

One obtains

$$y_1=\lambda\left(1-\sqrt{\frac{1-2\lambda-\xi}{1-2\lambda}}\right),\qquad y_2=\lambda\left(1+\sqrt{\frac{1-2\lambda-\xi}{1-2\lambda}}\right). \qquad (1.32)$$

The integrals that have to be calculated (see Refs. [7, 8, 15, 16]) are given by

$$
\begin{aligned}
\tilde{I}(l,m) &= \int\!\!\int y^l z^m dy\,dz, \qquad \tilde{S}(l,m)=\int\!\!\int\frac{y^l z^m dy\,dz}{\sqrt{(1-y)^2-\xi}}, \\
\tilde{J}(l,m) &= \int\!\!\int\frac{y^l z^m dy\,dz}{(1-y)^2-\xi}, \qquad \tilde{T}(l,m)=\int\!\!\int\frac{y^l z^m dy\,dz}{((1-y)^2-\xi)^{3/2}}. \qquad (1.33)
\end{aligned}
$$

For simplicity the principles of the new calculations should be presented only in case of the integration class $\tilde{I}$ where intermediate steps lead to a mixture with the class $\tilde{S}$. It is left to the skills of the reader to translate the present results to the other integral classes.

One can see that the integration can be done by dividing the integration range into two parts, a part from $y = 0$ to $y = y_1$ and a part from $y = y_1$ to $y = y_2$,

$$\tilde{I}(l,m) = \int_0^{y_1}\int_{z_-}^{z_+} y^l z^m dy\, dz + \int_{y_1}^{y_2}\int_{z_-}^{2\lambda-y} y^l z^m dy\, dz. \tag{1.34}$$

Note, though, that this only holds as long as y_2 has not reached its upper limit at $y = 1-\sqrt{\xi}$. This point is irrelevant at the moment but will be discussed later, the calculation of these diagrams is explained in Appendix C.1.

1.1.4 The gluon cut results

The results using the exact gluon energy cut read (again with $N = \alpha_s N_c C_F q^2/4\pi v$)

$$\begin{aligned}
H_{U+L}^1 = N\Bigg\{&(4-\xi)(2-\xi)(t_{0-}-t_{0+}-t_{1-}+t_{1+}) - \frac{1}{4}(4-\xi)(2+3\xi)v + \\
&+(4-\xi)\ell_{4+} - \frac{1}{16}(32+40\xi-12\xi^2+3\xi^3)\ell_{6+} + \\
&+\frac{1}{2}(32\lambda-16\lambda^2-4\xi-8\lambda\xi-4\lambda^2\xi+\xi^2)(\ell_{7+}-\ell_{8+}) + \\
&+\left(2(4-\xi)y_1 - \frac{1}{2}(4+\xi)y_1^2\right)\ell_1 - \left(2(4-\xi)y_2 - \frac{1}{2}(4+\xi)y_2^2\right)\ell_2 + \\
&-\frac{1}{16}(128-52\xi+5\xi^2-8\xi y_1)v_1 + (4+\xi)\frac{b\xi\sqrt{\xi}}{8(b-aw_1)} + \\
&-\frac{1}{16}(224-192\lambda-52\xi+16\lambda\xi+5\xi^2)y_1 + \frac{1}{2}\xi y_1^2 + \\
&-\frac{1}{16}(128-52\xi+5\xi^2-8\xi y_2)v_2 - (4+\xi)\frac{b\xi\sqrt{\xi}}{8(b+aw_2)} + \\
&+\frac{1}{16}(224-192\lambda-52\xi+16\lambda\xi+5\xi^2)y_2 - \frac{1}{2}\xi y_2^2\Bigg\}
\end{aligned} \tag{1.35}$$

$$\begin{aligned}
H_{U+L}^2 = \xi N\Bigg\{&3(2-\xi)(t_{0-}-t_{0+}-t_{1-}+t_{1+}) + \frac{3}{4}(10-\xi)v + \frac{1}{16}(8-32\xi+3\xi^2)\ell_{6+} + \\
&+3\ell_{4+} + \frac{1}{2}(24\lambda+4\lambda^2-3\xi)(\ell_{7+}-\ell_{8+}) + \left(6y_1+\frac{1}{2}y_1^2\right)\ell_1 - \left(6y_2+\frac{1}{2}y_2^2\right)\ell_2 + \\
&-\frac{1}{16}(152-5\xi+8y_1)v_1 - \frac{b\xi\sqrt{\xi}}{8(b-aw_1)} - \frac{1}{16}(152-16\lambda-5\xi)y_1 - \frac{1}{2}y_1^2 + \\
&-\frac{1}{16}(152-5\xi+8y_2)v_2 + \frac{b\xi\sqrt{\xi}}{8(b+aw_2)} + \frac{1}{16}(152-16\lambda-5\xi)y_2 + \frac{1}{2}y_2^2\Bigg\}
\end{aligned} \tag{1.36}$$

$$H_{U+L}^{4\ell} = N\Bigg\{4v\left((2-\xi)(t_{0-}+t_{0+}) - 2v(t_{1-}+t_{1+})\right) - \frac{1}{2}(1+\sqrt{\xi})(2-\sqrt{\xi})^2 + 4v\ell_{4-} +$$

$$
\begin{aligned}
&-2(3-\xi)v\ell_{5-}-\frac{\xi}{4}(20+3\xi)\ell_{6-}-\frac{2}{v_\lambda}(3-14\lambda+20\lambda^2-8\lambda^3-3\xi+4\lambda\xi)\ell_{7-}\ + \\
&+\frac{2}{v}(3-8\lambda+4\lambda^2-3\xi+8\lambda\xi-2\lambda^2\xi)\ell_{8-}-2(3-y_1-\xi)v_1\ell_1+2(3-y_2-\xi)v_2\ell_2\ + \\
&+\frac{1}{4}(48-48\lambda-5\xi-8y_1)v_1-\frac{4\lambda\xi v_1}{y_1}+\frac{b\xi\sqrt{\xi}}{2(b-aw_1)}+\frac{1}{4}(32-5\xi)y_1-2y_1^2\ + \\
&-\frac{1}{4}(48-48\lambda-5\xi-8y_2)v_2+\frac{4\lambda\xi v_2}{y_2}+\frac{b\xi\sqrt{\xi}}{2(b+aw_2)}+\frac{1}{4}(32-5\xi)y_2-2y_2^2\Big\} \qquad (1.37)
\end{aligned}
$$

$$
\begin{aligned}
H_L^1 = N\Big\{&\xi(2-\xi)(t_{0-}-t_{0+})-2\xi(2+\xi)(t_{1-}-t_{1+})-\sqrt{\xi}(1-\sqrt{\xi})(2+4\sqrt{\xi}-3\xi)t_w+ \\
&+\frac{1}{4}(16-54\xi+3\xi^2)v+\xi\ell_{4+}+16\xi\ell_{5+}+\frac{\xi}{16}(8+8\xi-3\xi^2)\ell_{6+}\ + \\
&-\frac{\xi}{2v_\lambda^2}(8\lambda-28\lambda^2+16\lambda^3+16\lambda^4+\xi-12\lambda\xi-8\lambda^2\xi-\xi^2)\ell_{7+}\ + \\
&-\frac{\xi}{2v^2}(8\lambda+4\lambda^2-\xi-8\lambda\xi+\xi^2)\ell_{8+}-\xi\left(2y_1+\frac{1}{2}y_1^2\right)\ell_1+\xi\left(2y_2+\frac{1}{2}y_2^2\right)\ell_2\ + \\
&+\frac{2(1-2\lambda-(1-\lambda)\sqrt{\xi})}{(1-2\lambda-\sqrt{\xi})(1-\sqrt{\xi})\sqrt{\xi}}(2-8\lambda+8\lambda^2-\xi+2\lambda\xi-4\lambda^2\xi+7\xi^2-3\lambda\xi^2\ + \\
&\qquad\qquad -(2-6\lambda+4\lambda^2+3\xi-3\lambda\xi-2\lambda^2\xi+3\xi^2)\sqrt{\xi})\ell_3\ + \\
&-\frac{1}{16}(32-72\xi+5\xi^2-8\xi y_1)v_1+\frac{b\xi^2\sqrt{\xi}}{8(b-aw_1)}-\frac{1}{16}(32-72\xi+16\lambda\xi+5\xi^2)y_1+\frac{\xi}{2}y_1^2\ + \\
&-\frac{1}{16}(32-72\xi+5\xi^2-8\xi y_2)v_2-\frac{b\xi^2\sqrt{\xi}}{8(b+aw_2)}+\frac{1}{16}(32-72\xi+16\lambda\xi+5\xi^2)y_2-\frac{\xi}{2}y_2^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
H_L^2 = \xi N\Big\{&(2-\xi)(t_{0-}-t_{0+})-2v^2(t_{1-}-t_{1+})-\sqrt{\xi}(1-\sqrt{\xi})t_w+\frac{1}{4}(22-3\xi)v+ \\
&+\ell_{4+}+\frac{1}{16}(8-8\xi+3\xi^2)\ell_{6+}+\frac{1}{2}(8\lambda+4\lambda^2-\xi)(\ell_{7+}-\ell_{8+})-\frac{2\lambda^2\xi}{v^2}\ell_{8+}\ + \\
&+\left(2y_1+\frac{1}{2}y_1^2\right)\ell_1-\left(2y_2+\frac{1}{2}y_2^2\right)\ell_2+2\frac{1-2\lambda-(1-\lambda)\sqrt{\xi}}{(1-\sqrt{\xi})\sqrt{\xi}}(1-2\lambda-\xi+\lambda\sqrt{\xi})\ell_3\ + \\
&-\frac{1}{16}(72-5\xi+8y_1)v_1-\frac{b\xi\sqrt{\xi}}{8(b-aw_1)}-\frac{1}{16}(72-16\lambda-5\xi)y_1-\frac{1}{2}y_1^2\ + \\
&-\frac{1}{16}(72-5\xi+8y_2)v_2+\frac{b\xi\sqrt{\xi}}{8(b+aw_2)}+\frac{1}{16}(72-16\lambda-5\xi)y_2+\frac{1}{2}y_2^2\Big\} \qquad (1.38)
\end{aligned}
$$

$$
\begin{aligned}
H_L^{4\ell} = N\Big\{&\xi(10+3\xi)(t_{1-}+t_{1+})-\frac{\xi}{2}(24-7\xi)\ell_{6-}-\frac{4\lambda^2\xi}{v}\ell_{8-}+ \\
&+\left(\frac{8\lambda^3\xi^2}{v_\lambda^3}+\frac{3\sqrt{\xi}}{2w_\lambda}(2-\sqrt{\xi})(1-\sqrt{\xi})^2-\frac{3\sqrt{\xi}w_\lambda}{2}(2+\sqrt{\xi})(1+\sqrt{\xi})^2-4\xi v_\lambda\right)\ell_{7-}\ + \\
&+\left(\frac{3\sqrt{\xi}}{2w_0}(2-\sqrt{\xi})(1-\sqrt{\xi})^2-\frac{3\sqrt{\xi}w_0}{2}(2+\sqrt{\xi})(1+\sqrt{\xi})^2-4\xi v\right)(\ell_{5-}-\ell_{8-})\ +
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{3\sqrt{\xi}}{2w_1}(2-\sqrt{\xi})(1-\sqrt{\xi})^2-\frac{3\sqrt{\xi}w_1}{2}(2+\sqrt{\xi})(1+\sqrt{\xi})^2-4\xi v_1\right)\ell_1\ + \\
&-\left(\frac{3\sqrt{\xi}}{2w_2}(2-\sqrt{\xi})(1-\sqrt{\xi})^2-\frac{3\sqrt{\xi}w_2}{2}(2+\sqrt{\xi})(1+\sqrt{\xi})^2-4\xi v_2\right)\ell_2\ + \\
&+\frac{1-2\lambda-(1-\lambda)\sqrt{\xi}}{(1-2\lambda-\sqrt{\xi})\sqrt{\xi}}\Big\{2-8\lambda+8\lambda^2-7\xi+5\lambda\xi+2\lambda^2\xi-3\xi^2\ + \\
&\qquad\qquad -(1-6\lambda+8\lambda^2-9\xi+3\lambda\xi)\sqrt{\xi}\Big\}\left(\frac{1}{w_1}-\frac{1}{w_2}\right)\ + \\
&-\frac{1-2\lambda+(1-\lambda)\sqrt{\xi}}{(1-2\lambda+\sqrt{\xi})\sqrt{\xi}}\Big\{2-8\lambda+8\lambda^2-7\xi+5\lambda\xi+2\lambda^2\xi-3\xi^2\ + \\
&\qquad\qquad +(1-6\lambda+8\lambda^2-9\xi+3\lambda\xi)\sqrt{\xi}\Big\}(w_1-w_2)\ + \\
&+(2+7\xi)v_1+(2+7\xi)y_1-(2+7\xi)v_2+(2+7\xi)y_2\Big\}
\end{aligned}
\tag{1.39}
$$

$$
\begin{aligned}
H_F^{1\ell} = N\Big\{&2(2-\xi)^2(t_{0-}-t_{0+})-(8+2\xi+\xi^2)(t_{1-}-t_{1+})+ \\
&+\sqrt{\xi}(1-\sqrt{\xi})(2-\sqrt{\xi})(4+\sqrt{\xi})t_w\ + \\
&-2(6+\xi)v+2(2-\xi)\ell_{4+}+8\xi\ell_{5+}-\frac{1}{2}(4+6\xi-3\xi^2)\ell_{6+}\ + \\
&+\frac{1}{v_\lambda^2}(16\lambda-72\lambda^2+96\lambda^3-32\lambda^4-2\xi-20\lambda\xi+40\lambda^2\xi-32\lambda^3\xi\ + \\
&\qquad\qquad +3\xi^2+8\lambda\xi^2+4\lambda^2\xi^2-\xi^3)\ell_{7+}\ + \\
&-\frac{1}{v^2}(16\lambda-8\lambda^2-2\xi-24\lambda\xi+4\lambda^2\xi+3\xi^2+8\lambda\xi^2+2\lambda^2\xi^2-\xi^3)\ell_{8+}\ + \\
&+4(1-2\lambda)(3-2\lambda)\ell_{9+}+(2(4-3\xi)y_1-2y_1^2)\ell_1-(2(4-3\xi)y_2-2y_2^2)\ell_2\ + \\
&-\frac{2(1-2\lambda-(1-\lambda)\sqrt{\xi})}{(1-2\lambda-\sqrt{\xi})(1-\sqrt{\xi})}\Big(6-16\lambda+8\lambda^2+11\xi-5\lambda\xi+2\lambda^2\xi-\xi^2\ + \\
&\qquad\qquad -(15-22\lambda+8\lambda^2+\xi+\lambda\xi)\sqrt{\xi}\Big)\ell_3\ + \\
&-\frac{3}{4}(4-7\xi-4y_1)v_1+\frac{b\xi\sqrt{\xi}}{2(b-aw_1)}-\frac{1}{4}(16+16\lambda-21\xi)y_1+3y_1^2\ + \\
&-\frac{3}{4}(4-7\xi-4y_2)v_2-\frac{b\xi\sqrt{\xi}}{2(b+aw_2)}+\frac{1}{4}(16+16\lambda-21\xi)y_2-3y_2^2\Big\}
\end{aligned}
\tag{1.40}
$$

$$
\begin{aligned}
H_F^{2\ell} = \xi N\Big\{&2(2-\xi)(t_{0-}-t_{0+})-(4-\xi)(t_{1-}-t_{1+})+\sqrt{\xi}(1-\sqrt{\xi})t_w+2v+2\ell_{4+}+ \\
&-\frac{3}{2}\xi\ell_{6+}+(8\lambda-\xi)\ell_{7+}-\frac{1}{v^2}(8\lambda-\xi-8\lambda\xi-2\lambda^2\xi+\xi^2)\ell_{8+}+4y_1\ell_1-4y_2\ell_2\ + \\
&-2\frac{1-2\lambda-(1-\lambda)\sqrt{\xi}}{(1-\sqrt{\xi})\sqrt{\xi}}(1-2\lambda-\xi+\lambda\sqrt{\xi})\ell_3-5v_1-5y_1-5v_2+5y_2\Big\}
\end{aligned}
\tag{1.41}
$$

$$
\begin{aligned}
H_F^4 = N\Big\{ & 4v(2-\xi)(t_{0-}+t_{0+}) - 2(4-5\xi)(t_{1-}+t_{1+}) - \frac{1}{2}(1+\sqrt{\xi})(2-\sqrt{\xi})^2 + \\
& +4v\ell_{4-} - 6v\ell_{5-} - 8\xi\ell_{6-} - \frac{2}{v_\lambda}(3-14\lambda+20\lambda^2-8\lambda^3-2\xi-\xi^2)\ell_{7-} + \\
& +\frac{2}{v}(3-8\lambda+4\lambda^2-2\xi+8\lambda\xi-\xi^2)\ell_{8-} + 4(1-2\lambda)(3-2\lambda)\ell_{9-} + \\
& -2(3-y_1)v_1\ell_1 + 2(3-y_2)v_2\ell_2 + \\
& +\frac{1}{4}(12+16\lambda+\xi-4y_1)v_1 + \frac{b\xi\sqrt{\xi}}{2(b-aw_1)} - \frac{4\lambda\xi v_1}{y_1} + \frac{1}{4}(24+\xi)y_1 - y_1^2 + \\
& -\frac{1}{4}(12+16\lambda+\xi-4y_2)v_2 + \frac{b\xi\sqrt{\xi}}{2(b+aw_2)} + \frac{4\lambda\xi v_2}{y_2} + \frac{1}{4}(24+\xi)y_2 - y_2^2\Big\}
\end{aligned}
\tag{1.42}
$$

$$
\begin{aligned}
H_I^{4\perp} = \sqrt{\xi}N\Big\{ & -2v(2-\xi)(t_{0-}+t_{0+}) + \frac{1}{2}(16+7\xi)(t_{1-}+t_{1+}) + \\
& +\frac{1}{4}(1+\sqrt{\xi})(2-\sqrt{\xi})^2 - 2v\ell_{4-} - \frac{1}{8}(72-30\xi-3\xi^2)\ell_{6-} + \\
& -\left(\frac{\xi v}{2} + \frac{3(2-\sqrt{\xi})(1-\sqrt{\xi})^2}{2w_0} + \frac{3(2+\sqrt{\xi})(1+\sqrt{\xi})^2 w_0}{2}\right)(\ell_{5-}-\ell_{8-}) + \\
& +\Bigg(\frac{\xi}{v_\lambda^3}(1-2\lambda-\xi)(1-4\lambda-\xi) + \\
& \qquad -\frac{\xi v_\lambda}{2} - \frac{3(2-\sqrt{\xi})(1-\sqrt{\xi})^2}{2w_\lambda} - \frac{3(2+\sqrt{\xi})(1+\sqrt{\xi})^2 w_\lambda}{2}\Bigg)\ell_{7-} + \\
& +\frac{1}{v}(8\lambda-4\lambda^2-\xi-8\lambda\xi-\lambda^2\xi+\xi^2)\ell_{8-} - 2(1-2\lambda)\ell_{9-} + \\
& -\left(\frac{\xi v_1}{2} + \frac{3(2-\sqrt{\xi})(1-\sqrt{\xi})^2}{2w_1} + \frac{3(2+\sqrt{\xi})(1+\sqrt{\xi})^2 w_1}{2}\right)\ell_1 + \\
& +\left(\frac{\xi v_2}{2} + \frac{3(2-\sqrt{\xi})(1-\sqrt{\xi})^2}{2w_2} + \frac{3(2+\sqrt{\xi})(1+\sqrt{\xi})^2 w_2}{2}\right)\ell_2 + \\
& -\frac{1-2\lambda-(1-\lambda)\sqrt{\xi}}{(1-2\lambda-\sqrt{\xi})\xi}\Big\{2-8\lambda+8\lambda^2-7\xi+5\lambda\xi+2\lambda^2\xi-3\xi^2 + \\
& \qquad\qquad -(1-6\lambda+8\lambda^2-9\xi+3\lambda\xi)\sqrt{\xi}\Big\}\left(\frac{1}{w_1}-\frac{1}{w_2}\right) + \\
& -\frac{1-2\lambda+(1-\lambda)\sqrt{\xi}}{(1-2\lambda+\sqrt{\xi})\xi}\Big\{2-8\lambda+8\lambda^2-7\xi+5\lambda\xi+2\lambda^2\xi-3\xi^2 + \\
& \qquad\qquad +(1-6\lambda+8\lambda^2-9\xi+3\lambda\xi)\sqrt{\xi}\Big\}(w_1-w_2) + \\
& +\frac{1}{8}(28+5\xi)v_1 + \frac{2\lambda\xi v_1}{y_1} - \frac{b\xi\sqrt{\xi}}{4(b-aw_1)} + \frac{1}{8}(28+5\xi)y_1 + \\
& -\frac{1}{8}(28+5\xi)v_2 - \frac{2\lambda\xi v_2}{y_2} - \frac{b\xi\sqrt{\xi}}{4(b+aw_2)} + \frac{1}{8}(28+5\xi)y_2\Big\}
\end{aligned}
\tag{1.43}
$$

$$
\begin{aligned}
H_A^{1\perp} = \sqrt{\xi}N\Big\{ & -2(2-\xi)(t_{0-}-t_{0+}) + \frac{1}{2}(16-3\xi)(t_{1-}-t_{1+}) + \\
& + \frac{1}{2}(4+\sqrt{\xi})(2-\sqrt{\xi})(1-\sqrt{\xi})t_w + \frac{1}{2}(16-3\xi)v \; + \\
& -2\ell_{4+} - 2(7-\xi)\ell_{5+} + \frac{1}{8}(8-6\xi+3\xi^2)\ell_{6+} - 2(1-2\lambda)\ell_{9+} - \frac{\xi}{2}y_1\ell_1 + \frac{\xi}{2}y_2\ell_2 \; + \\
& + \frac{\xi}{v_\lambda^2}(1-5\lambda+4\lambda^2-4\lambda^3-\xi+\lambda\xi)\ell_{7+} + \frac{1}{v^2}(8\lambda-4\lambda^2-\xi-8\lambda\xi+5\lambda^2\xi+\xi^2)\ell_{8+} \; + \\
& - \frac{1-2\lambda-(1-\lambda)\sqrt{\xi}}{(1-2\lambda-\sqrt{\xi})(1-\sqrt{\xi})\sqrt{\xi}}\Big(6-16\lambda+8\lambda^2+11\xi-5\lambda\xi+2\lambda^2\xi-\xi^2 \; + \\
& \qquad -(15-22\lambda+8\lambda^2+\xi+\lambda\xi)\sqrt{\xi}\Big)\ell_3 \; + \\
& + \frac{1}{8}(4+5\xi)v_1 - \frac{b\xi\sqrt{\xi}}{4(b-aw_1)} + \frac{1}{8}(4+5\xi)y_1 \; + \\
& + \frac{1}{8}(4+5\xi)v_2 + \frac{b\xi\sqrt{\xi}}{4(b+aw_2)} - \frac{1}{8}(4+5\xi)y_2\Big\}
\end{aligned} \tag{1.44}
$$

$$
\begin{aligned}
H_A^{2\perp} = \sqrt{\xi}N\Big\{ & -2(2-\xi)(t_{0-}-t_{0+}) + \frac{1}{2}(8-3\xi)(t_{1-}-t_{1+}) + \frac{\xi}{2}(1-\sqrt{\xi})t_w + \\
& - \frac{3}{2}(4-\xi)v - 2\ell_{4+} - 2(1+\xi)\ell_{5+} - \frac{1}{8}(2-\xi)(4-3\xi)\ell_{6+} - (8\lambda-\xi-\lambda\xi)\ell_{7+} \; + \\
& + \frac{1}{v^2}(8\lambda+4\lambda^2-\xi-8\lambda\xi-3\lambda^2\xi+\xi^2)\ell_{8+} + 2(1-2\lambda)\ell_{9+} \; + \\
& - \frac{1}{2}(8-\xi)y_1\ell_1 + \frac{1}{2}(8-\xi)y_2\ell_2 - \frac{1-2\lambda-(1-\lambda)\sqrt{\xi}}{1-\sqrt{\xi}}(1-2\lambda+\lambda\sqrt{\xi}-\xi)\ell_3 \; + \\
& + \frac{1}{8}(52-5\xi)v_1 + \frac{b\xi\sqrt{\xi}}{4(b-aw_1)} + \frac{1}{8}(52-5\xi)y_1 \; + \\
& + \frac{1}{8}(52-5\xi)v_2 - \frac{b\xi\sqrt{\xi}}{4(b+aw_2)} - \frac{1}{8}(52-5\xi)y_2\Big\}
\end{aligned} \tag{1.45}
$$

$$
\begin{aligned}
H_A^{3N} = \sqrt{\xi}N\Big\{ & 2v(2-\xi)(t_{0-}+t_{0+}) - \frac{1}{2}(8-13\xi)(t_{1-}+t_{1+}) + \\
& + \frac{1}{4}(2-\sqrt{\xi})^2(1+\sqrt{\xi}) + 2v\ell_{4-} - \frac{1}{2}(8+\xi)v\ell_{5-} + \frac{1}{8}(8-30\xi+3\xi^2)\ell_{6-} \; + \\
& - \left(\frac{8+\xi}{2}v_\lambda - \xi\frac{1-2\lambda-\xi}{v_\lambda}\right)\ell_{7-} + \frac{1}{2v}(8-16\lambda-8\lambda^2-5\xi+16\lambda\xi+2\lambda^2\xi-3\xi^2)\ell_{8-} \; + \\
& -2(1-2\lambda)\ell_{9-} - \frac{1}{2}(8+\xi)v_1\ell_1 + \frac{1}{2}(8+\xi)v_2\ell_2 \; + \\
& + \frac{1}{8}(52+5\xi)v_1 - \frac{2\lambda\xi v_1}{y_1} - \frac{b\xi\sqrt{\xi}}{4(b-aw_1)} + \frac{1}{8}(52+5\xi)y_1 \; + \\
& - \frac{1}{8}(52+5\xi)v_2 + \frac{2\lambda\xi v_2}{y_2} - \frac{b\xi\sqrt{\xi}}{4(b+aw_2)} + \frac{1}{8}(52+5\xi)y_2\Big\}
\end{aligned} \tag{1.46}
$$

A remark on divergent integrals

A subtle issue which occured for the class of integrals and gives rise to IR divergences has to be pointed out at this step. The integral

$$\tilde{I}_a(-1,-1)=\int_0^{y_1}\int_{z_-}^{z_+}\frac{dy\,dz}{yz}=\int_0^{y_1}\ln\left(\frac{z_+(y)}{z_-(y)}\right)\frac{dy}{y} \tag{1.47}$$

contains an IR divergence which can be regularized by a gluon mass $m_G=\sqrt{\Lambda q^2}$. This changes the lower limit in y from 0 to $y_-=\Lambda+\sqrt{\Lambda\xi}$, and the limits in z to

$$z_\pm(y)=\frac{1}{4y+\xi}\left(2y-2y^2-\xi y+2\Lambda y+2\Lambda\pm 2\sqrt{(y-\Lambda)^2-\Lambda\xi}\sqrt{(1-y)^2-\xi}\right). \tag{1.48}$$

The integration over z, therefore, gives rise to

$$\tilde{I}_a(-1,-1)=\int_{y_-}^{y_1}\ln\left(\frac{2y-2y^2-\xi y+2\Lambda y+2\Lambda+2\sqrt{(y-\Lambda)^2-\Lambda\xi}\sqrt{(1-y)^2-\xi}}{2y-2y^2-\xi y+2\Lambda y+2\Lambda-2\sqrt{(y-\Lambda)^2-\Lambda\xi}\sqrt{(1-y)^2-\xi}}\right)\frac{dy}{y}. \tag{1.49}$$

This integral is not analytically calculable for general values of Λ. However, one can divide it up into a divergent and a convergent part which can be calculated separately assuming that Λ is a small parameter. The divergent part is an integral where the integrand coincides with the original one at the divergent point, i.e. at the origin $y=0$ in the considered case. Therefore one can neglect higher powers in y in all cases where they do not get in conflict with the value y_- and obtain

$$D=\int_{\sqrt{\Lambda\xi}}^{y_1}\ln\left(\frac{(1+v^2)y+2v\sqrt{y^2-\Lambda\xi}}{(1+v^2)y-2v\sqrt{y^2-\Lambda\xi}}\right)\frac{dy}{y}. \tag{1.50}$$

Before doing this approximation, the integration has been shifted by the amount $-\Lambda$ in order to make it easier for expansions in the lower boundary. This integral can be calculated to obtain

$$\begin{aligned}D &= \ln\left(\frac{1+v}{1-v}\right)\ln\left(\frac{y_1^2}{\Lambda\xi}\right)-\mathrm{Li}_2\left(\frac{2v}{(1+v)^2}\right)+\mathrm{Li}_2\left(\frac{-2v}{(1-v)^2}\right)+\\ &\quad -\frac{1}{2}\mathrm{Li}_2\left(-\frac{(1+v)^2}{(1-v)^2}\right)+\frac{1}{2}\mathrm{Li}_2\left(-\frac{(1-v)^2}{(1+v)^2}\right)\;=\\ &=: t_p-\ln\left(\frac{1+v}{1-v}\right)\ln\Lambda.\end{aligned} \tag{1.51}$$

In the case $\Lambda\to 0$ one obtains the limiting value

$$D\to D_0=2\ln\left(\frac{1+v}{1-v}\right)\lim_{\varepsilon\to 0}\int_\varepsilon^{y_1}\frac{dy}{y} \tag{1.52}$$

which is an ill-defined quantity for $\varepsilon=0$. However, one can subtract it from the original integral also taken in the limit $\Lambda\to 0$ because the divergences cancel,

$$C=\lim_{\varepsilon\to 0}\left\{\int_\varepsilon^{y_1}\ln\left(\frac{2-2y-\xi+2\sqrt{(1-y)^2-\xi}}{2-2y-\xi-2\sqrt{(1-y)^2-\xi}}\right)\frac{dy}{y}-2\ln\left(\frac{1+v}{1-v}\right)\int_\varepsilon^{y_1}\frac{dy}{y}\right\}. \tag{1.53}$$

With the substitution mentioned above, one does the partial fractioning

$$\frac{dy}{y}=-\frac{dw}{w_0-w}+\frac{dw}{w_0+w}+\frac{dw}{1-w}-\frac{dw}{1+w}. \tag{1.54}$$

This gives rise to

$$\begin{aligned}\int_\varepsilon^{y_1}\ln\left(\frac{2-2y-\xi+2\sqrt{(1-y)^2-\xi}}{2-2y-\xi-2\sqrt{(1-y)^2-\xi}}\right)\frac{dy}{y}\ =&\\ =\ I_{0-}(w_0')-I_{0-}(w_1)-I_{0+}(w_0')+I_{0+}(w_1)\ +&\\ -I_{1-}(w_0')+I_{1-}(w_1)+I_{1+}(w_0')-I_{1+}(w_1),&\end{aligned} \tag{1.55}$$

$$\begin{aligned}\int_\varepsilon^{y_1}\frac{dy}{y}\ =&\\ =\ I^0_{0-}(w_0')-I^0_{0-}(w_1)-I^0_{0+}(w_0')+I^0_{0+}(w_1)\ +&\\ -I^0_{1-}(w_0')+I^0_{1-}(w_1)+I^0_{1+}(w_0')-I^0_{1+}(w_1)&\end{aligned} \tag{1.56}$$

where w_0' is close in value to w_0. Without going into detail concerning what the different parts are, it is instructive that the divergences are now contained in the parts $I_{0-}(w_0')$ and $I^0_{0-}(w_0')$ which contain the factor $(w_0-w)^{-1}$ in the integrand. One obtains

$$I_{0-}(w)=t_p^l-2\ln\left(\frac{1+v}{1-v}\right)\ln(w_0-w),\qquad I^0_{0-}(w)=-\ln(w_0-w) \tag{1.57}$$

where t_p^l is a dilogarithmic decay rate term which is finite in the limit $w\to w_0$ and actually vanishes there. Therefore, one can calculate this convergent part, add the divergent part of the whole expression and obtains a modified term,

$$\begin{aligned}I'_{0-}(w_0)\ :=&\ \lim_{w_0'\to w_0}\left(I_{0-}(w_0')-2\ln\left(\frac{1+v}{1-v}\right)I^0_{0-}(w_0')\right)+t_p-\ln\left(\frac{1+v}{1-v}\right)\ln\Lambda\ =\\ =&\ t_p-\ln\left(\frac{1+v}{1-v}\right)\ln\Lambda\end{aligned} \tag{1.58}$$

while in all other terms one can replace w_0' by w_0. t_p is found to be the decay rate term $t^{ba}_{0-}(w_0)$ as listed in Appendix C.2.2.

1.1.5 The soft gluon approximation

Using the results for an exact gluon energy cut is shown in the previous subsection, one can also deal with the soft gluon approximation as well. Basic ingredient for this approximation is the *eikonal approximation* where the gluon momentum is neglected in the numerators of Feynman diagram contributions. This approximation leads to the fact that the hadron tensor is proportional to the Born term, in this case

$$H^i_{\mu\nu}(soft)=g_s^2C_F\left(\frac{p_1^2}{(p_1p_3)^2}-\frac{2(p_1p_2)}{(p_1p_3)(p_2p_3)}+\frac{p_2^2}{(p_2p_3)^2}\right)H^i_{\mu\nu}(Born) \tag{1.59}$$

(cf. the soft gluon factor in Eq. (1.132)). The Born term is independent of the dimensionless three-body phase space variables $x=E_G/\sqrt{q^2}=p_3q/q^2$ and $u=(p_1-p_2)q/q^2$ and can therefore be taken out of the integral. The integral that remains is given by

$$h=\frac{\alpha_sC_F}{4\pi v}\int_{\sqrt{\Lambda}}^{\lambda}\int_{-u_+}^{u_+}h(x,u)dx\,du \tag{1.60}$$

where

$$h(x,u)=8\frac{(1-2x+\Lambda)(u^2-(x-\Lambda)^2)+\xi(x-\Lambda)^2}{(u^2-(x-\Lambda)^2)} \tag{1.61}$$

and

$$u_+^2=(x^2-\Lambda)\frac{1-2x+\Lambda-\xi}{1-2x+\Lambda} \tag{1.62}$$

where Λ can be set to zero for all cases without IR divergences. After the integration over w one obtains

$$h(x)=4\left(\frac{2u_+\xi}{(2-\Lambda)^2-u_+^2}-\frac{2-4x-2\Lambda-\xi}{x-\Lambda}\ln\left(\frac{x-\Lambda+u_+}{x-\Lambda-u_+}\right)\right) \tag{1.63}$$

and therefore

$$\begin{aligned} h &= \frac{\alpha_s C_F}{\pi v}\Bigg\{\left(2v-(2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right)\ln\left(\frac{2\lambda}{\sqrt{\Lambda}}\right)+4\left(\sqrt{1-2\lambda}\sqrt{1-2\lambda-\xi}-v\right)+\\ &\quad +2v\left(\ln\left(\frac{z_\lambda}{z_0}\right)+2\ln\left(\frac{z_0^2-1}{z_\lambda z_0-1}\right)\right)-\ln z_0+4\lambda\ln z_\lambda+\\ &\quad +(2-\xi)\Bigg(\frac{1}{2}\ln^2\left(\frac{z_\lambda}{z_0}\right)+2\ln z_0\ln\left(\frac{z_\lambda z_0-1}{z_0^2-1}\right)+\frac{1}{4}\ln^2 z_0+\\ &\quad +\mathrm{Li}_2\left(\frac{2v}{1+v}\right)+\mathrm{Li}_2\left(1-\frac{z_\lambda}{z_0}\right)+\mathrm{Li}_2(1-z_\lambda z_0)-\mathrm{Li}_2(1-z_0^2)\Bigg)\Bigg\} \end{aligned} \tag{1.64}$$

where

$$z_0=\frac{1+v}{1-v},\qquad z_\lambda=\frac{\sqrt{1-2\lambda}+\sqrt{1-2\lambda-\xi}}{\sqrt{1-2\lambda}-\sqrt{1-2\lambda-\xi}}. \tag{1.65}$$

For $\lambda\to 0$ one obtains

$$\begin{aligned} h &\to \frac{\alpha_s C_F}{\pi v}\Bigg\{\left(2v-(2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right)\ln\left(\frac{2\lambda}{\sqrt{\Lambda}}\right)+\\ &\quad +(2-\xi)\left(\frac{1}{4}\ln^2\left(\frac{1+v}{1-v}\right)+\mathrm{Li}_2\left(\frac{2v}{1+v}\right)\right)\Bigg\}. \end{aligned} \tag{1.66}$$

It can be shown explicitly that the hadron tensor components of the exact solution which were presented in the previous section can be written in this limit $\lambda\to 0$ as Born term contributions times the factor given in Eq. (1.66),

$$\begin{array}{cccc} H^1_{U+L}\approx(4-\xi)H & H^2_{U+l}\approx 3\xi H & H^3_{U+L}\approx 0 & H^4_{U+L}\approx 4vH\\ H^1_L\approx\xi H & H^2_L\approx\xi H & H^3_L\approx 0 & H^4_L\approx 0\\ H^1_F\approx 2(2-\xi)H & H^2_F\approx 2\xi H & H^3_F\approx 0 & H^4_F\approx 4vH\\ H^1_I\approx 0 & H^2_I\approx 0 & H^3_I\approx 0 & H^4_I\approx -2vH\\ H^1_E\approx 0 & H^2_E\approx 0 & H^3_E\approx 0 & H^4_E\approx 0\\ H^1_A\approx -2\sqrt{\xi}H & H^2_A\approx -2\sqrt{\xi}H & H^3_A\approx 0 & H^4_A\approx 0\\ H^1_R\approx 0 & H^2_R\approx 0 & H^3_R\approx -2v\sqrt{\xi}H & H^4_R\approx 0 \end{array} \tag{1.67}$$

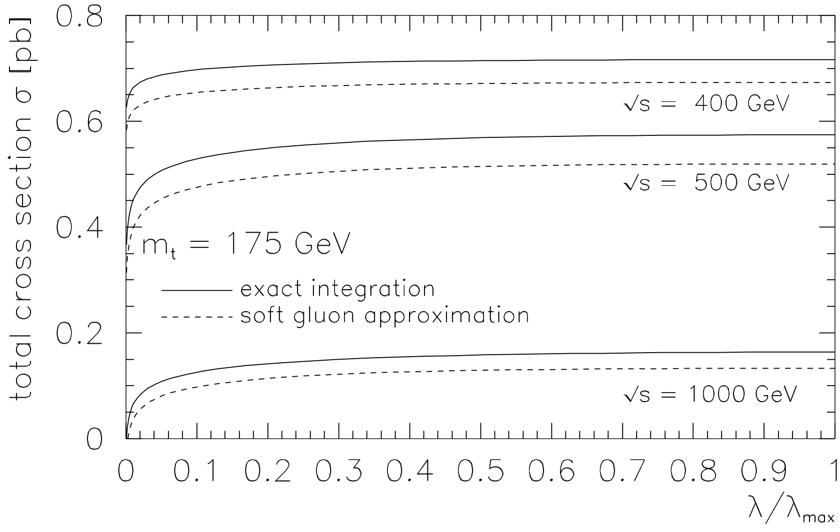


Figure 1.2: Total cross section in dependence on $\lambda/\lambda_{\max}$ where $\lambda_{\max} = (1-\xi)/2$ for the exact integration up to the gluon energy cut λ (solid curves) and the soft gluon approximation (dashed curves) for center-of-mass energies $\sqrt{s} = 400\,GeV$, $500\,GeV$, and $1000\,GeV$. For the (running) quark mass one takes the value $m_t = 175\,GeV$.

with

$$H = -2\left(2v - (2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right)\ln\left(\frac{2\lambda}{\sqrt{\Lambda}}\right) - (2-\xi)t_0 + 2\ln\left(\frac{1+v}{1-v}\right). \tag{1.68}$$

where the limits for the decay reate terms are found in Appendix C.3. A remark is in order here, related to the notation for the hadron tensor components. The Indices U, L and F represent the $\frac{1}{2}(1+\cos^2\theta)$, $\sin^2\theta$ and $\cos\theta$ dependent parts (θ is the polar angle). They will occur in the longitudinal polarized and unpolarized case. The other components contribute to the transverse polarized case in the beam plane. For the perpendicular polarization one has

$$H_I^\perp := H_I \quad \text{for the } \sin(2\theta) \text{ dependence}, \qquad H_A^\perp := H_A \quad \text{for the } \sin\theta \text{ dependence} \tag{1.69}$$

while for the normal polarization one has

$$H_I^N := H_E \quad \text{for the } \sin(2\theta) \text{ dependence}, \quad H_A^N := -H_R \quad \text{for the } \sin\theta \text{ dependence.} \tag{1.70}$$

Both calculations can be seen to have the same limit. In addition to these considerations, the exact result has also been compared with the result for the soft gluon approximation. In Fig. 1.2 the total cross section for the exact integration and the soft gluon approximation for three different center-of-mass energies is shown.

1.1.6 The full phase space limit

One may finally consider the second possible limit, namely $\lambda \to \lambda_{\max} = (1-\xi)/2$ in which case the whole phase space is used. However, as have been mentioned earlier, the limit cannot be taken for the exact expressions because the division of the phase space into the two parts does only work up to $\lambda=(1-\sqrt{\xi})/(2-\sqrt{\xi})$ where y_2 takes the maximal value $1-\sqrt{\xi}$. Starting from this point, the limit y_2 decreases again in order to finally coincide with the limit y_1 at $\lambda=y_1=y_2=(1-\xi)/2$. The splitting of the integration as defined before works only up to $\lambda=(1-\sqrt{\xi})/2$. For $\lambda=(1-\xi)/2$ the portion of the phase space to the right of y_2 have to be taken into account as well. Therefore, one has to add correcting integrals of the kind

$$\tilde{I}_c(l,m)=\int_{y_2}^{1-\sqrt{\xi}}\int_{z_-}^{z_+}y^lz^m dy\,dz. \tag{1.71}$$

This integral type has the same decomposition as $\tilde{I}_o$ and $\tilde{I}_{ab}$ because only the limits are changed. One therefore constructs

$$\begin{aligned}\hat{S}_c(k,l) &= \int_{y_2}^{1-\sqrt{\xi}}\frac{y^l dy}{(4y+\xi)^k} \;=\\ &= (1+\sqrt{\xi})^l\int_0^{w_2}\left(\frac{1-w^2}{b^2-a^2w^2}\right)^k\left(\frac{w_0^2-w^2}{1-w^2}\right)^l\frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2}, \qquad (1.72)\\ \hat{I}_c^{ba}(l) &= \int_{y_2}^{1-\sqrt{\xi}}\ln\left(\frac{z_+(y)}{z_-(y)}\right)y^l dy \;=\\ &= (1+\sqrt{\xi})^l\int_0^{w_2}\ln\left(\frac{(1+w)(b+aw)}{(1-w)(b-aw)}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l\frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2} \qquad (1.73)\end{aligned}$$

via $\hat{S}_c(k,l)=S_{ab}(k,l,w_2)-S_{ab}(k,l,0)$. It is no problem to add these issues to the procedure. However, one can already see from the very beginning that the result will be the same as for the old calculations without intersection because the integrals with range from y_1 to y_2 vanish and one is left with the integrals with subscripts o and c which sum up to the full integral,

$$\begin{aligned}\tilde{I}(l,m) &= \tilde{I}_o(l,m)+\tilde{I}_c(l,m) \;=\\ &= \int_0^{(1-\xi)/2}\int_{z_-}^{z_+}y^lz^m dy\,dz+\int_{(1-\xi)/2}^{1-\sqrt{\xi}}\int_{z_-}^{z_+}y^lz^m dy\,dz \;=\\ &= \int_0^{1-\sqrt{\xi}}\int_{z_-}^{z_+}y^lz^m dy\,dz. \qquad (1.74)\end{aligned}$$

It is expected, therefore, that the result will be the same as for the calculation taking the whole phase space without dividing it by a gluon energy cut [15, 16, 8]. In order to prove this expectation, the missing phase space integration has been calculated. Using corrected decay rate terms given in Appendix C.4, the results for this additional phase space portion (with general value of y_2) are given by

$$\begin{aligned}H_{U+L}^1 &= N\Bigg\{(4-\xi)(2-\xi)(t_{0-}^c-t_{0+}^c-t_{1-}^c+t_{1+}^c)-4(4-\xi)v\ell_{4+}^c+\\ &+\left(\frac{1}{16}(32+40\xi-12\xi^2+3\xi^3)+2(4-\xi)y_2-\frac{1}{2}(4+\xi)y_2^2\right)\ell_2^c+\\ &+\frac{1}{8}(128-52\xi+5\xi^2-8\xi y_2)v_2-(4+\xi)\frac{\xi(4-\xi)v_2}{8(4y_2+\xi)}\Bigg\}, \qquad (1.75)\end{aligned}$$

$$\begin{aligned}
H^2_{U+L} &= \xi N \Bigg\{ 3(2-\xi)(t^c_{0-} - t^c_{0+} - t^c_{1-} + t^c_{1+}) - 12v\ell^c_{4+} + \\
&- \left(\frac{1}{16}(8 - 32\xi + 3\xi^2) - 6y_2 - \frac{1}{2}y_2^2 \right) \ell^c_2 + \frac{1}{8}(152 - 5\xi + 8y_2)v_2 + \frac{\xi(4-\xi)v_2}{8(4y_2+\xi)} \Bigg\}, \quad (1.76)
\end{aligned}$$

$$\begin{aligned}
H^{4\ell}_{U+L} &= N \Bigg\{ 4(2-\xi)v(t^c_{0-} + t^c_{0+}) - 8(1-\xi)(t^c_{1-} + t^c_{1+}) - 8(1-\xi)\ell^c_{4-} + \\
&+ \frac{1}{2}(32 - 12\xi + 3\xi^2)\ell^c_{5-} + \frac{1}{4}\xi(20+3\xi)\ell^c_{6-} - 16(1-\xi)\ell^c_{7-} - 2(3 - y_2 - \xi)v_2\ell^c_2 + \\
&+ \frac{1}{2}(24 - 16\sqrt{\xi} - 13\xi + 7\xi\sqrt{\xi}) - \frac{1}{2}(32-5\xi)y_2 + 4y_2^2 - \xi(2-\sqrt{\xi})^2 \frac{1 - y_2 + \sqrt{\xi}}{2(4y_2+\xi)} \Bigg\},
\end{aligned} \quad (1.77)$$

$$\begin{aligned}
H^1_L &= N \Bigg\{ \xi(2-\xi)(t^c_{0-} - t^c_{0+}) - 2\xi(2+\xi)(t^c_{1-} - t^c_{1+}) - \sqrt{\xi}(2 + 2\sqrt{\xi} - 7\xi + 3\xi\sqrt{\xi})t^c_w + \\
&- 4\xi v\ell^c_{4+} + 16\xi\ell^c_{5+} - \xi \left(\frac{1}{16}(8 + 8\xi - 3\xi^2) + 2y_2 + \frac{1}{2}y_2^2 \right) \ell^c_2 + \\
&+ \frac{1}{8}(32 - 72\xi + 5\xi^2 - 8\xi y_2)v_2 - \frac{\xi^2(4-\xi)v_2}{8(4y_2+\xi)} \Bigg\}, \quad (1.78)
\end{aligned}$$

$$\begin{aligned}
H^2_L &= \xi N \Bigg\{ (2-\xi)(t^c_{0-} - t^c_{0+}) - 2(1-\xi)(t^c_{1-} - t^c_{1+}) - \sqrt{\xi}(1-\sqrt{\xi})t^c_w - 4v\ell^c_{4+} + \\
&- \left(\frac{1}{16}(8 - 8\xi + 3\xi^2) - 2y_2 - \frac{1}{2}y_2^2 \right) \ell^c_2 + \frac{1}{8}(72 - 5\xi + 8y_2)v_2 + \frac{\xi(4-\xi)v_2}{8(4y_2+\xi)} \Bigg\}, \quad (1.79)
\end{aligned}$$

$$\begin{aligned}
H^{4\ell}_L &= N \Bigg\{ \xi(10+3\xi)(t^c_{1-} + t^c_{1+}) + \xi(24-7\xi)\ell^c_{5-} + \frac{1}{2}\xi(24-7\xi)\ell^c_{6-} + \\
&+ \left(\frac{3\sqrt{\xi}}{2w_2}(2-\sqrt{\xi})(1-\sqrt{\xi})^2 - \frac{3\sqrt{\xi}w_2}{2}(2+\sqrt{\xi})(1+\sqrt{\xi})^2 - 4\xi v_2 \right) \ell^c_2 + \\
&+ 2(1-\sqrt{\xi})(2 - 6\sqrt{\xi} + 13\xi) - 2(2+7\xi)y_2 \Bigg\}, \quad (1.80)
\end{aligned}$$

$$\begin{aligned}
H^{1\ell}_F &= N \Bigg\{ 2(2-\xi)^2(t^c_{0-} - t^c_{0+}) - (8 + 2\xi + \xi^2)(t^c_{1-} - t^c_{1+}) + \\
&+ \sqrt{\xi}(8 - 10\sqrt{\xi} + \xi + \xi\sqrt{\xi})t^c_w - 8(2-\xi)v\ell^c_{4+} + 8\xi\ell^c_{5+} + \\
&+ \left(\frac{1}{2}(4 + 6\xi - 3\xi^2) + 2(4-3\xi)y_2 - 2y_2^2 \right) \ell^c_2 + \frac{3}{2}(4 - 7\xi - 4y_2)v_2 - \frac{\xi(4-\xi)v_2}{2(4y_2+\xi)} \Bigg\},
\end{aligned} \quad (1.81)$$

$$\begin{aligned}
H^{2\ell}_F &= \xi N \Bigg\{ 2(2-\xi)(t^c_{0-} - t^c_{0+}) - (4-\xi)(t^c_{1-} - t^c_{1+}) + \\
&+ \sqrt{\xi}(1-\sqrt{\xi})t^c_w - 8v\ell_{4+} + \left(\frac{3}{2}\xi + 4y_2 \right) \ell^c_2 + 10v_2 \Bigg\}, \quad (1.82)
\end{aligned}$$

$$\begin{aligned}
H^4_F &= N \Bigg\{ 4(2-\xi)v(t^c_{0-} + t^c_{0+}) - 2(4-5\xi)(t^c_{1-} + t^c_{1+}) - 8(1-\xi)\ell^c_{4-} + \\
&+ 16\ell^c_{5-} + 8\xi\ell^c_{6-} - 16(1-\xi)\ell^c_{7-} - 2(3-y_2)v_2\ell^c_2 +
\end{aligned}$$

$$+\frac{1}{2}(20-16\sqrt{\xi}-3\xi+\xi\sqrt{\xi})-\frac{1}{2}(24+\xi)y_2+2y_2^2-\xi(2-\sqrt{\xi})^2\frac{1-y_2+\sqrt{\xi}}{2(4y_2+\xi)}\Big\}, \quad (1.83)$$

$$\begin{aligned}
H_I^{4\perp} &= \sqrt{\xi}N\Big\{-2(2-\xi)v(t^c_{0-}+t^c_{0+})+\frac{1}{2}(16+7\xi)(t^c_{1-}+t^c_{1+})+4(1-\xi)\ell^c_{4-}+\\
&+\frac{1}{4}(4-\xi)(10+3\xi)\ell^c_{5-}+\frac{1}{8}(72-30\xi-3\xi^2)\ell^c_{6-}+\\
&+8(1-\xi)\ell^c_{7-}-\left(\frac{\xi v_2}{2}+\frac{3}{2w_2}(2-\sqrt{\xi})(1-\sqrt{\xi})^2+\frac{3w_2}{2}(2+\sqrt{\xi})(1+\sqrt{\xi})^2\right)\ell^c_2+\\
&+\frac{1}{4}(76-124\sqrt{\xi}+53\xi-7\xi\sqrt{\xi})-\frac{1}{4}(28+5\xi)y_2+\xi(2-\sqrt{\xi})^2\frac{1-y_2+\sqrt{\xi}}{4(4y_2+\xi)}\Big\}, \quad (1.84)
\end{aligned}$$

$$\begin{aligned}
H_A^{1\perp} &= \sqrt{\xi}N\Big\{-2(2-\xi)(t^c_{0-}-t^c_{0+})+\frac{1}{2}(16-3\xi)(t^c_{1-}-t^c_{1+})+\\
&+\frac{1}{2}(8-10\sqrt{\xi}+\xi+\xi\sqrt{\xi})t^c_w+8v\ell^c_{4+}-2(7-\xi)\ell^c_{5+}+\\
&-\left(\frac{1}{8}(8-6\xi+3\xi^2)+\frac{1}{2}\xi y_2\right)\ell^c_2-\frac{1}{4}(4+5\xi)v_2+\frac{\xi(4-\xi)v_2}{4(4y_2+\xi)}\Big\}, \quad (1.85)
\end{aligned}$$

$$\begin{aligned}
H_A^{2\perp} &= \sqrt{\xi}N\Big\{-2(2-\xi)(t^c_{0-}-t^c_{0+})+\frac{1}{2}(8-3\xi)(t^c_{1-}-t^c_{1+})+\frac{1}{2}\xi(1-\sqrt{\xi})t^c_w+8v\ell^c_{4+}+\\
&-2(1+\xi)\ell^c_{5+}+\left(\frac{1}{8}(2-\xi)(4-3\xi)-\frac{1}{2}(8-\xi)y_2\right)\ell^c_2-\frac{1}{4}(52-5\xi)v_2-\frac{\xi(4-\xi)v_2}{4(4y_2+\xi)}\Big\}, \\
&\qquad (1.86)
\end{aligned}$$

$$\begin{aligned}
H_A^{3N} &= \sqrt{\xi}N\Big\{2(2-\xi)v(t^c_{0-}+t^c_{0+})-\frac{1}{2}(8-13\xi)(t^c_{1-}+t^c_{1+})-4(1-\xi)\ell^c_{4-}+\\
&+\frac{1}{4}(24-2\xi-3\xi^2)\ell^c_{5-}-\frac{1}{8}(8-30\xi+3\xi^2)\ell^c_{6-}-8(1-\xi)\ell^c_{7-}-\frac{1}{2}(8+\xi)v_2\ell^c_2+\\
&+\frac{1}{4}(52-52\sqrt{\xi}+5\xi-7\xi\sqrt{\xi})-\frac{1}{4}(52+5\xi)y_2+\xi(2-\sqrt{\xi})^2\frac{1-y_2+\sqrt{\xi}}{4(4y_2+\xi)}\Big\} \quad (1.87)
\end{aligned}$$

One can check that the sum of both contributions reproduces the result for the full phase space for $y_2=(1-\xi)/2$.

1.2 Longitudinal spin-spin correlation

In this section the $O(\alpha_s)$ radiative corrections to longitudinal spin-spin correlations and their polar angle dependence for massive quark pairs produced in e^+e^- annihilations is calcutated. This covers the case where the polar angle is averaged as well as the polar angle dependent calculation presented in Refs. [18, 24]. The interesting result is obtained that the longitudinal spin-spin correlation in heavy quark pair production is 100% at the Born term level in the forward and backward directions and remains very close to 100% after the radiative corrections are applied.

The longitudinal polarization of massive quarks affects the shape of the energy spectrum of their secondary decay leptons. Thus longitudinal spin-spin correlation effects in pair produced quarks and antiquarks will lead to correlation effects of the energy spectra of their secondary decay leptons and antileptons. As a byproduct of the calculation the $m \to 0$ limit and the role of the $O(\alpha_s)$ residual mass effects is discussed in this section, also for the single spin case. It will be delineated how residual mass effects contribute to the various spin-flip and no-flip terms in the $m \to 0$ limit for each of the three structure functions that describe the polar angle dependence.

1.2.1 Joint quark-antiquark density matrix

An extension of the no-spin and single spin cross sections considered in the previous section, the natural generalization consists in the *differential joint quark-antiquark density matrix*

$$d\boldsymbol{\sigma}^{\alpha} = d\sigma^{\alpha}_{\lambda_1\lambda_2;\lambda_1'\lambda_2'} \tag{1.88}$$

where λ_1 and λ_2 denote the *helicities* of the quark and antiquark, respectively. In this section the longitudinal polarization of the quark and antiquark is of main interest, in particular the longitudinal spin-spin correlations. Thus one specifies to the diagonal case $\lambda_1 = \lambda_1'$ and $\lambda_2 = \lambda_2'$. The label α specifies the polarization of the initial γ^*, Z or interference contributions thereof. The longitudinal spin-spin correlation and its polar angle dependence is expressed as before by the three polarization components U, L and F.

The diagonal part of the differential joint density matrix can be represented in terms of its components along the products of the unit matrix and the z-components of the Pauli matrix $\boldsymbol{\sigma}_3$ ($\boldsymbol{\sigma}_3 = \hat{p}_1\vec{\boldsymbol{\sigma}}$ for the quark and $\boldsymbol{\sigma}_3 = \hat{p}_2\vec{\boldsymbol{\sigma}}$ for the antiquark, $\hat{p}_i = \vec{p}_i/|\vec{p}_i|$). One has

$$d\boldsymbol{\sigma}^{\alpha} = \frac{1}{4}\left(d\sigma_{\alpha}\mathbb{1}\otimes\mathbb{1} + d\sigma_{\alpha}^{(\ell_1)}\boldsymbol{\sigma}_3\otimes\mathbb{1} + d\sigma_{\alpha}^{(\ell_2)}\mathbb{1}\otimes\boldsymbol{\sigma}_3 + d\sigma_{\alpha}^{(\ell_1\ell_2)}\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_3\right) \tag{1.89}$$

where the first and the second Pauli matrices stand for the quark and the antiquark, respectively. An alternative but equivalent representation of the longitudinal spin contributions can be written down in terms of the longitudinal spin components $s_1^\ell = 2\lambda_1$ and $s_2^\ell = 2\lambda_2$ with $s_1^\ell, s_2^\ell = \pm 1$ (or $s_1^\ell, s_2^\ell \in \{\uparrow,\downarrow\}$). One has

$$d\sigma(s_1^\ell, s_2^\ell) = \frac{1}{4}\left(d\sigma_{\alpha} + d\sigma_{\alpha}^{(\ell_1)}s_1^\ell + d\sigma_{\alpha}^{(\ell_2)}s_2^\ell + d\sigma_{\alpha}^{(\ell_1\ell_2)}s_1^\ell s_2^\ell\right). \tag{1.90}$$

Eq. (1.90) is easily inverted,

$$\begin{aligned}
d\sigma_{\alpha} &= d\sigma_{\alpha}(\uparrow\uparrow) + d\sigma_{\alpha}(\uparrow\downarrow) + d\sigma_{\alpha}(\downarrow\uparrow) + d\sigma_{\alpha}(\downarrow\downarrow),\\
d\sigma_{\alpha}^{(\ell_1)} &= d\sigma_{\alpha}(\uparrow\uparrow) + d\sigma_{\alpha}(\uparrow\downarrow) - d\sigma_{\alpha}(\downarrow\uparrow) - d\sigma_{\alpha}(\downarrow\downarrow),\\
d\sigma_{\alpha}^{(\ell_2)} &= d\sigma_{\alpha}(\uparrow\uparrow) - d\sigma_{\alpha}(\uparrow\downarrow) + d\sigma_{\alpha}(\downarrow\uparrow) - d\sigma_{\alpha}(\downarrow\downarrow),\\
d\sigma_{\alpha}^{(\ell_1\ell_2)} &= d\sigma_{\alpha}(\uparrow\uparrow) - d\sigma_{\alpha}(\uparrow\downarrow) - d\sigma_{\alpha}(\downarrow\uparrow) + d\sigma_{\alpha}(\downarrow\downarrow).
\end{aligned} \tag{1.91}$$

$O(\alpha_s)$ radiative corrections to the rate component $d\sigma_\alpha$ have been discussed in Refs. [7, 14], beam polarization effects [8] and beam-event correlation effects [8, 9] have been included. The $O(\alpha_s)$ radiative corrections to the longitudinal spin component $d\sigma_{\alpha}^{(\ell_1)}$ have been calculated in Ref. [7] including again beam polarization and beam-event correlation effects [8]. As concerns the longitudinal spin-spin correlation component $d\sigma_{\alpha}^{(\ell_1\ell_2)}$, the $O(\alpha_s)$ tree graph contributions have been determined in Ref. [25]. The full $O(\alpha_s)$ radiative corrections to the fully integrated spin-spin correlation component $\sigma_{\alpha}^{(\ell_1\ell_2)}$ are calculated in

Refs. [17] and [18] where beam-event correlation effects were averaged out while the polar angle dependence of the longitudinal spin-spin correlations, i.e. the polar angle structure induced by the rate and spin-spin correlation components $d\sigma_\alpha$ and $d\sigma_\alpha^{(\ell_1\ell_2)}$ have been determined in Ref. [24].

As before, the electro-weak cross section and the spin-spin correlation components are written in modular form in terms of two building blocks [3] (see Eq. (1.4)). Since the electro-weak model dependence and the polar angle dependence of $d\sigma_\alpha$ and $d\sigma_\alpha^{(\ell_1\ell_2)}$ are the same, a compact notation $d\sigma_\alpha^{\{\ell_1\ell_2\}}$ is introduced where $d\sigma_\alpha^{\{\ell_1\ell_2\}}$ stand for either $d\sigma_\alpha$ or $d\sigma_\alpha^{(\ell_1\ell_2)}$. Thus one writes

$$\begin{aligned}
\frac{d\sigma^{\{\ell_1\ell_2\}}}{d\cos\theta} &= \frac{3}{8}(1+\cos^2\theta)\left(g_{11}\sigma_U^{1\{\ell_1\ell_2\}}+g_{12}\sigma_U^{2\{\ell_1\ell_2\}}\right)\\
&\quad+\frac{3}{4}\sin^2\theta\left(g_{11}\sigma_L^{1\{\ell_1\ell_2\}}+g_{12}\sigma_L^{2\{\ell_1\ell_2\}}\right)\\
&\quad+\frac{3}{4}\cos\theta\left(g_{43}\sigma_F^{3\{\ell_1\ell_2\}}+g_{44}\sigma_F^{4\{\ell_1\ell_2\}}\right). \qquad (1.92)
\end{aligned}$$

The index $i=1,2,3,4$ in $\sigma_\alpha^{i\{\ell_1\ell_2\}}$ runs over the four possible linear combinations of bilinear products of vector and axial vector currents. θ is the polar angle between the electron beam direction and the top quark direction. The components g_{ij} $(i,j=1,2,3,4)$ of the electro-weak coupling matrix can be found in Appendix A.

1.2.2 Born term and loop contributions

Born term and loop contribution contribute to the longitudinal spin-spin correlation components given by

$$\sigma_\alpha^{\{\ell_1\ell_2\}}=\frac{\pi\alpha^2 v}{3q^4}H_\alpha^{i\{\ell_1\ell_2\}} \qquad \left(\text{with } v=\sqrt{1-4m_q^2/q^2}\right) \qquad (1.93)$$

where the helicity structure functions $H_\alpha^{i\{\ell_1\ell_2\}}$ $(\alpha=U,L,F)$ are obtained from the hadron tensor components $H_{\mu\nu}^{i\{\ell_1\ell_2\}}$ via covariant projection. While the Born term contributions for the unpolarized and single spin contributions are already given in the previous section, the Born term contributions to the spin correlated two-body hadron tensor components are given by

$$\begin{aligned}
H_U^{1(\ell_1\ell_2)}(Born) &= -2N_cq^2(1+v^2),\\
H_U^{2(\ell_1\ell_2)}(Born) &= -2N_cq^2(1-v^2),\\
H_L^{1(\ell_1\ell_2)}(Born) &= H_L^{2(\ell_1\ell_2)}(Born) \;=\; N_cq^2(1-v^2),\\
H_F^{4(\ell_1\ell_2)}(Born) &= -4N_cq^2v. \qquad (1.94)
\end{aligned}$$

The loop contributions for the longitudinal spin-spin components read

$$\begin{aligned}
H_U^{1(\ell_1\ell_2)}(loop) &= -4N_cq^2(\mathrm{Re}\,A+v^2\,\mathrm{Re}\,C),\\
H_U^{2(\ell_1\ell_2)}(loop) &= -4N_cq^2(\mathrm{Re}\,A-v^2\,\mathrm{Re}\,C),\\
H_L^{1(\ell_1\ell_2)}(loop) &= H_L^{2(\ell_1\ell_2)}(loop) \;=\; 2N_cq^2(\xi\,\mathrm{Re}\,A+v^2\,\mathrm{Re}\,B),\\
H_F^{4(\ell_1\ell_2)}(loop) &= -4N_cq^2v(\mathrm{Re}\,A+\mathrm{Re}\,C). \qquad (1.95)
\end{aligned}$$

There is also a contribution coming from the imaginary part of the vertex correction which multiplies the imaginary part of the Breit–Wigner function of the Z resonance as indicated in the electro-weak model parameters g_{43} and g_{13}. The relevant hadron tensor component results from the V/A interference term in the F projection. The contribution is given by

$$H_F^{3(\ell_1\ell_2)}(loop) = -8N_c q^2 v\,\mathrm{Im}\,B. \tag{1.96}$$

These contributions are rather small especially if one is far away from the Z resonance.

1.2.3 Three-body tree graph contributions

In the three-body case $e^+e^- \to q\bar{q}G$ the hadron tensor components $H_\alpha^{i\{\ell_1\ell_2\}}(y,z)$ are related to the components of the cross section by

$$\begin{aligned}
\frac{d\sigma_a^{i\{\ell_1\ell_2\}}}{dy\,dz} &= \frac{\pi\alpha^2 v}{3q^4}\left(\frac{q^2}{16\pi^2 v}\sum_{j=1}^{4} g_{1j}H_a^{j\{\ell_1\ell_2\}}(y,z)\right) \quad \text{for } a=U,L,I \\
\frac{d\sigma_a^{i\{\ell_1\ell_2\}}}{dy\,dz} &= \frac{\pi\alpha^2 v}{3q^4}\left(\frac{q^2}{16\pi^2 v}\sum_{j=1}^{4} g_{4j}H_a^{j\{\ell_1\ell_2\}}(y,z)\right) \quad \text{for } a=F,A
\end{aligned} \tag{1.97}$$

where $y = 1-2p_1q/q^2$ and $z = 1-2p_2q/q^2$ are the two energy-type phase space parameters defined earlier. Note that the three-body helicity structure functions $H_\alpha^{i\{\ell_1\ell_2\}}(y,z)$ have a different dimension than their two-body counterparts in Eq. (1.97), indicated by explicitly referring to the (y,z)-dependence of the three-body structure functions. The projection onto the three helicity structure functions U, L, F is done as for the two-body case. The $O(\alpha_s)$ spin dependent hadronic three-body tensor

$$H_{\mu\nu}(p_1,p_2,p_3,s_1,s_2) = \sum_{\text{gluonspin}} \langle q\bar{q}g|J_\mu|0\rangle\langle 0|J_\mu^\dagger|q\bar{q}g\rangle \tag{1.98}$$

can easily be calculated from the relevant Feynman diagrams. The longitudinal spin components of the quark and antiquark can be projected out with the help of the respective longitudinal spin vectors. They read

$$\begin{aligned}
(s_1^\ell)^\mu &= \frac{c_1}{\sqrt{\xi}}(\sqrt{(1-y)^2-\xi};0,0,1-y) \\
(s_2^\ell)^\mu &= \frac{c_2}{\sqrt{\xi}}(\sqrt{(1-z)^2-\xi};(1-z)\sin\theta_{12},0,(1-z)\cos\theta_{12})
\end{aligned} \tag{1.99}$$

with

$$\cos\theta_{12} = \frac{yz+y+z-1+\xi}{\sqrt{(1-y)^2-\xi}\sqrt{(1-z)^2-\xi}} \tag{1.100}$$

where $\xi = 4m_q^2/q^2 = 1-v^2$. In combination with the four-momenta constructed in the previous section, this representation of spin vectors gives rise to scalar products

$$\begin{aligned}
p_1p_2 &= \frac{1}{2}q^2(1-y-z)-m^2, \qquad p_1p_3 = \frac{1}{2}q^2 z, \qquad p_2p_3 = \frac{1}{2}q^2 y, \\
p_1s_1 &= 0, \qquad p_1s_2 = c_2\sqrt{q^2}\frac{2(1-z)(1-y-z)-\xi(2-y-z)}{2\sqrt{\xi}\sqrt{(1-z)^2-\xi}}, \qquad s_1^2 = -1
\end{aligned}$$

$$
\begin{aligned}
p_2s_1 &= c_1\sqrt{q^2}\frac{2(1-y)(1-y-z)-\xi(2-y-z)}{2\sqrt{\xi}\sqrt{(1-y)^2-\xi}}, \qquad p_2s_2 = 0, \qquad s_2^2 = -1\\
p_3s_1 &= c_1\sqrt{q^2}\frac{2(1-y)z-\xi(y+z)}{2\sqrt{\xi}\sqrt{(1-y)^2-\xi}}, \qquad p_3s_2 = c_1\sqrt{q^2}\frac{2(1-z)y-\xi(y+z)}{2\sqrt{\xi}\sqrt{(1-z)^2-\xi}},\\
s_1s_2 &= c_1c_2\frac{2(1-y)(1-z)(1-y-z)-\xi(3(1-y-z)+y^2+yz+z^2)+\xi^2}{\xi\sqrt{(1-y)^2-\xi}\sqrt{(1-z)^2-\xi}}. \qquad (1.101)
\end{aligned}
$$

1.2.4 The full $O(\alpha_s)$ result

After integrating the tree-graph contributions over the phase space and adding the one-loop contributions, the full $O(\alpha_s)$ result can again be expressed by an extended set of decay rate terms which are given in Appendix B. In adding these two components, the infrared singularities which are parametrized by the gluon mass cancel out. The finite result is independent of the specific choice for the regularization procedure. The results read

$$
\begin{aligned}
H^1_{U+L}(\alpha_s) &= N\Bigg\{-\frac{1}{4}(2+\xi)(20-3\xi)+\frac{1}{4}\sqrt{\xi}(32+12\xi+3\xi^2)+\\
&+(4-3\xi)\left(\frac{2-\xi}{2}(t_8-t_{16})+v(t_{10}+2t_{12})-\frac{1}{8v^2}(4-\xi)(8-3\xi-\xi^2)t_{13}\right)+\\
&+\frac{1}{8v}(88-78\xi-5\xi^2+3\xi^3)+\frac{1}{4v^3}(32-88\xi+76\xi^2-19\xi^3)t_{15}+\\
&-\frac{1}{2}(8-10\xi+\xi^2)t_{14}-\frac{1}{4v^2}\left(16-42\xi+31\xi^2-4\xi^3+8v^3(4-3\xi)\right)t_3\Bigg\}, \qquad (1.102)
\end{aligned}
$$

$$
\begin{aligned}
H^2_{U+L}(\alpha_s) &= \xi N\Bigg\{\frac{1}{4}(58-3\xi)-\frac{1}{4}\sqrt{\xi}(56+3\xi)-\frac{1}{8v}(54-65\xi+3\xi^2)+\\
&+\frac{2-\xi}{2}(t_8-t_{16})+v(t_{10}+2t_{12})+\frac{1}{8v^2}(96-140\xi+35\xi^2-3\xi^3)t_{13}+\\
&-\frac{1}{2}(10+3\xi)t_{14}+\frac{1}{4v^3}(8-20\xi+13\xi^2)t_{15}+\frac{1}{4v^2}(2+\xi-4\xi^2-8v^3)t_3\Bigg\}, \qquad (1.103)
\end{aligned}
$$

$$
\begin{aligned}
H^1_L(\alpha_s) &= N\Bigg\{-\frac{3}{4}(24+22\xi-\xi^2)+\frac{1}{4}\sqrt{\xi}(48+112\xi+3\xi^2)+\\
&-\xi\left(\frac{2-\xi}{2}(t_8-t_{16})+v(t_{10}+2t_{12})\right)-\frac{1}{8v^2}(64-32\xi-164\xi^2+51\xi^3-3\xi^4)t_{13}+\\
&+\frac{1}{8v}(128-106\xi-81\xi^2+3\xi^3)-\frac{\xi}{4v^3}(28-24\xi+3\xi^2)t_{15}+\\
&+\xi(6+\xi)t_{14}+\frac{\xi}{4v^2}(30-27\xi+4\xi^2+8v^3)t_3\Bigg\}, \qquad (1.104)
\end{aligned}
$$

$$
\begin{aligned}
H^2_L(\alpha_s) &= \xi N\Bigg\{\frac{1}{4}(34-3\xi)-\frac{3}{4}\sqrt{\xi}(12+\xi)-\frac{1}{8v}(46-65\xi+3\xi^2)+\\
&-\left(\frac{2-\xi}{2}(t_8-t_{16})+v(t_{10}+2t_{12})\right)+\frac{1}{8v^2}(80-136\xi+35\xi^2-3\xi^3)t_{13}+
\end{aligned}
$$

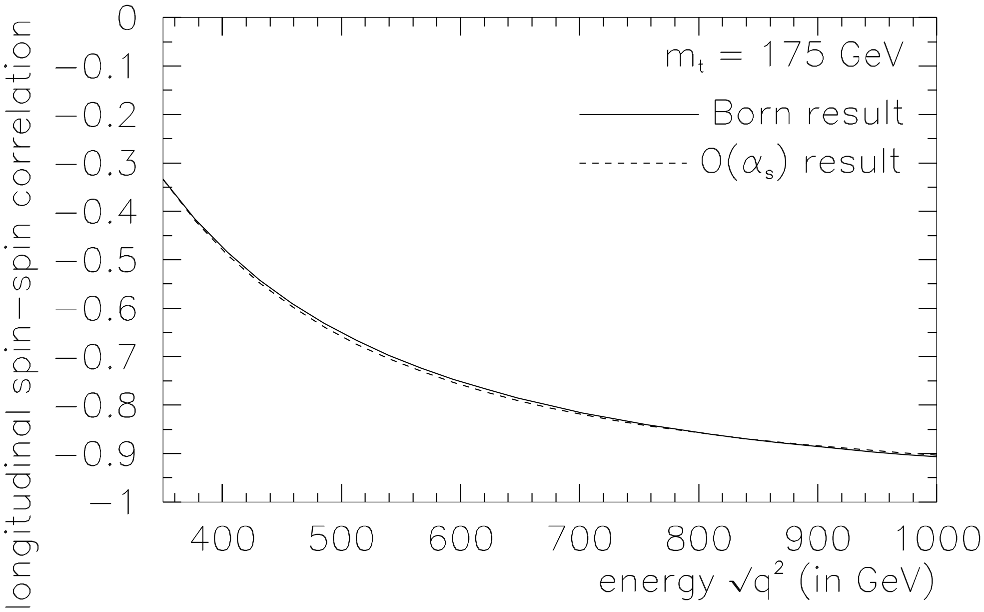


Figure 1.3: Energy dependence of $O(1)$ and $O(\alpha_s)$ mean longitudinal spin-spin correlations $\langle P^{\ell\ell}\rangle$ in $e^+e^- \to t\bar{t}(g)$

$$-(1+\xi)t_{14} - \frac{1}{2v^3}(4-9\xi+4\xi^2)t_{15} + \frac{1}{2v^2}(1-4\xi+2\xi^2+4v^3)t_3\Bigg\}, \tag{1.105}$$

$$H_F^3(\alpha_s) \;=\; N\Bigg\{2\pi v\xi\Bigg\}, \tag{1.106}$$

$$\begin{aligned} H_F^4(\alpha_s) \;=\; & N\Bigg\{2\sqrt{\xi}+4v^2+2v(2-\xi)(t_8-t_{16}+t_{22})+\frac{1}{4v^2}(16-32\xi+18\xi^2-\xi^3)t_{21}+ \\ & +\frac{\xi}{v^3}(5-8\xi+4\xi^2)(t_{20}-t_{19})+\frac{\sqrt{\xi}}{2v^3}(8-11\xi+3\xi^2+2\xi^3)(t_{20}+t_{19}) + \\ & +\frac{\xi}{4v^2}(20-22\xi+\xi^2)(t_{18}-t_{17})+\frac{\sqrt{\xi}}{4v^2}(16-10\xi-5\xi^2)(t_{18}+t_{17}) + \\ & -\frac{\sqrt{\xi}}{2v^2}(2+7\xi-5\xi^2)(t_{12}-t_{10})+\frac{1}{v^2}(2-\xi)(1+\xi)(t_{12}+t_{10}+2t_1-4\ln 2) + \\ & +4v^2t_{12}-(4+\xi)t_{14}+\frac{1}{v^3}(4-5\xi+2\xi^3)t_{15}-\frac{1}{2v^2}(2-\xi)(4-5\xi+5\xi^2)t_{13} + \\ & +\frac{1}{4v^2}\Big((2-\xi)(4-5\xi+5\xi^2)-2v(8-14\xi+7\xi^2)\Big)\,t_3\Bigg\}. \end{aligned} \tag{1.107}$$

A detailed analysis of the result can be found in Refs. [18, 24]. Here only two results are shown, namely the energy dependence of the mean longitudinal spin-spin correlation in Fig. 1.3 and the polar angle dependence of the angle dependent spin-spin correlation in Fig. 1.4. The correlation function in general is given by

$$\langle P^{\ell\ell}\rangle = \frac{\sigma^{(\ell_1\ell_2)}}{\sigma} = \frac{\sigma(\uparrow\uparrow)-\sigma(\uparrow\downarrow)-\sigma(\downarrow\uparrow)+\sigma(\downarrow\downarrow)}{\sigma(\uparrow\uparrow)+\sigma(\uparrow\downarrow)+\sigma(\downarrow\uparrow)+\sigma(\downarrow\downarrow)}. \tag{1.108}$$

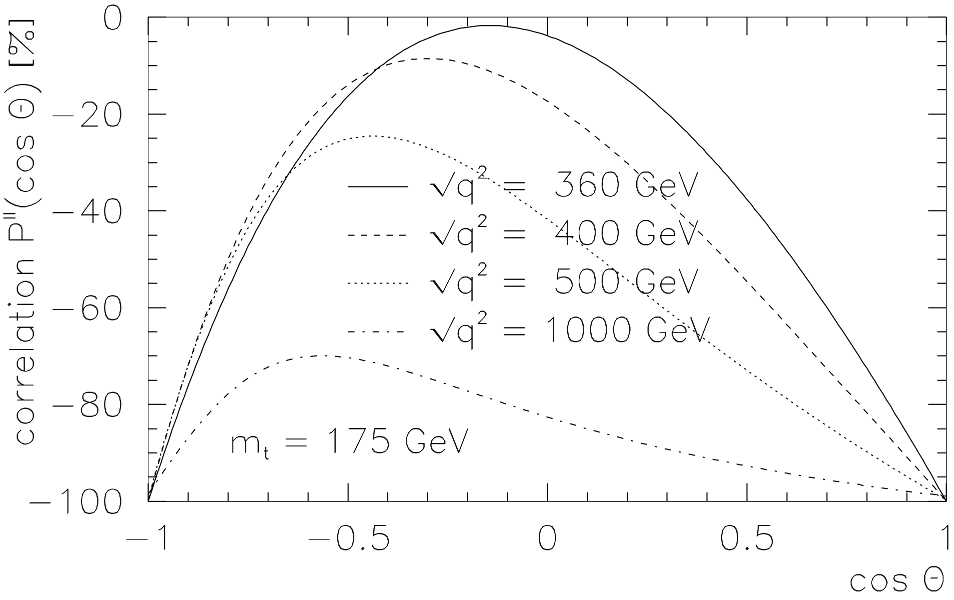


Figure 1.4: $O(\alpha_s)$ polar angle dependence of the longitudinal spin-spin asymmetry in $e^+e^- \to t\bar{t}(g)$ at $\sqrt{q^2} = 360$, 400, 500, and 1000 GeV

For the mean spin-spin correlation function one integrates the differential cross section in Eq. (1.92) over the polar angle θ and obtains

$$\sigma^{\{\ell_1\ell_2\}} = \left(g_{11}\sigma_U^{1\{\ell_1\ell_2\}} + g_{12}\sigma_U^{2\{\ell_1\ell_2\}}\right) + \left(g_{11}\sigma_L^{1\{\ell_1\ell_2\}} + g_{12}\sigma_L^{2\{\ell_1\ell_2\}}\right) = \left(g_{11}\sigma_{U+L}^{1\{\ell_1\ell_2\}} + g_{12}\sigma_{U+L}^{2\{\ell_1\ell_2\}}\right) \tag{1.109}$$

which is used in Eq. (1.108). For the polar angle dependent spin-spin correlation one replaces the total cross sections in Eq. (1.108) by the differential cross sections in Eq. (1.92).

1.2.5 Massless QCD and the zero-mass limit

As is well-known by now, the $m_q \to 0$ limit of spin-flip contributions does not coincide with that of massless QCD ($m_q = 0$). This can be easily obtained for the spin-spin correlation results obtained in this section. The $m_q = 0$ expressions can be calculated in dimensional regularization as described in Ref. [15]. One obtains

$$\begin{aligned} H_U^1(\alpha_s) &= 4N_cq^2\frac{\alpha_sC_F}{4\pi}, \qquad H_U^{1(\ell_1\ell_2)}(\alpha_s) = -4N_cq^2\frac{\alpha_sC_F}{4\pi}, \\ H_L^1(\alpha_s) &= 8N_cq^2\frac{\alpha_sC_F}{4\pi}, \qquad H_L^{1(\ell_1\ell_2)}(\alpha_s) = -8N_cq^2\frac{\alpha_sC_F}{4\pi}, \\ H_F^4(\alpha_s) &= 0, \qquad H_F^{4(\ell_1\ell_2)}(\alpha_s) = 0. \end{aligned} \tag{1.110}$$

Using the limiting expressions for the decay rate terms given in Appendix B, for the mass dependent results in the limit $m_q \to 0$ one obtains

$$H_U^1(\alpha_s) = 4N_cq^2\frac{\alpha_sC_F}{4\pi}, \quad H_L^1(\alpha_s) = 8N_cq^2\frac{\alpha_sC_F}{4\pi}, \quad H_F^4(\alpha_s) = 0 \tag{1.111}$$

for the unpolarized structure functions and

$$
\begin{aligned}
H_U^{1(\ell_1,\ell_2)}(\alpha_s) &= 12N_cq^2\frac{\alpha_s C_F}{4\pi} = (-4+[16])N_cq^2\frac{\alpha_s C_F}{4\pi}, \\
H_L^{1(\ell_1,\ell_2)}(\alpha_s) &= -8N_cq^2\frac{\alpha_s C_F}{4\pi}, \\
H_F^{4(\ell_1,\ell_2)}(\alpha_s) &= 16N_cq^2\frac{\alpha_s C_F}{4\pi} = [16]N_cq^2\frac{\alpha_s C_F}{4\pi}
\end{aligned}
\tag{1.112}
$$

for the longitudinal spin-spin correlation functions. With the square bracket notation the difference between these results and the results for massless QCD are indicated, known as *anomalous spin-flip terms*. The current-current structures for $i = 2, 3$ can be seen to be zero in this limit.

By adding in the Born term contributions in Eqs. (1.21) and (1.94) and the longitudinal single-spin hadron tensor components, one finally obtains in the $m_q \to 0$ limit ($C_F = 4/3$ is made explicit here)

$$
\begin{aligned}
H_U^1(s_1^\ell,s_2^\ell) &= \frac{1}{4}\Big(H_U^1 + H_U^{1(\ell_1\ell_2)}s_1^\ell s_2^\ell\Big) \\
&= N_cq^2\left((1-s_1^\ell s_2^\ell)\left(1+\frac{1}{3}\times\frac{\alpha_s}{\pi}\right)+\left[\frac{4}{3}\times\frac{\alpha_s}{\pi}s_1^\ell s_2^\ell\right]\right), \\
H_L^1(s_1^\ell,s_2^\ell) &= \frac{1}{4}\Big(H_L^1 + H_L^{1(\ell_1\ell_2)}s_1^\ell s_2^\ell\Big) \\
&= N_cq^2(1-s_1^\ell s_2^\ell)\left(0+\frac{2}{3}\times\frac{\alpha_s}{\pi}+[0]\right), \\
H_F^1(s_1^\ell,s_2^\ell) &= \frac{1}{4}\Big(H_F^{1(\ell_1)}s_1^\ell + H_F^{1(\ell_2)}s_2^\ell\Big) \\
&= N_cq^2(s_1^\ell-s_2^\ell)\left(1+0\times\frac{\alpha_s}{\pi}-\left[\frac{2}{3}\times\frac{\alpha_s}{\pi}\right]\right), \\
H_U^4(s_1^\ell,s_2^\ell) &= \frac{1}{4}\Big(H_U^{1(\ell_1)}s_1^\ell + H_U^{1(\ell_2)}s_2^\ell\Big) \\
&= N_cq^2(s_1^\ell-s_2^\ell)\left(1+\frac{1}{3}\times\frac{\alpha_s}{\pi}-\left[\frac{2}{3}\times\frac{\alpha_s}{\pi}\right]\right), \\
H_L^4(s_1^\ell,s_2^\ell) &= \frac{1}{4}\Big(H_L^{1(\ell_1)}s_1^\ell + H_L^{1(\ell_2)}s_2^\ell\Big) \\
&= N_cq^2(s_1^\ell-s_2^\ell)\left(0+\frac{2}{3}\times\frac{\alpha_s}{\pi}+[0]\right), \\
H_F^4(s_1^\ell,s_2^\ell) &= \frac{1}{4}\Big(H_F^4 + H_F^{4(\ell_1\ell_2)}s_1^\ell s_2^\ell\Big) \\
&= N_cq^2\left((1-s_1^\ell s_2^\ell)\left(1+0\times\frac{\alpha_s}{\pi}\right)+\left[\frac{4}{3}\times\frac{\alpha_s}{\pi}s_1^\ell s_2^\ell\right]\right).
\end{aligned}
\tag{1.113}
$$

In Table 1.1 all $m \to 0$ contributions to the various spin configurations for the parity-even (VV) and parity-odd (VA) current contributions are listed where again the $m = 0$ no-flip contributions has been split off by using the square bracket notation for the anomalous spin-flip contributions.

The anomalous spin-flip contributions have their origin in the collinear limit where the spin-flip contribution proportional to m survives since it is multiplied by the $1/m$

VV	U	L	F
$(\uparrow\uparrow)$	$0+[4/3]$	$0+[0]$	$0+[0]$
$(\uparrow\downarrow)$	$2/3-[4/3]$	$4/3+[0]$	$0-[4/3]$
$(\downarrow\uparrow)$	$2/3-[4/3]$	$4/3+[0]$	$0+[4/3]$
$(\downarrow\downarrow)$	$0+[4/3]$	$0+[0]$	$0+[0]$
VA	U	L	F
$(\uparrow\uparrow)$	$0+[0]$	$0+[0]$	$0+[4/3]$
$(\uparrow\downarrow)$	$2/3-[4/3]$	$4/3+[0]$	$0-[4/3]$
$(\downarrow\uparrow)$	$-2/3+[4/3]$	$-4/3+[0]$	$0-[4/3]$
$(\downarrow\downarrow)$	$0+[0]$	$0+[0]$	$0+[4/3]$

Table 1.1: $O(\alpha_s)$ corrections to specific spin configurations in QCD($m_q = 0$) and QCD($m_q \to 0$) for the parity-even (upper table) and odd current contributions (lower table). The entries are given in units of $N_c q^2 \alpha_s/\pi$.

collinear mass singularity. Because the anomalous spin-flip terms are associated with the collinear singularity, the flip contributions are universal and factorize into the Born term contribution and an universal spin-flip bremsstrahlung function [10]. This explains why there is no anomalous contribution for instance to H_L^4 and why the anomalous flip contributions to e.g. H_U^4 and H_F^1 are equal.

1.3 Decay of the polarized top quark

After having considered the pair production process of polarized top quarks, it is a natural consequence to also look at the decay of the polarized top quark. The top quark is very short-lived and therefore retains its full polarization content when it decays. Therefore, the polarization can be "read off" from the decay products. In the decay of an unpolarized or polarized top quark to the W^+ gauge boson and a bottom quark the W^+ is strongly polarized, or, phrased in a different language, the W^+ has a nontrivial spin density matrix. Thus being already polarized even for unpolarized top quarks, the spin density matrix of the W^+ can be tuned by changing the polarization of the top quark. The polarization of the W^+ will reveal itself in the angular decay distribution of its subsequent decays $W^+ \to l^+ + \nu_l$ or $W^+ \to \bar{q} + q$.

The CDF collaboration has already presented some results on the measurement of the longitudinal component of the W^+ based on the limited RUN I data [19]. The measurement has confirmed the expected dominance of the longitudinal mode. The error on this measurement is quite large ($\approx 45\%$) but is expected to be reduced significantly during RUN II at the TEVATRON started in March 2001. In RUN II one will produce $(5-6)\times 10^3$ top quark pairs per year and detector.

This number will be boosted to $10^7 - 10^8$ top quark pairs per year and detector at the LHC starting in 2005/2006. It is conceivable that the errors on the structure function measurements can be reduced to the $1-2\%$ level in the next few years [26]. If such an accuracy can, in fact, be achieved and, having in mind that the $O(\alpha_s)$ corrections to the top decay rate amount to 8.5% [27, 28, 29, 30, 31, 32], it is quite evident that one needs to improve on the known theoretical Born level predictions for the structure functions by calculating their next-to-leading order radiative corrections. At a later stage, when the data sample of polarized top quarks has become sufficiently large, one will be able to also analyze the decays of polarized top quarks.

Polarized top decay brings in again five additional polarized structure functions which can be measured through an analysis of spin-momentum correlations between the polarization vector of the top quark and the momenta of its decay products. Polarized top quarks will become available at hadron colliders through single top production which occurs at the 33% level of the top quark pair production rate [33]. Future e^+e^- colliders will also be copious sources of polarized top quark pairs [7, 8, 15, 16, 34, 35]. For example, at the proposed TESLA collider one expects rates of $(1-4)\times 10^5$ top quark pairs per year. The polarization of these can be easily tuned through the availability of polarized beams (see e.g. Ref. [36]). Further, there is a high degree of correlation between the polarization of top and anti-top quarks produced in pairs either at e^+e^- colliders [17, 18, 37, 38] or at hadron colliders [39] which can be probed through the joint decay distributions of the top and the anti-top quark.

In this section the momentum-momentum and spin-momentum correlations in the cascade decay process $t\to W^+ + b$ followed by $W^+\to l^+ + \nu_l$ (or $W^+\to \bar q + q$) is analyzed. For the decay $t\to W^+ + b$ the spin-momentum correlation between the spin of the top and the momentum of the W^+ is analyzed in the top quark rest frame. In the subsequent decay of the W^+ the rest frame of the gauge boson is chosen to analyze the correlation between the momentum of the lepton (or antiquark) and the initial momentum direction of the W^+. This must be contrasted with the *center of mass* analysis of polarized top decay where the spin-momentum correlations are all analyzed in the rest system of the top quark (for an $O(\alpha_s)$ analysis of this kind see Ref. [40]).

The decay distribution of unpolarized top quarks (or the average over its polarizations) is governed again by the three structure functions H_U, H_L and H_F known from the previous sections while the complete angular decay distribution is governed by altogether eight structure functions which are calculate analytically including their $O(\alpha_s)$ radiative corrections. One of the motivations for calculating the $O(\alpha_s)$ radiative corrections is the fact that the radiative QCD corrections populate helicity configurations that are not accessible at the Born level. However, it turns out that the $O(\alpha_s)$ population of such structure functions is rather small [41].

In order to retain full control over the b mass dependence, and having also other applications in mind, a finite mass value for the b quark has been kept in Ref. [42], improving on earlier calculations of polarized top decay where the b quark mass was neglected and where the attention was limited to the six (diagonal) structure functions that govern the polar angle distribution in the cascade decay [36]. The additional two (non-diagonal) structure functions calculated in Ref. [42] describe the azimuthal correlation of the plane of the top quark's polarization and the plane defined by the final fermions.

1.3.1 Structure functions for the top quark decay

It turns out that it is rather convenient from the computational point of view to represent the helicity projections defined by the gauge boson polarization vectors and the top polarization vector in covariant form. One has

$$\begin{aligned} H_i &= \Pi_i^{\mu\nu} H_{\mu\nu} \qquad i = U, L, F, \\ H_{i^P} &= \Pi_i^{\mu\nu} H_{\mu\nu}(s_t^\ell) \qquad i = U, L, F, \\ H_{i^P} &= \Pi_i^{\mu\nu} H_{\mu\nu}(s_t^\perp) \qquad i = I, A. \end{aligned} \tag{1.114}$$

Where the covariant projectors onto the diagonal density matrix elements are given by

$$\Pi_L^{\mu\nu} = \frac{m_W^2}{m_t^2}\frac{1}{|\vec{q}|^2}\Big(p_t^\mu - \frac{p_t\cdot q}{m_W^2}q^\mu\Big)\Big(p_t^\nu - \frac{p_t\cdot q}{m_W^2}q^\nu\Big), \tag{1.115}$$

$$\Pi_{U+L}^{\mu\nu} = -g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2}, \tag{1.116}$$

$$\Pi_F^{\mu\nu} = \frac{1}{m_t}\frac{1}{|\vec{q}|} i\epsilon^{\mu\nu\alpha\beta} p_{t,\alpha} q_\beta \tag{1.117}$$

where $\epsilon^{0123} = -1$. Again the projector for the unpolarized-transverse component U is not written out. Note that it can be obtained from the combination $\Pi_{U+L}^{\mu\nu} - \Pi_L^{\mu\nu}$. The projectors onto the transverse-longitudinal non-diagonal density matrix elements are given by

$$\Pi_I^{\mu\nu} = +\frac{m_W}{m_t}\frac{1}{|\vec{q}|}\Big\{\epsilon^\mu(x)\Big(p_t^\nu - \frac{p_t\cdot q}{m_W^2}q^\nu\Big) + \mu \leftrightarrow \nu\Big\}, \tag{1.118}$$

$$\Pi_A^{\mu\nu} = -\frac{m_W}{m_t^2}\frac{1}{|\vec{q}|^2}\Big\{i\epsilon^{\mu\alpha\beta\gamma}\epsilon_\alpha(x) p_{t,\beta} q_\gamma \Big(p_t^\nu - \frac{p_t\cdot q}{m_W^2}q^\nu\Big) - \mu \leftrightarrow \nu\Big\}. \tag{1.119}$$

These definitions involve the transverse polarization vector of the gauge boson $\epsilon_\alpha(x) = (0;1,0,0)$ pointing in the x-direction. The covariant representation of the longitudinal component of the polarization vector of the top spin vector s_t^ℓ is given by

$$s_t^{\ell,\mu} = \frac{1}{|\vec{q}|}\Big(q^\mu - \frac{p_t\cdot q}{m_t^2}p_t^\mu\Big), \tag{1.120}$$

whereas its transverse component $s_t^\perp$ reads

$$s_t^{\perp,\mu} = (0;1,0,0). \tag{1.121}$$

Note the inverse powers of $|\vec{q}| = \sqrt{q_0^2 - m_W^2}$ that enter the L, F, I and A projectors and the longitudinal polarization vector. They come in for normalization reasons. These inverse powers of $|\vec{q}|$ will make the necessary tree graph integrations to be dealt with in the full polar angle dependence somewhat more complicated than the total $(U + L)$ rate integration which has a rather simple projector.

In terms of there structure functions and including the appropriate normalization factor the four-fold decay distribution is given by

$$\begin{aligned} \frac{d\Gamma}{dq_0\, d\cos\theta_P\, d\cos\theta\, d\chi} &= \frac{1}{4\pi}\frac{G_F|V_{tb}|^2 m_W^2}{\sqrt{2}\pi}|\vec{q}|\Big\{\frac{3}{8}(H_U + P\cos\theta_p H_{U^P})(1+\cos^2\theta) + \\ &+\frac{3}{4}(H_L + P\cos\theta_p H_{L^P})\sin^2\theta + \frac{3}{4}(H_F + P\cos\theta_p H_{F^P})\cos\theta + \\ &+\frac{3}{4}P\sin\theta_p H_{I^P}\sin\theta\cos\theta\cos\chi + \frac{3}{4}P\sin\theta_p H_{A^P}\sin\theta\cos\chi\Big\} \end{aligned} \tag{1.122}$$

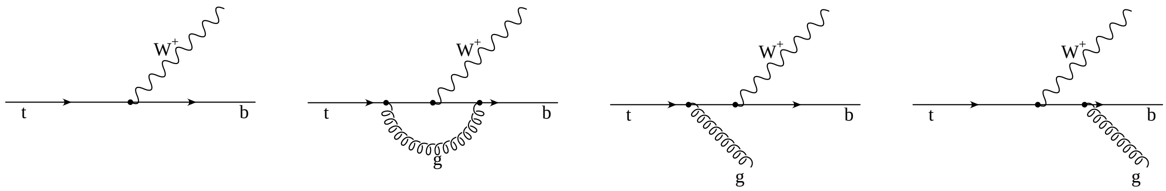


Figure 1.5: Born term contribution (left) and radiative corrections

where θ_P is the angle between the momentum of the W^+ in the top quark rest frame and the top quark polarization vector while P is the length of this polarization vector or the magnitude of the polarization of the top quark. θ and χ are the polar and azimuthal angle of the decay of the W^+ boson in the rest frame of this gauge boson, relative to the rest frame of the top quark decay. The freedom is taken to normalize the differential rate such that one obtains the total $t \to W^+ + b$ rate upon integration and not the total rate multiplied by the branching ratio of the respective W^+ decay channel.

As Eq. (1.122) shows, the non-diagonal structure functions H_{I^P} and H_{A^P} are associated with azimuthal measurements. This necessitates the definition of a hadron plane which is only possible through the availability of the x-component of the polarization vector of the top. This is the physical explanation of why the two structure functions H_{I^P} and H_{A^P} are functions only of the transverse component of the polarization vector of the top quark. For similar reasons the polarization dependent structure functions H_{U^P}, H_{L^P} and H_{F^P} depend only on the longitudinal component of the polarization vector. Setting $P = 0$ in Eq. (1.122) one obtains the decay distribution for unpolarized top decay. As in Ref. [41] the angular decay distribution can then also be sorted in terms of decays into transverse-plus and transverse-minus W^+ bosons given by the structure function combinations $(U + F)/2$ and $(U - F)/2$ which multiply the angular factors $(1 + \cos\theta)^2$ and $(1 - \cos\theta)^2$, respectively.

1.3.2 Born term results

The Born term tensor is calculated from the square of the Born term amplitude (see Fig. 1.5(a)) given by

$$M^\mu = V_{tb}\frac{g}{\sqrt{2}}\bar{u}_b\gamma^\mu\frac{1}{2}(1-\gamma_5)u_t. \tag{1.123}$$

Omitting the coupling factor $V_{tb}g/\sqrt{2} = 2m_W V_{tb}(G_F/\sqrt{2})^{1/2}$ one can write the Born term tensor (the spin of the b quark is summed) in a compact way,

$$\begin{aligned} B^{\mu\nu} &= \frac{1}{4}\mathrm{Tr}\left((\not{p}_b + m_b)\gamma^\mu(1-\gamma_5)(\not{p}_t + m_t)(1+\gamma_5\not{s}_t)\gamma^\nu(1-\gamma_5)\right) = \\ &= 2\left(\bar{p}_t^\nu p_b^\mu + \bar{p}_t^\mu p_b^\nu - g^{\mu\nu}\bar{p}_t\cdot p_b + i\epsilon^{\mu\nu\alpha\beta}p_{b,\alpha}\bar{p}_{t,\beta}\right) \end{aligned} \tag{1.124}$$

since only even-numbered γ-matrix strings survive between the two $(1-\gamma_5)$ with

$$\bar{p}_t^\mu = p_t^\mu - m_t s_t^\mu. \tag{1.125}$$

The relation of the Born term tensor $B^{\mu\nu}$ to the hadron tensor $H^{\mu\nu}$ defined earlier is found to be given by [42]

$$H^{\mu\nu}(Born) = \frac{1}{4m_t^2}\delta\left(q_0 - \frac{m_t^2 + m_W^2 - m_b^2}{2m_t}\right)B^{\mu\nu}. \tag{1.126}$$

using the scaled variables $x = m_W/m_t$ and $y = m_b/m_t$ and the abbreviation

$$|\vec{q}| = \frac{m_t}{2}\sqrt{\lambda} \quad \text{with} \quad \lambda = \lambda(1, x^2, y^2) = 1 - 2x^2 - 2y^2 + x^4 - 2x^2y^2 + y^4 \tag{1.127}$$

the Born term results read

$$\begin{aligned}
B_{U+L} &= m_t^2\frac{1}{x^2}\Big((1-y^2)^2 + x^2(1-2x^2+y^2)\Big) \quad\to\quad m_t^2\frac{1}{x^2}(1-x^2)(1+2x^2),\\
B_{U^P+L^P} &= m_t^2\sqrt{\lambda}\frac{1}{x^2}(1-2x^2-y^2) \quad\to\quad m_t^2\frac{1}{x^2}(1-x^2)(1-2x^2),\\
B_L &= m_t^2\frac{1}{x^2}\Big((1-y^2)^2 - x^2(1+y^2)\Big) \quad\to\quad m_t^2\frac{1}{x^2}(1-x^2),\\
B_{L^P} &= m_t^2\sqrt{\lambda}\frac{1}{x^2}(1-y^2) \quad\to\quad m_t^2\frac{1}{x^2}(1-x^2),\\
B_F &= -2m_t^2\sqrt{\lambda} \quad\to\quad -2m_t^2(1-x^2),\\
B_{F^P} &= 2m_t^2(1-x^2+y^2) \quad\to\quad 2m_t^2(1-x^2),\\
B_{I^P} &= -2m_t^2\sqrt{\lambda}\frac{1}{x} \quad\to\quad -2m_t^2\frac{1}{x}(1-x^2),\\
B_{A^P} &= 2m_t^2\frac{1}{x}(1-x^2-y^2) \quad\to\quad 2m_t^2\frac{1}{x}(1-x^2)
\end{aligned} \tag{1.128}$$

where the right hand side expressions are the limiting values for vanishing bottom quark mass (i.e. for $y \to 0$).

1.3.3 The one-loop contributions

The one-loop contributions to fermionic $(V-A)$ transitions have a long history. Since QED and QCD have the same structure at the one-loop level the history even dates back to QED times. The reference used here will be the work of Gounaris and Paschalis [43] (see also Ref. [44]) who used a gluon mass regulator to regularize the gluon IR singularity. The one-loop amplitudes (see Fig. 1.5(b)) are defined by the covariant expansion ($J_\mu^V = \bar{q}_b\gamma_\mu q_t$, $J_\mu^A = \bar{q}_b\gamma_\mu\gamma_5 q_t$)

$$\begin{aligned}
\langle b(p_b)|J_\mu^V|t(p_t)\rangle &= \bar{u}_b(p_b)\Big\{\gamma_\mu F_1^V + p_{t,\mu}F_2^V + p_{b,\mu}F_3^V\Big\}u_t(p_t),\\
\langle b(p_b)|J_\mu^A|t(p_t)\rangle &= \bar{u}_b(p_b)\Big\{\gamma_\mu F_1^A + p_{t,\mu}F_2^A + p_{b,\mu}F_3^A\Big\}\gamma_5 u_t(p_t).
\end{aligned} \tag{1.129}$$

In the Standard Model the appropriate current combination is given by $J_\mu^V - J_\mu^A$.

Keeping only the finite terms and the relevant mass (M) ($\ln y$ and $\ln^2 y$) and infrared (IR) ($\ln(\Lambda_t)$) singular logarithmic terms, one obtains the rather simple result

$$F_1^V = F_1^A = 1 - \frac{\alpha_s(q^2)}{4\pi}C_F\Big(4 + \frac{1}{x^2}\ln(1-x^2) + \ln\Big(\frac{y}{1-x^2}\frac{\Lambda_t^2}{(1-x^2)^2}\Big) +$$

$$
\begin{aligned}
& +2\ln\Big(\frac{\Lambda_t}{y}\frac{1}{1-x^2}\Big)\ln\Big(\frac{y}{1-x^2}\Big)+2\,\mathrm{Li}(x^2)\Big), \\
F_2^V &= -F_2^A = \frac{1}{m_t}\frac{\alpha_s(q^2)}{4\pi}C_F\frac{2}{x^2}\Big(1+\frac{1-x^2}{x^2}\ln(1-x^2)\Big), \\
F_3^V &= -F_3^A = \frac{1}{m_t}\frac{\alpha_s(q^2)}{4\pi}C_F\frac{2}{x^2}\Big(-1+\frac{2x^2-1}{x^2}\ln(1-x^2)\Big),
\end{aligned}
\tag{1.130}
$$

where the scaled gluon mass has been denoted by $\Lambda_t = m_G^2/m_t^2$. It is worth mentioning at this point that the gluon mass regulator scheme can be converted to the dimensional reduction scheme by the replacement $\ln(\Lambda_t) \to 1/\varepsilon - \gamma_E + \ln(4\pi\mu^2/m_t^2)$ where μ is the renormalization scale parameter.

1.3.4 The tree-graph contributions

The tree graph contribution results from the square of the real gluon emission graphs shown in Fig. 1.5(c) and 1.5(d). Omitting again the weak coupling factor $V_{tb}g/\sqrt{2}$ for the time being, the corresponding hadron tensor is given by

$$
\begin{aligned}
H^{\mu\nu}(k_0,q_0) &= -4\pi\alpha_s C_F\,\frac{8}{(k\cdot p_t)(k\cdot p_b)}\Big\{-\frac{k\cdot p_t}{k\cdot p_b}\Big[(p_b\cdot p_b)\Big(k^\mu\bar p_t^\nu+k^\nu\bar p_t^\mu-k\cdot\bar p_t g^{\mu\nu}\Big)+ \\
&+i\Big(\epsilon^{\alpha\beta\mu\nu}(p_b-k)\cdot\bar p_t-\epsilon^{\alpha\beta\gamma\nu}(p_b-k)^\mu\bar p_{t,\gamma}+\epsilon^{\alpha\beta\gamma\mu}(p_b-k)^\nu\bar p_{t,\gamma}\Big)k_\alpha p_{b,\beta}\Big]+ \\
&+\frac{k\cdot p_b}{k\cdot p_t}\Big[(\bar p_t\cdot p_t)\Big(k^\mu p_b^\nu+k^\nu p_b^\mu-k\cdot p_b g^{\mu\nu}-i\epsilon^{\alpha\beta\mu\nu}k_\alpha p_{b,\beta}\Big)+ \\
&-(\bar p_t\cdot k)\Big((p_t-k)^\mu p_b^\nu+(p_t-k)^\nu p_b^\mu-(p_t-k)\cdot p_b g^{\mu\nu}-i\epsilon^{\alpha\beta\mu\nu}(p_t-k)_\alpha p_{b,\beta}\Big)\Big]+ \\
&-(\bar p_t\cdot p_b)\Big(k^\mu p_b^\nu+k^\nu p_b^\mu-k\cdot p_b g^{\mu\nu}-i\epsilon^{\alpha\beta\mu\nu}k_\alpha p_{b,\beta}\Big)+ \\
&+(p_t\cdot p_b)\Big(k^\mu\bar p_t^\nu+k^\nu\bar p_t^\mu-k\cdot\bar p_t g^{\mu\nu}\Big)-(k\cdot p_b)\Big(p_t^\mu\bar p_t^\nu+p_t^\nu\bar p_t^\mu-p_t\cdot\bar p_t g^{\mu\nu}\Big)+ \\
&+(k\cdot p_t)\Big((p_b+k)^\mu\bar p_t^\nu+(p_b+k)^\nu\bar p_t^\mu+(p_b+k)\cdot\bar p_t g^{\mu\nu}\Big)+ \\
&+(k\cdot\bar p_t)\Big(2p_b^\mu p_b^\nu-p_b\cdot p_b g^{\mu\nu}\Big)-i\Big(\epsilon^{\alpha\beta\mu\nu}(k\cdot\bar p_t)+\epsilon^{\alpha\beta\gamma\mu}k^\nu\bar p_{t,\gamma}-\epsilon^{\alpha\beta\gamma\nu}k^\mu\bar p_{t,\gamma}\Big)p_{b,\alpha}p_{t,\beta}+ \\
&+i\Big(\epsilon^{\alpha\beta\mu\nu}(p_t\cdot\bar p_t)+\epsilon^{\alpha\beta\gamma\mu}p_t^\nu\bar p_{t,\gamma}-\epsilon^{\alpha\beta\gamma\nu}p_t^\mu\bar p_{t,\gamma}\Big)k_\alpha p_{b,\beta}\Big\}+B^{\mu\nu}\cdot\Delta_{\mathrm{SGF}}
\end{aligned}
\tag{1.131}
$$

where k is the 4-momenta of the emitted gluon. The hadron tensor is written in dependence on the two phase-space variables k_0 and q_0. This notation distinguishes this three-particle hadron tensor in Eq. (1.131) from the two-particle hadron tensor for the Born and one-loop contributions. The IR divergent *soft gluon factor* Δ_{SGF} (cf. Eq. (1.59)) is given by

$$
\Delta_{\mathrm{SGF}} := -4\pi\alpha_s C_F\Big(\frac{m_b^2}{(k\cdot p_b)^2}+\frac{m_t^2}{(k\cdot p_t)^2}-2\frac{p_b\cdot p_t}{(k\cdot p_b)(k\cdot p_t)}\Big). \tag{1.132}
$$

The IR-singular part of the tree-graph contribution, therefore, is isolated by splitting off an universal soft gluon factor which multiplies the lowest order Born term tensor $B^{\mu\nu}$. This facilitates the treatment of the soft gluon singularity to be regularized by a (small) gluon mass m_G. Since the soft gluon factor is universal in that it multiplies the lowest order Born contribution, the requisite soft gluon integration has to be done only once and is identical for all eight structure functions.

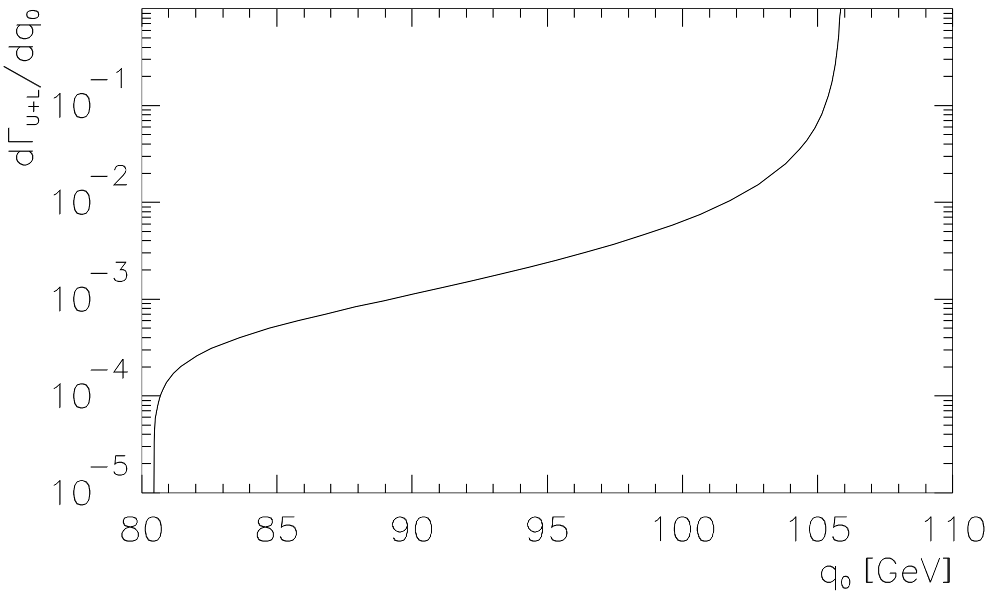


Figure 1.6: Differential W^+ boson energy distribution $d\Gamma_{U+L}/dq_0$ for the total rate resulting from $O(\alpha_s)$ gluon emission ($m_b = 4.8\,GeV$)

1.3.5 The full $O(\alpha_s)$ result for $m_b = 0$

To relate the two-particle and the three-particle hadron tensors one has to do the appropiate phase space integration on the hadron tensor of the three-particle case. The limits of the phase space integration are given by

$$k_{0-} \leq k_0 \leq k_{0+}, \qquad m_W \leq q_0 \leq \frac{m_t^2 + m_W^2 - (m_b + m_G)^2}{2m_t} \tag{1.133}$$

where

$$k_{0\pm} = \frac{(m_t - q_0)(M_+^2 - 2q_0 m_t) \pm \sqrt{q_0^2 - m_W^2}\sqrt{(M_-^2 - 2q_0 m_t)^2 - 4m_G^2 m_b^2}}{2(m_t^2 + m_W^2 - 2q_0 m_t)} \tag{1.134}$$

and $M_\pm^2 := m_t^2 + m_W^2 - m_b^2 \pm m_G^2$. The integration over the gluon energy is simple and the results will not be presented here in explicit analytical form. Instead some representative results on the differential W^+ boson energy distribution are shown that result from the real gluon emission graphs Fig. 1.5(c) and 1.5(d) in graphical form in Figs. 1.6 and 1.7. Fig. 1.6 shows the W^+ boson energy distribution for the total rate $d\Gamma_{U+L}/dq_0$. The energy distribution rises sharply from the lower energy limit, where the W^+ boson is produced at rest, then increases rapidly over the intermediate range of W^+ boson energies and finally rises sharply again towards the end of the spectrum, where the soft gluon singularity is located. In Fig. 1.7 the same distribution for the partial rate into positive helicity W bosons, $d\Gamma_+/dq_0$ ($\Gamma_+ = \frac{1}{2}(\Gamma_U + \Gamma_F)$) is shown for $m_b = 0$ and $m_b \neq 0$. As mentioned before there is no Born term contribution to $d\Gamma_+/dq_0$ for $m_b = 0$ and thus $d\Gamma_+/dq_0$ possesses no IR singularity in this limit. The absence of the IR singularity in the $m_b = 0$ case (dashed line) is quite apparent in Fig. 1.6. The distribution rises moderately fast from the lower end of the spectrum, then turns down over the intermediate range of energies and finally tends to zero at the end of the spectrum where the phase space closes. The

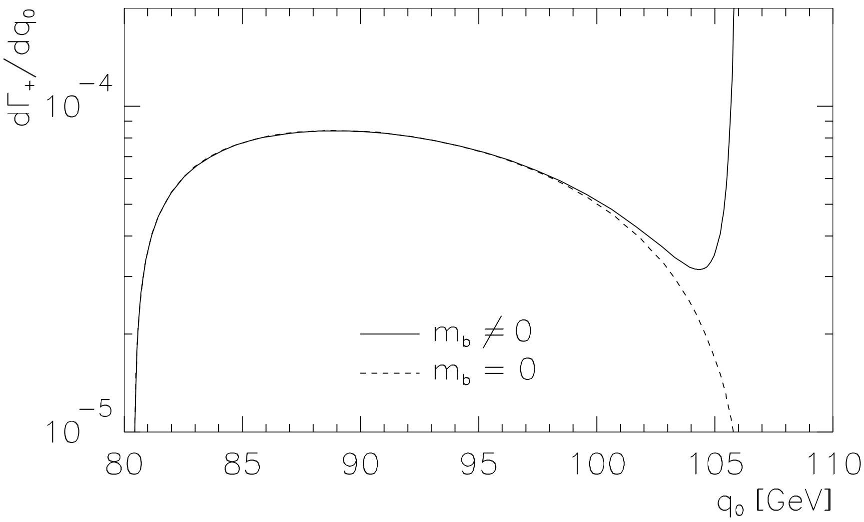


Figure 1.7: Differential W^+ boson energy distribution $d\Gamma_+/dq_0$ for the partial rate into positive helicity W^+ bosons resulting from $O(\alpha_s)$ gluon emission for $m_b = 4.8\,GeV$ (solid line) and for $m_b = 0$ (dashed line).

$m_b = 0$ (dashed line) and $m_b \neq 0$ (full line) distributions lie on top of each other for most of the lower part of the spectrum. Starting at around $4.8\,GeV$ below the upper phase space boundary the two distributions begin to diverge from each other. Whereas the $m_b = 0$ curve turns down and goes to zero at the end of the spectrum, the $m_b \neq 0$ curve starts to rise again and, in fact, tends to infinity at the end of the spectrum due to its IR singular behaviour. Note the huge differences in scale of the $d\Gamma_{U+L}/dq_0$ and the $d\Gamma_+/dq_0$ distributions which will be reflected in big differences in the total α_s-corrections for the two respective rates.

The second integration over the energy of the W boson is more difficult. Details can be found in Ref. [42]. As it turns out, the analytical $m_b \neq 0$ results are quite lengthy. For this reason only the $m_b = 0$ results are shown here since they are sufficiently simple to be presented in compact form. They have been obtained by taking the $m_b \to 0$ limit of the $m_b \neq 0$ results in Ref. [42]. For practical purposes the $m_b = 0$ results are sufficiently accurate for top decays since $m_b \neq 0$ effects are generally quite small. This is particularly true if a running b quark mass at the top mass scale is used. Quantitative results on the α_s and $m_b \neq 0$ corrections are given in Refs. [41, 42]. In combining the Born term and the $O(\alpha_s)$ one-loop and tree-graph contributions, the mass and infrared singular terms cancel among the $O(\alpha_s)$ contributions as they must according to the Lee-Nauenberg theorem and one remains with a finite result. This result is presented in terms of scaled rate functions defined by $\hat{\Gamma}_i := \Gamma_i/\Gamma_0$ ($i = U+L$, $U^P + L^P$, L, L^P, F, F^P, I^P, and A^P) with

$$\Gamma_0 = \Gamma_{U+L}(Born) = \frac{G_F m_W^2 m_t}{8\sqrt{2}\pi}|V_{tb}|^2 \frac{(1-x^2)^2(1+2x^2)}{x^2} \tag{1.135}$$

(with $x = m_W/m_t$) and read

$$\begin{aligned}\hat{\Gamma}_{U+L} \;&=\; 1 + \hat{\Gamma}_1(x)\Bigg\{ \frac{(1-x^2)(5+9x^2-6x^4)}{2x^2} - \frac{2(1-x^2)^2(1+2x^2)\pi^2}{3x^2} + \\ &-\frac{(1-x^2)^2(5+4x^2)}{x^2}\ln(1-x^2) - 4(1+x^2)(1-2x^2)\ln(x) \;+\end{aligned}$$

$$-\frac{4(1-x^2)^2(1+2x^2)}{x^2}\ln(x)\ln(1-x^2)-\frac{4(1-x^2)^2(1+2x^2)}{x^2}\,\mathrm{Li}_2(x^2)\Bigg\},$$

$$\begin{aligned}\hat{\Gamma}_{U^P+L^P} &= \frac{1-2x^2}{1+2x^2}+\hat{\Gamma}_1(x)\Bigg\{-\frac{(1-x)^2(15+2x-5x^2-12x^3+2x^4)}{2x^2}+\frac{(1+4x^2)\pi^2}{3x^2}+\\ &-\frac{(1-x^2)^2(1-4x^2)}{x^2}\ln(1-x)-\frac{(1-x^2)(3-x^2)(1+4x^2)}{x^2}\ln(1+x)\;+\\ &-\frac{4(1-x^2)^2(1-2x^2)}{x^2}\,\mathrm{Li}_2(x)+\frac{4(2+5x^4-2x^6)}{x^2}\,\mathrm{Li}_2(-x)\Bigg\},\end{aligned} \tag{1.136}$$

$$\begin{aligned}\hat{\Gamma}_L &= \frac{1}{1+2x^2}+\hat{\Gamma}_1(x)\Bigg\{\frac{(1-x^2)(5+47x^2-4x^4)}{2x^2}-\frac{2(1+5x^2+2x^4)\pi^2}{3x^2}+\\ &-\frac{3(1-x^2)^2}{x^2}\ln(1-x^2)-\frac{2(1-x)^2(2-x+6x^2+x^3)}{x^2}\ln(1-x)\ln(x)\;+\\ &+16(1+2x^2)\ln(x)-\frac{2(1+x)^2(2+x+6x^2-x^3)}{x^2}\ln(x)\ln(1+x)\;+\\ &-\frac{2(1-x)^2(4+3x+8x^2+x^3)}{x^2}\,\mathrm{Li}_2(x)-\frac{2(1+x)^2(4-3x+8x^2-x^3)}{x^2}\,\mathrm{Li}_2(-x)\Bigg\},\end{aligned} \tag{1.137}$$

$$\begin{aligned}\hat{\Gamma}_{L^P} &= \frac{1}{1+2x^2}+\hat{\Gamma}_1(x)\Bigg\{-(15-22x+105x^2-24x^3+4x^4)\frac{(1-x)^2}{2x^2}+\\ &+\frac{(1+24x^2+10x^4)\pi^2}{3x^2}-\frac{3(1-x^2)^2}{x^2}\ln(1-x)-\frac{(1-x^2)(17+53x^2)}{x^2}\ln(1+x)\;+\\ &-\frac{4(1-x^2)^2}{x^2}\,\mathrm{Li}_2(x)+\frac{4(2+22x^2+11x^4)}{x^2}\,\mathrm{Li}_2(-x)\Bigg\},\end{aligned} \tag{1.138}$$

$$\begin{aligned}\hat{\Gamma}_F &= \frac{-2x^2}{1+2x^2}+\hat{\Gamma}_1(x)\Bigg\{-2(1-x)^2(3-4x)+\frac{2(2+x^2)\pi^2}{3}+\\ &+\frac{2(1-x^2)^2(1+2x^2)}{x^2}\ln(1-x)+\frac{2(1-x^2)(1-9x^2+2x^4)}{x^2}\ln(1+x)\;+\\ &+8(1-x^2)^2\,\mathrm{Li}_2(x)+8(1+3x^2-x^4)\,\mathrm{Li}_2(-x)\Bigg\},\end{aligned} \tag{1.139}$$

$$\begin{aligned}\hat{\Gamma}_{F^P} &= \frac{2x^2}{1+2x^2}+\hat{\Gamma}_1(x)\Bigg\{2(1-x^2)(4+x^2)-\frac{2(1+x^2+2x^4)\pi^2}{3}+\\ &-\frac{2(1-x^2)^2(1+2x^2)}{x^2}\ln(1-x^2)-\frac{4(1-x)^2(1+3x+2x^2+2x^3)}{x}\ln(x)\ln(1-x)\;+\\ &-4(2-5x^2-2x^4)\ln(x)+\frac{4(1+x)^2(1-3x+2x^2-2x^3)}{x}\ln(x)\ln(1+x)\;+\\ &-\frac{4(1-x)^2(1+5x+6x^2+4x^3)}{x}\,\mathrm{Li}_2(x)+\frac{4(1+x)^2(1-5x+6x^2-4x^3)}{x}\,\mathrm{Li}_2(-x)\Bigg\},\end{aligned} \tag{1.140}$$

$$\begin{aligned}\hat{\Gamma}_{I^P} &= \frac{-2x}{1+2x^2}+\hat{\Gamma}_1(x)\Bigg\{\frac{2(1-x)^2(12-7x+12x^2)}{x}-\frac{(5+19x^2+2x^4)\pi^2}{3x}+\\&+\frac{(1-x^2)^2(1+5x^2)}{x^3}\ln(1-x)+\frac{(1-x^2)(1+30x^2+21x^4)}{x^3}\ln(1+x)\ +\\&+\frac{8(1-x^2)^2}{x}\,\mathrm{Li}_2(x)-\frac{4(7+15x^2+4x^4)}{x}\,\mathrm{Li}_2(-x)\Bigg\},\end{aligned} \tag{1.141}$$

$$\begin{aligned}\hat{\Gamma}_{A^P} &= \frac{2x}{1+2x^2}+\hat{\Gamma}_1(x)\Bigg\{\frac{2(1-x^2)(1+2x^2)}{x}-\frac{(3-5x^2+6x^4)\pi^2}{3x}+\\&-\frac{(1-x^2)^2(1+5x^2)}{x^3}\ln(1-x^2)-\frac{2(1-x)^2(3+7x+6x^2)}{x}\ln(x)\ln(1-x)\ +\\&-2x(5-11x^2)\ln(x)-\frac{2(1+x)^2(3-7x+6x^2)}{x}\ln(x)\ln(1+x)\ +\\&-\frac{2(1-x)^2(7+15x+10x^2)}{x}\,\mathrm{Li}_2(x)-\frac{2(1+x)^2(7-15x+10x^2)}{x}\,\mathrm{Li}_2(-x)\Bigg\}\end{aligned} \tag{1.142}$$

where

$$\hat{\Gamma}_1(x) := \frac{\alpha_s C_F x^2}{2\pi(1-x^2)^2(1+2x^2)}. \tag{1.143}$$

Detailed numerical results and a discussion of these results are given in Ref. [42]. The numerical results for the rate functions for $m_b = 4.8\,GeV$ [45] read

$$\begin{aligned}\hat{\Gamma}_{U+L} &= 1-0.0854, & \hat{\Gamma}_{(U+L)^P} &= 0.406(1-0.1162),\\ \hat{\Gamma}_U &= 0.297(1-0.0624), & \hat{\Gamma}_{U^P} &= -0.297(1-0.0689),\\ \hat{\Gamma}_L &= 0.703(1-0.0951), & \hat{\Gamma}_{L^P} &= 0.703(1-0.962),\\ \hat{\Gamma}_F &= -0.297(1-0.0687), & \hat{\Gamma}_{F^P} &= 0.297(1-0.0639),\\ \hat{\Gamma}_S &= 0.703(1-0.0895), & \hat{\Gamma}_{S^P} &= 0.703(1-0.0922),\\ && \hat{\Gamma}_{I^P} &= -0.228(1-0.0810),\\ && \hat{\Gamma}_{A^P} &= 0.228(1-0.0820)\end{aligned} \tag{1.144}$$

where the Born term results are factored out. Therefore, the second contribution in the parentheses is the numerical value for the $O(\alpha_s)$ radiative correction.

Chapter 2

Renormalization and resummation

In the previous chapter a running coupling constant as well as a running mass has been used. This chapter addresses the question about the origin of such running quantities, i.e. quantities which depend on the center-of-mass energy of the system under consideration. As will be shown later on, the running[1] is a consequence of the (necessary) renormalization of the (otherwise) divergent perturbation series. It can be understood as taking into account the clouds of virtual particles created in the neighbourhood of a particle which changes it's mass and weakens resp. strengthens the interaction of this particle with others, depending on the spatial distance. For QED, the cloud screens the particle and therefore weakens the interaction for increasing distances, while the opposite is valid for QCD. On the other hand, approaching the colour charge (by using higher and higher center-of-mass energies) will weaken the interaction. This phenomenon is known as *asymptotic freedom*. The running is mathematically expressed by the *renormalization group equation* which will be dealt with in the first section.

The asymptotic freedom of QCD has an important consequence for perturbation theory. It means that the perturbation series is quite well convergent in the region of high energies but badly convergent in the low energy region. Therefore, perturbation theory can only be applied to a part of the cross section observed for particle collisions. On the other hand, the coupling is only a parameter of theory. One is therefore led to the attempt to replace this parameter by other parameters which might bring in better convergence of the perturbation series. These other parameters can be obtained by summing up parts of the perturbation series which are known to a high extend. This procedure is known as *resummation* of the perturbation series and is equivalent with the change of the renormalization scheme, as will be seen later on. However, these changes do not "cure" the problem that at very low energies QCD will remain undescribable by perturbation series.

This chapter will introduce the basic quantities inportant also for the following chapters, namely the two-point or correlator function and its spectral density. Finally, the applicability of different kinds of moments are considered, leading to the conclusion that only so-called direct moments can correctly analyse the shape of the spectral density in order to compare it with the theory prediction. Different kinds of effective couplings can be constructed, depending on the special point of view.

[1] The general term "running" will be used in the following for the running of all kinds of quantities, not only for the coupling and the mass.

2.1 The renormalization group equation

The first considerations deal with the Callan–Symanzik equation for Green functions and the renormalization group equations for the parameters of the Green function. The Callan–Symanzik equation for a Green function Γ is obtained by comparing its dependence before and after being renormalized by a renormalization factor Z_Γ. Starting with the dependence

$$\Gamma^0 = \Gamma^0(\alpha_s, \alpha_g, m_i; p_i) \tag{2.1}$$

of the bare (i.e. unrenormalized) quantity on parameters as the strong coupling α_s, the gauge parameter α_g, the masses m_i of the included particles and their momenta p_i, this quantity can be renormalized by dividing it by a renormalization factor. This renormalization is neccessary because the calculation of higher orders within perturbation theory lead to IR- or UV-divergent integrals (or both).

2.1.1 Regularization

Both kinds of singularities have first to be parametrized. This first step is called *regularization* (see e.g. Ref. [46]). The regularization method which is normally used here is called *dimensional regularization* and includes both kind of singularities by replacing the four-dimensional space-time dimension by a $D = 4 - 2\varepsilon$ dimensional one. The singularities are therefore parametrized by inverse powers of the parameter ε which tends to zero (some effort is necessary to distinguish between IR- and UV-singularities in this regularization, but this should not be of any concern for the moment).

2.1.2 Subtraction

The next step in the renormalization procedure is the *subtraction*, i.e. the absorption of the parametrized singular parts in counter terms. Here one makes use of the multiplicative subtraction by a *renormalization factor* Z_Γ. By demanding that the renormalized quantity Γ in

$$\Gamma^0 = Z_\Gamma \Gamma \tag{2.2}$$

is finite one can determine the renormalization factor by expanding it in a double series in powers of both the perturbation series parameter and the inverse of ε and performing a comparison of coefficients. The decision whether finite terms are or aren't included in this renormalization factor and the selection of these finite terms distinguishes between the different *schemes*. The simplest one, the *minimal subtraction scheme (MS-scheme)*, includes no finite terms while the *modified minimal subtraction scheme ($\overline{MS}$-scheme)* associates every factor $1/\varepsilon$ by two finite terms to obtain a “shifted” parameter ε',

$$\frac{1}{\varepsilon'} = \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \tag{2.3}$$

where γ_E is Euler's constant ($\gamma_E \approx 0.577\dots$) which comes from the series expansion of Euler's gamma function. This scheme is used to simplify the final result by using the ambiguity of the choice for the renormalization factor. The parametrized singularities are incorporated in the renormalization factor, one is left with a renormalized quantity.

2.1.3 Scale dependence

The subtracted quantity, however, has to depend on some additional scale, the *renormalization scale* μ which has a mass dimensionality and compensates the disproportion in the dimensions resulting from the dimensional regularization. Therefore one has

$$\Gamma^0(\alpha_s,\alpha_g,m_i;p_i)=Z_\Gamma\Gamma(\mu,\alpha_s,\alpha_g,m_i;p_i). \tag{2.4}$$

This is the starting point for the Callan–Symanzik equation. Because the left hand side does not depend on the scale μ, one has

$$0=\mu\frac{d}{d\mu}\Gamma^0=\mu\frac{d}{d\mu}(Z_\Gamma\Gamma)=\mu\frac{dZ_\Gamma}{d\mu}\Gamma+\mu Z_\Gamma\frac{d\Gamma}{d\mu}. \tag{2.5}$$

The dependence on μ is given either by the explicit dependence or by the implicit dependence of the parameters. Only the momentum as observable quantity cannot be scale dependent which is the reason for separating the arguments by a semicolon. One thus obtains

$$\left(\mu\frac{\partial}{\partial\mu}+\mu\frac{d\alpha_s}{d\mu}\frac{\partial}{\partial\alpha_s}+\mu\frac{d\alpha_g}{d\mu}\frac{\partial}{\partial\alpha_g}+\sum_i\mu\frac{dm_i}{d\mu}\frac{\partial}{\partial m_i}+\frac{\mu}{Z_\Gamma}\frac{dZ_\Gamma}{d\mu}\right)\Gamma(\mu,\alpha_s,\alpha_g,m_i;p_i)=0. \tag{2.6}$$

One defines

$$\begin{aligned}\gamma_\alpha &:= -\frac{\mu}{\alpha_s}\frac{d\alpha_s}{d\mu}, \qquad \gamma_g := \frac{\mu}{\alpha_g}\frac{d\alpha_g}{d\mu},\\ \gamma_{m_i} &:= -\frac{\mu}{m_i}\frac{dm_i}{d\mu} \quad\text{and}\quad \gamma_\Gamma := \frac{\mu}{Z_\Gamma}\frac{dZ_\Gamma}{d\mu}.\end{aligned} \tag{2.7}$$

Because of Eq. (2.5), the last definition is consistent with the others. Therefore, one ends up with

$$\left(\mu\frac{\partial}{\partial\mu}-\alpha_s\gamma_\alpha\frac{\partial}{\partial\alpha_s}-\alpha_g\gamma_g\frac{\partial}{\partial\alpha_g}-\sum_i m_i\gamma_{m_i}\frac{\partial}{\partial m_i}+\gamma_\Gamma\right)\Gamma(\mu,\alpha_s,a,m_i;p_i)=0 \tag{2.8}$$

which is the *Callan–Symanzik equation.*

2.1.4 Solution of the Callan–Symanzik equation

Eq. (2.8) becomes solvable if there is no explicit dependence on the renormalization parameter. In some sense one can "shift" this dependence to the parameters by creating a second differential equation for Γ. This is done by scaling the moments p_i by a factor $e^{\bar t}$ and using Euler's theorem for homogeneous functions. Looking at the different dimensionalities of the parameters, one obtains (the bar in $\bar t$ is chosen for later convenience)

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_i m_i\frac{\partial}{\partial m_i}+\frac{\partial}{\partial\bar t}-D_\Gamma\right)\Gamma(\mu,\alpha_s,\alpha_g,m_i;e^{\bar t}p_i)=0 \tag{2.9}$$

where D_Γ is the *mass dimension* of Γ. The subtraction of Eq. (2.8) and Eq. (2.9) leads to

$$\begin{aligned}\Bigg(&-\frac{\partial}{\partial\bar t}-\alpha_s\gamma_\alpha\frac{\partial}{\partial\alpha_s}-\alpha_g\gamma_g\frac{\partial}{\partial\alpha_g}+\\ &-\sum_i m_i(1+\gamma_{m_i})\frac{\partial}{\partial m_i}+D_\Gamma+\gamma_\Gamma(\alpha_s)\Bigg)\Gamma(\mu,\alpha_s,\alpha_g,m_i;e^{\bar t}p_i)=0,\end{aligned} \tag{2.10}$$

the partial derivative with respect to μ cancels out. The equation is formally solved by

$$\Gamma(\mu,\alpha_s,a,m_i;e^{\bar t}p_i)=\exp\left(\int_0^{\bar t}\left(D_\Gamma+\gamma_\Gamma(\bar\alpha_s(\bar t'))\right)d\bar t'\right)\bar\Gamma(\bar\alpha_s,\bar\alpha_g,\bar m_i;\bar p_i) \tag{2.11}$$

where the quantities $\bar\alpha_s$, $\bar\alpha_g$ and $\bar m_i$ depend on the new parameter $\bar t$. Because D_Γ is changed to $D_\Gamma+\gamma_\Gamma$, it is obvious why γ_Γ is called the *anomalous dimension* of Γ. Similarly, γ_α, γ_g and γ_{m_i} are the anomalous dimensions of the corresponding parameters. In general one has

$$A^0=Z_AA\quad\Rightarrow\quad\gamma_A:=\frac{\mu}{Z_A}\frac{\partial Z_A}{\partial\mu}=-\frac{\mu}{A}\frac{\partial A}{\partial\mu} \tag{2.12}$$

for a quantity A because the bare quantity A^0 is independent of μ. There is a great deal of confusion in these definitions but the chosen definition of the anomalous dimension should be kept throughout this work.

2.1.5 Running parameters

Inserting the final equation of the previous subsection into the renormalization group equation (2.10), one obtains differential equations in $\bar t$ for the various parameters,

$$\begin{aligned}\frac{d\bar\alpha_s}{d\bar t}&=-\bar\alpha_s\gamma_\alpha\qquad\text{where }\bar\alpha_s(0)=\alpha_s,\\ \frac{d\bar\alpha_g}{d\bar t}&=-\bar\alpha_g\gamma_g\qquad\text{where }\bar\alpha_g(0)=\alpha_g,\qquad\text{and}\\ \frac{d\bar m_i}{d\bar t}&=-(1+\gamma_{m_i}(\bar\alpha_s))\bar m_i\qquad\text{where }\bar m_i(0)=m_i.\end{aligned} \tag{2.13}$$

In the following only the first of these equations will be considered, which is known as the *renormalization group equation for the running coupling* α_s,

$$\frac{d\bar\alpha_s}{d\bar t}=-\bar\alpha_s\gamma_\alpha. \tag{2.14}$$

This equation will not be solved explicitly at this moment because there will be better methods afterwards to do this. But the renormalization group equation can be solved formally. This is done by separating the variables,

$$d\bar t=-\frac{d\bar\alpha_s}{\bar\alpha_s\gamma_\alpha}\quad\Rightarrow\quad\bar t=-\int_{\alpha_s}^{\bar\alpha_s}\frac{d\alpha}{\alpha\gamma_\alpha} \tag{2.15}$$

where $\bar\alpha_s(\bar t=0)=\alpha_s$ is used. As the first one of Eqs. (2.7) indicates, α_s depends on μ, while $\bar\alpha_s$ does not. Therefore, the total derivative of $\bar t$ results in

$$\frac{d\bar t}{d\mu}=\frac{1}{\alpha_s\gamma_\alpha}\frac{d\alpha_s}{d\mu}=-\frac{1}{\alpha_s\gamma_\alpha}\frac{\alpha_s}{\mu}\gamma_\alpha=-\frac{1}{\mu}. \tag{2.16}$$

2.1.6 Introduction of a second mass scale

The solution for $\bar t$ up to a constant is the negative logarithm of μ. But because μ is a dimensional variable, there is a scale necessary to define the logarithm properly. As scale one uses the *center-of-mass energy* q^2 of the system so that

$$\bar t=\frac12\ln\left(\frac{q^2}{\mu^2}\right). \tag{2.17}$$

The reason for this scale is as follows: One knows that at $\bar{t} = 0$ there is $\bar{\alpha}_s(\bar{t} = 0) = \alpha_s$ which is $\alpha_s(\mu^2)$ (the square is again chosen for convenience). At the end one wants $\bar{\alpha}_s$ to be a function of the center-of-mass energy which is the only mass scale applicable for a specific process. This is done in order to obtain $\bar{\alpha}_s(\bar{t}) = \alpha_s(q^2)$. Therefore, the renormalization group equation for the coupling can be written as

$$q^2 \frac{d\alpha_s(q^2)}{dq^2} = -\alpha_s(q^2)\gamma_\alpha =: \beta(\alpha_s(q^2)) \tag{2.18}$$

which defines the *beta function*. There are many different ways to define the beta function. The definition chosen here is one of the most common ones. The beta function is a function in α_s only and not explicitly in q^2 because it is expressible by a perturbative series expansion, and the expansion parameter is again α_s. Therefore, this differential equation is an *autonomous differential equation*.

2.1.7 The beta function coefficients

As mentioned before, the beta function can be calculated by means of perturbation theory and can thus be expressed as a series in α_s. Actually, the beta function is known up to the fourth order in α_s. There are again different ways to denote the coefficients. The convention used here, rather common in the literature [48], is

$$\begin{aligned}\beta\left(\frac{\alpha_s}{\pi}\right) &:= -\sum_{m=0}^{\infty} \beta_m \left(\frac{\alpha_s}{\pi}\right)^{m+2} = \\ &= -\left(\frac{\alpha_s}{\pi}\right)^2 \left(\beta_0 + \beta_1 \left(\frac{\alpha_s}{\pi}\right) + \beta_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \beta_3 \left(\frac{\alpha_s}{\pi}\right)^3\right) + O(\alpha_s^6)\end{aligned} \tag{2.19}$$

where the expansion parameter is changed from α_s to α_s/π and

$$\begin{aligned}&\beta_0 = \frac{1}{4}\left[11 - \frac{2}{3}N_f\right], \quad \beta_1 = \frac{1}{16}\left[102 - \frac{38}{3}N_f\right], \quad \beta_2 = \frac{1}{64}\left[\frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2\right], \\ &\beta_3 = \frac{1}{256}\left[\frac{149753}{6} + 3564\zeta(3) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta(3)\right)N_f \right. \\ &\qquad \left. + \left(\frac{50065}{162} + \frac{6472}{81}\zeta(3)\right)N_f^2 + \frac{1093}{729}N_f^3\right].\end{aligned} \tag{2.20}$$

ζ is Riemann's zeta-function and N_f is the number of active flavours involved at the specific energy. The energy which will be interesting in the following is the energy corresponding to the rest mass of the τ lepton, $m_\tau \approx 1.777\,GeV$. This energy is located between the thresholds of the strange and the charm quark. Therefore, one has to use $N_f = 3$ and obtains

$$\beta_0 = \frac{9}{4}, \qquad \beta_1 = 4, \qquad \beta_2 = \frac{3863}{384}, \qquad \beta_3 = \frac{140599}{4608} + \frac{445}{32}\zeta(3). \tag{2.21}$$

One ends up with an differential equation of the form

$$q^2 \frac{d}{dq^2}\left(\frac{\alpha_s}{\pi}\right) = -\left(\frac{\alpha_s}{\pi}\right)^2 \left(\beta_0 + \beta_1 \left(\frac{\alpha_s}{\pi}\right) + \beta_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \beta_3 \left(\frac{\alpha_s}{\pi}\right)^3\right) + O(\alpha_s^6). \tag{2.22}$$

Considering the fact that μ^2 and q^2 play complementary roles in $\bar{t}$, there is an apparent contradiction to the first of Eqs. (2.7). Note, however, that α_s is *not* a function of $\bar{t}$ (as $\bar{\alpha}_s$ is) but a function of its argument alone, i.e. either μ^2 or q^2. Therefore, one can use either q^2 or μ^2 as argument, and by using the derivative with respect to the same variable one ends up with two differential equations which are formally equal. The equation with μ^2 instead of q^2 is the one used in Ref. [47] as well as in Ref. [48]. At the same time this frees one from keeping track on a scaling. Again one has to introduce a mass scale if one wants to write the left hand side of the renormalization equation (2.22) like before as a derivative with respect to some logarithm. This will be done in the next subsection by introducing a new scale Λ.

2.1.8 Introduction of a third mass scale

By defining

$$t = \ln\left(\frac{q^2}{\Lambda^2}\right) \tag{2.23}$$

one obtains

$$\frac{d}{dt}\left(\frac{\alpha_s}{\pi}\right) = \beta\left(\frac{\alpha_s}{\pi}\right) = -\sum_{m=0}^{\infty}\beta_m\left(\frac{\alpha_s}{\pi}\right)^{m+2} \tag{2.24}$$

with $\alpha_s = \alpha_s(\Lambda^2 e^t)$. There is still a possibility to rescale this equation. If one defines a new parameter $\bar{a}(t) = \beta_0\alpha_s(\Lambda^2 e^t)/\pi = 9\alpha_s(\Lambda^2 e^t)/4\pi$, one ends up with

$$\frac{d\bar{a}(t)}{dt} = -\bar{a}^2(t)\Big(1 + c_1\bar{a}(t) + c_2\bar{a}^2(t) + c_3\bar{a}^3(t)\Big) + O(\bar{a}^6) \tag{2.25}$$

where $c_i := \beta_i/(\beta_0)^{i+1}$, thus

$$c_1 = \frac{64}{81}, \qquad c_2 = \frac{3863}{4374}, \qquad c_3 = \frac{140599}{118098} + \frac{3560}{6561}\zeta(3). \tag{2.26}$$

This is the form of the renormalization group equation for the coupling that will be used in the following (see Ref. [49]. The methods in Ref. [50] are more involved, resumming the coupling to an effective coupling. This will be dealt with later on).

2.2 Correlator function and spectral density

As mentioned in the introduction of this chapter, the coupling parameter α_s is a quantity with a priori no physical meaning. It is a tool of perturbation theory which enables one to calculate measurable quantities as power series in this parameter. Its value depends on the methods used, e.g. the different subtraction schemes. The main topic, therefore, has to be to express one physical observable, in the previous case the total e^+e^- annihilation cross section, by another one, for which one can use the semileptonic τ decay ratio which is well measured. Therefore, the main task is building a bridge from the one to the other quantity by passing by the ambiguous field of perturbation series in α_s as safely as possible. This is done in Ref. [47].

2.2.1 Vector and axial-vector contributions

The semileptonic (hadronic) τ lepton decay is mediated by the charged weak hadronic current of the form

$$j^w_\mu(x) = V_{ud}\bar{u}\gamma_\mu(1-\gamma_5)d + V_{us}\bar{u}\gamma_\mu(1-\gamma_5)s \tag{2.27}$$

where V_{ud} and V_{us} are Cabbibo–Kobayashi–Maskawa matrix elements (elements of the weak mixing matrix). The *correlator for the weak hadronic currents* in Eq. (2.27) has the general form

$$\Pi_{\mu\nu}(q^2) = 12\pi^2 i \int \langle T j_\mu(x) j^\dagger_\nu(0)\rangle e^{iqx}dx = q_\mu q_\nu \Pi_q(q^2) + g_{\mu\nu}\Pi_g(q^2) \tag{2.28}$$

where $\Pi_q(q^2)$ and $\Pi_g(q^2)$ are invariant scalar functions. These scalar functions are further specified depending on which current is considered. In case of the $(\bar{u}d)$ quark current $j_\mu(x) = \bar{u}\gamma_\mu(1-\gamma_5)d$ (denoted as light quark case) the massless limit is assumed in which case the correlator is transverse, i.e. both invariant functions $\Pi_{q,g}(q^2)$ are expressible through a single scalar correlator function $\Pi_{ud}(q^2)$

$$\Pi_q(q^2) = \Pi_{ud}(q^2), \qquad \Pi_g(q^2) = -q^2\Pi_{ud}(q^2). \tag{2.29}$$

The correlator in case of the $(\bar{u}s)$ quark current (the term proportional to V_{us}, also referred to as the strange quark case) is slightly different as the nonvanishing strange quark mass is taken into account (see Ref. [51] and references therein). The remaining parts of the consideration are rather similar to the light quark case. The specification $(\bar{u}d)$ will be skipped in the following and $\Pi_{ud}(q^2) \equiv \Pi(q^2)$ will be used. In addition to the different current specifications as in Eqs. (2.27) and (2.29) it is convenient in some cases to consider the vector and axial parts of the correlator separately,

$$\Pi_{V+A}(q^2) = \Pi_V(q^2) + \Pi_A(q^2) \tag{2.30}$$

with $\Pi_V(q^2)$ being related to the vector part and $\Pi_A(q^2)$ being related to the axial part. The spectral density intorduced later on will split accordingly into vector and axial-vector parts, $\rho_{V+A}(s) = \rho_V(s) + \rho_A(s)$.

Because one mainly concentrates on the massless limit for the correlator, the above mentioned simplifications apply. In the following the indices V and A will be omitted in the generic case as well if no confusion arises. The correlator in Eq. (2.28) is normalized to the number of colours N_c (the same holds true for the spectral density which means that $\rho(s) \to N_c$ for $s \to \infty$) in the leading parton model approximation with massless quarks. In the following occasionally also a slightly different normalization will be used which explicitly accounts for the number of colours, resulting in a correlator which is normalized to unity.

The above considerations are general and are also used by experimentalists to classify the appropiate channels: strange particles (K mesons) form the strange channel, non-strange axial-vector mesons (as a_1 and the like) form the axial-vector channel, and the classical vector meson ρ represents the non-strange vector channel. In this respect the pion is somewhat special. It is a Goldstone boson for the lightest flavours with spin zero and gives a contribution to the axial correlator $\Pi_A(q^2)$ in the massless limit.

2.2.2 The relative τ lepton decay rate

Before specifying the theoretical calculations the general form of an important observable of τ decays will be given. The basic observable is the *relative τ lepton decay rate* for the decay of the τ lepton into hadrons written in the standard form as

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau + l + \bar{\nu}_l)} = N_c S_{\text{EW}} \Big(|V_{ud}|^2 (1 + \delta_{ud}) + |V_{us}|^2 (1 + \delta_{us}) \Big) . \tag{2.31}$$

The leading terms in Eq. (2.31) are the *parton model results* while the terms δ_{ud} and δ_{us} represent the effects of QCD interactions and mass effects (in case of nonvanishing quark masses) [52, 53]. V_{ud} and V_{us} are elements of the weak mixing matrix as defined in Eq. (2.27), and S_{EW} describes the electroweak radiative corrections to the τ decay rate [54].

2.2.3 The relative hadronic cross section

In a similar manner the hadronic cross section $\sigma_h^{(0)} = \sigma(e^+e^- \to \text{hadrons})$ can be expressed in relation to the muonic cross section $\sigma(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha^2/3s$ as

$$\sigma_h^{(0)}(s) = \frac{4\pi\alpha^2}{3s} R(s), \qquad R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{2.32}$$

where α is the fine structure constant of QED and $s = q^2$. On the other hand, the *relative hadronic cross section* $R(s)$ can be expressed as the discontinuity of the two-point correlator $\Pi(q^2)$ along its cut at the negative real axis,

$$R(s) \sim \rho(s) \tag{2.33}$$

where $\rho(s)$ is the *spectral density*. To understand the meaning of the spectral density, one has to introduce the *two-point correlator for hadronic currents* also in the e^+e^- *channel*. It is given by

$$12\pi^2 i \int \langle 0 | j_\mu^{\text{em}}(x) j_\nu^{\text{em}}(0) | 0 \rangle e^{iqx} d^4x = (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi(-q^2). \tag{2.34}$$

The *scalar two-point correlator function* $\Pi(q^2)$ is connected with the spectral density by the *dispersion relation* (see e.g. Ref. [55])

$$\Pi(q^2) = \int_{s_0}^{\infty} \frac{\rho(s) ds}{s + q^2} \tag{2.35}$$

where $s_0 = 4m_\pi^2$ is the production threshold of the light flavours. The dispersion relation implies the reverse relation

$$\rho(s) = \frac{1}{2\pi i} \operatorname{Disc} \Pi(s) \tag{2.36}$$

where the *discontinuity* is given by

$$\operatorname{Disc} \Pi(s) := \Pi(se^{-i\pi}) - \Pi(se^{i\pi}) \tag{2.37}$$

and where $e^{\pm i\pi}$ stands for the approach to the negative real axis from the positive and negative imaginary half plane, resp. Note further that the argument s on the left hand side is an argument of $\mathrm{Disc}\,\Pi$ and *not* of Π only. The dispersion relation has its origin in *Cauchy's theorem*. One calculates the circle integral with center at $t = q^2$ and radius r and enlarges this circle, avoiding the cut of $\Pi(q^2)$ along the negative real axis starting at $t = -s_0$. One then obtains

$$\begin{aligned}
2\pi i\Pi(q^2) &= \oint_{|t-q^2|=r} \frac{\Pi(t)dt}{t-q^2} = \\
&= \int_{-\pi}^{\pi} \frac{\Pi(Re^{i\varphi})iRe^{i\varphi}d\varphi}{Re^{i\varphi}-q^2} + \int_{Re^{i\pi}}^{s^0 e^{i\pi}} \frac{\Pi(t)dt}{t-q^2} + \int_{s_0 e^{-i\pi}}^{Re^{-i\pi}} \frac{\Pi(t)dt}{t-q^2} = \\
&\approx i\int_{-\pi}^{\pi} \Pi(Re^{i\varphi})d\varphi + \int_{R}^{s_0} \frac{\Pi(se^{i\pi})e^{i\pi}ds}{se^{i\pi}-q^2} + \int_{s_0}^{R} \frac{\Pi(se^{-i\pi})e^{-i\pi}ds}{se^{-i\pi}-q^2} = \\
&= i\int_{-\pi}^{\pi} \Pi(Re^{i\varphi})d\varphi + \int_{R}^{s_0} \frac{\Pi(se^{i\pi})ds}{s+q^2} + \int_{s_0}^{R} \frac{\Pi(se^{-i\pi})ds}{s+q^2} = \\
&= i\int_{-\pi}^{\pi} \Pi(Re^{i\varphi})d\varphi + \int_{s_0}^{R} \frac{\left(\Pi(se^{-i\pi}) - \Pi(se^{i\pi})\right) ds}{s+q^2} \qquad (2.38)
\end{aligned}$$

where the change from t to $s = -t$ has been used. The integral over the infinite circle path with radius R vanishes. Using the definition of the discontinuity, one thus ends up with the dispersion relation.

2.2.4 Adler's function

There is still another representation for $\Pi(q^2)$. If one uses $\tilde{\Pi}(q^2) := \Pi(q^2)/q^2$ instead of $\Pi(q^2)$ and constructs a dispersion relation for this quantity, one has to take into account the additional pole at $q^2 = 0$. Therefore, instead of the integral starting at $s = s_0$ the integral starts at $s = 0$. The spectral density is replaced by

$$\tilde{\rho}(s) = \frac{1}{2\pi i}\,\mathrm{Disc}\,\tilde{\Pi}(s) = \frac{\mathrm{Disc}\,\Pi(s)}{-2\pi i s}. \qquad (2.39)$$

Inserting $\tilde{\Pi}(q^2)$ one ends up with [56]

$$\Pi(q^2) = \Pi(0) - q^2\int_0^\infty \frac{\rho(s)ds}{s(s+q^2)} = \Pi(0) + \int_0^\infty \left(\frac{1}{s+q^2} - \frac{1}{s}\right)\rho(s)ds. \qquad (2.40)$$

The first, constant part is singular in general. One has to subtract this singularity. But if one calculates the derivative of $\Pi(q^2)$, one obtains either

$$-\frac{d\Pi(q^2)}{dq^2} = \int_{s_0}^\infty \frac{\rho(s)ds}{(s+q^2)^2} \quad \text{or} \quad -\frac{d\Pi(q^2)}{dq^2} = \int_0^\infty \frac{\rho(s)ds}{(s+q^2)^2}. \qquad (2.41)$$

Both quantities are positive definite and finite [57]. Thus, one can define an invariant with respect to the renormalization group equation (see Refs. [58, 59]), known as *Adler's function*

$$D(q^2) := -q^2\frac{d\Pi(q^2)}{dq^2} = q^2\int_0^\infty \frac{\rho(s)ds}{(s+q^2)^2} \qquad (2.42)$$

where the second form of the derivative has been used, knowing that the ambiguity means that the integral up to s_0 vanishes identically. This is obviously clear when one takes a "natural glimpse" at the spectrum because there is no hadron production possible below the light flavour threshold (see also Ref. [60] as well as restricting remarks in Chapter 7).

2.2.5 Coefficients for Adler's function

Adler's function has been calculated as a perturbation series up to three-loop order for the mass-zero part [61, 62]. The contributions for non-vanishing mass are done up to $O(m^2, \alpha_s^2)$ and also for non-perturbative contributions in Ref. [63]. In the present applications the main emphasis is put on the massless part. At the same scale q^2 as the one taken for the running coupling α_s one obtains

$$D(q^2) = 1 + K_0\left(\frac{\alpha_s}{\pi}\right) + K_1\left(\frac{\alpha_s}{\pi}\right)^2 + K_2\left(\frac{\alpha_s}{\pi}\right)^3 + O(\alpha_s^4) \tag{2.43}$$

which is the contribution *only for a single quark colour C and flavour f* with

$$\begin{aligned} K_0 &= 1, \qquad K_1 = \frac{365}{24} - 11\zeta(3) - \left(\frac{11}{12} - \frac{2}{3}\zeta(3)\right) N_f, \\ K_2 &= \frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5) + \\ &\qquad - \left(\frac{7847}{216} - \frac{262}{9}\zeta(3) + \frac{25}{9}\zeta(5)\right) N_f + \left(\frac{151}{162} - \frac{19}{27}\zeta(3)\right) N_f^2. \end{aligned} \tag{2.44}$$

Note that $K_0 = 1$, so that the series starts with $1+\alpha_s/\pi$ (which is transformed to $1+9a/4$ for the rescaled parameter).

2.2.6 Scaling property of the coupling to first order

When quantities are considered at different energy scales, the running of the parameters have to be taken into account. The running of the coupling is determined by the solution of the renormalization group equation, considered here in leading order only. The equation one has to solve is simply given by $\bar{a}'(t) = -\bar{a}^2$ which can be integrated to result in

$$\frac{1}{\bar{a}(t)} - \frac{1}{\bar{a}(0)} = t. \tag{2.45}$$

There is the parameter Λ left in the definition of t which can be specified in order to simplify this equation. If one fixes Λ so that $\bar{a}(0) = \infty$, one obtains

$$\bar{a}(t) = \frac{1}{t} \quad \Rightarrow \quad \alpha_s(q^2) = \frac{\pi}{\beta_0 \ln(q^2/\Lambda^2)}. \tag{2.46}$$

This scale Λ is the *QCD-scale* $\Lambda_{\rm QCD}$, in this case to the first order. It depends on the order one uses for the expansion of the beta function and depends on the selected subtraction scheme as well, in this case the $\overline{\rm MS}$-scheme. Therefore, one should more exactly name this parameter $\Lambda^{(1)}_{\overline{\rm MS}}$. Hoewever, for brevity the former notation is kept. If one expands this result for the coupling constant at some fixed scale μ^2, one obtains

$$\begin{aligned} \alpha_s(q^2) &= \frac{\pi}{\beta_0 \ln(q^2/\Lambda^2_{\rm QCD})} = \frac{\pi}{\beta_0\left(\ln(\mu^2/\Lambda^2_{\rm QCD}) - \ln(\mu^2/q^2)\right)} = \\ &\approx \alpha_s(\mu^2)\left(1 + \frac{\ln(\mu^2/q^2)}{\ln(\mu^2/\Lambda^2_{\rm QCD})} + \dots\right). \end{aligned} \tag{2.47}$$

2.3 Expressions to compare with the e^+e^- data

In the previous subsection a special case of the solution for the coupling has been considered, namely the solution to first order. But even in the general case the expansion will be expressible in terms of powers of logarithms. One can choose an ansatz for Adler's function in replacing the coefficients K_i by a power series in $\ln(\mu^2/q^2)$ and match the coefficients by using the invariance property of $D(q^2)$. Starting with this section, one considers only reduced parts of the quantities.

2.3.1 The reduced part of Adler's function

As seen before, Adler's function starts with $1+9a/4$. This fact will be used to define a *reduced part* $d(q^2)$ of it by setting

$$D(q^2) = 1 + \frac{9}{4}d(q^2). \tag{2.48}$$

$D(q^2)$ is invariant with respect to the renormalization group equation which means that

$$\mu^2\frac{dD(q^2)}{d\mu^2} = 0 \quad \Rightarrow \quad \mu^2\frac{d}{d\mu^2}d(q^2) = 0, \quad d(\mu^2) = a(1+k_1a+k_2a^2+k_3a^3+k_4a^4)+O(a^6) \tag{2.49}$$

with $k_i := K_i/(\beta_0)^i$ where $a = a(\mu^2) := \bar{a}(\ln(\mu^2/\Lambda^2))$ is taken at the scale μ^2. The unknown coefficients k_3 and k_4 are included as well. Note that the invariance property is given by a derivative with the (not explicitly indicated) dependence on μ^2, *not* on q^2. To solve the differential equation for $d(q^2)$, one uses the ansatz

$$\begin{aligned} d(q^2) &= a + (k_1 + k_{11}\ell)a^2 + (k_2 + k_{21}\ell + k_{22}\ell^2)a^3 + \\ &\quad +(k_3 + k_{31}\ell + k_{32}\ell^2 + k_{33}\ell^3)a^4 + (k_4 + k_{41}\ell + k_{42}\ell^2 + k_{43}\ell^3 + k_{44}\ell^4)a^5 \end{aligned} \tag{2.50}$$

with

$$\ell := \ln\left(\frac{\mu^2}{q^2}\right) \quad \Rightarrow \quad \mu^2\frac{d\ell}{d\mu^2} = 1, \qquad \mu^2\frac{da(\mu^2)}{d\mu^2} = \beta(a). \tag{2.51}$$

Therefore, one obtains

$$\begin{aligned} 0 &\stackrel{!}{=} \mu^2\frac{d}{d\mu^2}d(q^2) = \\ &= \beta(a) + k_{11}a^2 + 2(k_1 + k_{11}\ell)a\beta(a) + \\ &\quad +(k_{21} + 2k_{22}\ell)a^3 + 3(k_2 + k_{21}\ell + k_{22}\ell^2)a^2\beta(a) + \\ &\quad +(k_{31} + 2k_{32}\ell + 3k_{33}\ell^2)a^4 + 4(k_3 + k_{31}\ell + k_{32}\ell^2 + k_{33}\ell^3)a^3\beta(a) + \\ &\quad +(k_{41} + 2k_{42}\ell + 3k_{43}\ell^3 + 4k_{44}\ell^3)a^5 + \\ &\qquad +5(k_4 + k_{41}\ell + k_{42}\ell^2 + k_{43}\ell^3 + k_{44}\ell^4)a^4\beta(a). \end{aligned} \tag{2.52}$$

The comparison for the coefficients of different powers of a results in

$$k_{11} = 1, \qquad k_{21} + 2k_{22}\ell = 2(k_1 + k_{11}\ell) + c_1 \quad \Rightarrow \quad k_{21} = 2k_1 + c_1, \quad k_{22} = k_{11} = 1,$$
$$k_{31} + 2k_{32}\ell + k_{33}\ell^2 = 3(k_2 + k_{21}\ell + k_{22}\ell^2) + 2c_1(k_1 + k_{11}\ell) + c_2 \quad \Rightarrow$$

$$\begin{aligned}
k_{31} &= 3k_2 + 2c_1k_1 + c_2,\\
2k_{32} &= 3k_{21} + 2c_1k_{11} = 6k_1 + 3c_1 + 2c_1 = 6k_1 + 5c_1,\\
3k_{33} &= 3k_{22} = 3 \quad \Rightarrow \quad k_{33} = 1,\\
k_{41} + 2k_{42}\ell + 3k_{43}\ell^2 + 4k_{44}\ell^3 &= 4(k_3 + k_{31}\ell + k_{32}\ell^2 + k_{33}\ell^3) \; +\\
&\quad +3c_1(k_2 + k_{21}\ell + k_{22}\ell^2) + 2c_2(k_1 + k_{11}\ell) + c_3 \quad \Rightarrow\\
k_{41} &= 4k_3 + 3c_1k_2 + 2c_2k_1 + c_3,\\
2k_{42} &= 4k_{31} + 3c_1k_{21} + 2c_2k_{11} = 12k_2 + 8c_1k_1 + 4c_2 + 6c_1k_1 + 3c_1^2 + 2c_2 =\\
&= 12k_2 + 14c_1k_1 + 3c_1^2 + 6c_2,\\
3k_{43} &= 4k_{32} + 3c_1k_{22} = 12k_1 + 10c_1 + 3c_1 = 12k_1 + 13c_1,\\
4k_{44} &= 4k_{33} = 4 \quad \Rightarrow \quad k_{44} = 1.
\end{aligned}$$

The solution is thus given by

$$\begin{aligned}
d(q^2) &= a + (k_1 + \ell)a^2 + (k_2 + (2k_1 + c_1)\ell + \ell^2)a^3 \; +\\
&\quad + \left(k_3 + (3k_2 + 2c_1k_1 + c_2)\ell + \frac{1}{2}(6k_1 + 5c_1)\ell^2 + \ell^3\right) a^4 \; +\\
&\quad + \Big(k_4 + (4k_3 + 3c_1k_2 + 2c_2k_1 + c_3)\ell \; + \\
&\qquad + \frac{1}{2}(12k_2 + 14c_1k_1 + 3c_1^2 + 6c_2)\ell^2 + \frac{1}{3}(12k_1 + 13c_1)\ell^3 + \ell^4\Big)a^5 + O(a^6).
\end{aligned} \tag{2.53}$$

2.3.2 The reduced part of the correlator function

The function $d(q^2)$ or equally $D(q^2)$ are functions in q^2 only by the powers of ℓ. To get from Adler's function to the correlator function, one has to reverse the differentiation in Eq. (2.42) by integrating correspondingly. In the same manner one can define a relative part of the correlator function, simply called *relative correlator function* $p(q^2)$, by

$$\Pi(q^2) = 1 + \frac{9}{4}p(q^2) \tag{2.54}$$

again for a single quark colour and flavour. Then

$$d(q^2) = -q^2\frac{d}{dq^2}p(q^2) = \frac{d}{d\ell}p(q^2) \quad \Rightarrow \quad p(q^2) = p(\mu^2) + \int_0^\ell d(q^2)d\ell' \tag{2.55}$$

where $d(q^2)$ and $p(q^2)$ are understood as (polynomial) functions in ℓ or (for the integrand) in ℓ'. Integration gives

$$\begin{aligned}
p(q^2) &= p(\mu^2) + a\ell + \left(k_1\ell + \frac{1}{2}\ell^2\right) a^2 + \left(k_2\ell + \frac{1}{2}(2k_1 + c_1)\ell^2 + \frac{1}{3}\ell^3\right) a^3 \; +\\
&\quad + \left(k_3\ell + \frac{1}{2}(3k_2 + 2c_1k_1 + c_2)\ell^2 + \frac{1}{6}(6k_1 + 5c_1)\ell^3 + \frac{1}{4}\ell^4\right) a^4 \; +\\
&\quad + \Big(k_4\ell + \frac{1}{2}(4k_3 + 3c_1k_2 + 2c_2k_1 + c_3)\ell^2 \; +\\
&\qquad + \frac{1}{6}(12k_2 + 14c_1k_1 + 3c_1^2 + 6c_2)\ell^3 + \frac{1}{12}(12k_1 + 13c_1)\ell^4 + \frac{1}{5}\ell^5\Big)a^5 + O(a^6).
\end{aligned} \tag{2.56}$$

2.3.3 The reduced part of the spectral density

Again one defines a reduced part, the *relative spectral density* $r(s)$, by

$$\rho_f(s) = 1 + \frac{9}{4}r(s) = \frac{1}{2\pi i}\,\mathrm{Disc}\,\Pi(s). \tag{2.57}$$

In order to obtain this quantity, one first considers the discontinuities of the different powers of $\ell = \ell(q^2)$,

$$\begin{aligned}
\mathrm{Disc}\,\ell(s) &= \ln\left(\frac{\mu^2}{se^{-i\pi}}\right) - \ln\left(\frac{\mu^2}{se^{i\pi}}\right) = \ln\left(\frac{\mu^2}{s}e^{i\pi}\right) - \ln\left(\frac{\mu^2}{s}e^{-i\pi}\right) = \\
&= \ln\left(\frac{\mu^2}{s}\right) + i\pi - \left(\ln\left(\frac{\mu^2}{s}\right) - i\pi\right) = i\pi - (-i\pi) = 2\pi i
\end{aligned} \tag{2.58}$$

and so

$$\begin{aligned}
\frac{1}{2\pi i}\,\mathrm{Disc}\,\ell(s) &= 1, \qquad \text{similarly} \\
\frac{1}{2\pi i}\,\mathrm{Disc}\,\ell^2(s) &= 2\ln\left(\frac{\mu^2}{s}\right), \\
\frac{1}{2\pi i}\,\mathrm{Disc}\,\ell^3(s) &= 3\ln^2\left(\frac{\mu^2}{s}\right) - \pi^2, \\
\frac{1}{2\pi i}\,\mathrm{Disc}\,\ell^4(s) &= 4\ln^3\left(\frac{\mu^2}{s}\right) - 4\pi^2\ln\left(\frac{\mu^2}{s}\right), \\
\frac{1}{2\pi i}\,\mathrm{Disc}\,\ell^5(s) &= 5\ln^4\left(\frac{\mu^2}{s}\right) - 10\pi^2\ln^2\left(\frac{\mu^2}{s}\right) + \pi^4.
\end{aligned} \tag{2.59}$$

One can define $L := \ln(\mu^2/s)$ and obtains

$$\begin{aligned}
r(s) &= a + (k_1 + L)a^2 + \left(k_2 - \frac{\pi^2}{3} + (2k_1 + c_1)L + L^2\right)a^3 + \\
&+ \left(k_3 - \frac{1}{6}(6k_1 + 5c_1)\pi^2 + (3k_2 + 2c_1k_1 + c_2 - \pi^2)L + \frac{1}{2}(6k_1 + 5c_1)L^2 + L^3\right)a^4 + \\
&+ \Bigg(k_4 - (12k_2 + 14c_1k_1 + 3c_1^2 + 6c_2)\frac{\pi^2}{6} + \frac{\pi^4}{5} + \\
&\quad + \left(4k_3 + 3c_1k_2 + 2c_2k_1 + c_3 - (12k_1 + 13c_1)\frac{\pi^2}{3}\right)L + \\
&\quad + \frac{1}{2}(12k_2 + 14c_1k_1 + 3c_1^2 + 6c_2 - 4\pi^2)L^2 + \frac{1}{3}(12k_1 + 13c_1)L^3 + L^4\Bigg)a^5 + O(a^6).
\end{aligned} \tag{2.60}$$

2.3.4 The relative hadronic e^+e^- annihilation cross section

As for the perturbative series of Adler's function in Eq. (2.43), the spectral density is given for a single quark colour and flavour. In order to be commensurate with the relative hadronic cross section one started with, all flavours occuring at the specified center-of-mass energy have to be added. Assuming that all three flavours used up to now are massless, the contributions are the same except for a factor Q_f^2 denoting the square of the electric charge of the respective quark with flavour f in units of the elementary charge. This factor comes from the two vertices of the quark loop with the Z_0 boson. Therfore, the matching with the experimental side is done by calculating

$$\begin{aligned} R_{e^+e^-}(s) &= N_c \sum_{f=u,d,s} \rho(s) = N_c \sum_{f=u,d,s} Q_f^2 \left(1+\frac{9}{4}r(s)\right) = \\ &= 3\left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right)\left(1+\frac{9}{4}r(s)\right) = 2\left(1+\frac{9}{4}r(s)\right). \end{aligned} \tag{2.61}$$

Note that there is in principle an additional term which has its origin in QCD diagrams in analogy to the QED *light-by-light diagrams*.[2] This additional term contributing to $R_{e^+e^-}$ independently from the scheme is given by

$$-\left(\sum_{f=u,d,s} Q_f\right)^2 \left(\frac{35}{108}-\frac{5}{3}\zeta(3)\right) a^3 \tag{2.62}$$

which of course vanishes in our case because the total charge is zero.

Finally one can note that the considered terms only contain the perturbative contribution. Also the long-distance non-perturbative contributions, expanded in a power series in $1/s$ by using the *Operator Product Expansion (OPE)* and factorized in vacuum matrix elements should principally have been taken into account. But as noteed in Ref. [66], the magnitudes of these power corrections fall off rapidly with s, and they can be completely neglected for center-of-mass energies beyond a few *GeV*. Thus one is done with the theory side for the relative hadronic e^+e^- annihilation cross section at this point.

2.4 The relative semihadronic τ decay rate

As explained in Ref. [66], the semihadronic (or semileptonic) decay rate

$$R_\tau = \frac{\Gamma(\tau\to\nu_\tau+\text{hadrons})}{\Gamma(\tau\to\nu_\tau+e^-+\bar\nu_e)} \tag{2.63}$$

can be expressed as an integral over hadronic spectral functions,

$$\begin{aligned} R_\tau &= \frac{1}{\pi i}\int_0^{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+\frac{2s}{m_\tau^2}\right)\mathrm{Disc}\,\Pi^{(1)}(s)+\mathrm{Disc}\,\Pi^{(0)}(s)\right]\frac{ds}{m_\tau^2} = \\ &= \frac{1}{\pi i}\int_0^{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+\frac{2s}{m_\tau^2}\right)\mathrm{Disc}\,\Pi^{(1+0)}(s)-\frac{2s}{m_\tau^2}\mathrm{Disc}\,\Pi^{(0)}(s)\right]\frac{ds}{m_\tau^2} \end{aligned} \tag{2.64}$$

where the appropriate combinations of correlators are

$$\Pi^{(J)}(q^2) = |V_{ud}|^2\left(\Pi^{(J)}_{ud,V}(q^2)+\Pi^{(J)}_{ud,A}(q^2)\right)+|V_{us}|^2\left(\Pi^{(J)}_{us,V}(q^2)+\Pi^{(J)}_{us,A}(q^2)\right), \tag{2.65}$$

[2] For details see Ref. [64] where the whole set of diagrams and the calculational techniques are described together with the results up to order a^3. Mistakes in the leading term are corrected in Refs. [62, 65], see the discussion in Ref. [66]. Cf. also considerations in Chapter 7.

$\Pi^{(1+0)}(q^2) = \Pi^{(1)}(q^2) + \Pi^{(0)}(q^2)$, and

$$\Pi^{\mu\nu}_{ij,V/A}(q^2) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{(1)}_{ij,V/A}(q^2) + q^\mu q^\nu \Pi^{(0)}_{ij,V/A}(q^2) \tag{2.66}$$

for

$$\begin{aligned} \Pi^{\mu\nu}_{ij,V}(q^2) &= 12\pi^2 i \int e^{iqx} \langle 0|T(V^\mu_{ij}(x)V^\nu_{ij}(0)^\dagger)|0\rangle, \\ \Pi^{\mu\nu}_{ij,A}(q^2) &= 12\pi^2 i \int e^{iqx} \langle 0|T(A^\mu_{ij}(x)A^\nu_{ij}(0)^\dagger)|0\rangle. \end{aligned} \tag{2.67}$$

Here $J = 0, 1$ is the angular momentum in the hadronic rest frame, and $V^\mu_{ij} = \bar\psi_j \gamma^\mu \psi_i$, $A^\mu_{ij} = \bar\psi_j \gamma^\mu \gamma_5 \psi_i$ are the vector resp. axial vector colour singlet quark currents. At first sight this seems to be a rather complicated construction scheme. Fortunately it can be shown that the non-perturbative contributions to the correlator functions are again quite small, while the perturbative contributions are the same for all lower indices. For the $J = 1$ case as in Ref. [67], one ends up with

$$R_\tau = \frac{1}{\pi i}(|V_{ud}|^2 + |V_{us}|^2) \int_0^{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \mathrm{Disc}\,\Pi(s) \frac{ds}{m_\tau^2}. \tag{2.68}$$

One can write the integral in a dimensionless form by substituting $x = s/m_\tau^2$,

$$R_\tau = \frac{1}{\pi i}(|V_{ud}|^2 + |V_{us}|^2) \int_0^1 (1-x)^2(1+2x)\, \mathrm{Disc}\,\Pi(m_\tau^2 x) dx. \tag{2.69}$$

This expression is the starting point for the following considerations.

2.4.1 Introducing moments

The correlator $\Pi(q^2)$ used in Eq. (2.68) is the the one which previously occured in the e^+e^- annihilation cross section. Therefore, one can insert Eq. (2.61) and obtains

$$R_\tau = 2(|V_{ud}|^2 + |V_{us}|^2) \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) R_{e^+e^-}(s). \tag{2.70}$$

This equation is the first connection between the τ decay rate and the e^+e^- annihilation cross section. If one defines *moments*

$$R_n(s_0) := (n+1) \int_0^{s_0} \frac{ds}{s_0} \left(\frac{s}{s_0}\right)^n R_{e^+e^-}(s) = (n+1) \int_0^1 x^n R_{e^+e^-}(s_0 x) dx, \tag{2.71}$$

one can write

$$R_\tau = (|V_{ud}|^2 + |V_{us}|^2) \left(2R_0(m_\tau^2) - 2R_2(m_\tau^2) + R_3(m_\tau^2)\right). \tag{2.72}$$

Knowing that

$$(n+1) \int_0^1 x^n dx = \left[x^{n+1}\right]_{x=0}^1 = 1 \qquad \text{for all } n \geq 0, \tag{2.73}$$

one can again change to relative quantities and obtain

$$R_\tau = R_\tau^0 \left(1 + \frac{9}{4} r_\tau\right) \qquad \text{where } r_\tau = 2r_0(m_\tau^2) - 2r_2(m_\tau^2) + r_3(m_\tau^2), \tag{2.74}$$

$$r_n(s_0) \;:=\; (n+1) \int_0^1 x^n r(s_0 x) dx. \tag{2.75}$$

Besides the factor $(|V_{ud}|^2 + |V_{us}|^2) \approx 3$, R_τ^0 can also contain some other corrections as the electroweak and the nonperturbative corrections. The relative quantity r_τ which is to be used in the perturbative analysis in what follows will change accordingly.

2.4.2 Perturbative series for decay rate and moments

The following two subsections will mainly explain the features of Ref. [47]. As explained before, the main goal is to express the perturbative series expansion of the e^+e^- annihilation cross section as a series in the τ decay rate. Both quantities are physical observables, and it will be shown that also the coefficients of this series are of this kind, as expected. To be more precise, the different moments r_n will be expressed in terms of r_τ. To start with, r_n is given by

$$\begin{aligned} r_n(m_\tau^2) &= (n+1)\int_0^1 x^n r(m_\tau^2 x)dx \ = \\ &= a + \left(k_1 + \tilde{I}(1,n)\right)a^2 + \left(k_2 - \frac{\pi^2}{3} + (2k_1+c_1)\tilde{I}(1,n) + \tilde{I}(2,n)\right)a^3 + O(a^4) \end{aligned} \tag{2.76}$$

where Eq. (2.60) has been inserted. This insertion is done up to the third order because at this point the path of calculation is illustrated only. The full calculation is done by a MATHEMATICA program [68]. The integral $\tilde{I}(m,n)$ is defined by

$$\tilde{I}(m,n) := (n+1)\int_0^1 x^n \ln\left(\frac{m_\tau^2}{m_\tau^2 x}\right)dx = (n+1)\int_0^1 x^n \ln\left(\frac{1}{x}\right)dx \tag{2.77}$$

where $\mu = m_\tau$. Similarly, r_τ is given by

$$\begin{aligned} r_\tau &= 2\int_0^1 (1-x)^2(1+2x)r(m_\tau^2 x)dx \ = \\ &= a + (k_1 + I_\tau(1))\,a^2 + \left(k_2 - \frac{\pi^2}{3} + (2k_1+c_1)I_\tau(1) + I_\tau(2)\right)a^3 + O(a^4) \end{aligned} \tag{2.78}$$

with

$$I_\tau(m) := 2\int_0^1 (1-x)^2(1+2x)\ln\left(\frac{1}{x}\right)dx = 2\tilde{I}(m,0) - 2\tilde{I}(m,2) + \tilde{I}(m,3). \tag{2.79}$$

The integral $\tilde{I}(m,n)$ can be calculated analytically by using the substitution $x = e^{-t}$,

$$\begin{aligned} \tilde{I}(m,n) &= (n+1)\int_0^1 x^n \ln^m\left(\frac{1}{x}\right)dx \ = \ (n+1)\int_0^\infty t^m e^{-(n+1)t}dt \ = \\ &= \Big[-t^m e^{-(n+1)t}\Big]_{t=0}^\infty + m\int_0^\infty t^{m-1}e^{-(n+1)t}dt \ = \\ &= \frac{m(m-1)}{n+1}\int_0^\infty t^{m-2}e^{-(n+1)t}dt \ = \ \ldots \\ \ldots &= \frac{m!}{(n+1)^{m-1}}\int_0^\infty e^{-(n+1)t}dt \ = \ \frac{m!}{(n+1)^m}\Big[-e^{-(n+1)t}\Big]_{t=0}^\infty \ = \ \frac{m!}{(n+1)^m} \end{aligned} \tag{2.80}$$

and

$$I_\tau(m) = \left(\frac{2}{1^m} - \frac{2}{3^m} + \frac{1}{4^m}\right)m! \tag{2.81}$$

as quoted in Ref. [47].

2.4.3 Inversion and insertion

As mentioned earlier, the main task is to invert the series expansion of r_τ in terms of a and insert this result into the series expansion of r_n. The inversion of the series

$$r_\tau = a + \tilde{k}_1 a^2 + \tilde{k}_2 a^3 + \tilde{k}_3 a^4 + O(a^5) \tag{2.82}$$

can be obtained by the ansatz

$$\begin{aligned}
a &= r_\tau + a_1 r_\tau^2 + a_2 r_\tau^3 + a_3 r_\tau^4 + O(r_\tau^5) \qquad \Rightarrow \\
a^2 &= r_\tau^2 + 2a_1 r_\tau^3 + 2a_2 r_\tau^4 + a_1^2 r_\tau^4 + O(r_\tau^5), \\
a^3 &= r_\tau^3 + 3a_1 r_\tau^4 + O(r_\tau^5), \\
a^4 &= r_\tau^4 + O(r_\tau^5).
\end{aligned} \tag{2.83}$$

Inserting this, one ends up with

$$\begin{aligned}
r_\tau &= r_\tau + (a_1 + \tilde{k}_1) r_\tau^2 + (a_2 + 2a_1\tilde{k}_1 + \tilde{k}_2) r_\tau^3 + \\
&\qquad + \left(a_3 + (2a_2 + a_1^2)\tilde{k}_1 + 3a_1\tilde{k}_2 + \tilde{k}_3\right) r_\tau^4 + O(r_\tau^5).
\end{aligned} \tag{2.84}$$

A comparison of coefficients then results in

$$\begin{aligned}
a_1 &= -\tilde{k}_1, \\
a_2 &= -\tilde{k}_2 - 2a_1\tilde{k}_1 \;=\; -\tilde{k}_2 + 2\tilde{k}_1^2, \\
a_3 &= -\tilde{k}_3 - 3a_1\tilde{k}_2 - (2a_2 + a_1^2)\tilde{k}_1 \;= \\
&= -\tilde{k}_3 + 3\tilde{k}_1\tilde{k}_2 + 2\tilde{k}_1\tilde{k}_2 - 4\tilde{k}_1^3 - \tilde{k}_1^3 \;=\; -\tilde{k}_3 + 5\tilde{k}_1\tilde{k}_2 - 5\tilde{k}_1^3.
\end{aligned} \tag{2.85}$$

After having obtained these coefficients, the result for a in terms of powers of r_τ is inserted into the series expansion of r_n and results in

$$r_n(m_\tau^2) = f_{0n} r_\tau + f_{1n} r_\tau^2 + f_{2n} r_\tau^3 + f_{3n} r_\tau^4 + f_{4n} r_\tau^5 + O(r_\tau^6) \tag{2.86}$$

where one has returned to the previous order using

$$\begin{aligned}
f_{0n} &= I(0,n), \\
f_{1n} &= I(1,n), \\
f_{2n} &= I(2,n) + \rho_1 I(1,n), \\
f_{3n} &= I(3,n) + \left(I_\tau(2) - I_\tau(1)^2 - \frac{\pi^2}{3}\right) I(1,n) \;+ \\
&\quad + \frac{5}{2}\rho_1 I(2,n) + \rho_2 I(1,n), \\
f_{4n} &= I(4,n) - 3\left(I_\tau(2) - I_\tau(1)^2 - \frac{\pi^2}{3}\right) I(2,n) \;+ \\
&\quad + 2\left(I_\tau(3) - 3I_\tau(1)I_\tau(2) + 2I_\tau(1)^3\right) I(1,n) \;+ \\
&\quad + \rho_1\left(\frac{13}{3} I(3,n) + 5\left(I_\tau(2) - I - \tau(1)^2 - \frac{\pi^2}{3}\right) I(1,n)\right) \;+ \\
&\quad + 3\rho_2 I(2,n) + \rho_3 I(1,n).
\end{aligned} \tag{2.87}$$

The terms $I(m,n)$ occuring in these coefficients are *not* exactly the integrals $\tilde{I}(m,n)$ defined earlier but related to them by

$$\tilde{I}(m,n) = \sum_{p=0}^{m} \binom{m}{p} I_\tau(p) I(m-p,n). \tag{2.88}$$

This Binomial-like relation was found to express the resulting coefficients in a more concise and closed form. ρ_1, ρ_2 and ρ_3 are scheme independent quantities which will appear later on when all quantities are expressed in a power series of Adler's function. Their values are

$$\begin{aligned}
\rho_1 &= c_1 \approx 0.79012, \\
\rho_2 &= c_2 - c_1 k_1 - k_1^2 + k_2 \approx 1.03463, \\
\rho_3 &= c_3 - 2c_2 k_1 + c_1 k_1^2 + 4k_1^3 - 6k_1 k_2 + 2k_3 \approx -2.9795 + 2k_3
\end{aligned} \tag{2.89}$$

where one is left with the unknown coefficient k_3. This is the situation decribed in Ref. [47]. The Padé approximation mentioned there to estimate k_3 is discussed later on.

2.4.4 Introducing a circle integral

Again one uses Cauchy's theorem to rewrite the integral. In this case one uses the "trivial form" of this theorem which means that the integral of the function vanishes when it is taken on a closed path encircling a region where this function is holomorphic. This is true for the product of the correlator function with some polynomial function except for the above mentioned cut along the negative real axis starting from the negative value of the light flavour production threshold $s_0 = 4m_\pi^2$. But in extending the closed path to the already known form of a cut disk, one does even more, i.e. one has to leave out the cut starting at $s = 0$. Therefore, for the polynomial function

$$P_\tau(x) := 2(1+x)^2(1-2x) \tag{2.90}$$

one obtains

$$\begin{aligned}
0 &\stackrel{!}{=} \oint_{|x|=1} P_\tau(x)\Pi(m_\tau^2 x)dx + \int_{e^{i\pi}}^{0} P_\tau(x)\Pi(m_\tau^2 x)dx + \int_0^{e^{-i\pi}} P_\tau(x)\Pi(m_\tau^2 x)dx \;= \\
&= \oint_{|x|=1} P_\tau(x)\Pi(m_\tau^2 x)dx \;+ \\
&\qquad + \int_1^0 P_\tau(xe^{i\pi})\Pi(m_\tau^2 xe^{i\pi})e^{i\pi}dx + \int_0^1 P_\tau(xe^{-i\pi})\Pi(m_\tau^2 xe^{-i\pi})e^{-i\pi}dx \;= \\
&= \oint_{|x|=1} P_\tau(x)\Pi(m_\tau^2 x)dx - \int_0^1 P_\tau(-x)\left(\Pi(m_\tau^2 xe^{-i\pi}) - \Pi(m_\tau^2 xe^{i\pi})\right)dx \;= \\
&= \oint_{|x|=1} P_\tau(x)\Pi(m_\tau^2 x)dx - \int_0^1 P_\tau(-x)\,\mathrm{Disc}\,\Pi(m_\tau^2 x)dx.
\end{aligned} \tag{2.91}$$

and ends up with

$$R_\tau = -6\pi i(|V_{ud}|^2 + |V_{us}|^2) \oint_{|x|=1} P_\tau(x)\Pi(m_\tau^2 x)dx. \tag{2.92}$$

2.4.5 Introducing a weight function

As a next step Adler's function is introduced. One can do this by defining a *weight function*

$$W_\tau(x) := \frac{1}{x}\int_{-1}^{x} P_\tau(x')dx' = \frac{2}{x}\int_{-1}^{x}(1+x')^2(1-2x')dx' = \frac{1}{x}(1+2x-2x^3-x^4). \quad (2.93)$$

Using this weight function, one can perform a partial integration where the surface term vanishes because $W_\tau(-1)$ vanishes. One obtains

$$\begin{aligned} \oint_{|x|=1} W_\tau(x)D(m_\tau^2 x)dx &= -\oint_{|x|=1} xW_\tau(x)\frac{d\Pi(m_\tau^2 x)}{dx}dx = \quad (2.94)\\ = \oint_{|x|=1}\frac{d(xW_\tau(x))}{dx}\Pi(m_\tau^2 x)dx &- \Big[xW_\tau(x)\Pi(m_\tau^2 x)\Big]_{x=e^{-i\pi}}^{e^{i\pi}} = \oint_{|x|=1} P_\tau(x)\Pi(m_\tau^2 x)dx. \end{aligned}$$

Therefore, one finally obtains

$$R_\tau = -6\pi i(|V_{ud}|^2 + |V_{us}|^2)\oint_{|x|=1} W_\tau(x)D(m_\tau^2 x)dx. \quad (2.95)$$

2.4.6 Oscillatory and circle part

Eq. (2.95) gives the connection between Adler's function and the decay rate. This equation can be written in reduced quantities where the different moments are split off,

$$r_n(m_\tau^2) = \frac{1}{2\pi i}\oint_{|x|=1} W_n(x)d(m_\tau^2 x)dx \quad (2.96)$$

where $W_n(x)$ is the weight function corresponding to $P_n(x) = (n+1)(-x)^n$,

$$W_n(x) = \frac{n+1}{x}\int_{-1}^{x}(-x')^n dx' = \frac{(-1)^n}{x}\Big[x'^{n+1}\Big]_{x'=-1}^{x} = \frac{1}{x} + (-1)^n x^n. \quad (2.97)$$

Therefore, the reduced moment splits off into two parts, the *circle part* $r_{\rm circ}$ and the *oscillatory part* Δ_n,

$$\begin{aligned} r_n(m_\tau^2) &= r_{\rm circ}(m_\tau^2) + \Delta_n(m_\tau^2) \qquad \text{where} & (2.98)\\ r_{\rm circ}(m_\tau^2) &= \frac{1}{2\pi i}\oint_{|x|=1}\frac{1}{x}d(m_\tau^2 x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} d(m_\tau^2 e^{i\varphi})d\varphi, & (2.99)\\ \Delta_n(m_\tau^2) &= \frac{(-1)^n}{2\pi i}\oint x^n d(m_\tau^2 x)dx = \frac{(-1)^n}{2\pi}\int_{-\pi}^{\pi} e^{i(n+1)\varphi}d(m_\tau^2 e^{i\varphi})d\varphi, & (2.100) \end{aligned}$$

the circle part being the same for all moments, while the oscillatory part is n dependent. Because of its oscillatory behaviour, this contribution is suppressed but not negligible. Again one can write

$$r_\tau = r_{\rm circ}(m_\tau^2) + 2\Delta_0(m_\tau^2) - 2\Delta_2(m_\tau^2) + \Delta_3(m_\tau^2) =: r_{\rm circ}(m_\tau^2) + \Delta_\tau. \quad (2.101)$$

2.5 Resummation techniques

Different power series have appeared up to now, from the power series of the beta function up to the power series of the τ decay rate and the different moments discussed just before. In the last section the power series of the strong coupling have been replaced by a power series in the τ decay rate. The construction of the power series of r_τ was done by using the explicit series expansion of the beta function and inserting this into Adler's function which then was integrated using a specific weight function. But there is another possibility, namely to *solve* the renormalization group equation for the strong coupling exactly up to a given order of the beta function accuracy. This procedure is called *resummation*. The name is somehow misleading because one does not resum the power series expansion of r_τ with respect to the beta function coefficients but leaves the expression unexpanded.

2.5.1 Solution of the renormalization equation for the coupling

The renormalization group equation

$$\bar{a}'(t) = -\bar{a}^2(1 + c_1\bar{a} + c_2\bar{a}^2 + c_3\bar{a}^3) + O(\bar{a}^6), \qquad \bar{a}(0) = \infty \tag{2.102}$$

has been solved already for the leading order, the solution was $\bar{a}(t) = t^{-1}$. But one can go a few steps further. In order to do this, one choses a more appropriate parameter $z = 1/\bar{a}$ for which the renormalization group equation simplifies to

$$z'(t) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + O(z^{-4}), \qquad z(0) = 0. \tag{2.103}$$

The leading order solution is simply $z(t) = t$. For the first order one obtains

$$\begin{aligned} t &= t_0 + \int \frac{dx}{1 + c_1/z} = t_0 + \int \frac{z\,dz}{z + c_1} = t_0 + \int \left(1 - \frac{c_1}{z + c_1}\right) dz = \\ &= t_0 + z - c_1 \ln(z + c_1) = z - c_1 \left(\ln(z + c_1) - \ln c_1\right) \end{aligned} \tag{2.104}$$

where in the last step the initial condition to determine $t_0 = c_1 \ln c_1$ has been used. For the second order one obtains

$$\begin{aligned} t &= t_0 + \int \frac{dz}{1 + c_1/z + c_2/z^2} = t_0 + \int \frac{z^2 dz}{z^2 + c_1 z + c_2} = \\ &= t_0 + \int \left(1 - \frac{c_1}{2}\,\frac{2z + c_1}{z^2 + c_1 z + c_2} + \left(\frac{c_1^2}{2} - c_2\right)\frac{1}{z^2 + c_1 z + c_2}\right) dz = \\ &= t_0 + z - \frac{c_1}{2} \ln(z^2 + c_1 z + c_2) - \frac{2c_2 - c_1^2}{\sqrt{4c_2 - c_1^2}} \arctan\left(\frac{2z + c_1}{\sqrt{4c_2 - c_1^2}}\right) = \\ &= z - \frac{c_1}{2}\left(\ln(z^2 + c_1 z + c_2) - \ln c_2\right) + \\ &\quad - \frac{2c_2 - c_1^2}{\sqrt{4c_2 - c_1^2}} \left(\arctan\left(\frac{2z + c_1}{\sqrt{4c_2 - c_1^2}}\right) - \arctan\left(\frac{c_1}{\sqrt{4c_2 - c_1^2}}\right)\right). \end{aligned} \tag{2.105}$$

Both equations are implicit equations for z and thus $\bar{a}$. But this does not matter when one only wants to determine numerical dependences. Moreover, there is a possibility given by the command `NDSolve` in MATHEMATICA to solve a given differential equation by pure numerical means resulting in an interpolating function. This will be used in the following.

2.5.2 Determination of $\alpha_s(m_\tau^2)$ at different orders

The value $a_\tau := a(m_\tau^2)$ determined by solving the renormalization group equation still depends on the parameter Λ. One can close this gap left by the theory by fitting it to the well-known τ decay rate parameter r_τ. For this purpose one has to rewrite the moments in terms of a_τ which one uses as initial value for solving the renormalization group equation. In this way one avoids the parameter Λ. Although done for higher orders using MATHEMATICA, the considerations at this point are limited again to the solution of the renormalization group equation to leading order. One rewrites

$$a(m_\tau^2x)^{-1}=\ln\left(\frac{m_\tau^2x}{\Lambda^2}\right)=\ln\left(\frac{m_\tau^2}{\Lambda^2}\right)+\ln x=a_\tau^{-1}+\ln x\quad\Leftrightarrow\quad a(m_\tau^2x)=\frac{a_\tau}{1+a_\tau\ln x}. \tag{2.106}$$

In using the renormalization group equation together with the defining differential equation for the weight function $W_n(x)$,

$$q^2\frac{da}{dq^2}=x\frac{da}{dx}=-a^2,\qquad \frac{d}{dx}(xW_n(x))=P_n(x)=(n+1)(-x)^n, \tag{2.107}$$

one can calculate the nth moment at the τ mass scale to be

$$\begin{aligned} r_n(m_\tau^2) &= \frac{1}{2\pi i}\oint_{|x|=1}W_n(x)d(m_\tau^2x)dx = \\ &= a_\tau\left(r_{0n}+k_1a_\tau r_{1n}+k_2a_\tau^2r_{2n}+k_3a_\tau^3r_{3n}+k_4a_\tau^4r_{4n}\right)+O(a_\tau^6) \end{aligned} \tag{2.108}$$

where $d=a(1+k_1a+k_2a^2+k_3a^3+k_4a^4)$ is used at the scale q^2, and

$$a_\tau^{i+1}r_{in}:=\frac{1}{2\pi i}\oint_{|x|=1}W_n(x)a(m_\tau^2x)^{i+1}dx. \tag{2.109}$$

As an example r_{10} shall be calculated,

$$\begin{aligned} a_\tau^2r_{10} &= \frac{1}{2\pi i}\oint_{|x|=1}W_0(x)a^2(m_\tau^2x)dx = -\frac{1}{2\pi i}\oint_{|x|=1}W_0(x)x\frac{da(m_\tau^2x)}{dx}dx = \\ &= \frac{1}{2\pi i}\oint_{|x|=1}P_0(x)a(m_\tau^2x)dx = \frac{1}{2\pi i}\oint_{|x|=1}\frac{a_\tau dx}{1+a_\tau\ln x} = \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a_\tau e^{i\varphi}d\varphi}{1+ia_\tau\varphi}\qquad\Rightarrow\qquad r_{10}=\frac{1}{2\pi a_\tau}\int_{-\pi}^{\pi}\frac{e^{i\varphi}d\varphi}{1+ia_\tau\varphi}. \end{aligned} \tag{2.110}$$

The specific combination $2r_0-2r_2+r_3$ is then fitted to the experimental value at r_τ by some systematic trial and error method.

2.5.3 Singularities and convergence radius

The use of $a(m_\tau^2x)$ on the whole circle parametrized by $x=e^{i\varphi}$ leads to the question whether this function is defined on the whole circle or whether there are singularities on this circle which restrict the convergence radius. In terms of the angle φ, the renormalization group equation is written as

$$-i\frac{da(m_\tau^2e^{i\varphi})}{d\varphi}=-a^2(1+c_1a+c_2a^2+c_3a^3)+O(a^6) \tag{2.111}$$

which is solved by

$$\varphi(a;a_\tau) = -i\int_{a_\tau}^{a}\frac{da'}{\beta(a')}. \tag{2.112}$$

$\varphi = \varphi(a;a_\tau)$ describes the angle corresponding to a parameter a which is evolved from the generally complex initial value a_τ. Singularities in the evolution process are encountered at angles

$$\varphi_s(a_\tau) := \varphi(a+\infty;a_\tau) = -i\int_{a_\tau}^{\infty}\frac{da'}{\beta(a')}. \tag{2.113}$$

Even though there are no singularities for real values of a_τ, one has to extend the search to the whole complex plane because the convergence radius is determined by the singularities which are located most close to the origin. The somehow unhandy equation (2.113) can again be rewritten in terms of $z = 1/a$ to give

$$\varphi_s(a_\tau) := i\int_0^{1/a_\tau}\frac{dz}{1+c_1/z+c_2/z^2+c_3/z^3} \tag{2.114}$$

which then will be integrated by MATHEMATICA. After inversion, the singular starting values $a_\tau(\varphi_s)$ appear as lines in the complex a_τ plane which are parametrized by the angle φ_s at which the singularity $a = \infty$ occurs. They are shown Fig. 2.1, together with the convergence circle. The obtained values are given in the Ref. [49], the resummed value $a_\tau^{(4)} = 0.2704$ extracted from the τ decay rate lies outside the region of convergence.

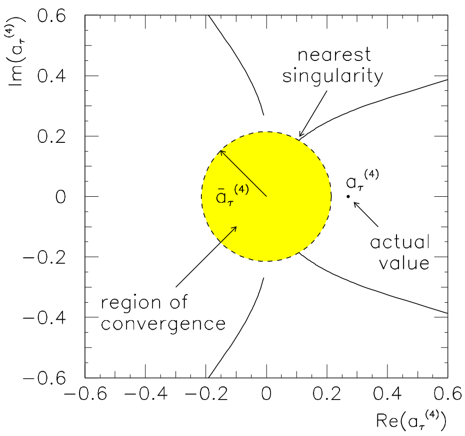


Figure 2.1: Singular starting values $a_\tau^{(4)}(\phi_s)$ ($\phi_s \in [-\pi,\pi]$) as parametric curves in the complex plane. The radius of the dashed circle determines the convergence radius of the resummed series.

2.5.4 The advantage of resummation techniques

There is a very simple example given in Ref. [50] to explain the advantage of the resummation technique. Take two observables f and g given by perturbative series in powers of a coupling a,

$$f(a) = a(1-a+a^2-\dots) = \frac{a}{1+a}, \tag{2.115}$$

$$g(a) = a(1-2a+4a^2-\dots) = \frac{a}{1+2a}. \tag{2.116}$$

In this example one explicitly knows the functional dependence as shown as last term on the left. The two observables are related by

$$g(f) = \frac{f}{1+f} = f(1-f+f^2-\dots). \tag{2.117}$$

Now assume that one wants to fit the right hand side of Eq. (2.115) to an experimental value of about $f = 0.6$. The exact formula results in $a = 1.5$ for which the series in Eq. (2.115) diverges. Therefore, one cannot obtain a from this fit without a proper resummation procedure which in this case is trivially given by the appended exact formula. With the same reason one cannot obtain a value of g in terms of a using the perturbative series in Eq. (2.116). On the other hand, the direct relation in terms of the series in Eq. (2.117) converges perfectly for $f = 0.6$ and gives an unambiguous result for g in terms of the measured f. Such an improvement does only occur when the underlying theory and the origin of the series are analyzed in detail. One is able to do so with the above resummation procedure, but one cannot answer the question about the convergence or divergence of the series obtained by reexpanding the moments in terms of r_τ.

2.5.5 Scheme dependence and a new beta function

It has to be emphasized that the renormalization group equation for the coupling and therefore the resummation procedure is invariant under a one-parameter subgroup parametrized by γ which describes the rescaling of the renormalization scale μ. This rescaling is connected with a change of the selected subtraction scheme. To see this connection, remember that the renormalization scale is introduced in dimensional regularization to compensate the "loss of dimensionality" when going from a four dimensional to a D dimensional space-time with $D = 4-2\varepsilon$. A one-loop integral thus results in a compensation factor $\mu^{-2\varepsilon}$. On the other hand, the integral itself in general produces a logarithmic singularity which is parametrized by $1/\varepsilon$. Therefore, the singular term is proportional to $\mu^{-2\varepsilon}/\varepsilon$ in the MS-scheme. If one changes to the $\overline{\rm MS}$-scheme, the singularity $1/\varepsilon$ together with constants shown in Eq. (2.3) is regarded as $1/\varepsilon'$. But instead of changing the dimensional parameter one can absorb this change in the renormalization scale μ,

$$(\mu^2)^{-\varepsilon}\left(\frac{1}{\varepsilon}-\gamma_E+\ln 4\pi\right)=\frac{(\bar\mu^2)^{-\varepsilon}}{\varepsilon},\qquad \bar\mu^2=e^\gamma\mu^2,\qquad \gamma=\gamma_E-\ln 4\pi \tag{2.118}$$

where the expansion of the additional factor in ε is used. Every change of the subtraction scheme is expressible in this form by using an appropriate γ. The rescaling of the renormalization scale does not change the renormalization group equation itself for it is an autonomeous differential equation, but it changes the coupling parameter,

$$\tilde a(\mu^2)=a(e^\gamma\mu^2)=a-\gamma a^2+(\gamma^2-c_1\gamma)a^3-(\gamma^3-\frac{5}{2}c_1\gamma^2+c_2\gamma)a^4+\ldots \tag{2.119}$$

and is called the *one parameter subgroup of the renormalization group*. This transformation under the subgroup can be obtained by calculating a Taylor series expansion of $a(e^\gamma\mu^2)$,

$$\tilde a(\mu^2)=a(e^\gamma\mu^2)=a(\Lambda^2e^{t+\gamma})=\bar a(t+\gamma)=\sum_{n=0}^\infty\frac{\gamma^n}{n!}\frac{d^n\bar a(t)}{dt^n} \tag{2.120}$$

where the terms of the Taylor series can be calculated iteratively by using the beta function,

$$\bar a_0:=\bar a(t),\qquad \bar a_i:=\frac{d\bar a_{i-1}(t)}{dt}=\frac{d\bar a(t)}{dt}\frac{d\bar a_{i-1}}{d\bar a}=\beta(\bar a)\frac{d\bar a_{i-1}}{d\bar a}. \tag{2.121}$$

The invariance of the beta function itself is rather evident, as can be shown as follows. For this purpose the series in Eq. (2.119) is inverted by the "standard ansatz"

$$a \;=\; \tilde a+b_1\tilde a^2+b_2\tilde a^3+b_3\tilde a^4+O(\tilde a^5)\qquad\Rightarrow$$

$$\begin{aligned} a^2 &= \tilde{a}^2 + 2b_1\tilde{a}^3 + 2b_2\tilde{a}^4 + b_1^2\tilde{a}^4 + O(\tilde{a}^5), \\ a^3 &= \tilde{a}^3 + 3b_1\tilde{a}^4 + O(\tilde{a}^5), \\ a^4 &= \tilde{a}^4 + O(\tilde{a}^5). \end{aligned} \tag{2.122}$$

One obtains

$$\begin{aligned} b_1 &= \gamma, \\ b_2 &= 2b_1\gamma - \gamma^2 + c_1\gamma \ = \ 2\gamma^2 - \gamma^2 + c_1\gamma \ = \ \gamma^2 + c_1\gamma, \\ b_3 &= 2b_2\gamma + b_1^2\gamma - 3b_1(\gamma^2 - c_1\gamma) + \gamma^3 - \frac{5}{2}c_1\gamma^2 + c_2\gamma \ = \\ &= 2\gamma^3 + 2c_1\gamma^2 + \gamma^3 - 3\gamma^3 + 3c_1\gamma^2 + \gamma^3 - \frac{5}{2}c_1\gamma^2 + c_2\gamma \ = \\ &= \gamma^3 + \frac{5}{2}c_1\gamma^2 + c_2\gamma \end{aligned} \tag{2.123}$$

and thus

$$a = \tilde{a} + \gamma\tilde{a}^2 + (\gamma^2 + c_1\gamma)\tilde{a}^3 + \left(\gamma^3 + \frac{5}{2}c_1\gamma^2 + c_2\gamma\right)\tilde{a}^4 + O(\tilde{a}^5). \tag{2.124}$$

This expression can be inserted in the beta function in terms of a and gives rise to the beta function for $\tilde{a}$,

$$\beta(\tilde{a}) = \frac{d\tilde{a}}{dt} = \frac{da}{dt}\frac{d\tilde{a}}{da} = \beta(a)\frac{d\tilde{a}}{da} \tag{2.125}$$

which turns out to be invariant. On the other hand, if one rewrites the reduced function $d(q^2)$ in terms of $\tilde{a}$, the coefficients k_i are rescaled while the coefficients c_i remain fixed. Therefore, a suitable choice for a scheme could lead to a value $\tilde{a}$ *inside* the convergence region while at the same time the series in k_i might have deteriorated. This again underlines the fact that the coupling taken in a specific scheme has no physical meaning. To get rid of this ambiguity, one might use a physical observable, namely the reduced Adler's function itself as an *effective coupling* [69, 70, 71], and construct a corresponding beta function

$$\beta_d(d) = -d^2(1 + \rho_1 d + \rho_2 d^2 + \rho_3 d^3) + O(d^6) \tag{2.126}$$

with the coefficients

$$\begin{aligned} \rho_1 &= c_1, \qquad \rho_2 \ = \ c_2 - c_1k_1 - k_1^2 + k_2, \\ \rho_3 &= c_3 - 2c_2k_1 + c_1k_1^2 + 4k_1^3 - 6k_1k_2 + 2k_3. \end{aligned} \tag{2.127}$$

Because the new beta function is expressed in terms of a physical observable, these coefficients are renormalization group invariants.

2.5.6 The Padé approximation

The problem that the coefficient k_3 is not known up to now still remains. But there are some methods to obtain an estimate for it. The most popular method is described in this subsection. It is called the *Padé approximation* and is based on the assumption that the approximation of the present series is given by a ratio of polynomial functions (see e.g. Ref. [72]),

$$P_{[M/N]}(x) := \frac{A_M(x)}{B_N(x)} = \frac{1 + a_1 x + \ldots + a_M x^M}{1 + b_1 x + \ldots + b_N x^N}, \tag{2.128}$$

called *Padé approximand*. In this expression a general factor of the given series as well as a factorized power of x are omitted. The coefficients can be determined by fitting the series expansion of this ratio to the given series up to order $M + N$, one then can use this series expansion to estimate the coefficient of order $M + N + 1$. It is a rather simple method, and to understand it to the full extend it is the best to apply it directly to the actual problem. The series one has to estimate is given by

$$d = a(1 + k_1 a + k_2 a^2 + k_3 a^3 + k_4 a^4) + O(a^6) \tag{2.129}$$

which is acually known up to the coefficient k_2, i.e. to the second order where the general factor a is omitted. To estimate this series one therefore has to select a Padé approximand with total order $M + N = 2$. The mostly used approximands are those in the mean range, in this case $P_{[1/1]}(x)$ (note that $P_{[2/0]}(x)$ actually does not work). Therefore, one expands this approximand into a series,

$$P_{[1/1]}(a) = \frac{1 + a_1 a}{1 + b_1 a} = 1 + (a_1 - b_1)a + (b_1^2 - a_1 b_1)a^2 + (a_1 b_1^2 - b_1^3)a^3 + O(a^4). \tag{2.130}$$

If one then compare this to the actual series, one obtains

$$\begin{aligned} k_1 &= a_1 - b_1 &&\Rightarrow \quad a_1 = b_1 + k_1 \\ k_2 &= b_1^2 - a_1 b_1 = b_1^2 - b_1^2 - b_1 k_1 &&\Rightarrow \quad b_1 = -\frac{k_2}{k_1}, \quad a_1 = k_1 - \frac{k_2}{k_1} \end{aligned} \tag{2.131}$$

and so finally as [1/1] Padé approximation

$$k_3 \approx a_1 b_1^2 - b_1^3 = k_1 b_1^2 = \frac{k_2^2}{k_1} \approx 2.17. \tag{2.132}$$

This estimate is also given in Refs. [67, 73].

2.5.7 The subgroup dependence of ρ_3

Although ρ_3 is scheme independent by definition, one introduces an artificial scheme dependence by the estimated dependent quantity k_3. The scheme dependence can be obtained by inserting the "shifted" coupling $\tilde{a}$ of Eq. (2.119) in Eq. (2.129) which leads to

$$\begin{aligned} \tilde{d} &= \tilde{a} + (k_1 - \gamma)\tilde{a}^2 + (k_2 - (2k_1 + c_1)\gamma + \gamma^2)\tilde{a}^3 + \\ &\qquad + \left(k_3 - (3k_2 + 2c_1 k_1 + c_2)\gamma + \left(3k_1 + \frac{5}{2}c_1\right)\gamma^2 - \gamma^3\right)\tilde{a}^4 + O(\tilde{a}^5) = \\ &=: \tilde{a} + \tilde{k}_1\tilde{a}^2 + \tilde{k}_2\tilde{a}^3 + \tilde{k}_3\tilde{a}^4 + O(\tilde{a}^5). \end{aligned} \tag{2.133}$$

One obtains

$$\begin{aligned}
\tilde\rho_1 &= c_1 \;= \rho_1, \\
\tilde\rho_2 &= c_2 - c_1\tilde k_1 - \tilde k_1^2 + \tilde k_2 \;= \\
&= c_2 - c_1(k_1-\gamma) - (k_1-\gamma)^2 + k_2 - (2k_1+c_1)\gamma + \gamma^2 \;= \\
&= c_2 - c_1k_1 - k_1^2 + k_2 + (c_1 + 2k_1 - (2k_1+c_1))\gamma - \gamma^2 + \gamma^2 \;=\; \rho_2
\end{aligned} \tag{2.134}$$

which indicates the invariance of the first two coefficients, and

$$\begin{aligned}
\tilde\rho_3 &= c_3 - 2c_2\tilde k_1 + c_1\tilde k_1^2 + 4\tilde k_1^3 - 6\tilde k_1\tilde k_2 + 2\tilde k_3 \;= \\
&= c_3 - 2c_2k_1 + c_1k_1^2 + 4k_1^3 - 6k_1k_2 + k_3 \;+ \\
&\quad + \Big(2c_2 - 2c_1k_1 - 12k_1^2 + 6k_2 + 6k_1(2k_1+c_1) - 6k_2 - 4c_1k_1 - 2c_2\Big)\gamma \;+ \\
&\quad + \left(c_1 + 12k_1 - 6(2k_1+c_1) - 6k_1 + 2\left(3k_1 + \frac{5}{2}c_1\right)\right)\gamma^2 - 4\gamma^3 + 6\gamma^3 - 2\gamma^3 \;=\; \rho_3.
\end{aligned} \tag{2.135}$$

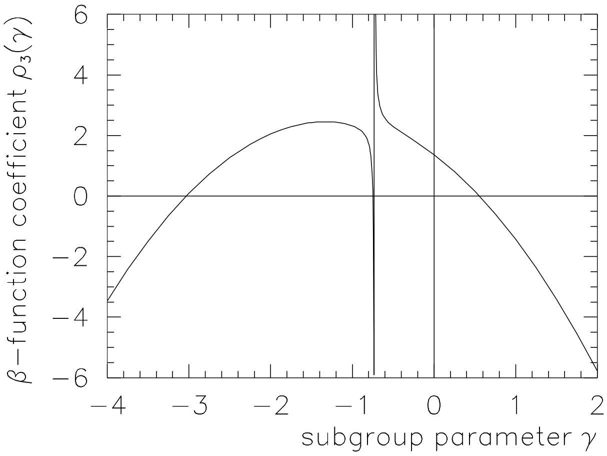


Figure 2.2: Dependence of the beta function coefficient ρ_3 on the subgroup parameter γ which specifies the choice of the renormalization scheme

This would be the situation if one would know k_3. But one does not know it up to now. Instead one estimates some value for it by using the Padé approximation. For instance, one estimates a value in the shifted system as $\tilde k_3 = \tilde k_2^2/\tilde k_1$. Then $\tilde\rho_3$ will indeed become dependent on γ [50]. The dependence is shown in Fig. 2.2 and corresponds to the $\overline{\text{MS}}$-scheme. For this one obtains a value $\rho_3 = 1.36$. If one is led by the *principle of minimal sensitivity* [74] which works in many applications (see e.g. [75]), one would select a value in the region around $\gamma = -1.3288$, at which value one has $\rho_3 = 2.4530$. Note that this value is not very far away from the value $\gamma = -2$ for the G-scheme [61] with $\rho_3 = 2.0518$. Finally, an estimate for k_3 independent from the Padé method is given. The assumption is that one uses a subtraction scheme in which Adler's function is expressible as pure geometrical series (called *"GS scheme"*),

$$d = a_{\rm GS}(1 + ka_{\rm GS} + k^2a_{\rm GS}^2 + k^3a_{\rm GS}^3) + O(a_{\rm GS}^5). \tag{2.136}$$

Note that the scheme dependence gives the additional degree of freedom to fit the second order coefficient to the geometrical series. Then the assumption fixes the third order coefficient. Numerically one obtains

$$d = a_{\rm GS}(1 - 0.1917a_{\rm GS} + 0.0367a_{\rm GS}^2 + (k_3 - 2.602)a_{\rm GS}^3) + O(a_{\rm GS}^5) \tag{2.137}$$

which gives an estimate $k_3 = 2.595$.

2.6 Moments and effective couplings

The ordinary criterion for the "quality" of moments as in Eq. (2.71) is the explicit convergence of the perturbation series. However, this convergence can be concealed by the use of a particular scheme. In order to get rid of artificial scheme-dependent constants in the perturbation theory expressions, one can express the spectral density in terms of an *effective coupling*. The really invariant measure is given by the mutual relations between moments while the effective coupling is still a parameter which one needs for constructing a scheme-independent convergence criterion. The arbitrareness of its choice became obvious in the last section. There is no special physical meaning given to such kind of a parameter. Despite this fact, the introduction of a natural internal coupling parameter allows one to extend the perturbation theory series available for the description of the relations between observables by one more term as compared to the analysis in e.g. the $\overline{\rm MS}$-scheme (see e.g. Refs. [47, 76]). In this section (which covers a part of the calculations presented in Ref. [51]) one returns to the unreduced quantities.

2.6.1 The effective coupling on the cut

Starting with Adler's function given in Eq. (2.43) one deduces the spectral density $\rho(s)$ within the $\overline{\rm MS}$-scheme,

$$\rho(s)=1+\frac{\alpha_s(s)}{\pi}+K_1\left(\frac{\alpha_s(s)}{\pi}\right)^2+\left(K_2-\frac{\pi^2}{3}\beta_0^2\right)\left(\frac{\alpha_s(s)}{\pi}\right)^3+\ \dots \tag{2.138}$$

where the term proportional to π^2 is a result of the analytic continuation from the Euclidean domain (see the calculations in Chapter 2.3). An *effective coupling* $a_M(s)$ is defined on the physical cut for sufficiently large values of s by the relation

$$a_M(s)=\frac{\alpha_s(s)}{\pi}+K_1\left(\frac{\alpha_s(s)}{\pi}\right)^2+\left(K_2-\frac{\pi^2}{3}\beta_0^2\right)\left(\frac{\alpha_s(s)}{\pi}\right)^3+\dots \tag{2.139}$$

such that

$$\rho(s)=1+a_M(s). \tag{2.140}$$

The subscript "M" stands for a Minkowskian definition of the effective coupling, i.e. the definition on the physical cut. The decomposition of the spectral density in Eq. (2.140) reflects the fact that within perturbation theory the correlator contains the parton part which is independent of α_s. All the constants that may appear in the perturbation theory expression for the spectral density $\rho(s)$ due to a particular choice of the renormalization scheme are absorbed into the definition of the effective charge (see e.g. Refs. [69, 77, 78]), so that only effects of the running of the coupling itself remain. The solution of the evolution equation for the effective coupling,

$$s\frac{da_M(s)}{ds}=\beta(a_M(s))=-a_M(s)\Big(\beta_0a_M(s)+\beta_1a_M(s)^2+\beta_2a_M(s)^3+O(a_M(s)^4)\Big) \tag{2.141}$$

resulting from the renormalization group analysis of the correlator with a given effective beta function $\beta(a)$ can be obtained by quadrature,

$$a_M(s)\ =\ a_M+\int_{M_\tau^2}^s\beta(a_M(s'))\frac{ds'}{s'}\ = \tag{2.142}$$

$$
\begin{aligned}
&= a_M + \beta_0 L a_M^2 + (\beta_1 L + \beta_0^2 L^2) a_M^3 + (\beta_2 L + \frac{5}{2}\beta_1\beta_0 L^2 + \beta_0^3 L^3) a_M^4 + O(a_M^5) = \\
&= a_M - \beta(a_M) L + \frac{1}{2}\beta(a_M)\frac{\partial\beta(a)}{\partial a}\bigg|_{a=a_M} L^2 - \frac{1}{6}\beta(a_M)\frac{\partial}{\partial a}\beta(a)\frac{\partial\beta(a)}{\partial a}\bigg|_{a=a_M} L^3 + O(L^4) = \\
&= \exp\left(-L\beta(a)\frac{\partial}{\partial a}\right) a\bigg|_{a=a_M}
\end{aligned}
\tag{2.143}
$$

where $a_M = a_M(M_\tau^2)$ and $L = \ln(M_\tau^2/s)$. In the last line the solution of Eq. (2.141) is written in a symbolic operator form. Defining the effective coupling $a_M(s)$ directly through the spectral density $\rho(s)$ itself one obtains perturbative corrections to the moments in Eq. (2.145) only because of running. Without running one would have

$$
M_l = 1 + a_M(M_\tau^2) \quad \text{or} \quad m_l = a_M(M_\tau^2) \tag{2.144}
$$

with m_l defined in Eq. (2.145). Thus, moments just allow one to study the evolution or beta function of the effective coupling $a_M(s)$.

2.6.2 Direct moments and improvement of the convergence

The contributions of powers of logarithms (from Eq. (2.143)) to normalized moments

$$
M_l = (l+1)\int_0^{M_\tau^2} \rho(s)\frac{s^l ds}{(M_\tau^2)^{l+1}} \equiv 1 + m_l. \tag{2.145}
$$

are given by

$$
(l+1)\int_0^{M_\tau^2} \ln\left(\frac{M_\tau^2}{s}\right)\frac{s^l ds}{(M_\tau^2)^{l+1}} = \frac{1}{l+1}, \tag{2.146}
$$

$$
(l+1)\int_0^{M_\tau^2} \ln^2\left(\frac{M_\tau^2}{s}\right)\frac{s^l ds}{(M_\tau^2)^{l+1}} = \frac{2}{(l+1)^2}, \ \ldots \tag{2.147}
$$

A general formula for an arbitrary (integer) power of the logarithm reads

$$
(l+1)\int_0^{M_\tau^2} \ln^p\left(\frac{M_\tau^2}{s}\right)\frac{s^l ds}{(M_\tau^2)^{l+1}} = \frac{p!}{(l+1)^p}. \tag{2.148}
$$

Therefore, at any fixed order of the perturbation series expansion the effects of running die out for large values of l improving the (explicit) asymptotic structure of the perturbative series for the moments in Eq. (2.145). This is obvious and expected in QCD with its property of asymptotic freedom because the moments with weight functions s^l suppress the infrared (small s) region of integration where perturbation theory is not applicable. Such moments are called *direct moments*. However, emphasizing the high-energy region is not preferable from the experimental point of view. Therefore, one modifies the moments in order to balance the precision requirements.

2.6.3 Modified moments and deprovement of the convergence

As the simplest modification done in order to suppress experimental errors from the high-energy end of the spectrum, the system of *modified moments*

$$
M_{kl} = \int w_{kl}(s)\frac{\rho(s)ds}{M_\tau^2} = \frac{(k+l+1)!}{k!l!}\int_0^{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^k\left(\frac{s}{M_\tau^2}\right)^l\frac{\rho(s)ds}{M_\tau^2} =: 1 + m_{kl}. \tag{2.149}
$$

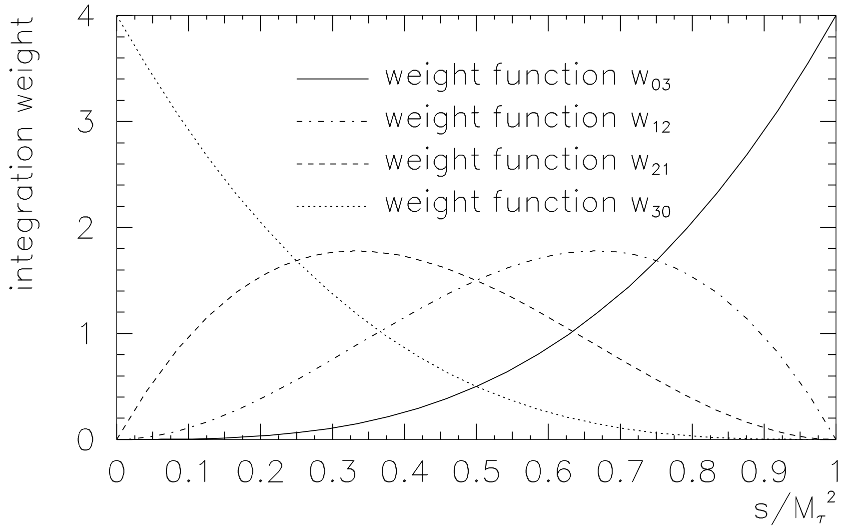


Figure 2.3: Different weight functions $w_{kl}(s)$ for $(k,l) = (3,0)$, $(2,1)$, $(1,2)$, and $(0,3)$.

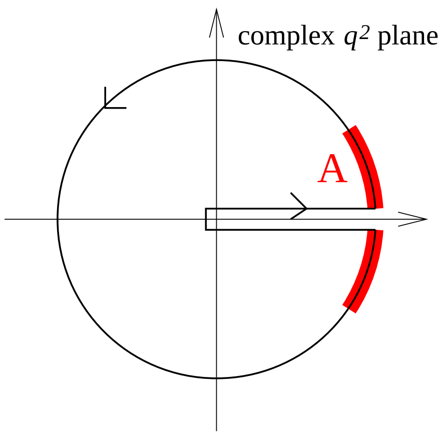


Figure 2.4: contour of integration in the complex q^2-plane

has been introduced. Some examples for the weight function $w_{kl}(s)$ are shown in Fig. 2.3. Within the set given in Eq. (2.149) the best choice from the experimental point of view is to use large values for k and small values for l. This choice was also advocated to be justified theoretically as improving the precision based on the integration over the contour in the complex q^2-plane (see Fig. 2.4 and Ref. [79]). The reasoning was that the weight functions $(1 - s/M_\tau^2)^k$ suppress the contribution of that part of the contour that is close to the real positive semi-axis where OPE is not applicable (region A in Fig. 2.4) which in turn, according to standard wisdom, can improve the accuracy of theoretical predictions. But this is not the whole story, as will be shown later. In order to observe the effect of modified moments, one again considers powers of logarithms,

$$(k+1)\int_0^{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^k \ln\left(\frac{M_\tau^2}{s}\right)\frac{ds}{M_\tau^2} = \sum_{j=1}^{k+1}\frac{1}{j} = \gamma_E + \psi(k+2), \tag{2.150}$$

$$\begin{aligned}(k+1)\int_0^{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^k \ln^2\left(\frac{M_\tau^2}{s}\right)\frac{ds}{M_\tau^2} &= \left(\sum_{j=1}^{k+1}\frac{1}{j}\right)^2 + \sum_{j=1}^{k+1}\frac{1}{j^2} = \\ &= (\gamma_E + \psi(k+2))^2 + \psi'(k+2)\end{aligned} \tag{2.151}$$

$$\ldots$$

where γ_E is Euler's constant, $\psi(z)$ is the digamma function, and $\psi'(z)$ is the first derivative of the digamma function (cf. Appendix D.5). For large values of k, Eq. (2.150) grows as $\ln(k)$, while Eq. (2.151) grows as $\ln^2(k)$.

2.6.4 The effective scale as a new criterion

The property of convergence of the resulting perturbation series for the moments can be reformulated in the language of effective scales for the moments themselves. This is similar to the approach where moments are reexpressed by moments [47]. Indeed, taking the expansion of Eq. (2.142) up to next-to-leading order,

$$a_M(s) = a_M + \beta_0 L a_M^2 + O(a_M^3) = a_M + \beta_0 a_M^2 \ln(M_\tau^2/s) + O(a_M^3), \tag{2.152}$$

for the s^l-moments (i.e. the moments M_{0l} of Eq. (2.149)) one has

$$m_{0l} = a_M + \frac{\beta_0}{l+1} a_M^2 + O(a_M^3) \tag{2.153}$$

which translates into the effective scale $M_\tau^2 e^{-1/(l+1)}$ through the relation

$$a_M + \frac{\beta_0 a_M^2}{l+1} + O(a_M^3) = a_M(M_\tau^2 e^{-1/(l+1)}) + O(a_M^3). \tag{2.154}$$

On the other hand, for the $(M_\tau^2 - s)^k$-moments given by Eq. (2.149) with large values for k and $l = 0$ the result of integrating Eq. (2.152) leads to

$$m_{k0} = a_M + \beta_0 a_M^2 \ln(k) + O(a_M^3) = a_M(M_\tau^2/k) + O(a_M^3). \tag{2.155}$$

This leads to the effective scale M_τ^2/k for large k. The results of these explicit calculations agree with a qualitative estimate based on the observation that the essential region of integration where integrals for the moments are saturated for reasonably smooth functions $\rho(s)$ is located around

$$s_{\max} = M_\tau^2 \frac{l}{l+k}. \tag{2.156}$$

Obviously, the quantity $a_M(M_\tau^2/k)$ cannot be evaluated in perturbative QCD for large values of k. But even for finite values $k = 2$ and 3 one obtains

$$\begin{aligned} m_{20} &= a_M + \beta_0 a_M^2 \frac{11}{6} + O(a_M^3) = a_M(M_\tau^2 e^{-11/6}) + O(a_M^3) = a_M(0.16 M_\tau^2) + O(a_M^3), \\ m_{30} &= a_M + \beta_0 a_M^2 \frac{25}{12} + O(a_M^3) = a_M(M_\tau^2 e^{-25/12}) + O(a_M^3) = a_M(0.12 M_\tau^2) + O(a_M^3). \end{aligned} \tag{2.157}$$

The effective scale Λ_M for the effective coupling a_M defined in Eq. (2.139) is given by

$$\Lambda_M^2 = \exp(k_1/\beta_0)\Lambda_{\overline{\rm MS}}^2 = 2.07 \Lambda_{\overline{\rm MS}}^2 \approx 0.1\, GeV^2 \tag{2.158}$$

where the value $\Lambda_{\overline{\rm MS}} = 350\, MeV$ [80] as determined from τ decays is used. Keeping this in mind, a value $0.12 \times M_\tau^2 = 0.36\, GeV^2$ for the effective coupling of the moment M_{30} is definitely located in the nonperturbative region for the process under consideration.

2.6.5 Integration of running effects to all orders

According to Cauchy's theorem, the integration along a contour of a closed curve in the complex plane is completely equivalent to the integration along the cut – if the cut can be identified. Applying this general statement to the correlator functions and spectral densities, the condition stated here is no problem at all. For finite-order perturbation theory expressions at least the cut is determined by the analyticity properties of the functions $\ln^p(-M_\tau^2/q^2)$ for positive integers p. This leads to the assumption that it would also work if one would use the renormalization group improved correlator function. Under this assumption the integration along the contour including a full renormalization group resummation has developed to be the most popular technique of accounting for the running of perturbative quantities: it efficiently resums an infinite number of terms generated by the evolution of the coupling constant [81, 67].

However, the analyticity structure of the resummed correlator function is different from the one in finite-order perturbation theory. Additional cuts lead to the inclusion of nonperturbative features which have to be taken special care of [82]. Such a situation will be met in the considerations that follow.

2.6.6 The effective coupling on the contour

As in the case of the analysis on the physical cut, the main concern is to account for the running. Therefore, the introduction of an effective coupling is helpful also in the Euclidean domain. Starting with Adler's function in Eq. (2.43), one introduces again an *effective coupling* $a_E(q^2)$ by

$$D(q^2) = -q^2\frac{d}{dq^2}\Pi(q^2) = 1 + a_E(q^2). \tag{2.159}$$

This equation can be solved for $\Pi(q^2)$ by using the ansatz

$$\Pi(q^2) = \ln\left(\frac{\mu^2}{q^2}\right) + \ln\Big(f(a_E(q^2))\Big). \tag{2.160}$$

Inserting this ansatz, one obtains

$$-q^2\frac{d}{dq^2}\Pi(q^2) = 1 - \frac{1}{f(a_E(q^2))}q^2\frac{d}{dq^2}f\left(a_E(q^2)\right) = 1 - \frac{f'(a_E(q^2))}{f(a_E(q^2))}q^2\frac{da_E(q^2)}{dq^2} \tag{2.161}$$

The effective coupling $a_E(q^2)$ obeys the renormalization group equation (see e.g. [88])

$$q^2\frac{d}{dq^2}a_E(q^2) = \beta(a_E(q^2)). \tag{2.162}$$

Taking this into account, one obtains

$$1 + a_E(q^2) \stackrel{!}{=} 1 - \frac{f'(a_E(q^2))}{f(a_E(q^2))}\beta\left(a_E(q^2)\right) \;\Rightarrow\; f'\left(a_E(q^2)\right)\beta\left(a_E(q^2)\right) = -f\left(a_E(q^2)\right)a_E(q^2). \tag{2.163}$$

This differential equation for f can be solved by separation,

$$\ln f = \int\frac{df}{f} = -\int\frac{a_E da_E}{\beta(a_E)}. \tag{2.164}$$

For $\beta(a) = -\beta_0 a^2$ one obtains

$$\beta(a) = -\beta_0 a^2 \quad \Rightarrow \quad \ln f = \frac{1}{\beta_0} \ln a_E, \tag{2.165}$$

for $\beta(a) = -\beta_0 a^2 - \beta_1 a^3$ one obtains

$$\begin{aligned}
\ln f &= \int \frac{a_E da_E}{\beta_0 a_E^2 + \beta_1 a_E^3} = \int \frac{da_E}{\beta_0 a_E + \beta_1 a_E^2} = \\
&= \frac{1}{\beta_0} \int \frac{da_E}{a_E + c_1 a_E^2} = \frac{1}{\beta_0} (\ln a_E - \ln(1 + c_1 a_E)) = \qquad \left(c_1 = \frac{\beta_1}{\beta_0}\right) \\
&= \frac{1}{\beta_0} \ln\left(\frac{a_E}{1 + c_1 a_E}\right) = \frac{1}{\beta_0} \ln\left(\frac{\beta_0 a_E}{\beta_0 + \beta_1 a_E}\right).
\end{aligned} \tag{2.166}$$

It is obvious that this scheme can be generalized to arbitary finite parts of the beta function series. For the considerations that are aimed for, however, it is enough to use only the leading order in the beta function which contains already the bulk of the whole effect. Effects due to higher order corrections of the beta function are quite small and do not change the basic picture. They only slightly affect the conclusions numerically [49, 50]. Therefore, the resummed correlation function reads

$$\Pi(q^2) = \ln\left(\frac{\mu^2}{q^2}\right) + \frac{1}{\beta_0} \ln(a_E(q^2)) + \text{subtractions} \tag{2.167}$$

where

$$a_E(q^2) = \frac{\alpha_\tau/\pi}{1 + (\beta_0 \alpha_\tau/\pi) \ln(q^2/M_\tau^2)} \tag{2.168}$$

with $a_E(M_\tau^2) = \alpha_\tau/\pi$. Parameterizing the contour by $q^2 = M_\tau^2 e^{i\varphi}$ one obtains

$$\Pi(M_\tau^2 e^{i\varphi}) = -i\varphi - \frac{1}{\beta_0} \ln(1 + i\beta_0 \alpha_\tau \varphi/\pi) + \text{subtractions} \tag{2.169}$$

where appropriate subtractions are added.

2.6.7 The moments on the circle

Now the modified moments in Eq. (2.149) can be analyzed. According to

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{|z|=1} \Pi(M_\tau^2 z) z^l dz &= \frac{-1}{2\pi i} \left(\int_{e^{i\pi}}^{0e^{i\pi}} + \int_{0e^{-i\pi}}^{e^{-i\pi}} \right) \Pi(M_\tau^2 z) z^l dz = \\
&= \frac{-1}{2\pi i} \int_0^1 \Big(\Pi(M_\tau^2 x e^{-i\pi}) - \Pi(M_\tau^2 x e^{-i\pi}) \Big) (-x)^l (-dx) = \\
&= (-1)^l \int_0^1 \rho(M_\tau^2 x) x^k dx = \int_0^{M_\tau^2} \frac{\rho(s) s^l ds}{(M_\tau^2)^{l+1}}
\end{aligned} \tag{2.170}$$

they can be expressed by

$$\begin{aligned}
M_{kl} &= 1 + m_{kl} = \frac{(-1)^l}{2\pi i} \frac{(k+l+1)!}{k!l!} \oint_{|z|=1} \Pi(M_\tau^2 z)(1+z)^k z^l dz = \\
&= \frac{(-1)^l}{2\pi} \frac{(k+l+1)!}{k!l!} \int_{-\pi}^{\pi} \Pi(M_\tau^2 e^{i\varphi})(1 + e^{i\varphi})^k e^{i(l+1)\varphi} d\varphi.
\end{aligned} \tag{2.171}$$

Taking the correlator function as given in Eq. (2.169), the first part leads to the *parton result* $M_{kl} = 1$ while the second, α_τ-dependent part gives the moment m_{kl} where

$$m_{kl} = \frac{(-1)^{l+1}}{2\pi\beta_0}\frac{(k+l+1)!}{k!l!}\int_{-\pi}^{\pi}(1+e^{i\varphi})^k e^{il\varphi}\ln(1+i\beta_0\alpha_\tau\varphi/\pi)e^{i\varphi}d\varphi. \tag{2.172}$$

Note that the estimate of the saturation region for the moments in this form is more complicated because the measure of the integral is a rapidly oscillating function, especially

$$\int_{-\pi}^{\pi}(1+e^{i\varphi})^k e^{il\varphi}e^{i\varphi}d\varphi = 0. \tag{2.173}$$

Additional care is necessary to identify essential regions of integration. In the following subsections the zeroth order moment m_{00} will be considered for reasons of simplicity.

2.6.8 The appearence of nonperturbative contributions

One can treat the moment

$$m_{00} = \frac{1}{2\pi\beta_0}\int_{-\pi}^{\pi}\ln(1+i\beta_0\alpha_\tau\varphi/\pi)e^{i\varphi}d\varphi \tag{2.174}$$

simply by expanding the logarithm into a power series in α_τ, the effective coupling in τ decays. This gives nothing new in comparison with the finite-order perturbation theory case. At least one gets the chance to find out the convergence condition of this series which is the convergence condition of the logarithm series, $\beta_0\alpha_\tau < 1$. With

$$\frac{\alpha_\tau^{\rm exp}}{\pi} = 0.14, \qquad \frac{1}{\pi\beta_0} = \frac{4}{9\pi} = 0.1415\ldots, \tag{2.175}$$

the convergence criterion is fulfilled, though only marginally. But one can also proceed with the analysis of the moments in a different way by constructing just an efficient computational scheme. Integrating n times by parts one obtains

$$\begin{aligned}
m_{00} &= \frac{-1}{2\pi\beta_0}\int_{-\pi}^{\pi}\ln(1+i\beta_0\alpha_\tau\varphi/\pi)e^{i\varphi}d\varphi \;=\\
&= \frac{-1}{2\pi i\beta_0}\left\{\Big[\ln(1+i\beta_0\alpha_\tau\varphi/\pi)e^{i\varphi}\Big]_{-\pi}^{\pi} - \left(\frac{\beta_0\alpha_\tau}{\pi}\right)\int_{-\pi}^{\pi}\frac{ie^{i\varphi}d\varphi}{1+i\beta_0\alpha_\tau\varphi/\pi}\right\} =\\
&= \frac{-1}{2\pi i\beta_0}\left\{\Big[\ln(1+i\beta_0\alpha_\tau\varphi/\pi)e^{i\varphi}\Big]_{-\pi}^{\pi} - \left(\frac{\beta_0\alpha_\tau}{\pi}\right)\left[\frac{e^{i\varphi}}{1+i\beta_0\alpha_\tau\varphi/\pi}\right]_{-\pi}^{\pi} + \right.\\
&\qquad\left. - \left(\frac{\beta_0\alpha_\tau}{\pi}\right)^2\int_{-\pi}^{\pi}\frac{ie^{i\varphi}d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^2}\right\} =\\
&= \frac{-1}{2\pi i\beta_0}\left\{\Big[\ln(1+i\beta_0\alpha_\tau\varphi/\pi)e^{i\varphi}\Big]_{-\pi}^{\pi} - \left(\frac{\beta_0\alpha_\tau}{\pi}\right)\left[\frac{e^{i\varphi}}{1+i\beta_0\alpha_\tau\varphi/\pi}\right]_{-\pi}^{\pi} + \right.\\
&\qquad\left. - \left(\frac{\beta_0\alpha_\tau}{\pi}\right)^2\left[\frac{e^{i\varphi}}{(1+i\beta_0\alpha_\tau\varphi/\pi)^2}\right]_{-\pi}^{\pi} - 2\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^3\int_{-\pi}^{\pi}\frac{ie^{i\varphi}d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^3}\right\} = \ldots\\
&= \frac{-1}{2\pi i\beta_0}\left\{\Big[\ln(1+i\beta_0\alpha_\tau\varphi/\pi)e^{i\varphi}\Big]_{-\pi}^{\pi} - \sum_{j=1}^{n-1}(j-1)!\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^j\left[\frac{e^{i\varphi}}{(1+i\beta_0\alpha_\tau\varphi/\pi)^j}\right]_{-\pi}^{\pi} + \right.\\
&\qquad\left. -(n-1)!\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^n\int_{-\pi}^{\pi}\frac{ie^{i\varphi}d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^n}\right\}.
\end{aligned} \tag{2.176}$$

where the derivatives of $u = \ln(1 + i\beta_0\alpha_\tau\varphi/\pi)$ and $v = e^{i\varphi}$ have been used as a chain to accomblish this integration by parts. Using the polar coordinate functions r and ϕ (note the difference to φ!) defined by

$$1 \pm i\beta_0\alpha_\tau = re^{\pm i\phi}, \qquad r = \sqrt{1+\beta_0^2\alpha_\tau^2}, \qquad \phi = \arctan(\beta_0\alpha_\tau) \tag{2.177}$$

one obtains

$$\begin{aligned} \Big[\ln(1+i\beta_0\alpha_\tau\varphi/\pi)e^{i\varphi}\Big]_{-\pi}^{\pi} &= -(\ln(re^{i\phi} - \ln(re^{-i\phi})) \;=\; -2i\phi, \qquad (2.178)\\ \left[\frac{e^{i\varphi}}{(1+i\beta_0\alpha_\tau\varphi/\pi)^j}\right]_{-\pi}^{\pi} &= -\left(\frac{1}{r^je^{ij\phi}} - \frac{1}{r^je^{-ij\phi}}\right) \;=\; \frac{-1}{r^j}(-2i\sin(j\phi)). \end{aligned}$$

Therefore finally, after n-fold integration by part one has

$$\begin{aligned} m_{00} &= \frac{-1}{2\pi i\beta_0}\bigg\{-2i\phi + \sum_{j=1}^{n-1}(j-1)!\left(\frac{\beta_0\alpha_\tau}{\pi r}\right)^j(-2i\sin(j\phi)) + \\ &\qquad -(n-1)!\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^n\int_{-\pi}^{\pi}\frac{ie^{i\varphi}d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^n}\bigg\} = \qquad (2.179)\\ &= \frac{1}{\pi\beta_0}\bigg\{\phi + \sum_{j=1}^{n-1}(j-1)!\left(\frac{\beta_0\alpha_\tau}{\pi r}\right)^j\sin(j\phi) + \frac{(n-1)!}{2}\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^n\int_{-\pi}^{\pi}\frac{e^{i\varphi}d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^n}\bigg\}. \end{aligned}$$

The n-fold integration by parts removes a polynomial of order n from the expansion of the logarithm in Eq. (2.169). This result is an asymptotic expansion, the last term in Eq. (2.179) is the residual term of formal order α_τ^n. But this result is not an ordinary series expansion but an asymptotic expansion valid in the sense of Poincaré. The meaning of this becomes obvious when the residual term is replaced by using an identity valid for any n,

$$\begin{aligned} &(n-1)!\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^n\int_{-\pi}^{\pi}\frac{e^{i\varphi}d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^n} = \\ &= 2\pi e^{-\pi/\beta_0\alpha_\tau} - (n-1)!\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^n\left(\int_{-\infty}^{-\pi} + \int_{\pi}^{\infty}\right)\frac{e^{i\varphi}d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^n} \end{aligned} \tag{2.180}$$

Then the zeroth order moment reads

$$\begin{aligned} m_{00} &= \frac{1}{\pi\beta_0}\bigg\{\pi e^{-\pi/\beta_0\alpha_\tau} + \phi + \sum_{j=1}^{n-1}(j-1)!\left(\frac{\beta_0\alpha_\tau}{\pi r}\right)^j\sin(j\phi) \\ &\qquad -\frac{(n-1)!}{2}\left(\frac{\beta_0\alpha_\tau}{\pi}\right)^n\left(\int_{-\infty}^{-\pi} + \int_{\pi}^{\infty}\right)\frac{e^{i\varphi}d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^n}\bigg\} \end{aligned} \tag{2.181}$$

which is a valid change because the new residual term is again well-defined. But in doing this change, an explicit nonperturbative term $e^{-\pi/\beta_0\alpha_\tau}$ appeared. Eq. (2.181) and Eq. (2.179) are formally different but actually identical. Therefore, the choice for the expansion (or representation) for the moment is a question of calculating the residual term. Any conclusions about the precision or the analytic structure of the sum of the series based on the terms of the series only without a specification of the residual term are rather useless. This residual term will be calculated in the following.

2.6.9 A new spectral density

Using $u = \ln(1 + i\beta_0\alpha_\tau\varphi/\pi)$ and $v = 1 + e^{i\varphi}$ to build up a one-fold integration by parts (the boundary term vanishes because of v) one obtains an equivalent representation for the zeroth order moments,

$$m_{00} = \frac{\alpha_\tau}{2\pi^2}\int_{-\pi}^{\pi}\frac{(1+e^{i\varphi})d\varphi}{1+i\beta_0\alpha_\tau\varphi/\pi}. \qquad (2.182)$$

One now can extend the integration range from the interval $[-\pi, \pi]$ on the real axis to a rectangular contour in the complex φ-plane (see Fig. 2.5). The vertical length R tends to infinity. Using the residue theorem one obtains

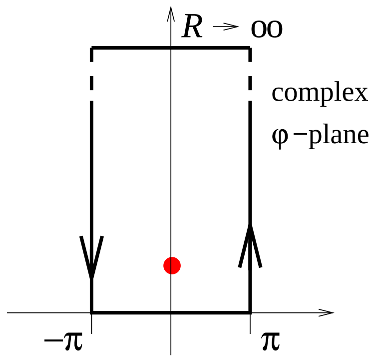


Figure 2.5: the integration contour in the complex φ-plane

$$\begin{aligned} m_{00} &= \frac{\alpha_\tau}{2\pi^2}\int_{-\pi}^{\pi}\frac{(1+e^{i\varphi})d\varphi}{1+i\beta_0\alpha_\tau\varphi/\pi} = \frac{\alpha_\tau}{2\pi^2}2\pi i\,\mathrm{Res}\left[\frac{1+e^{i\varphi}}{1+i\beta_0\alpha_\tau\varphi/\pi};\varphi=\frac{i\pi}{\beta_0\alpha_\tau}\right]+ \\ &\quad -\frac{\alpha_\tau}{2\pi^2}\left(\int_{\pi+i0}^{\pi+i\infty}+\int_{\pi+i\infty}^{-\pi+i\infty}+\int_{-\pi+i\infty}^{-\pi+i0}\right)\frac{(1+e^{i\varphi})d\varphi}{1+i\beta_0\alpha_\tau\varphi/\pi} = \\ &= \frac{i\alpha_\tau}{\pi}\frac{1+e^{i\varphi}}{i\beta_0\alpha_\tau/\pi}\bigg|_{\varphi=i\pi/\beta_0\alpha_\tau} + \\ &\quad -\frac{\alpha_\tau}{2\pi^2}\int_0^\infty\frac{(1+e^{i\pi-\xi})id\xi}{1+i\beta_0\alpha_\tau-\beta_0\alpha_\tau\xi/\pi} + \qquad (\varphi=\pi+i\xi) \\ &\quad +\frac{\alpha_\tau}{2\pi^2}\int_0^\infty\frac{(1+e^{-i\pi-\xi})id\xi}{1-i\beta_0\alpha_\tau-\beta_0\alpha_\tau\xi/\pi} + \qquad (\varphi=-\pi+i\xi) \\ &\quad +\frac{\alpha_\tau}{2\pi^2}\lim_{R\to\infty}\int_{-\pi}^{\pi}\frac{(1+e^{i\tilde\varphi-R})d\tilde\varphi}{1+i\beta_0\alpha_\tau\tilde\varphi/\pi-\beta_0\alpha_\tau R/\pi}. \qquad (\varphi=\tilde\varphi+iR) \end{aligned} \qquad (2.183)$$

The last part vanishes because the integrand vanishes for $R \to \infty$ while the integration range is finite. Therefore, one obtains

$$\begin{aligned} m_{00} &= \frac{1}{\beta_0}(1+e^{-\pi/\beta_0\alpha_\tau}) + \\ &\quad -\frac{i\alpha_\tau}{2\pi^2}\int_0^\infty\left(\frac{(1-e^{-\xi})d\xi}{1-\beta_0\alpha_\tau\xi/\pi+i\beta_0\alpha_\tau}-\frac{(1-e^{-\xi})d\xi}{1-\beta_0\alpha_\tau\xi/\pi-i\beta_0\alpha_\tau}\right) = \\ &= \frac{1}{\beta_0}(1+e^{-\pi/\beta_0\alpha_\tau}) - \frac{\beta_0\alpha_\tau^2}{\pi^2}\int_0^\infty\frac{(1-e^{-\xi})d\xi}{(1-\beta_0\alpha_\tau\xi/\pi)^2+\beta_0^2\alpha_\tau^2} = \\ &= \frac{1}{\beta_0}(1+e^{-\pi/\beta_0\alpha_\tau}) - \frac{1}{\beta_0}\int_0^\infty\frac{(1-e^{-\xi})d\xi}{\pi^2+(\xi-\pi/\beta_0\alpha_\tau)^2}. \end{aligned} \qquad (2.184)$$

One can replace $-\pi/\beta_0\alpha_\tau = \ln(\Lambda^2/M_\tau^2)$ and substitute $-\xi = \ln(s/M_\tau^2)$, so $e^{-\xi} = s/M_\tau^2$ and $-d\xi = ds/s$, to obtain

$$m_{00} = \frac{1}{\beta_0}\left(1+\frac{\Lambda^2}{M_\tau^2}\right) - \frac{1}{\beta_0}\int_0^{M_\tau^2}\frac{(1-s/M_\tau^2)ds}{(\pi^2+\ln^2(s/\Lambda^2))s}. \qquad (2.185)$$

A single integration by parts can be performed, using the elements

$$u = 1 - x \quad \Rightarrow \quad u' = -1, \qquad v = \frac{1}{\pi}\arctan\left(\frac{\ln(x/a)}{\pi}\right) \quad \Rightarrow \quad v' = \frac{1}{(\pi^2 + \ln^2(x/a))x}, \tag{2.186}$$

with $x = s/M_\tau^2$ and $a = \Lambda^2/M_\tau^2$ to obtain

$$\begin{aligned}
\beta_0 m_{00} &= 1 + a - \int_0^1 \frac{(1-x)dx}{(\pi^2 + \ln^2(x/a))x} = \\
&= 1 + a + \frac{1}{\pi}\left[(1-x)\arctan\left(\frac{\ln(x/a)}{\pi}\right)\right]_0^1 + \frac{1}{\pi}\int_0^1 \arctan\left(\frac{\ln(x/a)}{\pi}\right)dx = \\
&= 1 + a - \frac{1}{2} + \frac{1}{\pi}\int_0^1 \arctan\left(\frac{\ln(x/a)}{\pi}\right)dx = \\
&= 1 + a - \frac{1}{2} - \frac{1}{2}\int_0^1 dx - \frac{1}{\pi}\int_0^1 \arctan\left(\frac{\pi}{\ln(x/a)}\right)dx = \\
&= \frac{\Lambda^2}{M_\tau^2} - \frac{1}{\pi}\int_0^{M_\tau^2} \arctan\left(\frac{\pi}{\ln(s/\Lambda^2)}\right)\frac{ds}{M_\tau^2}
\end{aligned} \tag{2.187}$$

where for the last steps the identity

$$y = \tan z = \frac{1}{\cot z} = \frac{-1}{\tan(z + \pi/2)} \quad \Leftrightarrow \quad z = \arctan y = -\frac{\pi}{2} - \arctan\left(\frac{1}{y}\right) \tag{2.188}$$

is employed. Finally,

$$\arccos\left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}}\right) = \arcsin\left(\frac{\pi}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}}\right) = \arctan\left(\frac{\pi}{\ln(s/\Lambda^2)}\right) \tag{2.189}$$

leading to

$$m_{00} = \frac{1}{\beta_0}\left(\frac{\Lambda^2}{M_\tau^2}\right) + \frac{1}{\pi\beta_0}\int_0^{M_\tau^2} \arccos\left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln(s/\Lambda^2)}}\right)\frac{ds}{M_\tau^2} \tag{2.190}$$

which stays on the same Riemann sheet for the range of values that is of interest here. One easily recognizes this representation as an integration over the singularities of $\Pi(q^2)$ in Eq. (2.167). In addition to a cut along the positive semi-axis there appears also a part of the singularity on the negative real s-axis. This part is a pure mathematical feature of the concrete approximation chosen for $\Pi(q^2)$ and is not related to the physical content of the problem. Indeed, if moments are written as explicit functions one can calculate them in the way found most convenient for a concrete application. The result reads

$$m_{00} = \int_{-\Lambda^2}^{M_\tau^2} \frac{\sigma(s)ds}{M_\tau^2} \tag{2.191}$$

with

$$\sigma(s) = \frac{1}{\beta_0}\theta(\Lambda^2 + s)\theta(-s) + \frac{1}{\pi\beta_0}\theta(s)\arccos\left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}}\right). \tag{2.192}$$

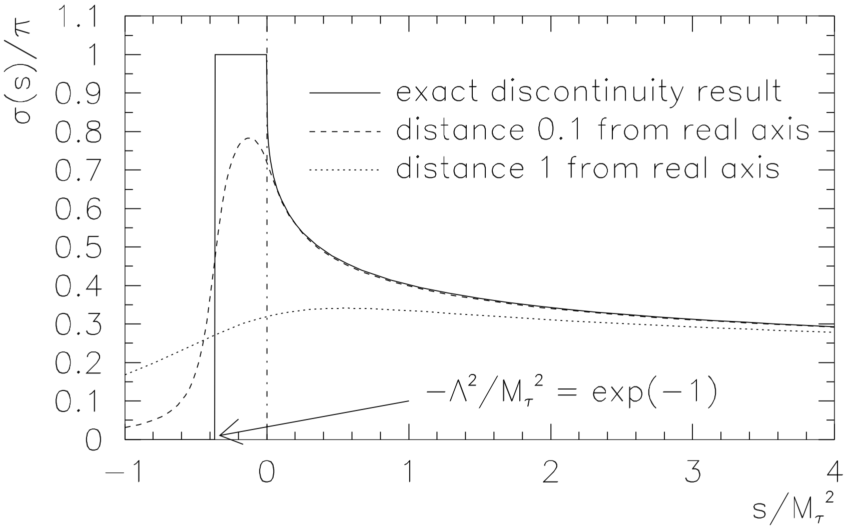


Figure 2.6: The spectral function $\sigma(s)$ as a function of s. Shown is the exact result for the discontinuity (full line) as well as results for non-closed circle integrals approaching the axis along a full circle from both sides, see Fig. 2.2. The numerical values for the distances orthogonal to the real axis towards the positive and negative imaginary semi-plane in s are given in units of M_τ^2.

This formal result can be reformulated as integration over the spectrum $\sigma(s)$ using Cauchy's theorem. Indeed, from Eq. (2.167) one readily deduces the singularity of the resummed polarization function with the discontinuity

$$\mathrm{Disc}\,\Pi(s) = \frac{2\pi i}{\beta_0}\left\{\theta(\Lambda^2+s)\theta(-s) + \frac{1}{\pi}\theta(s)\arccos\left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2+\ln^2(s/\Lambda^2)}}\right)\right\} \tag{2.193}$$

which coincides with $\sigma(s)$ in Eq. (2.192) and, thereby, simply represents the spectrum.

2.6.10 A comment on the universality of the IR fixed point

The spectrum is displayed in Fig. 2.6. The part of the spectral density along the negative real axis is the afore mentioned nonperturbative contribution coming from the resummation procedure. Using appropriate branches of the cut for the relevant functions, the continuum part of the spectral density can be rewritten as

$$\sigma_c(s) = \frac{1}{\pi\beta_0}\arctan(\beta_0\alpha(s)), \qquad \alpha(s) = \frac{\pi}{\beta_0\ln(s/\Lambda^2)} \tag{2.194}$$

By differentiation one can construct a differential equation for the continuum part,

$$s\frac{d}{ds}\sigma_c(s) = -\frac{1}{\pi^2\beta_0}\sin^2(\pi\beta_0\sigma_c(s)) \quad \text{for } s>0. \tag{2.195}$$

This equation can be considered as an evolution equation for the spectral density $\sigma_c(s)$ determining $\sigma_c(s)$ through its initial value $\sigma(M_\tau^2)$ which in turn can be understood as an *effective coupling*. Because of the initial value taken to be $\sigma(M_\tau^2)$, one now defines the

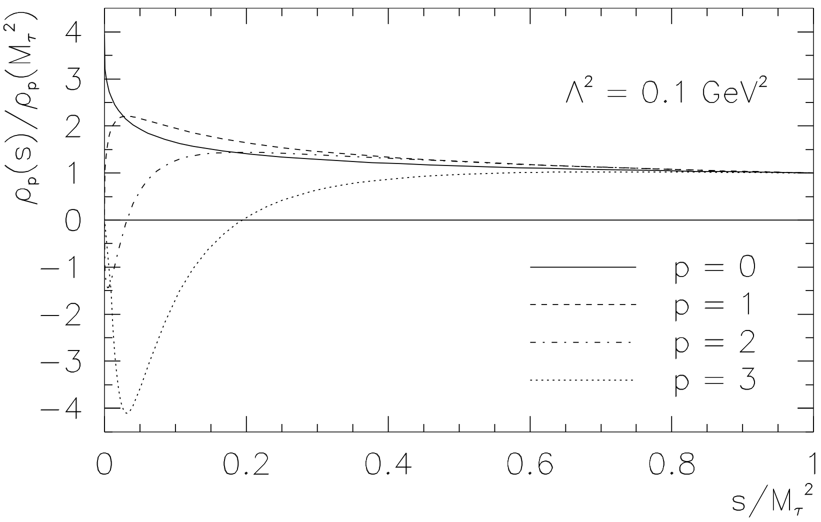


Figure 2.7: The spectral function $\rho_p(s)$ as a function of s, given by Eq. (2.198). All spectral functions for $p > 0$ are going to zero for $s \to 0$, for $p \geq 2$ they have nontrivial zeros and a fluctuating behaviour near the origin.

coupling as the value of the spectral density $\sigma_c(s)$ on the cut far from the IR region. This is a perturbation theory definition. The evolution of this coupling, however, is calculated by taking into account the analytic continuation. Then it has an IR fixed point with the coupling value

$$\sigma_c(0) = 1/\beta_0. \tag{2.196}$$

This IR fixed point and the behaviour of the spectral density as solution of the initial value problem, however, is not universal. If Adler's function starts with a proper power of the coupling constant as it is the case for gluonic observables, for instance, this picture will change. For

$$D_p(Q^2) = \left(\frac{\alpha_E(Q^2)}{\pi}\right)^{p+1} \tag{2.197}$$

one obtains

$$\rho_p(s) = \frac{1}{2\pi i}\left(\Pi_p(se^{-i\pi}) - \Pi_p(se^{i\pi})\right) = \frac{1}{\pi\beta_0}\,\frac{\sin(p\phi(s))}{pr^p(s)} \tag{2.198}$$

where

$$r(s) = \beta_0\sqrt{\ln^2(s/\Lambda^2) + \pi^2}, \qquad \tan\phi(s) = \frac{\pi}{\ln(s/\Lambda^2)} = \beta_0\alpha(s). \tag{2.199}$$

For $p = 0$ one retains Eq. (2.194),

$$\rho_0(s) = \frac{1}{\pi\beta_0}\phi(s) = \frac{1}{\pi\beta_0}\arctan(\beta_0\alpha(s)). \tag{2.200}$$

But for large values of p the function $\rho_p(s)$ starts to fluctuate when starting at $\rho(M_\tau^2)$ and approaching $s = 0$ where it finally reaches zero. Because of this behaviour, shown in Fig. 2.7 in some detail for the first four values of p, $\rho_p(s)$ cannot be interpreted as a (positive) spectral density in a region which enlarges with increasing values of p.

For the special case $p = 1$ a more explicit calculation can be performed. Starting from $D_1(Q^2) = (\alpha_s(Q^2)/\pi)^2$ one obtains

$$\begin{aligned}
\rho_1(s) &= \frac{1}{2\pi i\beta_0}\left(\frac{1}{\beta_0\ln(se^{-i\pi}/\Lambda^2)} - \frac{1}{\beta_0\ln(se^{i\pi}/\Lambda^2)}\right) = \\
&= \frac{1}{2\pi i\beta_0^2}\,\frac{\ln(s/\Lambda^2)+i\pi-\ln(s/\Lambda^2)+i\pi}{\ln^2(s/\Lambda^2)+\pi^2} = \frac{1}{\beta_0^2}\,\frac{1}{\ln^2(s/\Lambda^2)+\pi^2} = \\
&= \frac{1}{\beta_0^2}\,\frac{1}{\pi^2/\beta_0^2\alpha_s^2(s)+\pi^2} = \frac{\alpha_s^2(s)}{\pi^2(1-\beta_0^2\alpha_s^2(s))}.
\end{aligned} \tag{2.201}$$

Therefore, one defines a new effective coupling

$$\bar{\sigma}_c(s) = \frac{\alpha_s(s)}{\pi\sqrt{1+\beta_0^2\alpha_s^2(s)}}. \tag{2.202}$$

The beta function for this effective coupling $\bar{\sigma}_c(s)$ is calculated by taking the derivative of

$$\frac{1}{\bar{a}^2(s)} = \frac{1}{a^2(s)} + \pi^2\beta_0^2, \qquad a(s) = \frac{\alpha_s(s)}{\pi}. \tag{2.203}$$

One obtains

$$\frac{-2}{\bar{\sigma}_c^3(s)}\bar{\beta}_M\left(\bar{\sigma}_c(s)\right) = \frac{-2}{\bar{\sigma}_c^3(s)}s\frac{d\bar{\sigma}_c(s)}{ds} = \frac{-2}{a^3(s)}s\frac{da(s)}{ds} = \frac{-2}{a^3(s)}\beta\left(a(s)\right) = \frac{2\beta_0}{a(s)}, \tag{2.204}$$

so that

$$\bar{\beta}_M\left(\bar{\sigma}_c(s)\right) = -\frac{\bar{\sigma}_c^3(s)}{a(s)}\beta_0 = -\beta_0\bar{\sigma}_c^2(s)\sqrt{1-\pi^2\beta_0^2\bar{\sigma}_c^2(s)}. \tag{2.205}$$

In contrast to the case $p = 0$, the spectral density runs to zero for $s \to 0$. The infrared fixed point of $\bar{\sigma}_c(s)$ is therefore not given by $1/\beta_0$ but by $\bar{\sigma}_c(0) = 0$ while the comparison of the beta functions and their series expansions result in

$$\begin{aligned}
\beta_M(\sigma_c) &= -\frac{1}{\pi^2\beta_0}\sin^2(\pi\beta_0\sigma_c) = -\beta_0\sigma_c^2\left(1-\frac{1}{3}(\pi\beta_0\sigma_c)^2+\ldots\right), \\
\bar{\beta}_M(\bar{\sigma}_c) &= -\beta_0\bar{\sigma}_c^2\sqrt{1-\pi^2\beta_0^2\bar{\sigma}_c^2} = -\beta_0\bar{\sigma}_c^2\left(1-\frac{1}{2}(\pi\beta_0\bar{\sigma}_c)^2+\ldots\right).
\end{aligned} \tag{2.206}$$

2.6.11 Moments on the complex path

Especially the considerations of the last subsections show that the spectral density "feels" the IR region. This IR region is located around the origin but might have different (and sometimes unpredictable) forms, according to the kind of resummation. The zeros of the beta function, for instance, are responsible for the location of these poles. But because the beta function is only known up to finite order, nothing can be said about the exact location of IR singularities and cuts. On the other hand, from the point of view of closed integrals in the complex plane, the location of singularities does not matter for the result, as long as the critical region is not touched by the path (see Fig. 2.8). Because of the considerations done before, one indicates the critical region as the interiour of a circle of

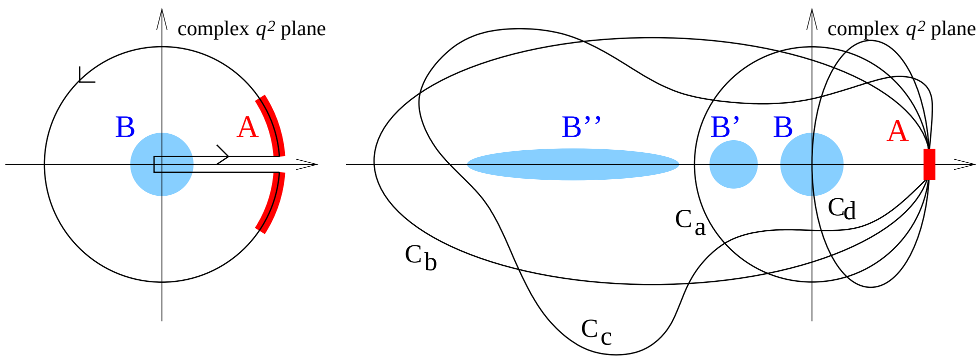


Figure 2.8: Contours in the complex plane with a fixation in the region A, taking into account possible occurrences of singularities. The figure part on the left hand side shows the standard circle path which circumvents the singular region B while the occurence of other singularity regions as discussed in the text (regions B′ and B″) may lead to different possibilities for choosing a path (C_a, C_b, C_c). The path C_d crosses the singularity region and, therefore, cannot be used from the perturbation theory point of view.

radius Λ^2 about the origin, and the moments are again an indicator for how much this quantitatively not penetrable region contributes to the result. The moments are given by

$$\begin{aligned} m_{kl} &= \frac{(k+l+1)!}{k!l!}\int_{-\Lambda^2}^{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^k\left(\frac{s}{M_\tau^2}\right)^l\frac{\sigma(s)ds}{M_\tau^2} = \\ . &= \frac{(k+l+1)!}{k!l!}\int_{-\Lambda^2/M_\tau^2}^{\Lambda^2/M_\tau^2}(1-x)^k x^l\sigma(M_\tau^2 x)dx + \\ &\quad +\frac{(k+l+1)!}{k!l!}\int_{\Lambda^2/M_\tau^2}^{1}(1-x)^k x^l\sigma_c(M_\tau^2 x)dx. \end{aligned} \tag{2.207}$$

The question to pose is how much the non-perturbative first part contributes to the second part. In Table 2.1 values for the contribution of the interval range $[-\Lambda^2,\Lambda^2]$ to the contribution for the whole range $[-\Lambda^2, M_\tau^2]$ are given for different values of k and l, taking $\Lambda^2 = 0.2\,GeV^2$. A strong enhancement can be observed especially for the case $l = 0$ and large values for k. One observes that the first term is larger than the second one already for $k > 3$. This observation confirms the statement that the large k moments are sensitive to the badly known parameter Λ. A more detailed analysis is found in Ref. [51].

$l=3$	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.002	-0.003
$l=2$	$+0.000$	$+0.002$	$+0.004$	$+0.008$	$+0.013$	$+0.020$	$+0.030$	$+0.042$
$l=1$	-0.003	-0.009	-0.018	-0.032	-0.052	-0.078	-0.113	-0.160
$l=0$	$+0.186$	$+0.305$	$+0.396$	$+0.469$	$+0.530$	$+0.582$	$+0.626$	$+0.665$
	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$

Table 2.1: Contribution of the integral taken over the interval from $-\Lambda^2$ to $+\Lambda^2$ for the moments m_{kl} in Eq. (2.207) relative to the integral over to the whole integration range.

Chapter 3

Diagrams of the sunrise type

Two-point correlators, as they were introduced in the previous chapter, are the main topic also for the chapter that follows. Next-to-leading order corrections for the baryonic correlator function can be calculated analytically for one finite and two vanishing quark masses. However, for a special type of Feynman diagram topology, two-point functions can be treated in a more general way. In these cases it is possible to calculate Feynman diagrams for different masses, an arbitrary number of internal lines, and an unspecified space-time dimension. These are two-point functions diagrams where the two outer vertices are connected by an arbitrary number of lines. The basic form, namely the case of three lines, is known in the literature as *sunrise diagram*. The diagrams with an arbitrary number of internal lines shall be called *water melon diagrams* in the following.[1]

Of course, Feynman diagrams with an arbitrary number of legs meeting at a vertex are only possible if the concrete form of the field theory is abandoned for the moment – the electroweak theory, for instance, would only allow the connection between two fermion lines and one boson line, the QCD as field theory for the strong interaction also allows for the connection of three or four boson lines, and finally for effective theories there are still other constellations available. In this chapter the specification of the field theory should therefore be postponed. This means that scalar propagators are used as Feynman rule elements for all kinds of propagators, characterized only by the four-momentum of the corresponding particle and its (potentially vanishing) mass. In this simplified approach, vertices have only kinematic meaning as the momentum conservation have to be guaranteed at these points, they do not contribute anything (resp. only a factor 1) to the Feynman diagrams. In this simplified theory it is easy, therefore, to look at the momenta not in *Minkowskian domain* but in *Euclidean domain*, as it was done in the previous chapter. This simplifies the calculations and representations significantly.

The special issue of the considerations presented in this chapter is that the calculations will be done in *configuration space* and not in *momentum space*. By Fourier transforms, convolutions are changed to simple multiplications. Because the transition from momentum space to configuration space is done by a Fourier tranform, easier methods in calculating within configuration space can therefore be expected. Whereas conventionally for n internal lines a $(n-1)$-fold d-dimensional integration is needed, in this case only a single one-dimensional integration has to be performed.

[1]In the literature there are also other names such as *banana diagrams* [83] or *basketball diagrams* [84]. An alternative name for sunrise diagrams is *sunset diagrams*, depending on the mood of the author.

3.1 Basics for the configuration space calculation

In this section the tools shall be supplied which are necessary for a calculation of Feynman diagrams within configuration space. These are mainly the propagators and the two-point functions, the correlator functions of composite fields to be constructed from these.

3.1.1 Propagators and correlators in configuration space

Starting point are current operators which generally can be described as fields composed of elementary parts (e.g. quark and gluon fields in case of the QCD). But there could also be found derivatives of these elementary fields as in chiral perturbation theory. For the current at the space-time point x the monomial

$$j_n(x) = \mathcal{D}_{\{\mu\}_1}\phi_1(x,m_1)\cdots\mathcal{D}_{\{\mu\}_n}\phi_n(x,m_n) \tag{3.1}$$

can be used. Here $\{\mu\} = \{\mu_1,\ldots,\mu_k\}$ is a multi-index, so

$$\mathcal{D}_{\{\mu\}} = \frac{\partial^k}{\partial x_{\mu_1}\cdots\partial x_{\mu_k}}. \tag{3.2}$$

Water melon diagrams are contained in the leading order contribution of the correlator

$$\Pi(x) = \langle Tj_n(x)J_{n'}(0)\rangle \tag{3.3}$$

which in configuration space takes the simple form

$$\Pi(x) = \mathcal{D}_{\{\mu\}_1\{\nu\}_1}(x,m_1)\cdots\mathcal{D}_{\{\mu\}_n\{\nu\}_n}(x,m_n). \tag{3.4}$$

Here $\mathcal{D}_{\{\mu\}\{\nu\}}(x,m) = \mathcal{D}_{\{\mu\}}\mathcal{D}_{\{\nu\}}D(x,m)$ is the derivative of the propagator $D(x,m)$ with respect to the coordinate x with a pair $\{\{\mu\},\{\nu\}\}$ of multi-indices. But how does the propagator $D(x,m)$ itself looks like? Here the building blocks start to get concise form. The propagator of a particle with mass m in D-dimensional space-time is given by the Fourier transform of the (Euclidean) scalar propagator in momentum space,

$$D(x,m) = \int\frac{d^Dp}{(2\pi)^D}\frac{e^{ip_\mu x^\mu}}{p^2+m^2} = \frac{(mx)^\lambda K_\lambda(mx)}{(2\pi)^{\lambda+1}x^{2\lambda}}. \tag{3.5}$$

This expression depends only on $|x| = \sqrt{x^2} = \sqrt{x_\mu x^\mu}$ which is denoted by x. Furthermore, practical considerations lead one to express the space-time dimension as $D = 2\lambda+2$ by a parameter λ. $K_\lambda(z)$ is the *McDonald function*, a modified Bessel function of the third kind, cf. Ref. [85]. In Appendix D.1 the Bessel functions which are a central quantity in the calculations are looked at more closely.

In the limit $m\to 0$ the propagator in Eq. (3.5) simplifies to

$$D(x,0) = \int\frac{d^Dp}{(2\pi)^D}\frac{e^{ip_\mu x^\mu}}{p^2} = \frac{\Gamma(\lambda)}{4\pi^{\lambda+1}x^{2\lambda}} \tag{3.6}$$

where $\Gamma(\lambda)$ is Euler's gamma function.

3.1.2 Options for the configuration space representation

The configuration space representation as presented here admits a few options. Propagators for particles with non-vanishing spin, for instance, can be obtained by calculating derivatives at the space-time point x. The calculation of the derivative does not change the functional structure and leads only to minimal modifications of the basic technique. For example, the propagator of the fermion with spin 1/2 is given by

$$S(x,m)=\left(i\gamma^\mu\frac{\partial}{\partial x^\mu}+m\right)D(x,m), \tag{3.7}$$

where γ^μ is a matrix of the Clifford algebra of Dirac matrices. A further generalization is given by taking into account also soft particle radiations from internal lines. The change for an internal line with mass m and four-momentum p for a particle with momentum $q\to 0$ to be emitted is given by

$$\begin{aligned} D(p^2,m) &= \frac{1}{p^2+m^2} \to \frac{1}{p^2+m^2}V(p,q)\frac{1}{(p-q)^2+m^2} = \\ &\overset{q=0}{=} \frac{1}{p^2+m^2}V(p,0)\frac{1}{p^2+m^2} = -V(p,0)\frac{d}{dm^2}\left(\frac{1}{p^2+m^2}\right). \end{aligned} \tag{3.8}$$

The derivative with respect to the mass can be calculated either for the result without radiation at the very end or already for the propagator itself before the integration has taken place. The last choice, however, only leads to a change for the index of the McDonald function, the general formula reads

$$D^{(\mu)}(x,m)=\int\frac{d^Dp}{(2\pi)^D}\frac{e^{ip_\mu x^\mu}}{(p^2+m^2)^{\mu+1}}=\frac{1}{(2\pi)^{\lambda+1}2^\mu\Gamma(\mu+1)}\left(\frac{m}{x}\right)^{\lambda-\mu}K_{\lambda-\mu}(mx). \tag{3.9}$$

Also in this case only marginal modifications have to be applied in order to cover these cases. In the following, therefore, only the basic form should be considered, all other forms can be obtained by corresponding modifications. In addition, the generality of the representation up to now should be limited to the actual forms of interest for this thesis. First of all, it should be mentioned again that the propagator $D(x,m)$ depends only on the four-absolute value $x=\sqrt{x_\mu x^\mu}$. Finally, in Eq. (3.3) only the case $n'=n$ shall be considered, omitting the tadpole situations which do not contain any interesting aspect for the calculations presented here. Tadpole diagrams in this respect are connections between legs ending at the same vertex. Because of their form one can call them *leaves of the water melon*. They will be left out.

At large, Eq. (3.4) together with the explicit representation for the propagators in Eq. (3.5) already constitutes the final result for the correlator function in configuration space. However, for practical application the momentum space representation turns out to be more useful. This representation will be derived in the following subsection.

3.1.3 Translation back into momentum space

The translation of the configuration space result back to momentum space is done by the *inverse Fourier transform*,

$$\tilde\Pi(p^2)=\int\Pi(x)e^{ip_\mu x^\mu}d^Dx=\int\langle Tj_n(x)j_{n'}(0)\rangle d^Dx. \tag{3.10}$$

A few words about the history of the subject are in order here. Before calculations in momentum space came into being with the development of recursions relying on the integration by parts technique, the configuration space technique was already used. It proved to be successful for massless diagrams with rather general topology [61] and indicated a real breakthrough in this field. Also the case of massive diagrams was considered in configuration space representation [86]. But it turned out that the configuration space representation was not of much help in this case because the angular dependence did not decouple and thus no essential simplifications could be expected.

However, for the special topology of water melon diagrams the angular dependence decouples completely. The angular integration in Eq. (3.10) can be performed in D-dimensional space-time and gives

$$\int e^{ip_\mu x^\mu} d^D\hat{x} = 2\pi^{\lambda+1}\left(\frac{px}{2}\right)^{-\lambda} J_\lambda(px), \tag{3.11}$$

again with $x=\sqrt{x_\mu x^\mu}$ and $p=\sqrt{p_\mu p^\mu}$. $J_\lambda(z)$ is the *Bessel function of the first kind* (cf. Appendix D.1.1), and $d^D\hat{x}$ is the rotation invariant measure of the unit sphere in D-dimensional (Euclidean) space-time. The generalization of Eq. (3.11) to more complicated integrands is simple. After taking an angular average it leads to different orders of the Bessel function. These emerge through the expansion of plane waves $\exp(ip_\mu x^\mu)$ in a series of functions orthogonal on the D-dimensional unit sphere, the *Gegenbauer polynomials* $C_l^\lambda(z)$ (cf. Appendix D.4),

$$\exp(ip_\mu x^\mu) = \Gamma(\lambda)\left(\frac{px}{2}\right)^{-\lambda}\sum_{l=0}^{\infty} i^l(\lambda+l)J_{\lambda+l}(px)C_l^\lambda\left(\frac{p_\mu x^\mu}{px}\right). \tag{3.12}$$

Integration techniques including the Gegenbauer polynomials are described in some detail in Ref. [61] where also many useful relations between the polynomials are found. These are in part collected in Appendix D.4 (see also Ref. [87]). Again a restriction to the standard case is possible which is given by the one-dimensional integral

$$\tilde{\Pi}(p^2) = 2\pi^{\lambda+1}\int_0^\infty \left(\frac{px}{2}\right)^{-\lambda} J_\lambda(px)D(x,m_1)\cdots D(x,m_n)x^{2\lambda+1}dx. \tag{3.13}$$

This integral with a Bessel function kernel describes the *Hankel transform*, a special kind of integral transform.

3.1.4 Analyticity of the correlator function

Eq. (3.13) already leads to informations about the analyticity of the correlator function $\tilde{\Pi}(p^2)$. In order to obtain this information, the asymptotic behaviour of the relevant Bessel functions have to be examined. At this point, however, a rough estimate suffices. Therefore, according to Appendix D.1.4,

$$J_\lambda(z) \to \sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\pi}{2}\lambda-\frac{\pi}{4}\right), \qquad K_\lambda(z)\to\sqrt{\frac{\pi}{2z}}e^{-z}. \tag{3.14}$$

Only the exponential part is of interest here because it determines the suppression or enhancement of the integrand at large values of z. Each of the factors $D(x,m_i)$, being proportional to a McDonald function $K_\lambda(mx)$, contributes an exponential factor $e^{-m_i x}$. The entirety of all factors, therefore, results in a factor e^{-Mx} with $M = \sum_{i=1}^n m_i$. On the other hand, because of the cosine function the Bessel function $J_\lambda(px)$ has two complex exponential parts, e^{ipx} and e^{-ipx}. Therefore, the integrand is bounded only in the case where the real part of the total exponent is less than zero, so the requirement is given by

$$-\operatorname{Im} p - M < 0, \qquad \operatorname{Im} p - M < 0 \qquad \Rightarrow \qquad |\operatorname{Im} p\,| < M = \sum_{i=1}^n m_i. \tag{3.15}$$

In the strip of the complex plane parallel to the real axis given by this condition, the correlator function is analytic. Expressed by the relativistic invariant quantity p^2, this means that the correlator function is analytic for $\operatorname{Re}(p^2) > -M^2$ whereas, starting from energies $E = M$ in the Minkowskian domain (found at $\operatorname{Re}(p^2) < 0$), it becomes singular. Depending on the number of internal lines of the diagram, this singularity is either a single pole (for the most degenerate case of a single propagator) or a cut.

3.1.5 Momentum subtraction

In the previous subsection the behaviour for large values of x which corresponds to the infrared (IR) region was considered. Some attention has to be paid also to the behaviour at small values of x, i.e. the ultraviolet (UV) region. For space-time dimensions $D > 2$ (i.e. $\lambda > 0$) and a sufficiently high number of internal lines or propagator factors, the integral diverges in the ultraviolet region (i.e. for small values of x).[2] However, UV singularities for water melon diagrams are very simple because there is only a general divergence and no subdivergencies. The R operation, the prescription how the renormalization is performed (cf. Ref. [88]), is well-defined, simple and applicable to numerical calculations which is an important condition for practical applications. The subtraction method consists of an expansion of the Bessel function of the first kind in the integrand into a Taylor series about $x = 0$ which then is subtracted from the Bessel function term by term up to a given order. This procedure is a recipe for a general but finite singularity. Expressed explicitly, the subtracted integral kernel is given by

$$\left[\left(\frac{px}{2}\right)^{-\lambda} J_\lambda(px)\right]_N = \left(\frac{px}{2}\right)^{-\lambda} J_\lambda(px) - \sum_{k=0}^N \frac{(-1)^k}{k!\Gamma(\lambda+k+1)}\left(\frac{px}{2}\right)^{2k}. \tag{3.16}$$

Inserted into Eq. (3.13), this corresponds to the momentum subtraction

$$\tilde\Pi_{\rm mom}(p^2) = \tilde\Pi(p^2) - \sum_{k=0}^N \frac{p^{2k}}{k!}\left(\frac{d}{dp^2}\right)^k \tilde\Pi(p^2)\Big|_{p^2=0}. \tag{3.17}$$

Note that the subtraction at the point $p^2 = 0$ is possible only if the diagram contains at least one massive line. On the other hand, if the diagram consists of massless lines only, the diagram can be calculated analytically anyhow in simple fashion. The subtraction can then be solved trivially.

[2]For $\lambda = 0$ the singularities are only logarithmic ones which makes this case technically simpler. Also the strength of the singularity does not grow so rapidly with increasing number of lines.

3.1.6 An "unorthodox" dimensional regularization

To give a meaning to the single terms in the expansion of Eq. (3.13) a regularization has to be performed. In most of the cases the dimensional regularization will be used here. However, in some cases one can also apply so-called "unorthodox" renormalization methods which are related to the dimensional regularization method but are simpler. In the specific case it is enough to perform regularization by multiplying an additional factor $x^{2\omega}$ (in order to keep the mass dimension a factor $(\mu x)^{2\omega}$ is preferred instead). Therefore, all propagators are kept in integer space-time dimension, only the space-time dimension of the integration measure is changed. This regularization method shall be called *"unorthodox regularization method"* in the following [89]. Note that similar modifications of the dimensional regularization method are known also for other applications. In some supersymmetric theories one keeps the four-dimensional structure of tensor fields in order to save the Ward identities. The corresponding modification of the dimensional regularization method is known as *dimensional regularization by dimensional reduction.* Here as well as in the considered case substantial simplifications result for the calculations.

It remains to show that finite quantities like the subtracted correlator function in Eq. (3.17) are independent on whether the full or the unorthodox dimensional regularization method is used. This will be shown for the simplest case, the case of a diagram consisting of a massive and a massless internal line. The corresponding diagram needs only one single subtraction ($N = 0$). If one calculates

$$\begin{aligned}\tilde{\Pi}_D(p^2) &= \int D(x,m)D(x,0)e^{ip_\mu x^\mu}d^Dx = \\ &= \frac{\Gamma(1-\lambda)}{(4\pi)^{\lambda+1}\lambda}(m^2)^{\lambda-1}\ {}_2F_1\left(1,1-\lambda;1+\lambda;-\frac{p^2}{m^2}\right)\end{aligned} \tag{3.18}$$

by using the conventional method and

$$\begin{aligned}\tilde{\Pi}_\omega(p^2) &= \int D_4(x,m)D_4(x,0)e^{ip_\mu x^\mu}x^{2\omega}d^4x = \\ &= \frac{\Gamma(1+\omega)\Gamma(\omega)}{(4\pi)^2}\left(\frac{m^2}{4}\right)^{-\omega}\ {}_2F_1\left(1+\omega,\omega;2;-\frac{p^2}{m^2}\right)\end{aligned} \tag{3.19}$$

by using the unorthodox method where ${}_2F_1(a,b;c;x)$ is the hypergeometric function (cf. Appendix D.2), for the corresponding limits $\omega \to 0$ and $\lambda = 1-\varepsilon \to 1$ the differences $\tilde{\Pi}_D(p^2) - \tilde{\Pi}_D(0)$ and $\tilde{\Pi}_\omega(p^2) - \tilde{\Pi}_\omega(0)$ are both finite and coincide. Because both methods lead to the same result, one can use ε instead of ω in the following.

3.2 Comparison with known results

In some special cases the analytical calculation of the integral in Eq. (3.13) is possible. In these cases a comparison with results given in the literature is appropiate. Here the comparatively trivial case of a water melon diagram with massless internal lines will not be dealt with. Instead, the case of vanishing outer momentum p is considered, the so-called bubble water melon diagrams.[3] Then the Bessel factor disappears from Eq. (3.13), and one has

$$2\pi^{\lambda+1}\left(\frac{mx}{2}\right)^{-\lambda}J_\lambda(mx)\to\frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}\quad\mbox{for}\quad mx\to 0. \tag{3.20}$$

The complexity of examples which are dealt with increases with the number of internal lines in the diagram. Diagrams with a single massive line and an arbitrary number of massless lines are solved by the formula

$$\int_0^\infty x^\mu K_\nu(mx)dx=2^{\mu-1}m^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{3.21}$$

The simplest example is ($\mu=1-\lambda$, $\nu=\lambda$)

$$\begin{aligned}\tilde\Pi_{11}(0) &= \int D(x,m)D(x,0)d^Dx \;=\; \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}\int D(x,m)D(x,0)x^{2\lambda+1}dx \;=\\ &= \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}\int\left(\frac{(mx)^\lambda K_\lambda(mx)}{(2\pi)^{\lambda+1}x^{2\lambda}}\right)\left(\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}x^{2\lambda}}\right)x^{2\lambda+1}dx \;=\\ &= \frac{m^\lambda}{2\lambda(2\pi)^{\lambda+1}}\int_0^\infty x^{1-\lambda}K_\lambda(mx)dx=\left(\frac{m^2}{2}\right)^{\lambda-1}\frac{\Gamma(1-\lambda)}{4\lambda(2\pi)^{\lambda+1}}.\end{aligned} \tag{3.22}$$

The corresponding calculation in momentum space confirms this result. Diagrams with two massive and an arbitrary number of massless lines can be determined using

$$\int_0^\infty x^{2\alpha-1}K_\mu(mx)K_\mu(mx)dx=\frac{2^{2\alpha-3}}{m^{2\alpha}\Gamma(2\alpha)}\Gamma(\alpha+\mu)\Gamma(\alpha)^2\Gamma(\alpha-\mu) \tag{3.23}$$

(see Appendix D.3). Here the example of a three-loop diagram with two equally massive lines and two massless lines at vanishing total momentum is given,

$$\begin{aligned}\tilde\Pi_{22}(0) &= \int D(x,m)^2D(x,0)^2d^Dx \;=\; \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}\int D(x,m)^2D(x,0)^2x^{2\lambda+1}dx \;=\\ &= \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}\int\left(\frac{(mx)^\lambda K_\lambda(mx)}{(2\pi)^{\lambda+1}x^{2\lambda}}\right)^2\left(\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}x^{2\lambda}}\right)^2x^{2\lambda+1}dx \;=\\ &= \left(\frac{m^2}{2}\right)^{3\lambda-1}\frac{\Gamma(\lambda)^2\Gamma(1-\lambda)\Gamma(1-2\lambda)^2\Gamma(1+3\lambda)}{16(2\pi)^{3\lambda+3}\Gamma(\lambda+1)\Gamma(2-4\lambda)}.\end{aligned} \tag{3.24}$$

This result corresponds to the base element M_1 for the calculation of massive three-loop diagrams with general three-loop topology as it is dealt with in Ref. [83].

[3] Bubble diagrams in general also contain diagrams not belonging to the water melon topology.

3.2.1 The case of three different masses

A more complicated example is the sunrise diagram with three different massive lines. This example exceeds the previous ones in its complexity already because the integral is UV divergent. For a special choice of the outer momenta there exists an analytic expression for this diagram [90]. For the calculation in configuration space the starting expression is given by

$$\tilde{\Pi}_3(p^2) = 2\pi^{\lambda+1}\int_0^\infty \left(\frac{px}{2}\right)^{-\lambda} J_\lambda(px)D(x,m_1)D(x,m_2)D(x,m_3)x^{2\lambda+1}dx. \tag{3.25}$$

To extract the finite part, the momentum subtraction is used. Therefore, $\tilde{\Pi}_3(p^2)$ is split up into a finite part $\tilde{\Pi}_{\rm mom}(p^2)$ and a singular part $\tilde{\Pi}_{\rm sing}(p^2)$,

$$\tilde{\Pi}_3(p^2) = \tilde{\Pi}_{\rm mom}(p^2) + \tilde{\Pi}_{\rm sing}(p^2). \tag{3.26}$$

The momentum subtracted correlator function is then given by

$$\tilde{\Pi}_{\rm mom}(p^2) = 2\pi^{\lambda+1}\int_0^\infty \left[\left(\frac{px}{2}\right)^{-\lambda} J_\lambda(px)\right]_1 D(x,m_1)D(x,m_2)D(x,m_3)x^{2\lambda+1}dx \tag{3.27}$$

where the definition of the square brackets in Eq. (3.16) is used. The singular part is given correspondingly by

$$\begin{aligned}\tilde{\Pi}_{\rm sing}(p^2) \;&=\; A + p^2 B \;=\; \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}\int_0^\infty D(x,m_1)D(x,m_2)D(x,m_3)x^{2\lambda+1}dx \;+\\ &\quad -p^2\frac{2\pi^{2\lambda+1}}{4\Gamma(\lambda+2)}\int_0^\infty x^2 D(x,m_1)D(x,m_2)D(x,m_3)x^{2\lambda+1}dx.\end{aligned} \tag{3.28}$$

In the literature the diagram is considered at a so-called *pseudo threshold*, in this case $p = m_1 + m_2 - m_3$ [90]. For simplicity the special case $m_1 = m_2 = m_3/2 = m$ is used here. For the case $p = 0$ the part B in Eq. (3.28) does not contribute. Furthermore, the momentum subtracted correlator function vanishes totally because it is regular at $p = 0$, $\tilde{\Pi}_{\rm mom}(0) = 0$. Therefore, only the part A in Eq. (3.28) needs to be considered in order to compare with the result [90]

$$\tilde{\Pi}_3^{\rm ref}(0) = \pi^{4-2\varepsilon}\frac{m^{2-4\varepsilon}\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}\left[-\frac{3}{\varepsilon^2}+\frac{8\ln 2}{\varepsilon}-8\ln^2 2\right]+O(\varepsilon). \tag{3.29}$$

The part A is of the general form

$$\int_0^\infty x^\rho K_\mu(mx)K_\mu(mx)K_\mu(2mx)dx \tag{3.30}$$

($\mu = \lambda$ and $\rho = 1-\lambda$ in this case). This integral is calculable without having to take recourse to integral tables (e.g. Ref. [91]). Primarily the method consists in a further subtraction, with the help of which the integral can be reduced to known (already calculated) integrals. This subtraction is applied to the last of the McDonald functions which occur in the integrand as factors. Using the power series expansion

$$\left(\frac{\xi}{2}\right)^\lambda K_\lambda(\xi) = \frac{\Gamma(\lambda)}{2}\left[1+\frac{1}{1-\lambda}\left(\frac{\xi}{2}\right)^2-\frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)}\left(\frac{\xi}{2}\right)^{2\lambda}\right]+O(\xi^4,\xi^{2+2\lambda}). \tag{3.31}$$

the subtraction can be performed. Note that the part A is still singular at $x = 0$. This part can be split into a finite part F and a singular part S according to $(2\pi)^{2D}A = F + S$ (the same normalization as in Ref. [90] is used). As mentioned before, the singular part can be reduced to known and already calculated integrals with two McDonald functions and can be calculated according to

$$\begin{aligned} S &= \frac{(2\pi)^D m^{2\lambda}}{\Gamma(\lambda+1)} \int_0^\infty x^{2(1-\lambda)-1} K_\lambda(mx) K_\lambda(mx) \times \\ &\qquad \times \frac{\Gamma(\lambda)}{2}\left[1 + \frac{(mx)^2}{1-\lambda} - \frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)}(mx)^{2\lambda}\right] dx = \\ &= \frac{(2\pi)^D m^{2-4\varepsilon}}{\Gamma(\lambda+1)} \int_0^\infty \xi^{2\varepsilon-1} K_\lambda(\xi) K_\lambda(\xi) \frac{\Gamma(\lambda)}{2}\left[1 + \frac{\xi^2}{1-\lambda} - \frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)}\xi^{2\lambda}\right] d\xi = \\ &= \pi^{4-2\varepsilon} \frac{m^{2-4\varepsilon}\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}\left[-\frac{3}{\varepsilon^2} + \frac{8\ln 2}{\varepsilon} + 8(2 - 2\ln 2 - \ln^2 2)\right] + O(\varepsilon). \end{aligned} \tag{3.32}$$

The pole part of this contribution is in agreement with the one in Eq. (3.29), only the finite part is different. This will be corrected by the part F. Because

$$\begin{aligned} F &= \frac{(2\pi)^D m^{2\lambda}}{\Gamma(\lambda+1)} \int_0^\infty x^{2(1-\lambda)-1} K_\lambda(mx) K_\lambda(mx) \times \\ &\qquad \times \left\{(mx)^\lambda K_\lambda(2mx) - \frac{\Gamma(\lambda)}{2}\left[1 + \frac{(mx)^2}{1-\lambda} - \frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)}(mx)^{2\lambda}\right]\right\} dx \end{aligned} \tag{3.33}$$

is a finite contribution, one can take $D = 4$ (i.e. $\lambda = 1$). F can then be simplified which leads to

$$F = 16\pi^4 m^2 \int_0^\infty \frac{dz}{z} K_1(z) K_1(z) \left\{ z K_1(2z) - \frac{1}{2}\left[1 + z^2(-1 + 2\gamma_E + 2\ln z)\right]\right\}. \tag{3.34}$$

This (integer dimensional) integral can be calculated numerically, the result reads

$$F = 16\pi^4 m^2 [-0.306853\ldots] = 16\pi^4 m^2 [-(1 - \ln 2) \times 1.0000\ldots] \tag{3.35}$$

(the analytic form is shown only for illustrative reasons). Because F is finite, the overall factor can be adjusted to the one of S (the difference is absorbed into $O(\varepsilon)$). One obtains

$$F = \pi^{4-2\varepsilon} \frac{m^{2-4\varepsilon}\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)} [-16(1 - \ln 2) \times 1.0000\ldots] + O(\varepsilon), \tag{3.36}$$

and both parts (S and F) together result in the same expression as in Eq. (3.29). One should emphasize that for the case of non-vanishing external momentum p no new situation arrises. There are only different finite parts which emerge. A subtraction term corresponding to the part B in Eq. (3.28) is required. However, this case is easier to deal with because the singularity at $x = 0$ is weaker and there is only one subtraction term needed.

3.2.2 Further considerations in four space-time dimensions

A general remark on the calculation of the correlator function near the production threshold of quark-antiquark pairs is in order here. In Ref. [90] the correlator function was given at this threshold. Note, however, that the correlator function at the threshold is not analytic and therefore no Taylor series expansion can be given. Some higher derivatives of the correlator function as function of the outer squared momentum p^2 do not exist at the threshold. In Sec. 3.5 a method will be introduced how calculations near the threshold can be performed.

The examples shown here are well-known so far and were developed by others using methods which differ from the methods described here. While it is possible to calculate water melon diagrams with an arbitrary number of internal lines carrying different masses using configuration space techniques, one rarely finds analytical expressions in the literature with which a comparison can be made. Beyond two-loop calculations there are only a few in the literature. A three-loop example will be dealt with in the following section.

3.2.3 Examples in odd space-time dimensions

It is interesting that the calculation of the correlator function in Eq. (3.13) can be performed in a closed form for odd space-time dimensions and an arbitrary number of internal lines. As the simplest example this shall be shown here for $D \to D_0 = 3$ space-time dimensions. Applications of the results in three space-time dimensions can be found. In three space-time dimensions the results can be used to compute phase space integrals for particles in jets where the momentum along the direction of the jet is fixed [92]. Another application can be found in three-dimensional QCD which emerges as the high temperature limit of the ordinary theory of strong interactions for the quark-gluon plasma (see e.g. [93, 94, 95, 96]). Three-dimensional models are also used to study the question of dynamical mass generation and the infrared structure of the models of quantum field theory [97, 98, 99]. A further theoretical application consists in the investigation of properties of baryons in the limit of infinite number of colours $N_c \to \infty$ where one has to take into account the spin structure of internal lines.

For $\lambda_0 = (D_0 - 2)/2 = 1/2$ the propagator in Eq. (3.5) simplifies and results in

$$D(x,m) \to D_3(x,m) = \frac{\sqrt{mx}K_{1/2}(mx)}{(2\pi)^{3/2}x} = \frac{e^{-mx}}{4\pi x} \tag{3.37}$$

while the integral kernel is given by

$$\left(\frac{px}{2}\right)^{-1/2} J_{1/2}(px) = \frac{2}{\sqrt{\pi}}\frac{\sin(px)}{px}. \tag{3.38}$$

The explicit result for the water melon diagram with n internal lines is then given by the integral

$$\begin{aligned}\tilde{\Pi}(p^2) &= 4\pi\int_0^\infty \frac{\sin(px)}{px}\frac{e^{-Mx}}{(4\pi x)^{n-2}}(\mu x)^{2\varepsilon}dx = \\ &= \frac{\Gamma(2-n+2\varepsilon)}{2ip(4\pi)^{n-1}}\left[(M-ip)^{n-2-2\varepsilon} - (M+ip)^{n-2-2\varepsilon}\right]\mu^{2\varepsilon}\end{aligned} \tag{3.39}$$

where ε is the regularization parameter and $M = \sum m_i$. At this point the advantages of the unorthodox regularization method becomes obvious: it allows for the analytical calculation of an arbitrary water melon diagram in general odd space-time dimensions. While the general expression for the spectral density (for $n \geq 2$) reads

$$\rho(s) = \frac{1}{2\pi i}\,\mathrm{Disc}\,\tilde{\Pi}(s) = \frac{(\sqrt{s}-M)^{n-2}}{2(4\pi)^{n-1}(n-2)!\sqrt{s}}\,\theta(s-M^2), \tag{3.40}$$

a few special cases are shown here. For $n = 1$ the usual propagator function emerges,

$$\rho(s) = \frac{1}{2\pi i}\left(\tilde{\Pi}(se^{-i\pi}) - \tilde{\Pi}(se^{i\pi})\right) = \delta(s-m^2). \tag{3.41}$$

For $n = 2$ the correlator function is still finite,

$$\tilde{\Pi}(p^2) = \frac{1}{8\pi ip}\ln\left(\frac{M+ip}{M-ip}\right) \tag{3.42}$$

so there is no need of a regularization. The corresponding spectral density is given by

$$\rho(s) = \frac{1}{8\pi\sqrt{s}}\,\theta\left(s-(m_1+m_2)^2\right). \tag{3.43}$$

This is a special case of Eq. (3.40), moreover it is nothing else than the three-dimensional phase-space for two particles. Only the cases for $n > 2$ are interesting. For the proper sunrise diagram ($n = 3$) one obtains the correlator function

$$\tilde{\Pi}(p^2) = \frac{1}{32\pi^2}\left\{\frac{1}{\varepsilon} - \frac{M}{ip}\ln\left(\frac{M+ip}{M-ip}\right) - \ln\left(\frac{M^2+p^2}{\mu^2}\right)\right\}. \tag{3.44}$$

The arbitrary scale μ mirrors the regularization. But the spectral density is independent of this scale and finite,

$$\rho(s) = \frac{\sqrt{s}-M}{32\pi^2\sqrt{s}}\,\theta(s-M^2), \tag{3.45}$$

as is expected.

At this point the relation between the momentum subtraction and the unorthodox regularization method can be shown directly. Taking Eq. (3.39) for $n = 3$ with a momentum subtraction included, one obtains

$$\tilde{\Pi}(p^2) = \int_0^\infty \left(\frac{\sin(px)}{px} - 1\right)\frac{e^{-Mx}}{(4\pi)^2 x}(\mu^2x^2)^\epsilon dx \tag{3.46}$$

which is UV-finite even for $\epsilon = 0$ because there is no singularity at the origin. For practical computations it is convenient to keep the factor $(\mu^2x^2)^\epsilon$ in the integrand since this factor gives a meaning to each of the two terms in the round brackets in Eq. (3.46) separately. Then the direct calculation gives

$$\begin{aligned}
\tilde{\Pi}(p^2) &= \int_0^\infty \left(\frac{\sin(px)}{px} - 1\right)\frac{e^{-Mx}}{(4\pi)^2 x}(\mu^2x^2)^\epsilon dx = \\
&= \frac{\Gamma(-1+2\epsilon)}{2ip(4\pi)^2}\left[(M-ip)^{1-2\epsilon} - (M+ip)^{1-2\epsilon}\right]\mu^{2\epsilon} - \frac{\Gamma(2\epsilon)}{(4\pi)^2}\left(\frac{\mu}{M}\right)^{2\epsilon} = \\
&= -\frac{1}{32\pi^2}\left\{\frac{M}{ip}\ln\left(\frac{M+ip}{M-ip}\right) + \ln\left(\frac{M^2+p^2}{M^2}\right)\right\}.
\end{aligned} \tag{3.47}$$

The poles cancel in this expression and the arbitrary scale μ changes to M. This corresponds to a transition from MS-type of renormalization schemes to a momentum subtraction scheme (with subtraction at the origin in this particular case). Since the spectral density $\rho(s)$ is finite, it can be computed using any regularization scheme as can be seen by comparing Eqs. (3.44) and (3.47).

The method presented here is also applicable for odd space-time dimensions other than $D_0 = 3$. For $D_0 = 5$ ($\lambda_0 = 3/2$) for instance, the propagator reads

$$D(x,m) \to D_5(x,m) = \frac{(mx)^{3/2}K_{3/2}(mx)}{(2\pi)^{5/2}x^3} = \frac{e^{-mx}}{8\pi^2x^3}(1+mx) \tag{3.48}$$

which assures that the integration in Eq. (3.13) can be performed in terms of elementary functions (powers and logarithms) again.

Note that particular models of different space-time dimensions are very useful because their properties may be simpler, and this may therefore allow one to study general features of the underlying field theory. For example, in six-dimensional space-time the simplest model of quantum field theory ϕ^3 is asymptotically free and can be used for simulations of some features of QCD. Though five-dimensional models are less popular than others, still there are useful applications for Yang-Mills theory in five-dimensional space-time where the UV structure of the models can be analyzed [100].

3.2.4 Analytic continuation in momentum space

The spectral density, representing the n-particle phase space as mentioned before, can be calculated for an arbitary space-time dimension in a closed form. Using the result

$$K_\lambda(z) = \frac{i\pi}{2}e^{i\pi\lambda/2}H_\lambda^+(iz), \qquad \left(H_\lambda^+(x)\right)^* = H_\lambda^-(x) \tag{3.49}$$

for complex z and real x and λ, where $H_\lambda^+(z) = H_\lambda^{(1)}(z)$ and $H_\lambda^-(z) = H_\lambda^{(2)}(z)$ are the *Hankel functions* [101] (see Appendix D.1.2), an analytic continuation of Eq. (3.13) to the complex plane can be performed which is necessary for calculating the spectral density

$$\begin{aligned}\rho(s) \;=\; & \frac{i}{2\pi}\int_0^\infty \left(\frac{2\pi\xi}{s}\right)^{\lambda+1} J_\lambda(\xi)\Bigg[e^{-i\pi(\lambda+1)}\prod_{i=1}^n \frac{1}{4}\left(\frac{m_i\sqrt{s}}{2\pi\xi}\right)^\lambda e^{i\pi(\lambda+1/2)}H_\lambda^+\left(\frac{m_i\xi}{\sqrt{s}}\right) + \\ & -e^{i\pi(\lambda+1)}\prod_{i=1}^n \frac{1}{4}\left(\frac{m_i\sqrt{s}}{2\pi\xi}\right)^\lambda e^{-i\pi(\lambda+1/2)}H_\lambda^-\left(\frac{m_i\xi}{\sqrt{s}}\right)\Bigg]d\xi. \end{aligned}\tag{3.50}$$

It is understood in this expression that a proper momentum subtraction has been performed, according to Eq. (3.16). The one-dimensional integral representation in Eq. (3.50) is simple enough for further processing, so that one can easily discuss special mass configurations. The evaluation of the integral in Eq. (3.50), however, is not always straightforward. The integrand contains highly oscillating functions that require some care in the numerical treatment. This is to be expected since the discontinuity, or the spectral density, is a distribution rather than a smooth function. However, because the analytic structure and the asymptotic behaviour of the integrand in Eq. (3.50) is completely known, the numerical computation of $\rho(s)$ can be made reliable and fast in domains where $\rho(s)$ is smooth enough, in particular far from threshold. One recipe is to extract the oscillating asymptotics first and then to perform the integration analytically, or to integrate the oscillating asymptotics numerically using integration routines that have special options for the treatment of oscillatory integrands. Both ways were checked in simple examples with reliable results. The remaining non-oscillating part is a slowly changing function which can be integrated numerically without difficulties. With this extra care the integration can be easily made safe, reliable and fast even for an average personal computer. The general numerical procedures are checked in three-dimensional space-time ($D_0 = 3$) where exact results are available.

For this check the spectral density for the $n = 3$ water melon diagram in three-dimensional space-time is recomputed. The Hankel function for indices $j + 1/2$ (or for $D_0 = 2j + 1$) is a finite combination of powers and an exponential (cf. Appendix D.1.5) which makes possible the explicit computation of the integral in Eq. (3.50), with the result

$$\rho(s) = -\frac{1}{\pi}\int_0^\infty \left(\frac{\sin\xi}{\xi} - 1\right)\sin\left(\frac{M\xi}{\sqrt{s}}\right)\frac{d\xi}{(4\pi)^2\xi} = \frac{\sqrt{s}-M}{32\pi^2\sqrt{s}}\,\theta(s-M^2). \tag{3.51}$$

This form agrees with the explicit formula given by Eq. (3.45). For this special case the explicit calculation is shown. The starting point is

$$\begin{aligned}
\rho(s) &= \frac{1}{2\pi i}\left[\tilde\Pi(se^{-i\pi}) - \tilde\Pi(se^{+i\pi})\right] = \\
&= \frac{2\pi^{\lambda+1}}{2\pi i}\Bigg[\int_0^\infty \left(\frac{x\sqrt{s}e^{-i\pi/2}}{2}\right)^{-\lambda} J_\lambda(x\sqrt{s}e^{-i\pi/2})x^{2\lambda+1}\prod_{i=1}^n D(x,m_i)dx\ + \\
&\qquad - \int_0^\infty \left(\frac{x\sqrt{s}e^{+i\pi/2}}{2}\right)^{-\lambda} J_\lambda(x\sqrt{s}e^{+i\pi/2})x^{2\lambda+1}\prod_{i=1}^n D(x,m_i)dx\Bigg] = \\
&= -i\Bigg[\int_{-i0}^{-i\infty}\left(\frac{\xi}{2\pi}\right)^{-\lambda} J_\lambda(\xi)\left(\frac{\xi e^{+i\pi/2}}{\sqrt{s}}\right)^{2\lambda+1}\frac{e^{+i\pi/2}}{\sqrt{s}}\prod_{i=1}^n D\left(\frac{\xi e^{+i\pi/2}}{\sqrt{s}}, m_i\right)d\xi\ + \\
&\qquad - \int_{i0}^{i\infty}\left(\frac{\xi}{2\pi}\right)^{-\lambda} J_\lambda(\xi)\left(\frac{\xi e^{-i\pi/2}}{\sqrt{s}}\right)^{2\lambda+1}\frac{e^{-i\pi/2}}{\sqrt{s}}\prod_{i=1}^n D\left(\frac{\xi e^{-i\pi/2}}{\sqrt{s}}, m_i\right)d\xi\Bigg] = \\
&= \frac{1}{2\pi i}\Bigg[\int_{-i0}^{-i\infty}\left(\frac{2\pi\xi}{s}\right)^{\lambda+1} J_\lambda(\xi)e^{+i\pi(\lambda+1)}\prod_{i=1}^n D\left(\frac{\xi e^{+i\pi/2}}{\sqrt{s}}, m_i\right)d\xi\ + \\
&\qquad - \int_{+i0}^{+i\infty}\left(\frac{2\pi\xi}{s}\right)^{\lambda+1} J_\lambda(\xi)e^{-i\pi(\lambda+1)}\prod_{i=1}^n D\left(\frac{\xi e^{-i\pi/2}}{\sqrt{s}}, m_i\right)d\xi\Bigg].
\end{aligned} \tag{3.52}$$

Now $\lambda = 1/2$ and $n = 3$ is selected. Using

$$J_{1/2}(\xi) = \sqrt{\frac{2}{\pi\xi}}\ \sin\xi, \qquad H^\pm_{1/2}\left(\frac{m\xi}{\sqrt{s}}\right) = \sqrt{\frac{2\sqrt{s}}{\pi m\xi}}e^{\mp i\pi/2}e^{\pm im\xi/\sqrt{s}} \tag{3.53}$$

and

$$D\left(\frac{\xi e^{\pm i\pi/2}}{\sqrt{s}}, m\right) = \frac{1}{4}\sqrt{\frac{m\sqrt{s}}{2\pi\xi}}e^{\mp i\pi}H^\mp_{1/2}\left(\frac{m\xi}{\sqrt{s}}\right) = \frac{\sqrt{s}}{4\pi\xi}e^{\mp i\pi/2}e^{\mp im\xi/\sqrt{s}}, \tag{3.54}$$

the product of propagators is given by

$$e^{\pm 3i\pi/2}\prod_{i=1}^{3} D\left(\frac{\xi e^{\pm i\pi/2}}{\sqrt{s}}, m_i\right) = \left(\frac{\sqrt{s}}{4\pi\xi}\right)^3 e^{\mp iM\xi/\sqrt{s}}. \tag{3.55}$$

Therefore, the spectral density in this case is given by

$$\rho(s) = \frac{1}{2\pi i}\left[\int_{-i0}^{-i\infty} \frac{\sin\xi}{(4\pi\xi)^2} e^{-iM\xi/\sqrt{s}} d\xi - \int_{+i0}^{+i\infty} \frac{\sin\xi}{(4\pi\xi)^2} e^{+iM\xi/\sqrt{s}} d\xi\right]. \tag{3.56}$$

Because there are no singularities on the complex ξ-plane except for the point $\xi = 0$ and since the exponential factors suppress the integrand for arcs from $-i\infty$ to ∞ and from $+i\infty$ to ∞, the integrals can be replaced by the (common) third part of the respective closed paths,

$$\rho(s) = \frac{1}{2\pi i}\int_0^\infty \frac{\sin\xi}{(4\pi\xi)^2}\left[e^{-iM\xi/\sqrt{s}} - e^{+iM\xi/\sqrt{s}}\right] d\xi \;\; = \;\; = -\frac{1}{\pi}\int_0^\infty \frac{\sin\xi}{(4\pi\xi)^2}\sin\left(\frac{M\xi}{\sqrt{s}}\right) d\xi. \tag{3.57}$$

This result is the unsubtracted version of Eq. (3.51). In order to obtain the final result given in Eq. (3.51), the necessary integrations are performed by moving the contour back into the complex ξ-plane and regularizing the singularity at the origin by an infinitely small shift $\pm i0$. Then closing the contour in the upper or lower semi-plane according to the sign of regularization one finds the integral by computing the residue at the origin.

3.3 Integral transformation in configuration space

The analytic structure of the spectral density can be determined directly by using the correlator function in configuration space. First of all, the opposite holds, as one can see by translating the dispersion relation into configuration space,

$$\Pi(x) = \frac{1}{(2\pi)^D}\int \tilde{\Pi}(p^2) e^{-ip_\mu x^\mu} d^D p = \int \rho(s) ds \int \frac{d^D p}{(2\pi)^D}\frac{e^{-ip_\mu x^\mu}}{s + p^2} = \int \rho(s) D(x, \sqrt{s}) ds. \tag{3.58}$$

Here $D(x, \sqrt{s})$ is the propagator for a particle with "mass" $\sqrt{s}$. This representation was used for sum rule applications in Refs. [102, 103] where the spectral density for the two-loop sunrise diagram was found in two-dimensional space-time [104]. With the explicit form of the propagator in configuration space given by Eq. (3.5), the representation in Eq. (3.58) turns into a particular example of the Hankel transform, namely the K-*transform* [105, 106]. Up to inessential factors of x and m, Eq. (3.58) reduces to the generic form of the K-transform for a conjugate pair of functions f and g,

$$g(y) = \int_0^\infty f(x) K_\nu(xy)\sqrt{xy}\, dx. \tag{3.59}$$

The inverse of this transform is known to be given by

$$f(x) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} g(y) I_\nu(xy)\sqrt{xy}\, dy \tag{3.60}$$

where $I_\nu(x)$ is a modified Bessel function of the first kind and the integration runs along a vertical contour in the complex plane to the right of the right-most singularity of the function $g(y)$ [106]. In order to obtain a representation for the spectral density $\rho(s)$ of a water melon diagram in general D-dimensional space-time one needs to apply the inverse K-transform to the particular case given by Eq. (3.58). One has

$$\rho(s) = -i\left(\frac{2\pi}{\sqrt{s}}\right)^{\lambda} \int_{c-i\infty}^{c+i\infty} x^{\lambda+1}\Pi(x) I_\lambda(x\sqrt{s})dx. \tag{3.61}$$

The inverse transform given by Eq. (3.61) solves the problem of determining the spectral density of water melon diagrams by reducing it to the computation of a one-dimensional integral for the general class of water melon diagrams with any number of internal lines and different masses. Compared to the general solution given by Eq. (3.50), the above form is simpler. In the following some explicit examples of applying the technique of computing the spectral density of water melon diagrams on the basis of integral transforms in configuration space are given (see also Refs. [107, 108]).

3.3.1 The one-loop case

First a remark about the mass degenerate one-loop case is in order. All necessary integrals (both for the direct and the inverse K-transform) involve no more than the product of three Bessel functions which can be found in a standard collection of formulas for special functions (see e.g. Ref. [101]). The spectral density in D-dimensional space-time (for two internal lines with equal masses m) can be computed to be

$$\rho(s) = \frac{(s-4m^2)^{\lambda-1/2}}{2^{4\lambda+1}\pi^{\lambda+1/2}\Gamma(\lambda+1/2)\sqrt{s}}, \qquad \sqrt{s} > 2m. \tag{3.62}$$

This formula is useful since it can be used to test the limiting cases of more general results. The corresponding spectral density for the nondegenerate case with two different masses m_1 and m_2 reads

$$(2\pi)^{2\lambda+1}\rho(s) = \frac{\Omega_{2\lambda+1}}{4\sqrt{s}}\left(\frac{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2}{4s}\right)^{\lambda-1/2}, \qquad \sqrt{s} > m_1+m_2, \tag{3.63}$$

where

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{3.64}$$

is a volume of a unit sphere in d-dimensional space-time. Note the identity

$$(s-m_1^2-m_2^2)^2-4m_1^2m_2^2 = \left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right] \tag{3.65}$$

which immediately allows one to locate the two-particle threshold.

3.3.2 General considerations for the odd-dimensional case

For odd-dimensional space-time the representation in Eq. (3.58) reduces to the ordinary Laplace transformation. To obtain the spectral density one can use Eq. (3.61). For energies below threshold it is possible to close the contour of integration to the right. With the appropiate choice of the constant c as specified above, the closed contour integration gives zero due to the absence of singularities in the relevant domain of the right semi-plane. By closing the contour of integration to the left and keeping only that part of the function $I_\lambda(z)$ which is exponentially falling for $\mathrm{Re}(z) < 0$ one can obtain another convenient integral representation for the spectral density when the energy is above threshold. The only singularities within the closed contour are then poles at the origin (in odd-dimensional space-time) and the evaluation of the integral can be done by determining the corresponding residues. These are purely algebraic manipulations, the simplicity of which also explain the simplicity of the computations in odd-dimensional space-time. For a small number of internal lines n the spectral density can also be found by using the convolution formulas for the spectral densities of a smaller number of particles (see e.g. Ref. [109]). For large n the computations described in Ref. [109] become quite cumbersome and the technique suggested in Refs. [107, 108] is much more convenient.

3.3.3 Examples for three-dimensional space-time

As an example for the odd-dimensional case, calculations in three dimensions are presented here. Using the three-dimensional propagator

$$D_3(x,m) = \frac{\sqrt{mx}K_{1/2}(mx)}{(2\pi)^{3/2}x} = \frac{e^{-mx}}{4\pi x}, \tag{3.66}$$

Eq. (3.58) takes the form

$$\Pi(x) = \int_0^\infty \rho(s)\frac{e^{-x\sqrt{s}}}{4\pi x}ds \quad \Leftrightarrow \quad 4\pi x\Pi(x) = \int_0^\infty 2\sqrt{s}\rho(s)e^{-x\sqrt{s}}d\sqrt{s}. \tag{3.67}$$

Inverting Eq. (3.67) results in

$$2\sqrt{s}\rho(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 4\pi x\Pi(x)e^{x\sqrt{s}}dx \quad \Leftrightarrow \quad \rho(s) = -i\int_{c-i\infty}^{c+i\infty}\frac{xe^{x\sqrt{s}}}{\sqrt{s}}\Pi(x)dx. \tag{3.68}$$

This is a special case of Eq. (3.61) with

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sinh(z) \tag{3.69}$$

where only the e^z piece of the hyperbolic function has to be retained. The solution given by Eq. (3.68) has the appropiate support as distribution or, equivalently, as an *inverse Laplace transform.* It vanishes for $\sqrt{s} < M = \sum m_i$ since the contour of integration can be closed to the right where there are no singularities of the integrand. Recall that for large x with $\mathrm{Re}(x) > 0$ the asymptotic behaviour of the correlator function $\Pi(x)$ is governed by the sum of the masses of the propagators and reads $\Pi(x) \sim \exp(-Mx)$. For $\sqrt{s} > M$ one can close the contour to the left in the complex x-plane. The only singularities are then the poles of $\Pi(x)$ at the origin, since $\Pi(x)$ is a product of propagators only, given in three space-time dimensions by Eq. (3.66) as well. The integration in Eq. (3.68) then reduces to finding the residues of the poles at the origin. Indeed, the insertion of

$$\Pi(x) = \prod_{i=1}^n D_3(x,m_i) = \frac{e^{-Mx}}{(4\pi x)^n} \tag{3.70}$$

into Eq. (3.68) leads to

$$\rho(s) = \frac{-i}{(4\pi)^n} \int_{c-i\infty}^{c+i\infty} \frac{e^{(\sqrt{s}-M)x}}{x^{n-1}\sqrt{s}} dx, \tag{3.71}$$

and by closing the contour to the left one obtains

$$\begin{aligned}
\rho(s) &= \frac{-i}{(4\pi)^n} 2\pi i \operatorname{Res}\left[\frac{e^{(\sqrt{s}-M)x}}{x^{n-1}\sqrt{s}};\ x=0\right] \theta(\sqrt{s}-M) = \\
&= \frac{1}{2(4\pi)^{n-1}} \frac{1}{(n-2)!} \frac{d^{n-2}}{dx^{n-2}} \left(\frac{x^{n-1}e^{(\sqrt{s}-M)x}}{x^{n-1}\sqrt{s}}\right)\bigg|_{x=0} \theta(\sqrt{s}-M) = \\
&= \frac{(\sqrt{s}-M)^{n-2}}{2(4\pi)^{n-1}(n-2)!\sqrt{s}}\, \theta(\sqrt{s}-M).
\end{aligned} \tag{3.72}$$

This result is in full agreement with Eq. (3.40).

3.3.4 The even-dimensional case

For even-dimensional space-time the analytic structure of $\Pi(x)$ in Eq. (3.4) is more complicated. There is a cut along the negative axis in the complex x-plane which prevents a straightforward evaluation by simply closing the contour of integration to the left (with $\text{Re}(x) < 0$). The discontinuity along the cut, however, is well-known and includes only Bessel functions that appear in the product of propagators for the correlator function. Therefore, the representation in Eq. (3.61) is essentially equivalent to the direct analytic continuation of the Fourier transform [107] but may be more convenient for a numerical treatment because there is no oscillating integrand in Eq. (3.61).

3.3.5 The four-dimensional case in more detail

In even number of dimensions one is dealing with a genuine K-transform. The important case of four-dimensional space-time is discussed in some detail in the following. For $D = 4$ ($\lambda = 1$), Eqs. (3.5) and (3.58) give

$$\Pi(x) = \int \rho(s) D_4(x, \sqrt{s}) ds = \int \rho(s) \frac{x\sqrt{s}K_1(x\sqrt{s})}{4\pi^2 x^2} ds, \tag{3.73}$$

and Eq. (3.61) is written as

$$\rho(s) = \frac{1}{2\pi i\sqrt{s}} \int_{c-i\infty}^{c+i\infty} 4\pi^2 x^2 \Pi(x) I_1(x\sqrt{s}) dx. \tag{3.74}$$

All remarks about the behaviour at large x apply here as well. However, the structure of singularities is more complicated than in the odd-dimensional case. In addition to the poles at the origin there is a cut along the negative axis that renders the computation of the spectral density more difficult. The cut arises from the presence of the functions $K_1(m_i x)$ in the correlator function $\Pi(x)$. Also the asymptotic behaviour of the function $I_1(z)$ is more complicated than that of $I_{1/2}(z)$. In particular the extraction of the exponentially falling component on the negative real axis is not straightforward. Incidentally, the fall-off behaviour of the function $I_1(z)$ on the negative real axis can be taken as an example of Stokes' phenomenon of asymptotic expansions (see e.g. [85]). While the analytic structure of the representation is quite transparent and the integration can be performed along a contour in the complex plane, there are some subtleties when one wants to obtain a convenient form for a numerical treatment analogous to the odd-dimensional case [107].

After closing the contour to the left (for $\sqrt{s} > M$) using the appropiate part of the function $I_1(z)$ one obtains

$$\begin{aligned} & i\pi \int_{c-i\infty}^{c+i\infty} x^2 \Pi(x) I_1(x\sqrt{s}) dx \qquad (3.75) \\ &= -\int_\epsilon^\infty r^2 \left(\Pi(e^{i\pi} r) + \Pi(e^{-i\pi} r) \right) K_1(r\sqrt{s}) dr + 2\int_\epsilon^\infty r^2 \Pi(r) K_1(r\sqrt{s}) dr \\ &\quad + \int_{C_-} z^2 \Pi(z)(i\pi I_1(z\sqrt{s}) + K_1(z\sqrt{s})) dz + \int_{C_+} z^2 \Pi(z)(i\pi I_1(z\sqrt{s}) - K_1(z\sqrt{s})) dz \end{aligned}$$

for the quantity entering Eq. (3.74).

The contours C_+ and C_- are semi-circles of radius ϵ around the origin in the upper and lower complex semi-plane, respectively (see Fig. 3.1). For practical evaluations of $\Pi(e^{\pm i\pi} r)$ the rule

$$K_1(e^{\pm i\pi}\xi) = -K_1(\xi) \mp i\pi I_1(\xi) \qquad (3.76)$$

($\xi = mr > 0$) for the analytic continuation of the McDonald functions is used.

Some comments are in order. The correlator function $\Pi(z)$ at $z = e^{\pm i\pi} r$ is proportional to the product of propagators of the form

$$D_4(e^{\pm i\pi} r, m_i) \sim \frac{m_i}{r} \left[K_1(m_i r) \pm i\pi I_1(m_i r) \right]. \qquad (3.77)$$

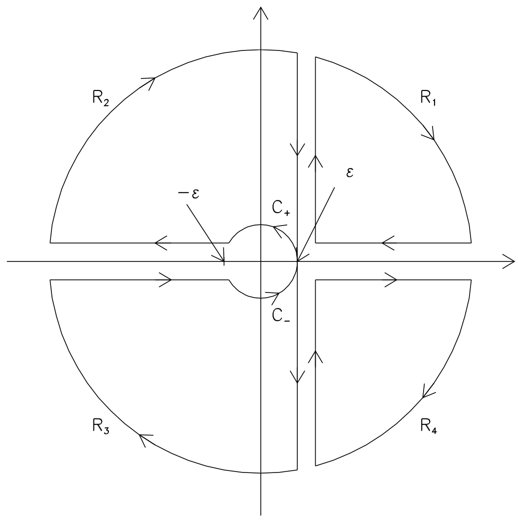


Figure 3.1: Integration contours used in the evaluation of Eq. (3.75). R_1, R_2, R_3 and R_4 are segments of a circle arround the origin where the radius of the circle is taken to infinity.

It is clear from this equation that the leading singular contribution proportional to the product of $K_1(m_i r)$ cancels in the sum in Eq. (3.75). Also the next-to-leading singular term disappears because of different signs in the product. Recall that the small ξ behaviour of the functions $K_1(\xi)$ and $I_1(\xi)$ is given by

$$K_1(\xi) = \frac{1}{\xi} + O(\xi \ln \xi), \quad I_1(\xi) = \frac{\xi}{2} + O(\xi^3). \qquad (3.78)$$

3.3.6 Remarks on the contour integration

A few remarks on the final representation Eq. (3.75) which is in a suitable form for numerical integration are in order at this point. An auxiliary regularization in terms of a circle of finite radius ϵ was introduced, the circle running around the origin with its pole-type singularities. The spectral density is independent of ϵ, and the parameter ϵ completely cancels in the full expression for the spectral density as given by Eq. (3.75). This is natural since the spectral density is finite for the class of water melon diagrams.

Eq. (3.75) contains no oscillatory integrands (cf. Eq. (3.50) to see the difference), and the integration can safely be done numerically. Thus Eq. (3.75) is an useful alternative representation for the spectral density. In practice the integration over the semi-circles is done by expanding the integrand in z for small z and keeping only terms singular in ϵ. The expansion requires only a finite number of terms and is a purely algebraic operation. Then the singularity in ϵ exactly cancels against those of the remaining integrals. This cancellation can also be done analytically leaving well-defined and smooth integrands for further numerical treatment.

Even if the full computation of the spectral function in the even-dimensional case described here is straightforward, it is nevertheless cumbersome. In order to exhibit the essential points in this calculation, the main lines of the calculation are illustrated by a simple and instructive example. Consider the calculation of the integral

$$\int_{c-i\infty}^{c+i\infty} \frac{\ln z}{z^2} e^z dz \tag{3.79}$$

which is rather close in structure to the real case. Due to the singularity at the origin one has to treat the vicinity of the origin carefully. One then proceeds by closing the contour to the left,

$$\begin{aligned}
\int_{c-i\infty}^{c+i\infty} \frac{\ln z}{z^2} e^z dz &= \int_{C_\epsilon} \frac{\ln z}{z^2} e^z dz - \int_\epsilon^\infty \frac{2i\pi}{x^2} e^{-x} dx \;= \\
&= -2\pi i \int_\epsilon^\infty \frac{e^{-x}dx}{x^2} + i\int_{-\pi}^{\pi} \frac{\ln\epsilon + i\phi}{\epsilon}(1+\epsilon e^{i\phi})e^{-i\phi} d\phi \;= \\
&= -2\pi i \int_\epsilon^\infty \frac{e^{-x}dx}{x^2} + i\int_{-\pi}^{\pi} \left(\frac{\ln\epsilon}{\epsilon}e^{-i\phi} + \frac{i\phi}{\epsilon}e^{-i\phi} + \ln\epsilon + i\phi\right) d\phi \;= \\
&= -2\pi i \int_\epsilon^\infty \frac{e^{-x}dx}{x^2} + \frac{\ln\epsilon}{\epsilon}0 + \frac{2\pi i}{\epsilon} + 2\pi i \ln\epsilon + 0 \;= \\
&= -2\pi i \left(\int_\epsilon^\infty \frac{e^{-x}dx}{x^2} - \frac{1}{\epsilon} - \ln\epsilon\right).
\end{aligned} \tag{3.80}$$

The combination in the brackets of the last equation remains finite as $\epsilon \to 0$. Now this limit should be considered in more detail. First the integration is split into two parts, from ϵ to 1 and from 1 to infinity,

$$\int_\epsilon^\infty \frac{e^{-x}dx}{x^2} = \int_1^\infty \frac{e^{-x}dx}{x^2} + \int_\epsilon^1 \frac{e^{-x}dx}{x^2}. \tag{3.81}$$

Then the first integral is just a number which can be found numerically with high precision. In the second integral one can expand the exponent in the integrand and finds

$$\int_\epsilon^1 \frac{e^{-x}dx}{x^2} = \int_\epsilon^1 \frac{dx}{x^2}\left(1 - x + \frac{x^2}{2} - \ldots\right) = \frac{1}{\epsilon} - 1 + \ln\epsilon + \frac{1}{2}(1-\epsilon) + \ldots \tag{3.82}$$

The singularity has the correct form and the finite series converges well. If one takes a value 0.1 instead of 1 for the splitting point in Eq. (3.81), the convergence of the finite series will be very fast. This procedure would be used for practical integration in a realistic case. An exact answer for the simple example after two integrations-by-part is given by

$$\int_\epsilon^\infty \frac{e^{-x}dx}{x^2} = e^{-\epsilon}\left(\frac{1}{\epsilon} + \ln\epsilon\right) - \int_\epsilon^\infty e^{-x}\ln x\, dx. \tag{3.83}$$

The last integral is finite at $\epsilon = 0$ and for the present purpose it suffices to compute it in this limit. The result is

$$\int_0^\infty e^{-x} \ln x \, dx = -\gamma_E \tag{3.84}$$

where γ_E is Euler's constant. All these manipulations can be easily done with a symbolic program. For the original integral one finds

$$\begin{aligned} -2\pi i \left(\int_\epsilon^\infty \frac{e^{-x} dx}{x^2} - \frac{1}{\epsilon} - \ln \epsilon \right) &= -2\pi i \left(e^{-\epsilon} \left(\frac{1}{\epsilon} + \ln \epsilon \right) + \gamma_E - \frac{1}{\epsilon} - \ln \epsilon \right) = \\ &= 2\pi i (1 - \gamma_E) \quad \text{at} \quad \epsilon = 0. \end{aligned} \tag{3.85}$$

Thus finally

$$\int_{c-i\infty}^{c+i\infty} \frac{\ln z}{z^2} e^z dz = 2\pi i (1 - \gamma_E). \tag{3.86}$$

This concludes the discussion of how to treat subtractions in this simplified case. The generalization to Bessel functions is straightforward (just expand near the origin and get expressions as in Eq. (3.80)). Then the form of the subtraction term depends on the number n of propagators in a water melon diagram. Writing down an explicit expression for some n is routine and can be left to the interested user. All the required expansions can be performed by a symbolic manipulation program.

3.3.7 The two-dimensional case

For two-dimensional space-time the representation analogous to Eq. (3.75) is simpler because there is no power singularity at the origin but only a logarithmic singularity which allows one to shrink the contour to a point (take the limit $\epsilon \to 0$). In this case one obtains

$$\rho(s) = \frac{1}{\pi} \int_0^\infty r(2\Pi(r) - \Pi(e^{i\pi} r) - \Pi(e^{-i\pi} r)) K_0(r\sqrt{s}) dr. \tag{3.87}$$

For $D_0 = 2$ the results are shown for the cases $n = 2$ and $n = 3$ in the following. For the one-loop case $n = 2$ one has

$$\rho(s) = \frac{1}{2\pi} \int_0^\infty r I_0(m_1 r) I_0(m_2 r) K_0(r\sqrt{s}) dr \tag{3.88}$$

which can be integrated explicitly (see Appendix D.3) and results in [91]

$$\rho(s) = \frac{1}{2\pi s} \sum_{k,l=0}^\infty \left(\frac{(k+l)!}{k!l!} \right)^2 \left(\frac{m_1^2}{s} \right)^k \left(\frac{m_2^2}{s} \right)^l. \tag{3.89}$$

This series can of course also be directly obtained by expanding Eq. (3.63).

For the case $n = 3$ one obtains

$$\begin{aligned} \rho(s) = \frac{1}{(2\pi)^2} \int_0^\infty & r(K_0(m_1 r) I_0(m_2 r) I_0(m_3 r) + I_0(m_1 r) K_0(m_2 r) I_0(m_3 r) \\ & + I_0(m_1 r) I_0(m_2 r) K_0(m_3 r)) K_0(r\sqrt{s}) dr. \end{aligned} \tag{3.90}$$

3.4 Configuration space based recurrence relations

In this section three-loop vacuum bubble diagrams with only one non-vanishing mass m are considered. A general three-loop vacuum diagram has the topology of a tetrahedron which can also thought of as "fish+propagator" topology, i.e. a fish-type diagram with an attached propagator (cf. Fig. 3.2). Using conventional recurrence relations based on the integration-by-parts technique, vacuum bubble diagrams can be reduced to master integrals, some of them having the water melon topology with a propagator attached (denoted by "melon+propagator" topology), some of them have a more complicated topology, the "spectacles+propagator" topology.

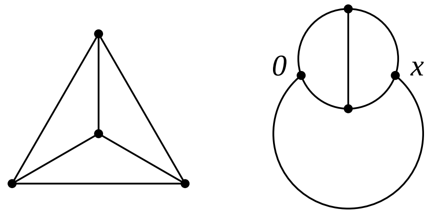


Figure 3.2: Three-loop vacuum bubble diagram in two different representations, namely the tetrahedron representation on the left hand side, and the "fish+propagator" representation on the right hand side where the configuration space points 0 and x are indicated.

The classification of the topology prototypes for three-loop vacuum bubbles was presented in Ref. [110] and shall be used in the following. The analytical computation of some missing master integrals has recently been completed [111]. However, the solution of the recurrence relations leading to the master integrals is complicated and time consuming, especially for large powers of propagators. In this section, therefore, new recurrence relations for a particular topology of vacuum bubbles are suggested which allows for an explicit solution [112]. The simplicity of the presented technique is manifest again in the configuration space representation for Feynman diagrams.

3.4.1 Explicit examples for the water melon topology

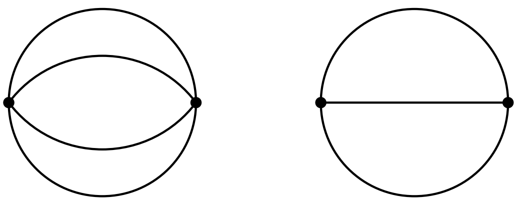


Figure 3.3: Three-loop (i.e. four-line) water melon diagram (left hand side) and two-loop water melon (three-line water melon, being the ordinary sunrise diagram, right hand side).

After a deliberate use of the recurrence relations for bubbles, in some cases two propagators can be removed. A typical situation of such a kind was analyzed in Ref. [83]. In such a case the diagrams become simple indeed. Even the most complicated ones belong to the subclass of water melon topologies which can be computed immediately (see Fig. 3.3). The most attractive feature of such a strategy is that for high derivatives of propagators (large powers of denominators) the corresponding recurrence relations for this particular topology can be solved very efficiently.

While the water melon class of diagrams can appear as part of the remnants of the general recursive procedure, there are some cases when they are just the final aim of the recurrence procedure. This is the case for the B_N subclass of diagrams [83]. Some of the master integrals (for instance, $D_3(0,1,0,1,1,1)$ in Ref. [110]) are exactly water melons. A further simplification of water melon diagrams can be achieved with the use of their particular properties. In the configuration space representation the water melon diagrams can be reduced to a specific basis set of simple integrals quite efficiently.

In the particular case of the subclass B_N of the bubble diagrams, the water melon topology diagrams emerge naturally and can be chosen as master configurations. The reduction of a general diagram of this subclass to the water melon topology is explicitly constructed in Ref. [83]. The expansion in ε within dimensional regularization is straightforward and is explicitly given for the evaluation of the numerical value of the integral B_4 [83] for which a new representation will be given here.

The starting point of the calculation is the definition of the B_N class of diagrams [83],

$$\begin{aligned} B_N(0,0,n_3,n_4,n_5,n_6) &= \int \frac{d^Dk\, d^Dl\, d^Dp}{m^{3D}(\pi^{D/2}\Gamma(1+\varepsilon))^3} \times \\ &\times \frac{m^{2n_3}}{((p+k)^2+m^2)^{n_3}} \frac{m^{2n_4}}{((p+l)^2+m^2)^{n_4}} \frac{m^{2n_5}}{((p+k+l)^2+m^2)^{n_5}} \frac{m^{2n_6}}{(p^2+m^2)^{n_6}} \end{aligned} \tag{3.91}$$

with two propagators absent ($n_1 = n_2 = 0$) to obtain a water melon topology for the three-loop case. In the following these first two indices in the notation for the B_N class diagrams will be suppressed. The configuration space expression for the generalized propagator (with crosses or having differentiated in its mass or momentum) is given by Eq. (3.9). This can be inserted into the above expression for B_N and after rearrangement of integrations leads to

$$\begin{aligned} B_N(n_3,n_4,n_5,n_6) &= \frac{m^{2(n_3+n_4+n_5+n_6)-3D}}{(\pi^{D/2}\Gamma(1+\varepsilon))^3} \times \\ &\times (2\pi)^{3D} \int D^{(n_3-1)}(x,m)D^{(n_4-1)}(x,m)D^{(n_5-1)}(x,m)D^{(n_6-1)}(x,m)d^Dx \end{aligned} \tag{3.92}$$

which can be reduced to a one-dimensional integral using the rotational invariance of the integration measure in Euclidean space-time,

$$d^Dx = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} x^{2\lambda+1} dx. \tag{3.93}$$

Note that various techniques of eliminating tensorial structures for vacuum diagrams were discussed in Refs. [108, 113]. On the other hand one has

$$\left(\frac{px}{2}\right)^{-\lambda} J_\lambda(px) \to \frac{1}{\Gamma(\lambda+1)} \quad \text{for} \quad p \to 0 \tag{3.94}$$

and therefore

$$\tilde{\Pi}(0) = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty D^{(n_3-1)}(x,m)D^{(n_4-1)}(x,m)D^{(n_5-1)}(x,m)D^{(n_6-1)}(x,m)x^{2\lambda+1}dx. \tag{3.95}$$

The comparison of these two formulas results in

$$B_N(n_3,n_4,n_5,n_6) = \frac{(2\pi)^{3D} m^{2(n_3+n_4+n_5+n_6)-3D}}{(\pi^{D/2}\Gamma(1+\varepsilon))^3} \tilde{\Pi}(0) \tag{3.96}$$

where the powers of the propagators occuring in $\tilde{\Pi}(0)$ have been appropiately adjusted. Some explicit applications of Eqs. (3.95) and (3.96) will be discussed in the following.

3.4.2 The integral $B_N(2,2,2,2)$

As a first example the integral $B_N(2,2,2,2)$ is considered for the case $\lambda = 1$ (four-dimensional space-time). The expressions for the generalized propagator is given by

$$D^{(1)}(x,m) = \frac{(x/m)^{1-\lambda}}{(2\pi)^{\lambda+1}2^1\Gamma(2)}K_{\lambda-1}(mx) = \frac{x^\varepsilon m^{-\varepsilon}}{2(2\pi)^{2-\varepsilon}}K_{-\varepsilon}(mx) \tag{3.97}$$

which results in

$$D^{(1)}(x,m) = \frac{1}{2(2\pi)^2}K_0(mx) \tag{3.98}$$

for $\varepsilon = 0$. One obtains

$$B_N(2,2,2,2) = \frac{(2\pi)^{12}m^4}{\pi^6}\;\frac{2\pi^2}{16(2\pi)^8}\int_0^\infty K_0^4(mx)x^3dx = 2\int_0^\infty K_0^4(x)x^3dx \tag{3.99}$$

where mx is replaced by a dimensionless x in the last step. Note that the function $K_0(x)$ is a propagator of a massive particle in two-dimensional space-time. Therefore, many results can be obtained by using two-dimensional field theory in Euclidean space-time (see e.g. Refs. [102, 104]). For the more general case of D-dimensional space-time one obtains

$$\tilde\Pi(0) = \frac{2\pi^{2-\varepsilon}m^{-4\varepsilon}}{16(2\pi)^{8-4\varepsilon}\Gamma(2-\varepsilon)}\int_0^\infty x^{4\varepsilon}K_{-\varepsilon}^4(mx)x^{3-2\varepsilon}dx \tag{3.100}$$

and

$$B_N(2,2,2,2) = \frac{2^{1-2\varepsilon}}{(1-\varepsilon)\Gamma(1+\varepsilon)^3\Gamma(1-\varepsilon)}\int_0^\infty K_{-\varepsilon}^4(x)x^{3+2\varepsilon}x. \tag{3.101}$$

To find higher orders in the ε-expansion necessary for computations within dimensional regularization, series expansions in ε of all quantities entering Eq. (3.101) are used. First one has the rather obvious results

$$\begin{aligned}\frac{2^{1-2\varepsilon}}{(1-\varepsilon)\Gamma(1+\varepsilon)^3\Gamma(1-\varepsilon)} &= 2(1+\varepsilon-2\varepsilon\ln 2+2\varepsilon\gamma_E)+O(\varepsilon^2),\\ x^{3+2\varepsilon} &= x^3(1+2\varepsilon\ln x)+O(\varepsilon^2).\end{aligned} \tag{3.102}$$

Within the dimensional regularization scheme the propagator in the configuration space contains the McDonald function with a non-integer index depending on the regularization parameter ε. To expand the McDonald function in the parameter ε entering its index, one uses the general formula [101]

$$\left[\frac{\partial K_\nu(z)}{\partial\nu}\right]_{\nu=\pm n} = \pm\frac{1}{2}n!\sum_{k=0}^{n-1}\left(\frac{z}{2}\right)^{k-n}\frac{K_k(z)}{k!(n-k)},\qquad n\in\{0,1,\dots\} \tag{3.103}$$

for the derivative of the McDonald function with respect to its index near integer values of this index. In this case one obtains

$$K_{-\varepsilon}(x) = K_0(x)+O(\varepsilon^2). \tag{3.104}$$

One ends up with

$$\begin{aligned}B_N(2,2,2,2) &= 2\int_0^\infty K_0^4(x)x^3dx+2\varepsilon(1+2\ln 2+2\gamma_E)\int_0^\infty K_0^4(x)x^3dx\\ &\qquad +4\varepsilon\int_0^\infty K_0^4(x)x^3\ln x\,dx+O(\varepsilon^2) =\\ &= 2I_0(3)+2\varepsilon(1-2\ln 2+2\gamma_E)I_0(3)+4\varepsilon I_0^l(3)+O(\varepsilon^2)\end{aligned} \tag{3.105}$$

where a general notation for the configuration space integrals

$$\begin{aligned} I_m(q) &= \int_0^\infty K_0^{4-m}(x) K_1^m(x) x^q dx, \\ I_m^l(q) &= \int_0^\infty K_0^{4-m}(x) K_1^m(x) x^q \ln x \, dx \end{aligned} \tag{3.106}$$

is introduced. The only new contribution in the ε-expansion up to the first order in Eq. (3.105) is connected with the logarithmic integral $I_0^l(3)$($I_m^l(q)$ from Eq. (3.106)). The part related to $I_0(3)$ in this order is a trivial kinematic contribution. The term $2(\ln 2 - \gamma_E)$ in Eq. (3.105) can be easily removed by redefining the logarithmic integral using $\ln x \to \ln(x e^{\gamma_E}/2)$.

Identifying the parameters B_3 and B_4 from [83] one finds

$$\begin{aligned} B_N(2,2,2,2) &= -\frac{3}{8} + \frac{7}{16} B_3 + \left(\frac{63}{32} B_3 + \frac{3}{16} B_4\right)\varepsilon + O(\varepsilon^2) = \\ &= -\frac{3}{8} + \frac{7}{16}\zeta(3) + \left(\frac{63}{32}\zeta(3) - \frac{63}{32}\zeta(4) + \frac{3}{16} B_4\right)\varepsilon + O(\varepsilon^2) \end{aligned} \tag{3.107}$$

where $B_3 = \zeta(3) - \frac{9}{2}\varepsilon\zeta(4) + O(\varepsilon^2)$ and $\zeta(n)$ is Riemann's zeta function. The comparison of the zeroth order term of Eq. (3.107) with Eq. (3.105) results in the relation

$$I_0(3) = -\frac{3}{16} + \frac{7}{32}\zeta(3) \tag{3.108}$$

which assigns a value to one of the initial terms of the recurrence relations which will dealt with in the following. The relation was checked numerically. In the first order part of the ε-expansion one solves for B_4 obtaining the representation

$$\begin{aligned} B_4 &= \frac{16}{3}\Big(2(1 - 2\ln 2 + 2\gamma_E) I_0(3) + 4 I_0^l(3) + \frac{63}{32}(\zeta(4) - \zeta(3))\Big) = \\ &= \frac{32}{3}\Big((1 - 2\ln 2 + 2\gamma_E) I_0(3) + 2 I_0^l(3)\Big) + \frac{21}{2}\Big(\zeta(4) - \zeta(3)\Big) \end{aligned} \tag{3.109}$$

which after substituting for $I_0^l(3)$ from Eq. (3.106) gives numerically $B_4 = -1.7628\ldots$ This numerical value expressed in terms of configuration space integrals of the technique presented here coincides with the result given in Ref. [83]. Taking the analytical expression for B_4 from Ref. [83],

$$B_4 = 16\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{2}{3}\ln^4(2) - \frac{2}{3}\pi^2\ln^2(2) - \frac{13}{180}\pi^4 \tag{3.110}$$

with $\mathrm{Li}_4(z)$ being a fourth order polylogarithm,

$$\mathrm{Li}_4(z) = \sum_{k=1}^\infty \frac{z^k}{k^4}, \qquad |z| < 1, \tag{3.111}$$

one obtains the result for the logarithmic integral $\tilde{I}_0^l(3)$,

$$\begin{aligned} \tilde{I}_0^l(3) &= \int_0^\infty K_0^4(x) \ln(x e^{\gamma_E}/2) x^3 \, dx = \\ &= \frac{3}{32} + \frac{3}{4}\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{17}{1920}\pi^4 - \frac{1}{32}\pi^2\ln^2(2) + \frac{1}{32}\ln^4(2) + \frac{49}{128}\zeta(3) \end{aligned} \tag{3.112}$$

which serves as the initial value for the recurrence relations for the set of logarithmic integrals. This value is checked numerically as well.

3.4.3 The integrals $B_N(2,2,2,1)$ and $B_N(2,3,3,4)$

For a further demonstration of the efficiency of the configuration space technique for fixed powers of the propagators, the two integrals $B_N(2,2,2,1)$ and $B_N(2,3,3,4)$ are calculated (the latter does not contain B_4 which is the reason for having selected this example). For the integral $B_N(2,2,2,1)$ in four-dimensional space-time ($\lambda = 1$) one has to include the propagator

$$D^{(0)}(x,m) = \frac{(x/m)^{-\lambda}}{(2\pi)^{\lambda+1}2^0\Gamma(1)}K_\lambda(mx) = \frac{x^{\varepsilon-1}m^{1-\varepsilon}}{(2\pi)^{2-\varepsilon}}K_{1-\varepsilon}(mx) \tag{3.113}$$

equal to $mK_1(mx)/4\pi^2 x$ for $\lambda = 1$ which is a standard propagator of a massive particle for $D = 4$. One obtains a representation of the form

$$B_N(2,2,2,1) = 4\int K_0^3(x)K_1(x)x^2dx = 4I_1(2) \tag{3.114}$$

(as before, the mass m is absorbed in x). For the integral $B_N(2,2,2,1)$ in the case $\lambda = 1-\varepsilon$ the generalization of Eq. (3.114) of the form

$$B_N(2,2,2,1) = \frac{2^{2-2\varepsilon}}{(1-\varepsilon)\Gamma(1+\varepsilon)^3\Gamma(1-\varepsilon)}\int_0^\infty K_{-\varepsilon}^3(x)K_{1-\varepsilon}(x)x^{2+2\varepsilon}dx \tag{3.115}$$

is obtained. The ε-expansion of the factor multiplying the integral is the same as in the former case except for an overall factor of 2. A similar statement is valid for the expansion of the power of x. What remains to be done is to expand the McDonald functions in the vicinity of integer values of their indices. To obtain this expansion one can use the relation

$$\frac{\partial K_\nu(x)}{\partial\nu}\Big|_{\nu=1} = \frac{1}{2}\left(\frac{x}{2}\right)^{-1}K_0(x) = \frac{1}{x}K_0(x) \tag{3.116}$$

which contributes to the power expansion as

$$K_{1-\varepsilon}(x) = K_1(x) - \frac{\varepsilon}{x}K_0(x) + O(\varepsilon^2). \tag{3.117}$$

Using these expansions one obtains the representation in terms of the basic integrals,

$$\begin{aligned} B_N(2,2,2,1) &= 4\int_0^\infty x^2K_0^3(x)K_1(x)dx\ + \\ &\quad +4\varepsilon(1-2\ln 2+2\gamma_E)\int_0^\infty x^2K_0^3(x)K_1(x)dx\ + \\ &\quad -4\varepsilon\int_0^\infty xK_0^4(x)dx + 8\varepsilon\int_0^\infty x^2\ln x\,K_0^3(x)K_1(x)dx + O(\varepsilon^2)\ = \\ &= 4I_1(2) + 4\varepsilon(1-2\ln 2+2\gamma_E)I_1(2) - 4\varepsilon I_0(1) + 8\varepsilon I_1^l(2) + O(\varepsilon^2). \end{aligned}$$

This result has to be compared with the output of the RECURSOR package [83] in terms of the explicit master integrals B_3 and B_4,

$$\begin{aligned} B_N(2,2,2,1) &= \frac{7}{4}B_3 + \frac{3}{4}B_4\varepsilon + O(\varepsilon^2)\ = \\ &= \frac{7}{4}\zeta(3) + \left(\frac{3}{4}B_4 - \frac{63}{8}\zeta(4)\right)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{3.118}$$

The zeroth order comparison gives the result $I_1(2) = 7\zeta(3)/16$ which has been verified numerically. The first order comparison results in

$$\begin{aligned} B_4 &= \frac{4}{3}(4(1-2\ln 2+2\gamma_E)I_1(2)-4I_0(1)+8I_1^l(2)+\frac{63}{8}\zeta(4)) = \\ &= \frac{16}{3}((1-2\ln 2+2\gamma_E)I_1(2)-I_0(1)+2I_1^l(2))+\frac{21}{2}\zeta(4) = -1.7628 \end{aligned} \tag{3.119}$$

as before (see Eq. (3.109)). For the integral $B_N(2,3,3,4)$ in the case $\lambda = 1-\varepsilon$ one needs to include further propagators. They are given by

$$\begin{aligned} D^{(2)}(x,1) &= \frac{x^{2-\lambda}}{(2\pi)^{\lambda+1}2^2\Gamma(3)}K_{\lambda-2}(x) = \frac{x^{1+\varepsilon}}{8(2\pi)^{2-\varepsilon}}K_{-1-\varepsilon}(x), \\ D^{(3)}(x,1) &= \frac{x^{3-\lambda}}{(2\pi)^{\lambda+1}2^3\Gamma(4)}K_{\lambda-3}(x) = \frac{x^{2+\varepsilon}}{48(2\pi)^{2-\varepsilon}}K_{-2-\varepsilon}(x). \end{aligned} \tag{3.120}$$

Both McDonald functions have to be expanded in their index. One has $K_{-1}(x) = K_1(x)$ and

$$\frac{\partial K_\nu(x)}{\partial\nu}\Big|_{\nu=-1} = -\frac{1}{2}\left(\frac{x}{2}\right)^{-1}K_0(x) = -\frac{1}{x}K_0(x), \tag{3.121}$$

thus

$$K_{-1-\varepsilon}(x) = K_1(x)+\frac{\varepsilon}{x}K_0(x)+O(\varepsilon^2), \tag{3.122}$$

and

$$\frac{\partial K_\nu(x)}{\partial\nu}\Big|_{\nu=-2} = -\frac{1}{2}2!\left(\frac{1}{2!}\left(\frac{x}{2}\right)^{-2}K_0(x)+\frac{1}{1!1!}\left(\frac{x}{2}\right)^{-1}K_1(x)\right) = -\frac{2}{x^2}K_0(x)-\frac{2}{x}K_1(x), \tag{3.123}$$

so that

$$K_{-2-\varepsilon} = K_2(x)+\frac{2\varepsilon}{x}K_1(x)+\frac{2\varepsilon}{x^2}K_0(x)+O(\varepsilon^2). \tag{3.124}$$

With these relations the ε-expansion for the integral in question is obtained,

$$\begin{aligned} B_N(2,3,3,4) &= \frac{2^{-2\varepsilon}}{192(1-\varepsilon)\Gamma(1+\varepsilon)^3\Gamma(1-\varepsilon)} \times \\ &\qquad \times \int_0^\infty K_{-\varepsilon}(x)K_{-1-\varepsilon}^2(x)K_{-2-\varepsilon}(x)x^{7+2\varepsilon}dx = \\ &= \frac{1}{192}I_{21}(7)+\frac{\varepsilon}{192}(1-2\ln 2+2\gamma_E)I_{21}(7)+\frac{\varepsilon}{96}I_{11}(6) \\ &\quad +\frac{\varepsilon}{96}I_3(6)+\frac{\varepsilon}{96}I_2(5)+\frac{\varepsilon}{96}I_{21}^l(7)+O(\varepsilon^2) \end{aligned} \tag{3.125}$$

where the generalized integral

$$I_{mn}(q) = \int_0^\infty K_0^{4-m-n}(x)K_1^m(x)K_2^n(x)x^q dx \tag{3.126}$$

is introduced. This expansion has to be compared with the representation through master integrals found in momentum space,

$$B_N(2,3,3,4) = \frac{1}{576}+\left(\frac{385}{65536}B_3-\frac{809}{884736}\right)\varepsilon+O(\varepsilon^2) \tag{3.127}$$

resulting in the identification

$$I_{21}(7) = \int_0^\infty K_0(x)K_1^2(x)K_2(x)x^7dx = \frac{1}{3} \tag{3.128}$$

which is surprisingly simple and contains no transcendental numbers usually present in such integrals. It is a curiosity that a similar identification allows one to express $\zeta(3)$ in terms of the basis integrals,

$$\begin{aligned} \zeta(3) &= \frac{1024}{1155}\Big((1-2\ln 2+2\gamma_E)I_{21}(7)+2I_{11}(6) \\ &\quad +2I_3(6)+2I_2(5)+2I_{21}^l(7)+\frac{809}{4608}\Big). \end{aligned} \tag{3.129}$$

Both equations are checked numerically to make certain that they are valid. These results serve as a hint that the standard basis may not be the simplest and most relevant basis for computations of massive three-loop diagrams.

3.4.4 The recursion

The preceding section has shown that three-loop water melon diagrams can be expressed as configuration space integrals of a product of at most four McDonald functions $K_\nu(x)$ where ν need not be an integer. In this subsection the three steps of how to reduce the set of necessary integrals to a smaller set are to be exhibited. First one can get rid of the non-integer dimensionality of the functions by using Eq. (3.103) to expand into powers of ε resulting in integrals containing a product of four McDonald functions with or without a factor $\ln(x)$. As a second step one uses the relation

$$K_n(x) = 2\frac{n-1}{x}K_{n-1}(x)+K_{n-2}(x) \tag{3.130}$$

for McDonald functions of different orders to further reduce the integrals to integrals only containing K_0^4, $K_0^3K_1$, $K_0^2K_1^2$, $K_0K_1^3$ and K_1^4 together with some positive powers of x and again with or without a factor $\ln(x)$. The last step consists in using

$$\frac{d}{dx}K_\nu(x) = -\frac{1}{2}(K_{\nu-1}(x)+K_{\nu+1}(x)) \tag{3.131}$$

for integer ν (which is valid for any complex ν as well) and a partial integration in order to reduce the necessary integrals to integrals containing only the McDonald function $K_0(x)$. For this purpose one uses two special cases of Eq. (3.131),

$$\begin{aligned} \frac{d}{dx}K_0(x) &= -K_1(x) \qquad \text{and} \\ \frac{d}{dx}K_1(x) &= -\frac{1}{2}(K_0(x)+K_2(x)) \;=\; -K_0(x)-\frac{1}{x}K_1(x). \end{aligned} \tag{3.132}$$

This recurrence procedure will now be considered for two different cases.

For integrals not containing logarithms, after some simple algebra one obtains the reduction relation

$$\begin{aligned} \int_0^\infty K_1^4(x)x^qdx &= -\int_0^\infty \frac{d}{dx}(K_0(x))K_1(x)^3x^qdx \;= \\ &= -\Big[K_0(x)K_1^3(x)x^q\Big]_0^\infty - 3\int_0^\infty K_0^2(x)K_1^2(x)x^qdx + (q-3)\int_0^\infty K_0(x)K_1^3(x)x^{q-1}dx \end{aligned} \tag{3.133}$$

for the most tedious case of four factors $K_1(x)$ in the integrand. The other cases are simpler and will not be written down here. For $q > m$, the surface terms of the form $[K_0^{(4-m)}(x)K_1^m(x)x^q]_0^\infty$ vanishes. Therefore, the only elements of this recursion are the integrals $I_m(q)$, and the recursion is expressed as

$$\begin{aligned}
I_4(q) &= (q-3)I_3(q-1) - 3I_2(q), \\
I_3(q) &= \frac{1}{2}((q-2)I_2(q-1) - 2I_1(q)), \\
I_2(q) &= \frac{1}{3}((q-1)I_1(q-1) - I_0(q)), \\
I_1(q) &= \frac{1}{4}qI_0(q-1),
\end{aligned} \tag{3.134}$$

which reduces the starting integrals to the basis integrals

$$I_0(q) = \int_0^\infty K_0^4(x)x^q dx. \tag{3.135}$$

For integrals containing a single logarithm, on the other hand, one finds

$$\begin{aligned}
\int_0^\infty K_1^4(x)x^q \ln x\, dx &= -\int_0^\infty K_1^3(x)\frac{dK_0(x)}{dx}x^q \ln x\, dx \\
= &-\Big[K_0(x)K_1^3(x)x^q \ln x\Big]_0^\infty - 3\int_0^\infty K_0^2(x)K_1^2(x)x^q \ln x\, dx\ + \\
&+(q-3)\int_0^\infty K_0(x)K_1^3(x)x^{q-1}\ln x\, dx + \int_0^\infty K_0(x)K_1^3(x)x^{q-1}dx
\end{aligned} \tag{3.136}$$

for the case with four functions $K_1(x)$ in the integrand. For integer $q > m$, the surface terms $[K_0^{4-m}(x)K_1^m(x)x^q \ln x]_0^\infty$ vanish again. Therefore, the recursion is expressed in terms of the integrals $I_m^l(q)$ and is given by

$$\begin{aligned}
I_4^l(q) &= (q-3)I_3^l(q-1) + I_3(q-1) - 3I_2^l(q), \\
I_3^l(q) &= \frac{1}{2}((q-2)I_2^l(q-1) + I_2(q-1) - 2I_1^l(q)), \\
I_2^l(q) &= \frac{1}{3}((q-1)I_1^l(q-1) + I_1(q-1) - I_0^l(q)), \\
I_1^l(q) &= \frac{1}{4}(qI_0^l(q-1) + I_0(q-1)).
\end{aligned} \tag{3.137}$$

Together with Eqs. (3.134) these relations give the complete set of one-parameter recurrence equations for reducing a general water melon integral to a set of master integrals of the form $I_0(q)$ from Eq. (3.135) and

$$I_0^l(q) = \int_0^\infty K_0^4(x)x^q \ln x\, dx. \tag{3.138}$$

All steps of the reduction as described before are at hand now. Therefore, Eqs. (3.135) and (3.138) constitute the basis for the evaluation of water melon diagrams with high powers of denominators.

3.4.5 The genuine sunrise case $N = 3$

The case of a two-loop water melon (genuine sunrise) is simple indeed and can be easily analyzed along the same lines. The corresponding basis set of configuration space integrals is quite analogous to the previous case and is simpler because it now includes only three McDonald functions,

$$\begin{aligned} J_n(q) &= \int_0^\infty K_0^{3-n}(x) K_1^n(x) x^q dx, \\ J_n^l(q) &= \int_0^\infty K_0^{3-n}(x) K_1^n(x) x^q \ln x \, dx. \end{aligned} \tag{3.139}$$

The reduction to the basis set of integrals analogous to the case of three-loop water melons given in Eqs. (3.135) and (3.137) can be readily obtained.

3.4.6 The efficiency of the reduction

The direct reduction of a water melon diagram to the master integrals is rather slow within the straightforward application of recurrence relations based on the momentum space representation. In practice the computation proceeds through the use of a table of integrals with given powers of the denominators. One would have a three-dimensional table for a given total power N if no modifications of the basic technique as developed in Ref. [83] have been introduced. Therefore, the number of entries grows as fast as N^3, compared to an increase as a first power of N for the reduction method introduced here (see Ref. [112] for more details). Note that in Refs. [114, 115] different recursion techniques have been described which also avoid the use of the three-dimensional tables in reducing the water-melon diagram. For instance, in the package MATAD [114] the three-loop water melon diagrams are reduced to a one-dimensional table of integrals using a dedicated set of (momentum-based) recurrence relations.

3.4.7 The generalization to the spectacle topology

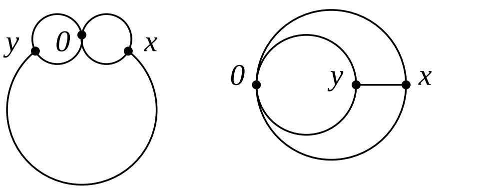


Figure 3.4: "spectacle+propagator" representation, also called "spectacle" topology diagram, in two different forms. The configurations space points 0, x and y are indicated.

If starting with the general tetrahedron topology, one propagator is removed, one ends up with the spectacle topology, as shown in Fig. 3.4 in two different representations. The configurations space expression of a spectacle topology diagram (with a different mass M on the frame) written in a form suitable for the actual purpose is given by

$$S = \int D(x-y, M) D(x, m)^2 D(y, m)^2 d^D x \, d^D y. \tag{3.140}$$

The key relation for a drastic simplification of the configuration space integral with the spectacle topology is the addition theorem for Bessel functions allowing one to perform some angular integration explicitly. One needs to integrate over the relative angle in the propagator $D(x-y, M)$. The relation used here is given by [101]

$$\frac{Z_\lambda(mR)}{R^\lambda} = 2^\lambda m^{-\lambda} \Gamma(\lambda) \sum_{k=0}^\infty (\lambda + k) \frac{J_{\lambda+k}(m\rho)}{\rho^\lambda} \frac{Z_{\lambda+k}(mr)}{r^\lambda} C_k^\lambda(\cos\varphi) \tag{3.141}$$

where C_k^λ are the Gegenbauer polynomials, Z is any of the Bessel functions J, N, or $H^\pm$,

$$R = \sqrt{r^2 + \rho^2 - 2r\rho\cos\varphi} \tag{3.142}$$

and $r > \rho$. For $r < \rho$ the arguments of the Bessel functions on the right hand side of Eq. (3.141) should be interchanged. For $Z = H^+$ and $m = e^{i\pi/2}M$ one can use

$$\begin{aligned} K_\lambda(z) &= \frac{i\pi}{2} e^{i\pi\lambda/2} H_\lambda^+(e^{i\pi/2}z), \\ I_\lambda(z) &= e^{-i\pi\lambda/2} J_\lambda(e^{i\pi/2}z) \qquad \text{for } -\pi < \arg z \le \frac{\pi}{2}, \\ I_\lambda(z) &= e^{3i\pi\lambda/2} J_\lambda(e^{-3i\pi/2}z) \qquad \text{for } \frac{\pi}{2} < \arg z \le \pi \end{aligned} \tag{3.143}$$

to obtain

$$\frac{K_\lambda(MR)}{R^\lambda} = 2^\lambda \Gamma(\lambda) \sum_{k=0}^{\infty} (\lambda + k) \frac{I_{\lambda+k}(M\rho)}{\rho^\lambda} \frac{K_{\lambda+k}(Mr)}{r^\lambda} C_k^\lambda(\cos\varphi), \qquad r > \rho. \tag{3.144}$$

The sum disappears after integration over the relative angle and only one term contributes,

$$\begin{aligned} \int \frac{K_\lambda(MR)}{R^\lambda} d\Omega_\rho &= 2^\lambda \Gamma(\lambda) \sum_{k=0}^{\infty} (\lambda + k) \frac{I_{\lambda+k}(M\rho)}{\rho^\lambda} \frac{K_{\lambda+k}(Mr)}{r^\lambda} \int C_k^\lambda(\cos\varphi) d\Omega_\varphi = \\ &= 2^\lambda \Gamma(\lambda) \lambda \frac{I_\lambda(M\rho)}{\rho^\lambda} \frac{K_\lambda(Mr)}{r^\lambda} \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} C_0^\lambda(1) = \\ &= (2\pi)^{\lambda+1} \frac{I_\lambda(M\rho)}{\rho^\lambda} \frac{K_\lambda(Mr)}{r^\lambda}, \qquad r > \rho, \end{aligned} \tag{3.145}$$

where the first equality is a consequence of the orthogonality relation for the Gegenbauer polynomials (cf. Appendix D.4) with the trivial factor $C_0^\lambda(1) = 1$. The result

$$\begin{aligned} &\int D(x - y, M) d\Omega_x d\Omega_y = \\ &= \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} \int D(R, M) d\Omega_x = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} \int \frac{(MR)^\lambda K_\lambda(MR)}{(2\pi)^{\lambda+1} R^{2\lambda}} d\Omega_x = \\ &= \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} M^\lambda \left(\frac{K_\lambda(Mx)}{x^\lambda} \frac{I_\lambda(My)}{y^\lambda} \theta(x - y) + \frac{K_\lambda(My)}{y^\lambda} \frac{I_\lambda(Mx)}{x^\lambda} \theta(y - x) \right) \end{aligned} \tag{3.146}$$

allows one to write down an expression for any spectacle-type diagram in the form of a two-fold integral with a simple integration measure,

$$\begin{aligned} S &= \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} M^\lambda \int_0^\infty D(x, m)^2 x^{2\lambda+1} dx \int_0^\infty D(y, m)^2 y^{2\lambda+1} dy \times \\ &\times \left(\frac{K_\lambda(Mx)}{x^\lambda} \frac{I_\lambda(My)}{y^\lambda} \theta(x - y) + \frac{K_\lambda(My)}{y^\lambda} \frac{I_\lambda(Mx)}{x^\lambda} \theta(y - x) \right) \end{aligned} \tag{3.147}$$

where $\theta(x)$ is the standard step-function distribution.

3.4.8 Computation of the integral basis for $N = 4$

A recursion only makes sense if starting values can be given. These starting values are discussed in the following. For even space-time dimensions, the general integral left over by the recursion has the form

$$I(q) = \int_0^\infty K_0^4(x) x^{2q+1} dx = I_0(2q+1). \tag{3.148}$$

First, the case $q = 0$ shall be considered. The integral $I(0)$ is the result for a four-line (three-loop) water melon diagram with a massive propagator $D(x,m) = K_0(mx)/2\pi$ within a two-dimensional theory with measure $2\pi x\,dx$. The corresponding two-line water melon (master one-loop diagram) in momentum space has the explicit form [102]

$$\tilde{\Pi}_2(p^2) = \frac{1}{2\pi\sqrt{p^2}\sqrt{p^2+4m^2}} \ln\left(\frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}}\right). \tag{3.149}$$

Using the substitution $p^2 = 4m^2\sinh^2(\eta/2)$, one obtains

$$\tilde{\Pi}_2(4m^2\sinh^2(\eta/2)) = \frac{1}{4\pi m^2}\frac{\eta}{\sinh\eta}, \qquad d^2p = 2\pi p\,dp = 2\pi m^2\sinh\eta\,d\eta \tag{3.150}$$

and therefore

$$I(0) = 2\pi m^2 \int \tilde{\Pi}_2(p)^2 d^2p = \frac{1}{4}\int_0^\infty \frac{\eta^2 d\eta}{\sinh\eta} = \frac{7}{8}\zeta(3) \tag{3.151}$$

where the standard integral [91]

$$\int_0^\infty \frac{\eta^{\alpha-1}d\eta}{\sinh\eta} = \frac{2^\alpha-1}{2^{\alpha-1}}\Gamma(\alpha)\zeta(\alpha) \tag{3.152}$$

has been used. Results for other values of q can be obtained by differentiating one of the two $\tilde{\Pi}_2$ in the integrand,

$$I(q) = 2\pi m^2 \int \tilde{\Pi}_2(p)(-m^2\Box_p)^q \tilde{\Pi}_2(p) d^2p \tag{3.153}$$

where $\Box_p$ is a two-dimensional d'Alembert operator in (Euclidean) momentum space, $\Box_p = \partial^2/\partial p_\mu \partial p^\mu$. There is a possibility to differentiate a separate line of this three-loop water melon diagram that leads to different representations for higher moments, but Eq. (3.153) is found to be the most convenient one. A general analytical solution to Eq. (3.153) for arbitrary large q is not yet available at present, but solutions for some first values of q can be found by reducing them to the standard set of master integrals, resulting in analytic expression such as $I(q) = A_q\zeta(3) - B_q$ where A_q and B_q are rational positive numbers. The details of these considerations can be found in Ref. [112]. However, one should mention that an analytic expression as $I(q) = A_q\zeta(3) - B_q$ is not very convenient because for high values the values for $A_q\zeta(3)$ and B_q become quite large while they nearly cancel in the difference. Therefore, it is more convenient in this case to give estimates for $I(q)$. Such estimates can be found in Ref. [112] as well.

3.4.9 Computation of the integral basis for $N = 3$

The basic initial integral (the basic sunrise diagram B_S) for the recurrence relation has the explicit form

$$B_S = \int \frac{\tilde{\Pi}_2(p^2)}{p^2 + M^2} d^2p \tag{3.154}$$

with $\tilde{\Pi}_2(p^2)$ given by Eq. (3.149). Here M is a mass of the third line which is kept different from the other two with masses m. By differentiating with respect to M, any positive power of the propagator (and/or power of x in configuration space representation) can be obtained. After changing the integration variable as in the preceding subsection one finds an explicit representation

$$B_S = \int_0^\infty \frac{\eta\, d\eta}{4m^2 \sinh^2(\eta/2) + M^2}. \tag{3.155}$$

After a change of variables the integration can be done and can be reduced to a polylogarithm function. Namely, for $t = e^{-\eta}$ one has

$$B_S = \int_0^\infty \frac{\eta\, d\eta}{4m^2 \sinh^2(\eta/2) + M^2} = -\frac{1}{m^2} \int_0^1 \frac{\ln t\, dt}{1 - 2\gamma t + t^2} = -\frac{\mathrm{Li}_2(1/t_1) - \mathrm{Li}_2(1/t_2)}{m^2(t_1 - t_2)} \tag{3.156}$$

where $\gamma = 1 - M^2/2m^2$, $t_{1,2} = \gamma \pm \sqrt{\gamma^2 - 1}$ and $\mathrm{Li}_2(z)$ is the dilogarithm function,

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \qquad |z| < 1. \tag{3.157}$$

The differentiation with respect to M is now straightforward and can be performed with a symbolic manipulation program. Because γ is a real number less or equal to one, t_1 and t_2 are either the same as γ (for $\gamma = \pm 1$) or complex conjugate numbers,

$$t_1 = \bar{t}_2 = \gamma + i\sqrt{1 - \gamma^2} =: \zeta. \tag{3.158}$$

Because $|\zeta| = 1$, possible values for the parameter ζ are referred to as *roots of unity* in the following, as it is done in the literature (cf. Ref. [111]).

In the case $M = 2m$ the integration simplifies because the two independent parameters M and $2m$ on which the integrand depends, coincide (degenerate case). The integral is then reduced to a special case of Eq. (3.152). One has $\gamma = -1$, $\zeta = -1$ and

$$\int_0^1 \frac{\ln t\, dt}{(1 + t)^2} = -\ln 2 \tag{3.159}$$

which leads to

$$B_S(M = 2m) = \frac{\ln 2}{m^2}. \tag{3.160}$$

For the case $M = m$ the standard result – *Clausen's polylogarithm* $\mathrm{Cl}_2(\pi/3)$ – is reproduced (see e.g. Ref. [111]). Indeed, $\gamma = 1/2$ and $\zeta = \exp(i\pi/3)$, so one obtains

$$B_S(M = m) = -\frac{1}{m^2} \int_0^1 \frac{\ln t\, dt}{1 - t + t^2} = \frac{2}{m^2\sqrt{3}} \mathrm{Im}\ \mathrm{Li}_2(e^{i\pi/3}) = \frac{2}{m^2\sqrt{3}} \mathrm{Cl}_2\left(\frac{\pi}{3}\right). \tag{3.161}$$

where the definition of Clausen's polylogarithm, $\mathrm{Cl}_2(\theta) = \mathrm{Im}\,\mathrm{Li}_2(e^{i\theta})$ has been used.

3.4.10 Transcendental numbers for three-loop bubbles

At this point a closer look at the water melon and spectacle topologies is in order. In calculating the general three-loop bubble topology (tetrahedron topology), many different transcendental numbers (the simplest being given by $\zeta(3)$ or $\ln(2)$) are found in the results. The argument of Ref. [116] is that all necessary transcendental numbers that appear in the tetrahedron case can already be found in the simpler spectacle and water melon topology. This argument will be explained in more detail in these subsections.

As before, the main building block for the treatment of three-loop vacuum bubbles is the one-loop two-line massive correlator $\Pi(p^2)$ in $D = 2 - 2\varepsilon$ dimensional (Euclidean) space-time,

$$\begin{aligned}\tilde{\Pi}(p^2) &= \int \frac{d^D k}{((p-k)^2+m^2)(k^2+m^2)} \\ &= \frac{2^{3+2\varepsilon}\pi^{1-\varepsilon}\Gamma(1+\varepsilon)}{(p^2+4m^2)^{1+\varepsilon}}\, {}_2F_1\left(1+\varepsilon, \frac{1}{2}; \frac{3}{2}; \frac{p^2}{p^2+4m^2}\right)\end{aligned} \tag{3.162}$$

with ${}_2F_1(a, b; c; z)$ being the *hypergeometric function* or, alternatively (cf. Eq. (3.63))

$$\tilde{\Pi}(p^2) = \int_{4m^2}^{\infty} \frac{\rho(s)ds}{s+p^2}, \qquad \rho(s) = \frac{(s-4m^2)^{\varepsilon}}{2\pi\sqrt{s(s-4m^2)}}\, \frac{\pi^{1/2-\varepsilon}}{2^{4\varepsilon}\Gamma(1/2+\varepsilon)}. \tag{3.163}$$

In order to reproduce the transcendental structure of the finite parts of the tetrahedron in four dimensions a first order ε expansion of water melons and spectacles near two-dimensional space-time is needed. Note that these diagrams are well-defined and ultraviolet finite in two dimensions and, formally, require no regularization. However, the sought-for transcendental structure appears only in higher orders of the ε expansion while the leading order is simple and contains only the standard transcendental numbers such as $\zeta(3)$ or $\ln(2)$. Therefore one writes

$$\tilde{\Pi}(p^2) = \tilde{\Pi}_2(p^2) + \varepsilon \Delta_\varepsilon \tilde{\Pi}_2(p^2) + O(\varepsilon^2) \tag{3.164}$$

and keeps only the first order in ε which happens to be sufficient for the goal of finding all the transcendental numbers appearing in the tetrahedron case.

The calculation of the zeroth and first order term in ε of the correlator function is done by using the expansion of the spectral density $\rho(s)$, given in Eq. (3.163),

$$\rho(s) = \frac{1}{2\pi\sqrt{s(s-4m^2)}}\left\{1 + \varepsilon \ln\left(\frac{s-4m^2}{m^2}\right) + O(\varepsilon^2)\right\} =: \rho_2(s) + \varepsilon\Delta_\varepsilon\rho_2(s) + O(\varepsilon^2). \tag{3.165}$$

For both parts one applies the substitution $s = 4m^2\cos^2(\xi/2)$. Even though the result for the leading order part was shown already, the calculation is detailed here also for this part in order to be able to take the same steps for the part of order ε. Inserting the leading order part into Eq. (3.163), one obtains

$$\tilde{\Pi}_2(p^2) = \int_{4m^2}^{s} \frac{\rho_2(s)ds}{s+p^2} = \frac{1}{2\pi}\int_0^\infty \frac{d\xi}{4m^2\cos^2(\xi/2)+p^2} = \frac{1}{2\pi}\int_0^\infty \frac{d\xi}{m^2(e^\xi+2+e^{-\xi})+p^2}. \tag{3.166}$$

Using a second substitution $t = e^{-\xi}$, one ends up with

$$\tilde{\Pi}_2(p^2) = \frac{1}{2\pi m^2} \int_0^1 \frac{dt}{1 + 2t + t^2 + p^2 t/m^2}. \tag{3.167}$$

At this point it is again natural to use $p^2 = 4m^2 \sinh^2(\eta/2)$ to obtain

$$\tilde{\Pi}_2(4m^2 \sinh^2(\eta/2) = \frac{1}{2\pi m^2} \int_0^1 \frac{dt}{1 + (e^\eta + e^{-\eta})t + t^2} = \frac{1}{2\pi m^2} \int_0^1 \frac{dt}{(t + e^\eta)(t + e^{-\eta})}. \tag{3.168}$$

One can use the partional fractioning

$$\frac{dt}{(t + e^\eta)(t + e^{-\eta})} = \frac{dt}{e^\eta - e^{-\eta}} \left(\frac{1}{t + e^\eta} - \frac{1}{t + e^{-\eta}} \right) = \frac{dt}{2\sinh\eta} \left(\frac{1}{t + e^\eta} - \frac{1}{t + e^{-\eta}} \right) \tag{3.169}$$

before performing the integration over t which results in

$$\begin{aligned} \tilde{\Pi}_2(4m^2 \sinh^2(\eta/2) &= \frac{1}{4\pi m^2 \sinh\eta} \Big[\ln(t + e^{-\eta}) - \ln(t + e^\eta) \Big]_0^1 = \\ &= \frac{1}{4\pi m^2 \sinh\eta} \left(\ln\left(\frac{1 + e^{-\eta}}{e^{-\eta}} \right) - \ln\left(\frac{1 + e^\eta}{e^\eta} \right) \right) = \\ &= \frac{1}{4\pi m^2 \sinh\eta} \left(\ln(e^\eta + 1) - \ln\left(\frac{1 + e^\eta}{e^\eta} \right) \right) = \\ &= \frac{1}{4\pi m^2 \sinh\eta} \ln(e^\eta) = \frac{\eta}{4\pi m^2 \sinh\eta}. \end{aligned} \tag{3.170}$$

With the same substitutions one has for the first order correction

$$\Delta_\varepsilon \tilde{\Pi}_2(4m^2 \sinh^2(\eta/2)) = \frac{1}{4\pi m^2 \sinh\eta} \int_0^1 \left(\frac{1}{t + e^{-\eta}} - \frac{1}{t + e^\eta} \right) (2\ln(1-t) - \ln t) dt. \tag{3.171}$$

Using

$$\int_0^1 \frac{\ln(1-t)dt}{t+y} = -\operatorname{Li}_2\left(\frac{1}{1+y} \right), \qquad \int_0^1 \frac{\ln t\, dt}{t+y} = \operatorname{Li}_2\left(\frac{-1}{y} \right) \tag{3.172}$$

and the fundamental dilogarithm relations (see Appendix E) one obtains

$$\begin{aligned} \operatorname{li}_2\left(\frac{1}{1+y} \right) &= -\operatorname{li}_2(1+y) - \frac{1}{2}\ln^2(-1-y) = \\ &= \operatorname{li}_2(-y) + \ln(1+y)\ln(-y) - \frac{1}{2}\ln^2(-1-y), \\ \operatorname{li}_2\left(\frac{-1}{y} \right) &= -\operatorname{li}_2(-y) - \frac{1}{2}\ln^2 y \end{aligned} \tag{3.173}$$

($\operatorname{li}_2(z)$ is the *indefinite dilogarithm*, see Appendix E.1) and therefore

$$\Delta_\varepsilon \tilde{\Pi}(4m^2 \sinh^2(\eta/2)) = \frac{-1}{4\pi m^2 \sinh\eta} \left[2\operatorname{li}_2\left(\frac{1}{1+y} \right) + \operatorname{li}_2\left(\frac{-1}{y} \right) \right]_{y=e^\eta}^{e^{-\eta}} = \frac{-f(e^{-\eta})}{4\pi m^2 \sinh\eta} \tag{3.174}$$

where

$$\begin{aligned} f(y) &:= \left[\mathrm{li}_2(-t)+2\ln(1+t)\ln(-t)-\ln^2(-1-t)-\frac{1}{2}\ln^2 t\right]_{1/y}^{y} = \\ &= 2\,\mathrm{Li}_2(-y)+2\ln y\ln(1+y)-\frac{1}{2}\ln^2 y+\frac{\pi^2}{6} = \\ &= 2\int_0^y \frac{\ln z}{1+z}dz-\frac{1}{2}\ln^2 y+\zeta(2). \end{aligned} \tag{3.175}$$

In the last step the two first terms were expressed by an integral again.

3.4.11 Water melons and spectacles from a new point of view

The function $f(y)$ is of central inportance for the following. Expanding the water melon diagram up to first order in ε,

$$W = 2\pi m^2\int\tilde{\Pi}(p^2)^2d^2p = W_0+\varepsilon W_1+O(\varepsilon^2), \tag{3.176}$$

the leading order contribution reproduces the previous result,

$$W_0 = 2\pi m^2\int\tilde{\Pi}_2(p^2)^2d^2p = \frac{7}{8}\zeta(3). \tag{3.177}$$

For the first order term one obtains

$$\begin{aligned} W_1 &= (4\pi m^2)^2\int_0^\infty\tilde{\Pi}_2(4m^2\sinh^2(\eta/2))\Delta_\varepsilon\tilde{\Pi}_2(4m^2\sinh^2(\eta/2))\sinh\eta\,d\eta = \qquad (3.178)\\ &= -\int_0^\infty\frac{\eta\,f(e^{-\eta})d\eta}{\sinh\eta} = \int_0^1\frac{2\ln t}{1-t^2}f(t)dt = \int_0^1 f(t)\ln t\left(\frac{1}{1-t}+\frac{1}{1+t}\right)dt. \end{aligned}$$

This result will be worked out later using a more general concept. If one proceeds the same way for the spectacle diagram

$$S(M) = \int\frac{d^2p}{p^2+M^2}\tilde{\Pi}_2(p)^2 = S_0(M)+\varepsilon S_1(M)+O(\varepsilon^2), \tag{3.179}$$

one obtains

$$\begin{aligned} S_1(M) &= 2\int\frac{d^2p}{p^2+M^2}\tilde{\Pi}_2(p)\Delta_\varepsilon\tilde{\Pi}_2(p) = \frac{1}{2\pi m^4}\int_0^1\frac{2tf(t)\ln t\,dt}{(1-t^2)(1-2\gamma t+t^2)} = \\ &= \frac{1}{2\pi m^4}\int_0^1 f(t)\ln t\left[\frac{1}{(1-\zeta)(1-\bar\zeta)(1-t)}-\frac{1}{(1+\zeta)(1+\bar\zeta)(1+t)}+\right.\\ &\qquad\left.-\frac{1}{\zeta-\bar\zeta}\left(\frac{2\zeta}{(1-\zeta^2)(\zeta-t)}-\frac{2\bar\zeta}{(1-\bar\zeta^2)(\bar\zeta-t)}\right)\right]dt \end{aligned} \tag{3.180}$$

where γ and ζ are the same quantities as before.

3.4.12 The sixth order roots of unity

There is a common integral for all these cases, given by

$$\int_0^1 \frac{dz}{\bar{\zeta}-z} f(z)\ln z = 2M(\zeta) + 3\,\mathrm{Li}_4(\zeta) - \zeta(2)\,\mathrm{Li}_2(\zeta). \tag{3.181}$$

While the second and third part on the right hand side can be given explicitly in terms of polylogarithms. The first contribution gives

$$M(\zeta) = \int_0^1 \frac{dz}{\bar{\zeta}-z_1}\ln z_1 \int_0^{z_1}\frac{dz_2}{1+z_2}\ln z_2 \tag{3.182}$$

which does not have such a simple form. Instead, using so-called *shuffling methods* explained in Appendix F, it can be reduced to the primitives

$$U_{3,1} = \sum_{m>n>0}\frac{(-1)^n}{m^3 n} = \frac{1}{2}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2(2) - \frac{1}{12}\ln^4(2) - 2\,\mathrm{Li}_4\left(\frac{1}{2}\right) \quad \text{and} \tag{3.183}$$

$$V_{3,1} = \sum_{m>n>0}(-1)^m \cos\left(\frac{2\pi n}{3}\right)\frac{1}{m^3 n} \tag{3.184}$$

occuring in Ref. [111] and the Clausen's polylogarithms

$$\mathrm{Cl}_2(\theta) = \mathrm{Im}\,\mathrm{Li}_2(e^{i\theta}), \qquad \mathrm{Cl}_4(\theta) = \mathrm{Im}\,\mathrm{Li}_4(e^{i\theta}). \tag{3.185}$$

One obtains

$$\begin{aligned}
M(1) &= \frac{17\pi^4}{1440} + 2U_{3,1},\\
M(e^{i\pi/3}) &= \frac{197\pi^4}{38880} - \frac{1}{3}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) + 2V_{3,1} + \frac{5i\pi^3}{162}\ln 3 + \frac{13}{108}i\pi^2\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{35i}{18}\mathrm{Cl}_4\left(\frac{\pi}{3}\right),\\
M(e^{2i\pi/3}) &= -\frac{79\pi^4}{12960} + \frac{1}{3}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) + \frac{7i\pi^2}{36}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{11i}{6}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) \quad \text{and}\\
M(-1) &= -\frac{\pi^4}{288} = -\frac{5}{16}\zeta(4)
\end{aligned} \tag{3.186}$$

as well as the complex conjugate of the second and third line. The *sixth order roots of unity* $\zeta = 1$, $e^{i\pi/3}$, $e^{2i\pi/3}$, and -1 used here correspond to $M = 0$, m, $\sqrt{3}m$, and $2m$, respectively. The first and last case occur in the water melon case as well. Here one obtains (cf. Appendix F)

$$W_1 = \frac{37\pi^4}{720} + 4U_{3,1}. \tag{3.187}$$

For the standard arrangement $M = m$ of masses for the spectacle diagram one obtains

$$S_1(M=m) = -\frac{251\pi^4}{58320} + 4U_{3,1} - \frac{16}{3}V_{3,1} + \frac{8}{9}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right). \tag{3.188}$$

3.5 The threshold expansion

With the method described so far the s-dependence of the spectral density can be calculated by a one-fold numerical integration according to Eq. (3.61). The numerical integration in Eq. (3.61) can be done for arbitrary space-time dimensions and a general number of lines with arbitrary masses. In this sense this is the most efficient representation for the spectral density of the water melon diagram. However, one can also develop an explicit expansion near the threshold with any desired accuracy. The corresponding expansion of Eq. (3.61) can then be compared with series expansions near the production threshold obtained with the traditional momentum space technique. It should be stressed that one is only interested in the spectral density because it is the main object for physical applications (see e.g. Refs. [45, 117, 118, 119, 120, 121, 122]). For practical reasons one starts with

$$\tilde{\Pi}(p^2)=2\pi^{\lambda+1}\int_0^\infty\left(\frac{px}{2}\right)^{-\lambda}J_\lambda(px)\Pi(x)x^{2\lambda+1}dx \tag{3.189}$$

(cf. Eq. (3.13)). The UV singularity of this correlation function can be subtracted as before by a power series expansion of the weight function $(px/2)^{-\lambda}J_\lambda(px)$ to an appropiate order which will be added and subtracted to this weight. But because the subtraction terms do not contribute to the spectral density, one can avoid this subtraction altogether. In order that the formally written expressions make sense they are supposed to be dimensionally regularized. Once again the unorthodox dimensional regularization method for water melon diagrams is used.

3.5.1 Continuation to the Minkowskian domain

The *threshold region* of a water melon diagram is determined by the condition $p^2+M^2\simeq 0$ where p is the Euclidean momentum and $M=\sum_i m_i$ is the threshold value for the spectral density. One introduces the Minkowskian momentum p_M defined by $p_M^2=-p^2$ which is an analytic continuation to the physical cut. Operationally this analytic continuation can be performed by replacing $p\to ip_M$. To analyze the region near the threshold one uses the parameter $\Delta=M-p_M$ which takes complex values. The parameter Δ is more convenient in the Euclidean domain while the parameter $E=-\Delta=p_M-M$ is the actual energy counted from threshold which is used in phenomenological applications. The analytic continuation of the Fourier transform in Eq. (3.189) to the Minkowskian domain has the form

$$\tilde{\Pi}(-p^2)=2\pi^{\lambda+1}\int_0^\infty\left(\frac{ipx}{2}\right)^{-\lambda}J_\lambda(ipx)\Pi(x)x^{2\lambda+1}dx \tag{3.190}$$

(the index M is dropped again because p is considered to be the Minkowskian momentum in the following). For the threshold expansion one has to analyze the large x behaviour of the integrand. It is this region that saturates the integral in the limit $p\to M$ or, equivalently, $E\to 0$. It is convenient to perform the analysis in a basis where the integrand has a simple large x behaviour. The most important part of the integrand is the Bessel function $J_\lambda(ipx)$ which contains both rising and falling pieces at large x,

$$J_\lambda(ipx)=\frac{1}{2}(H_\lambda^+(ipx)+H_\lambda^-(ipx)) \tag{3.191}$$

where the Hankel functions $H^\pm(ipx)$ show an asymptotic behaviour

$$H_\lambda^\pm(ipx)\sim z^{-1/2}e^{\pm px} \tag{3.192}$$

for large values of x (cf. Appendix D.1.4). This situation in a sense is quite analogous to the situation for the trigonometric function $\cos(z)$ for imaginary arguments z. Accordingly one splits up $\tilde{\Pi}(-p^2)$ into $\tilde{\Pi}(-p^2)=\tilde{\Pi}^+(-p^2)+\tilde{\Pi}^-(-p^2)$ with

$$\tilde{\Pi}^{\pm}(-p^2)=\pi^{\lambda+1}\int_0^\infty\left(\frac{ipx}{2}\right)^{-\lambda}H_\lambda^{\pm}(ipx)\Pi(x)x^{2\lambda+1}dx. \tag{3.193}$$

The two parts $\tilde{\Pi}^{\pm}(-p^2)$ of the correlation function $\tilde{\Pi}(-p^2)$ have completely different behaviour near threshold which allows one to analyze them independently.

3.5.2 The regular part $\tilde{\Pi}^+(-p^2)$

One first considers the contribution of the part $\tilde{\Pi}^+(-p^2)$ in a more qualitative way. The behaviour at large x is given by the asymptotic form of the functions which at leading order is

$$H^+(ipx)=\sqrt{\frac{2}{i\pi px}}e^{-px}(1+O(x^{-1})),\qquad K(mx)=\sqrt{\frac{\pi}{2mx}}e^{-mx}(1+O(x^{-1})). \tag{3.194}$$

The large x range of the integral (above a reasonably large cutoff parameter Λ) has the general form

$$\tilde{\Pi}^+_\Lambda(-p^2)\sim\int_\Lambda^\infty x^{-a}e^{-(2M-\Delta)x}dx \tag{3.195}$$

where $p=M-\Delta$ and

$$a=(n-1)(\lambda+1/2). \tag{3.196}$$

The right hand side of Eq. (3.195) is an analytic function in Δ in the vicinity of $\Delta=0$. It exhibits no cut or other singularities near the threshold and therefore does not contribute to the spectral density. That is the reason why the part $\tilde{\Pi}^+(-p^2)$ can be called the regular part. In more detail, one can use the relation

$$K_\lambda(z)=\frac{\pi i}{2}e^{i\lambda\pi/2}H_\lambda^+(iz) \tag{3.197}$$

between Bessel functions of different kinds in order to replace the Hankel function $H_\lambda^+(ipx)$ by the McDonald function $K_\lambda(px)$,

$$\begin{aligned}
\tilde{\Pi}^+(-p^2) &= \pi^{\lambda+1}\int_0^\infty\left(\frac{ipx}{2}\right)^{-\lambda}H_\lambda^+(ipx)\Pi(x)x^{2\lambda+1+2\varepsilon}dx \;=\\
&= -2i\pi^\lambda\int_0^\infty\left(\frac{px}{2}\right)^{-\lambda}e^{-i\lambda\pi/2}e^{-i\lambda\pi/2}K_\lambda(px)\Pi(x)x^{2\lambda+1+2\varepsilon}dx \;=\\
&= -2i(-\pi)^\lambda\int_0^\infty\left(\frac{px}{2}\right)^{-\lambda}K_\lambda(px)\Pi(x)x^{2\lambda+1+2\varepsilon}dx \;=\\
&= \frac{(-2\pi i)^{2\lambda+1}}{(p^2)^\lambda}\int_0^\infty\Pi_+(x)x^{2\lambda+1}dx
\end{aligned} \tag{3.198}$$

where $\Pi_+(x)=\Pi(x)D(x,p)$ is the polarization function of a new effective diagram which is equal to the initial polarization function multiplied by a propagator with p as mass parameter. One thus ends up with a vacuum bubble of the water melon type with one additional line compared to the initial diagram (see Fig. 3.5).

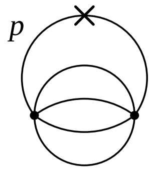


Figure 3.5: Representation of the regular part $\tilde{\Pi}^+(-p^2)$ as vacuum bubble with added line. The cross denotes an arbitrary number of derivatives of the specified line.

These diagrams have no singular behaviour at the production threshold $p = M$. As mentioned above, $\tilde{\Pi}^+(-p^2)$ is analytic in Δ near the origin $\Delta = 0$ and can therefore be omitted in the calculation of the spectral density. All derivatives of $\tilde{\Pi}^+(-p^2)$ with respect to $\Delta = M - p$ are represented as vacuum bubbles with one additional line carrying rising indices. Such diagrams can be efficiently calculated within the recurrence relation technique developed in Refs. [107, 108, 112]. If this part is found to be regular, one is left with the part $\tilde{\Pi}(-p^2)$.

3.5.3 The singular part $\tilde{\Pi}(-p^2)$

In contrast to the previous case, the integrand of $\tilde{\Pi}(-p^2)$ contains $H^-(ipx)$ which behaves like a rising exponential function at large x,

$$H^-(ipx) \sim x^{-1/2} e^{px}. \tag{3.199}$$

Therefore the integral is represented by

$$\tilde{\Pi}^-_\Lambda(-p^2) \sim \int_\Lambda^\infty x^{-a} e^{-\Delta x} dx \tag{3.200}$$

at $p = M - \Delta$. The function $\tilde{\Pi}^-(-p^2)$ is non-analytic near $\Delta = M - p = 0$ because for $\Delta < 0$ the integrand in Eq. (3.200) grows in the large x region and the integral diverges at the upper limit. Therefore the function which is determined by this integral is singular at $\Delta < 0$ ($E > 0$) and requires an interpretation for these values of the argument Δ. In the complex Δ plane with a cut along the negative axis the function is analytic. This cut corresponds to the physical positive energy cut. The discontinuity across the cut gives rise to the non-vanishing spectral density of the diagram.

Still the integration cannot be done analytically. In order to obtain an expansion for the spectral density near the threshold in an analytical form one makes use of the asymptotic series expansion for the function $\Pi(x)$ which crucially simplifies the integrands but still preserves the singular structure of the integral in terms of the variable Δ. The asymptotic series expansion to order N of the main part of each propagator, i.e. of the McDonald function, is given by [85]

$$K^{as}_{\lambda,N}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[\sum_{n=0}^{N-1}\frac{(\lambda,n)}{(2z)^n} + \theta\frac{(\lambda,N)}{(2z)^N}\right], \quad (\lambda,n) := \frac{\Gamma(\lambda+n-1/2)}{n!\Gamma(\lambda-n-1/2)} \tag{3.201}$$

($\theta \in [0,1]$). Therefore the asymptotic expansion of the function $\Pi(x)$ consists of an exponential factor e^{-Mx} and an inverse power series in x up to an order $\tilde{N}$ which is closely related to N. It is this asymptotic expansion that determines the singularity structure of the integral. The whole integral can be written as a sum of two terms,

$$\begin{aligned}\tilde{\Pi}^-(-p^2) &= \pi^{\lambda+1}\int\left(\frac{ipx}{2}\right)^{-\lambda} H^-_\lambda(ipx)\left(\Pi(x) - \Pi^{as}_N(x)\right) x^{2\lambda+1+2\varepsilon} dx \\ &\quad + \pi^{\lambda+1}\int\left(\frac{ipx}{2}\right)^{-\lambda} H^-_\lambda(ipx)\Pi^{as}_N(x) x^{2\lambda+1+2\varepsilon} dx = \tilde{\Pi}^{di}(-p^2) + \tilde{\Pi}^{as}(-p^2).\end{aligned} \tag{3.202}$$

The integrand of the first term $\tilde{\Pi}^{di}(-p^2)$ behaves as $1/x^{\tilde{N}}$ at large x while the integrand of the second term accumulates all lower powers of the large x expansion. Note that only the large x behaviour is essential for the near threshold expansion of the spectral density. This fact has been taken into account in Eqs. (3.195) and (3.200) where a cutoff Λ was introduced. A regularization is necessary even if the spectral density will be independent of it because the asymptotic expansion is not valid in the region near the origin in x. But because the cutoff regularization is quite inconvenient for practical calculations, the unorthodox dimensional regularization will be used in the following.

3.5.4 The difference part $\tilde{\Pi}^{di}(-p^2)$

The first part

$$\Pi^{di}(-p^2) = \pi^{\lambda+1} \int_0^\infty \left(\frac{ipx}{2}\right)^{-\lambda} H_\lambda^-(ipx) \left(\Pi(x) - \Pi^{\rm as}(x)\right) x^{2\lambda+1+2\varepsilon} dx \qquad (3.203)$$

contains the difference between the exact correlator in configuration space and its asymptotic approximation. If one takes the asymptotic expansion up to some order N, this difference will effectively be of order $o(x^{-N})$[4]. Because of this the Hankel function $H_\lambda^-(i(M-\Delta)x)$ can be expanded in $\eta = \Delta/M$. The effective power x^{-N} guarantees that the integrand decreases sufficiently fast for large values of x and the integral converges even at $\Delta = 0$. Therefore $\tilde{\Pi}^{di}(-p^2)$ gives no contributions to the spectral density up to a given order of the expansion in Δ, the term is inessential when the expansion of the spectral density is evaluated up to some given order.

One can readily determine the order of the expansion near $\Delta = 0$ at which a contribution to the spectral density appears when one uses further simplifications of the integrand of the term $\tilde{\Pi}^{di}(-p^2)$ in Eq. (3.202). Namely, one replaces the Hankel function under the integration sign by its asymptotic series expansion. The resulting exponential factor $e^{(p-M)t}$ can then be expanded in the parameter $\Delta = M - p$ and integrated together with the finite inverse power series in x. One obtains a finite power series in this parameter Δ which leads to a non-regular term of order Δ^N (for instance, $\Delta^N \ln\Delta$ or $\Delta^N\sqrt{\Delta}$). Therefore the part $\tilde{\Pi}^{di}(-p^2)$ is regular and gives no contribution to the spectral density up to the order Δ^N. For this reason one concentrates on the expansion of the second part $\tilde{\Pi}^{as}(-p^2)$ and finds that only this part contains the contribution to the spectral density up to the order N.

3.5.5 The asymptotic part $\tilde{\Pi}^{as}(-p^2)$

The expansion of the spectral density at small E is determined only by the integral

$$\Pi^{\rm as}(p^2) = \pi^{\lambda+1} \int_0^\infty \left(\frac{ipx}{2}\right)^{-\lambda} H_\lambda^-(ipx)\Pi^{\rm as}(x) x^{2\lambda+1+2\varepsilon} dx \qquad (3.204)$$

which is analytically calculable. It results in hypergeometric functions ${}_2F_1(a,b;c;z)$ accompained by Euler's Gamma functions where these as well as the parameters a and b

[4] The notation $o(x^{-N})$ indicates a contribution less than x^{-N}, in contrast to the notation $O(x^{-N-1})$ which means that the rest of the series starts with a term proportional to x^{-N-1}.

are in general linearly depending on ε. The argument z is

$$\text{either}\qquad \frac{p^2}{M^2}=(1-\eta)^2\qquad\text{or}\qquad \frac{M^2}{p^2}=\frac{1}{(1-\eta)^2},\qquad \eta=\frac{\Delta}{M}. \tag{3.205}$$

One therefore uses the relations

$$\begin{aligned}
{}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b-c+1;1-z)+\\
&\quad +(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(c-a,c-b;c-a-b+1;1-z)\\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1\left(a,a-c+1;a+b-c+1;1-\frac{1}{z}\right)+\\
&\quad +(1-z)^{c-a-b}z^{a-c}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1\left(c-a,1-a;c-a-b+1;1-\frac{1}{z}\right)
\end{aligned} \tag{3.206}$$

to convert the expression into an expression with hypergeometric functions at arguments close to zero (cf. Appendix D.2.2). Expanded in terms of η, this expression will result in a power series of the general form

$$A_0\left(\frac{1}{2\varepsilon}+\ln\left(\frac{\Delta}{\mu}\right)+C_0\right)+A_1\left(\frac{1}{2\varepsilon}+\ln\left(\frac{\Delta}{\mu}\right)+C_1\right)\Delta+A_2\left(\frac{1}{2\varepsilon}+\ln\left(\frac{\Delta}{\mu}\right)+C_2\right)\Delta^2+\dots \tag{3.207}$$

which will become unreliable at some finite power Δ^N because of the previous difference term.

3.5.6 The double-asymptotic part $\tilde\Pi^{das}(-p^2)$

Since the interesting singular behaviour of $\tilde\Pi^{as}(-p^2)$ is determined by the behaviour at large x, one can replace the first factor in the integrand of $\tilde\Pi^{as}(-p^2)$, i.e. the Hankel function, in the large x region by its asymptotic expansion up to some order N. One uses

$$H^{-as}_{\lambda,N}(iz)=\left(\frac{2}{\pi z}\right)^{1/2}e^{z+i\lambda\pi/2}\left[\sum_{n=0}^{N-1}\frac{(-1)^n(\lambda,n)}{(2z)^n}+\theta\frac{(-1)^N(\lambda,N)}{(2z)^N}\right] \tag{3.208}$$

(cf. Eq. (3.201) for the notation) to obtain a representation

$$\tilde\Pi^{das}(-p^2)=\pi^{\lambda+1}\int\left(\frac{ipx}{2}\right)^{-\lambda}H^{-as}_{\lambda,N}(ipx)\Pi^{as}_N(x)x^{2\lambda+1+2\varepsilon}dx. \tag{3.209}$$

The index "*das*" stands for "double asymptotic" and indicates that the integrand in Eq. (3.209) consists of a product of two asymptotic expansions: one for the correlator function $\Pi(x)$ and another for the Hankel function $H^-_\lambda(x)$ as weight (or kernel). Both asymptotic expansions are straightforward and can be obtained from standard handbooks on Bessel functions.

3.5.7 The result for the near threshold expansion

Because the integrand of the only integral relevant for the determination of the spectral density contains a product of two asymptotic series, one arrives at the conclusion that the integration necessary for evaluating the near threshold expansion of the water melon diagrams reduces to integrals of the type of Euler's Gamma function, i.e. integrals containing exponentials and powers. Indeed, the result of the expansion in Eq. (3.209) is an exponential function $e^{-\Delta x}$ times a power series in $1/x$, namely

$$x^{-a+2\varepsilon}e^{-\Delta x}\sum_{j=0}^{N-1}\frac{A_j}{x^j} \tag{3.210}$$

where a has already been defined in Eq. (3.196), and the coefficients A_j are simple functions of the momentum p and the masses m_i. The expression in Eq. (3.210) can be integrated analytically using

$$\int_0^\infty x^{-a+2\varepsilon}e^{-\Delta x}dx = \Gamma(1-a+2\varepsilon)\Delta^{a-1-2\varepsilon}. \tag{3.211}$$

The result is

$$\Pi^{das}(M-\Delta) = \sum_{j=0}^{N-1} A_j\Gamma(1-a-j+2\varepsilon)\Delta^{a+j-1-2\varepsilon}. \tag{3.212}$$

This expression is the final representation for the part of the polarization function of a water melon diagram necessary for the calculation of the spectral density near the production threshold.

The general structure of the expression in Eq. (3.212) can be discussed in detail. In the case where a takes integer values, these coefficients result in $1/\varepsilon$-divergences for small values of ε. The powers of Δ in Eq. (3.212) have to be expanded to first order in ε and give

$$\frac{1}{2\varepsilon}\Delta^{2\varepsilon} = \frac{1}{2\varepsilon} + \ln\Delta + O(\varepsilon). \tag{3.213}$$

Because of

$$\mathrm{Disc}\ln(\Delta) \equiv \ln(-E-i0) - \ln(-E+i0) = -2\pi i\theta(E) \tag{3.214}$$

$\Pi^{das}(M-\Delta)$ in Eq. (3.212) contributes to the spectral density. For half-integer values of a the power of Δ itself has a cut even for $\varepsilon = 0$. The discontinuity is then given by

$$\mathrm{Disc}\sqrt{\Delta} = -2\mathrm{i}\sqrt{\mathrm{E}}\,\theta(\mathrm{E}). \tag{3.215}$$

The method to construct a threshold expansion thus simply reduces to the analytical calculation of the integral in Eq. (3.209) which can be done for arbitrary dimension and an arbitrary number of lines with different masses (see Fig. 3.6). The technique is used in the following to work out some specific examples demonstrating both the simplicity and efficiency of the method.

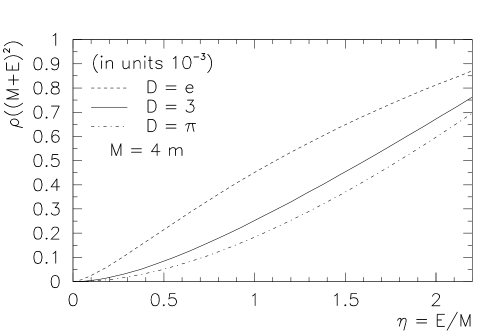


Figure 3.6: The spectral density for the four-line water melon diagram with equal masses for $D = e = 2.718\ldots$, $D = 3$, and $D = \pi = 3.14\ldots$ space-time dimensions, to demonstrate the practical convenience of the method.

3.6 Comparison of near threshold expansions

Even though a calculation of the spectral density is possible for any choice of mass parameters, the following three examples will deal with the case of equal masses in order to compare these special cases with the results given in the literature (see e.g. Ref. [130]). For the one-loop case the series can be compared with the series expansions of the exact expressions for the spectral densities as given by Eq. (3.62). But also the quite inconvenient case of one very small mass is considered for different diagrams.

3.6.1 Equal mass sunset diagram

The polarization function represented by the sunset diagram with three propagators with equal masses m in $D = 4$ space-time dimensions is given by

$$\Pi(x) = \frac{m^3 K_1(mx)^3}{(2\pi)^6 x^3}. \tag{3.216}$$

The exact spectral density is given by the integral representation in Eq. (3.61) which for this particular case reads

$$\rho(s) = \frac{2\pi}{i\sqrt{s}} \int_{c-i\infty}^{c+i\infty} I_1(x\sqrt{s})\Pi(x)x^2 dx. \tag{3.217}$$

In order to obtain a threshold expansion of the spectral density in Eq. (3.217) one uses Eq. (3.212) to calculate the expansion of the appropiate part of the polarization function. To illustrate the procedure the explicit form of the integrand in Eq. (3.209) is presented which is given by an asymptotic expansion at large x,

$$\begin{aligned} \pi^2 \left(\frac{ipx}{2}\right)^{-1} H^{as}_{1,N}(px)\Pi^{as}_N(x)x^{3+2\varepsilon} &= \frac{m^{3/2}e^{(p-3m)x}}{(4\pi)^3 p^{3/2}} x^{-3+2\varepsilon} \times \\ &\times \left\{1 + \frac{9}{8mx} - \frac{3}{8px} + \frac{9}{128m^2x^2} - \frac{27}{64mpx^2} - \frac{15}{128p^2x^2} + O(x^{-3})\right\}. \end{aligned} \tag{3.218}$$

From Eq. (3.218) one can easily read off the coefficients A_j that enter the expansion in Eq. (3.210). The spectral density is obtained by performing the term-by-term integration of the series in Eq. (3.218) and by evaluating the discontinuity across the cut along the positive energy axis $E > 0$. The result reads

$$\begin{aligned} \rho\left((M+E)^2\right) &= \frac{E^2}{384\pi^3\sqrt{3}} \left\{1 - \frac{1}{2}\eta + \frac{7}{16}\eta^2 - \frac{3}{8}\eta^3 + \frac{39}{128}\eta^4 - \frac{57}{256}\eta^5 + \right. \\ &\left. + \frac{129}{1024}\eta^6 - \frac{3}{256}\eta^7 - \frac{4047}{32768}\eta^8 + \frac{18603}{65536}\eta^9 - \frac{248829}{524288}\eta^{10} + O(\eta^{11})\right\} \end{aligned} \tag{3.219}$$

where $\eta = E/M$ and $M = 3m$ is used. The simplicity of the derivation is striking. With no cost it can be generalized to any number of lines, arbitrary masses, and any space-time dimension. The standard equal mass sunset is chosen for definiteness only. It also allows us to compare our results with results available in the literature. Eq. (3.219) reproduces the expansion coefficients $\tilde{a}_j$ obtained in Ref. [130] (the fourth column in Table 1 of Ref. [130]) by a direct integration in momentum space using the technique of region separation [123].

The case of the equal mass standard sunset diagram is the simplest one. There exists an analytical expression for the spectral density of the sunset diagram with three equal mass propagators in $D = 4$ space-time dimensions in terms of elliptic integrals [91] (see also Ref [124]). This expression can be used for a comparison with the exact result in Eq. (3.217) or with the expansion in Eq. (3.219). However, the result for $D = 2$ is presented here only in order to keep the resulting expressions in a reasonably short form (cf. Ref. [116]). In $D = 2$ space-time dimensions the spectral density for a sunset diagram with equal masses m can be readily obtained. One just uses the exact expression for the spectral density in the convolution representation [107, 108, 112] and proceed towards $n = 3$ equal masses. The convolution function for two spectral densities in $D = 2$ dimensional space-time ($\lambda = 0$) reads

$$\rho(s; s_1; s_2) = \frac{1}{2\pi\sqrt{(s - s_1 - s_2)^2 - 4s_1 s_2}}. \tag{3.220}$$

The two spectral densities one has to convolute are the spectral density of a correlator with two equal masses and the spectral density of a single massive line. While the latter is given by $\rho(s; m^2) = \delta(s - m^2)$, the former can be obtained from Eq. (3.220) by inserting $s_1 = s_2 = m^2$,

$$\rho(s; m^2; m^2) = \frac{1}{2\pi\sqrt{s(s - 4m^2)}}. \tag{3.221}$$

Therefore, the convolution leads to

$$\begin{aligned}\rho(s; m^2; m^2; m^2) &= \frac{1}{4\pi^2}\int_{4m^2}^{(\sqrt{s}-m)^2} \frac{dt}{\sqrt{(s - m^2 - t)^2 - 4m^2 t}\sqrt{t(t - 4m^2)}} = \\ &= \frac{1}{4\pi^2}\int_{4m^2}^{(\sqrt{s}-m)^2} \frac{dt}{\sqrt{t(t - 4m^2)((\sqrt{s} + m)^2 - t)((\sqrt{s} - m)^2 - t)}}.\end{aligned} \tag{3.222}$$

Now one use the relation (cf. Ref. [91])

$$\int_{t_1}^{t_2} \frac{dt}{\sqrt{(t - t_0)(t - t_1)(t_2 - t)(t_3 - t)}} = \frac{2}{\sqrt{(t_3 - t_1)(t_2 - t_0)}} K(k^2), \tag{3.223}$$

$$k^2 = \frac{(t_2 - t_1)(t_3 - t_0)}{(t_3 - t_1)(t_2 - t_0)} \tag{3.224}$$

with $t_3 > t_2 > t > t_1 > t_0$ and the definition of the complete elliptic integral of the first kind

$$K(k^2) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} = F\left(\frac{\pi}{2}, k^2\right) \tag{3.225}$$

(note the difference in the definition) for $t_0 = 0$, $t_1 = 4m^2$, $t_2 = (\sqrt{s} - m)^2$, and $t_3 = (\sqrt{s} + m)^2$ to perform the integration in Eq. (3.222). One obtains

$$k^2 = \frac{((\sqrt{s} - m)^2 - 4m^2)(\sqrt{s} + m)^2}{((\sqrt{s} + m)^2 - 4m^2)(\sqrt{s} - m)^2} \tag{3.226}$$

and finally ends up with

$$\rho(s;m^2;m^2;m^2) = \frac{K(k^2)}{2\pi^2(\sqrt{s}-m)\sqrt{(\sqrt{s}+m)^2-4m^2}}. \tag{3.227}$$

Therefore, the spectral density in terms of the energy E reads (see e.g. Ref. [125])

$$\begin{aligned} \rho\left((M+E)^2\right) &= \frac{1}{2\pi^2(2m+E)\sqrt{(4m+E)^2-4m^2}}K(k^2), \\ k^2 &= \frac{((2m+E)^2-4m^2)(4m+E)^2}{((4m+E)^2-4m^2)(2m+E)^2}, \qquad M=3m. \end{aligned} \tag{3.228}$$

By expanding the elliptic integral in terms of the threshold parameter E one reproduces the threshold expansion in Eq. (3.219). The result for $D=4$ space-time dimensions is expressible by the elliptic integrals with some rational functions as factors that makes the result a bit longer. Note that the representation in Eq. (3.228) is understood to be an analytical expression for the spectral density. However, it is a matter of taste whether the representation through the elliptic integrals as in Eq. (3.228) is considered simpler (or in a more analytical form) than the integral representation in Eq. (3.61). The only objection against the latter which one can find in the literature is that the Bessel functions are complicated (see e.g. Ref. [131]). But after more than a century of intensive investigation they are well-known and no more complicated than the square root of the fourth order polynomial which is used in Eq. (3.225) to define the elliptic integral.

3.6.2 Equal mass water melon diagrams with four or more propagators

The water melon diagrams with four or more propagators cannot be easily calculated by using the momentum space technique because this requires multiloop integration of entangled momenta. Within the configuration space technique the generalization to any number of lines (or loops) is immediate with no extra effort. Consider first a three-loop case of water melon diagrams. The polarization function of the equal mass water melon diagram with four propagators in $D=4$ space-time is given by

$$\Pi(x) = \frac{m^4 K_1(mx)^4}{(2\pi)^8 x^4}. \tag{3.229}$$

The exact spectral density of this diagram can be obtained from Eq. (3.217) while the near threshold expansion can be found using Eq. (3.212). One constructs the expansion of the spectral density near threshold explicitly and compares it with the exact result. The expansion of the integrand (cf. Eq. (3.209)) reads

$$\begin{aligned} \pi^2\left(\frac{ipx}{2}\right)^{-1} H^{as}_{1,N}(px)\Pi^{as}_N(x)x^{3+2\varepsilon} &= \frac{m^2 e^{(p-4m)x}}{(4\pi)^4\sqrt{2\pi}p^{3/2}} x^{-9/2+2\varepsilon} \times \\ \times \left\{1+\frac{3}{2mx}-\frac{3}{8px}+\frac{3}{8m^2x^2}-\frac{15}{128p^2x^2}-\frac{9}{16mpx^2}+O(x^{-3})\right\}. \end{aligned} \tag{3.230}$$

After the integration and the calculation of the discontinuity one obtains the expansion of the spectral density in the form

$$\begin{aligned}\rho\left((M+E)^2\right) &= \frac{E^{7/2}M^{1/2}}{26880\pi^5\sqrt{2}}\Big\{1-\frac{1}{4}\eta+\frac{81}{352}\eta^2-\frac{2811}{18304}\eta^3+\frac{17581}{292864}\eta^4+ \\ &+\frac{1085791}{19914752}\eta^5-\frac{597243189}{3027042304}\eta^6+\frac{4581732455}{12108169216}\eta^7-\frac{496039631453}{810146594816}\eta^8+O(\eta^9)\Big\}\end{aligned} \tag{3.231}$$

where $\eta = E/M$ and $M = 4m$ is the threshold value. One sees the difference with the previous three-line case. In Eq. (3.231) the cut represents the square root branch while in the three-line case it was a logarithmic cut. One can easily figure out the reason for this by looking at the asymptotic structure of the integrand. For even number of lines (i.e. odd number of loops) it is a square root branch, while for an odd number of lines (even number of loops) it is a logarithmic branch. This is true in even space-time dimensions. In the general case the structure of the cut depends on the dimensionality of the space-time as well. The general formula reads

$$\rho\left((M+E)^2\right) \sim E^{(\lambda+1/2)(n-1)-1}(1+O(E)). \tag{3.232}$$

For $D = 4$ space-time dimension (i.e. $\lambda = 1$) one can verify the result of Ref. [107, 108, 112] (cf. Eq. (3.232)),

$$\rho\left((M+E)^2\right) \sim E^{(3n-5)/2}(1+O(E)). \tag{3.233}$$

Numerically Eq. (3.231) reads

$$\begin{aligned}\rho\left((M+E)^2\right) &= 8.5962\cdot 10^{-5}E^{7/2}M^{1/2}\big\{1.000-0.250\eta+0.230\eta^2+ \\ &-0.154\eta^3+0.060\eta^4+0.055\eta^5-0.197\eta^6+0.378\eta^7-0.612\eta^8+O(\eta^9)\big\}\end{aligned} \tag{3.234}$$

where the coefficients have been written down up to three decimal places. It is difficult to say anything definite about the convergence of this series. By construction it is an asymptotic series. However, the practical (or explicit) convergence can always be checked by comparing series expansions like the one shown in Eq. (3.234) with the exact spectral density given in Eq. (3.217) by numerical integration.

To conclude this part of the section, one can state that the spectral density of the water melon diagram is most efficiently calculated within the configuration space technique. Whether it is the exact result or the expansion, the configuration space technique can readily deliver the desired result. The exact formula in Eq. (3.217) as well as the threshold expansion obtained from it can be used to calculate the spectral density for an arbitrarily large number of internal lines. Even not shown here, the case of different masses does not lead to any complications within the configuration space technique: the exact formula in Eq. (3.61) and/or the near threshold expansion work equally well for any arrangement of masses. Plots for general cases of different masses are not shown here because they are not very illustrative, showing only the common threshold. However, there is some interesting kinematic regime for different masses which is important for applications and which, to the best of my knowledge, have not been discussed earlier in the literature. An analytical solution for the expansion of the spectral density in this regime is given in the following.

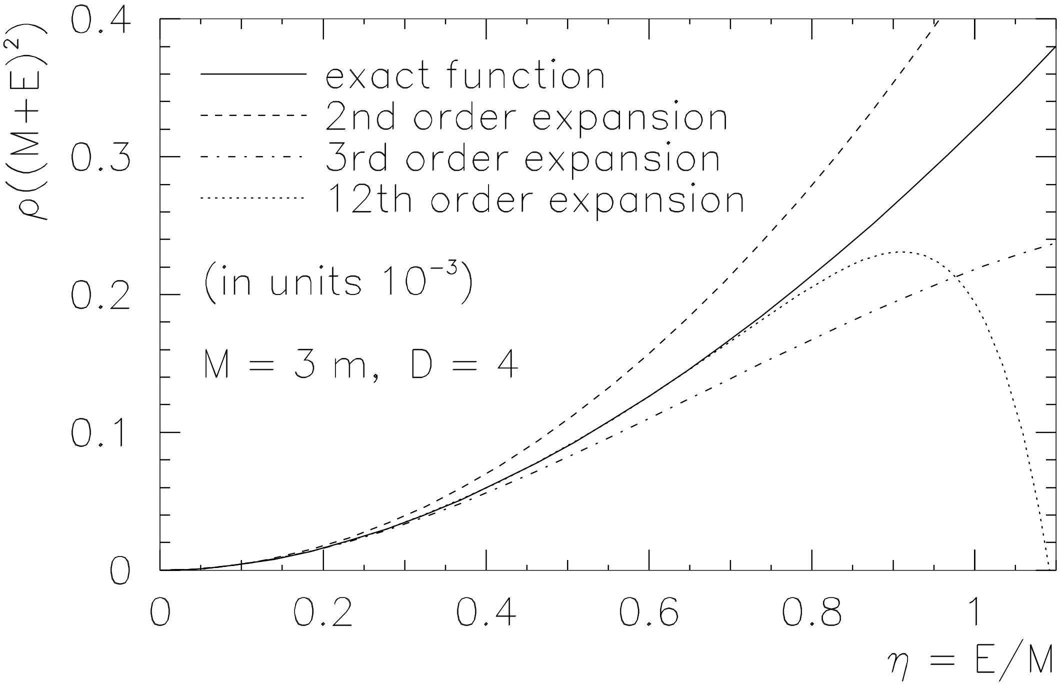


Figure 3.7: Various results for the spectral density for $n = 3$ equal masses in $D = 4$ space-time dimensions in dependence on the threshold parameter E/M. Shown are the exact solution obtained by using Eq. (3.217) (solid curve) and threshold expansions for different orders taken from Eq. (3.219) (dashed to dotted curves).

3.6.3 Strongly asymmetric case $m_0 \ll M$

The threshold expansion for equal (or close) masses breaks down for $E \approx M = \sum m_i$. The example is shown in Fig. 3.7 for the $D = 4$ proper sunset. However, if the masses are not equal, the region of the break-down of the expansion is determined by the mass with the smallest numerical value. The simplest example where one can see this phenomenon is the analytical expression for the spectral density of the single loop with two different masses m_1 and m_2. In $D = 4$ space-time dimensions (see e.g. Ref. [107, 108, 112]) one has

$$\rho\left((M+E)^2\right) = \frac{\sqrt{E(E+2m_1)(E+2m_2)(E+2M)}}{(4\pi(M+E))^2} \tag{3.235}$$

where $M = m_1 + m_2$. The threshold expansion is obtained by expanding the right hand side of Eq. (3.235) in E for small values of E. If m_2 is much smaller than m_1, the expansion breaks down at $E \approx 2m_2$. The break-down of the series expansion can also be observed in more general cases. If one of the masses (which is called m_0) is much smaller than the other masses, the threshold expansion is only valid in a very limited region $E \lesssim 2m_0$.

To generalize the expansion and to extend it to the region of $E \sim M$ one has to treat the smallest mass exactly. In this case one can use a method which is called the *resummation of the smallest mass contributions*. For this special resummation technique one starts with the representation

$$\tilde{\Pi}^{pas}(-p^2) = \pi^{\lambda+1} \int \left(\frac{ipx}{2}\right)^{-\lambda} H^{-as}_{\lambda,N}(ipx)\Pi^{as}_{m_0}(x)x^{2\lambda+1+2\varepsilon}dx \tag{3.236}$$

which is the part of the correlator function contributing to the spectral density. The integrand in Eq. (3.236) has the form

$$\Pi^{as}_{m_0}(x) = \Pi^{as}_{n-1}(x) D(m_0, x) \tag{3.237}$$

where the asymptotic expansions are substituted for all the propagators except for the one with the small mass m_0. This is indicated by the index "*pas*" in Eq. (3.236) which stands for "partial asymptotic". The main technical observation leading to the generalization of the expansion method is that $\Pi^{pas}(p)$ is still analytically computable in a closed form. Indeed, the integral to be computed has the form

$$\begin{aligned}\int_0^\infty x^{\mu-1} e^{-\tilde{\alpha}x} K_\nu(\beta x) dx \ &= \\ = \ \frac{\sqrt{\pi}(2\beta)^\nu}{(2\tilde{\alpha})^{\mu+\nu}} \frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+1/2)} \ &{}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \mu+\frac{1}{2}; 1-\frac{\beta^2}{\tilde{\alpha}^2}\right)\end{aligned} \tag{3.238}$$

where $\tilde{\alpha} = \Delta - m_0$ and $\beta = m_0$. The integral $\tilde{\Pi}^{pas}(-p^2)$ in Eq. (3.236) is thus expressible in terms of hypergeometric functions [101, 126]. For the construction of the spectral density, being our main concern as mentioned before, one has to find the discontinuity of the right hand side of Eq. (3.238). There are several ways to do this. For instance one can proceed by applying the discontinuity operation to the integrand of the integral representation of the hypergeometric function. The resulting integrals are calculated again in terms of hypergeometric functions. Indeed (cf. Appendix D.2.2),

$$\begin{aligned}\frac{1}{2\pi i}\,\mathrm{Disc}\int_0^\infty x^{\mu-1} e^{\alpha x} K_\nu(\beta x) dx \ &= \\ = \ \frac{2^\mu(\alpha^2-\beta^2)^{1/2-\mu}}{\alpha^{1/2-\nu}\beta^\nu} \frac{\Gamma(3/2)}{\Gamma(3/2-\mu)} \ &{}_2F_1\left(\frac{1-\mu-\nu}{2}, \frac{2-\mu-\nu}{2}; \frac{3}{2}-\mu; 1-\frac{\beta^2}{\alpha^2}\right)\end{aligned} \tag{3.239}$$

where $\alpha = E + m_0$. The final expression in Eq. (3.239) completely solves the problem of the generalization of the near threshold expansion technique. For integer values of μ there are no singular Gamma functions (with negative integer argument). Therefore one can remove the regularization and set $\varepsilon = 0$ when using this expression. Thus a direct transition from the correlation function as expressed through the integral to the spectral density in terms of one hypergeometric function has been found for each genuine integral. There is no need to use the recurrence relations available for hypergeometric functions.

In the following subsections explicit examples are given for $D = 4$ compared with the exact result in Eq. (3.217) and the pure expansion near the threshold. In the following the standard threshold expansion without resummation is called the *pure threshold expansion*.

3.6.4 The two-line water melon with a small mass

One starts with a (over)simplified example of the two-line diagram with masses m and $m_0 \ll m$ in four space-time dimensions. In this example the expansion of the spectral density and its generalized expansion can be readily compared analytically with the exact result in Eq. (3.235). This is the feature that justifies the discussion in this section. The results for the spectral density of this diagram are shown in Fig. 3.8. The solid curve displays the exact result obtained by using Eq. (3.217) (which reproduces the analytical expression in Eq. (3.235)). This result is compared with the two expansions.

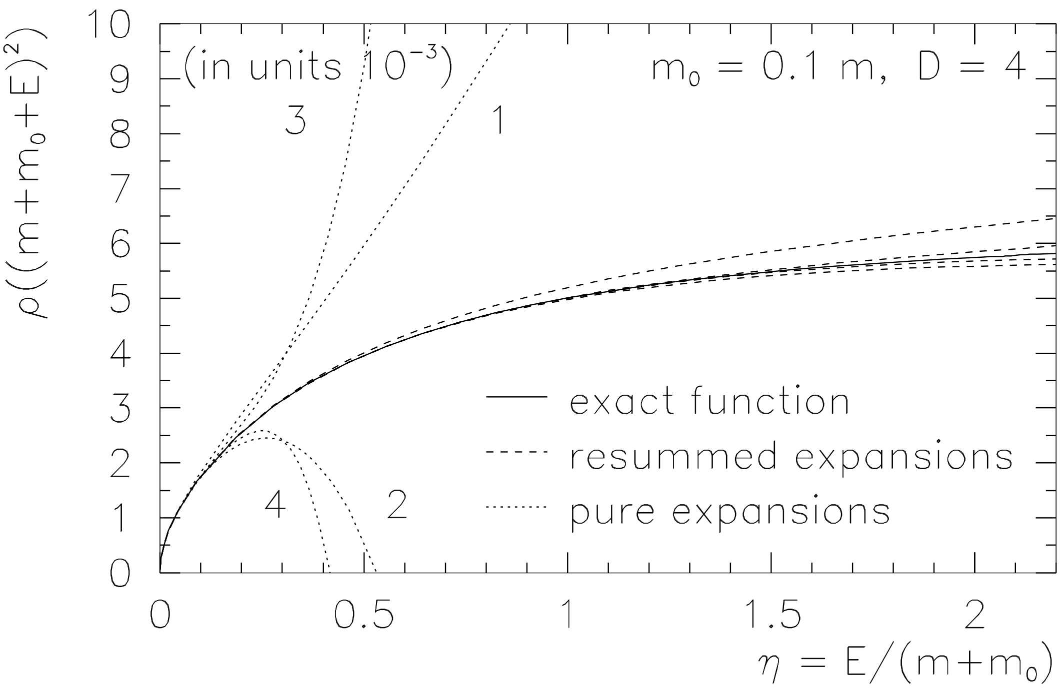


Figure 3.8: Various solutions for the spectral density for two masses m and $m_0 \ll m$ and $D = 4$ space-time dimensions. Shown are the exact solution which is obtained by using Eq. (3.217) (solid curve), the pure threshold expansions using Eq. (3.240) (dotted curves), and the solutions for the resummation of the smallest mass contributions like in Eq. (3.241) (dashed curves), both expansions from the first up to the fourth order in the asymptotic expansion. For the pure threshold expansion the order is indicated explicitly.

The pure expansion of the spectral density near threshold (the second order asymptotic expansion should suffice to show the general features in a short and concise form) reads

$$\begin{aligned}\rho^{das}\left((M+E)^2\right) &= \frac{\sqrt{2m_0 mE}}{8\pi^2 M^{3/2}}\left\{1+\left(\frac{1}{m}+\frac{1}{m_0}-\frac{7}{M}\right)\frac{E}{4}+\right.\\ &\left.-\left(\frac{1}{m_0^2}+\frac{1}{m^2}+\frac{12}{m_0 m}-\frac{79}{M^2}\right)\frac{E^2}{32}+O(E^3)\right\} \qquad (3.240)\end{aligned}$$

where $M = m + m_0$. As mentioned above, this series breaks down for $E > 2m_0$ (see Eq. (3.235)). If one looks at the dotted curves in Fig. 3.8 this becomes obvious. Here the series expansions is plotted up to the fourth order with the mass arrangement $m_0 = m/10$. The dashed lines represent the resummation of the smallest mass contributions. The analytical expression for the spectral density of the polarization function in Eq. (3.236) for the generalized asymptotic expansion based on Eq. (3.239) is given by

$$\begin{aligned}\rho^{pas}\left((M+E)^2\right) &= \frac{\sqrt{mE(E+2m_0)}}{8\pi^2(E+M)^{3/2}}\left\{{}_2F_1\left(0,\frac{1}{2};\frac{3}{2};1-\frac{m_0^2}{(E+m_0)^2}\right)+\right.\\ &+\frac{E(E+2m_0)}{8m(E+M)}\,{}_2F_1\left(\frac{1}{2},1;\frac{5}{2};1-\frac{m_0^2}{(E+m_0)^2}\right)+ \qquad (3.241)\\ &\left.-\frac{E^2(E+2m_0)^2}{128m^2(E+M)^2}\left(1+\frac{16m(E+M)}{5(E+m_0)^2}\right)\,{}_2F_1\left(1,\frac{3}{2};\frac{7}{2};1-\frac{m_0^2}{(E+m_0)^2}\right)+\ldots\right\}.\end{aligned}$$

The regularization parameter has been set to $\varepsilon = 0$ because the spectral density is finite. With $\varepsilon = 0$ the resulting expressions for the hypergeometric functions in Eq. (3.239) simplify. The first term in the curly brackets of Eq. (3.241) is obviously equal to 1 in this limit because the first parameter of the hypergeometric function vanishes for $\varepsilon = 0$. However, Eq. (3.241) is kept in its given form to show the structure of the contributions. The generalized threshold expansion has the form

$$\rho^{pas}\left((M+E)^2\right) = g_0(E,m_0) + Eg_1(E,m_0) + E^2 g_2(E,m_0) + \dots \tag{3.242}$$

where the functions $g_j(E,m_0)$ represent effects of the resummation of the smallest mass and are not polynomials in the threshold parameter E. In the simple two-line case the hypergeometric functions reduce to elementary functions. For instance,

$$\begin{aligned} {}_2F_1\left(\frac{1}{2},1;\frac{5}{2};1-\frac{m_0^2}{(E+m_0)^2}\right) &= \\ &= \frac{3(E+m_0)}{2E(E+2m_0)}\left(E+m_0-\frac{m_0^2}{2\sqrt{E(E+2m_0)}}\ln\left(\frac{E+m_0+\sqrt{E(E+2m_0)}}{E+m_0-\sqrt{E(E+2m_0)}}\right)\right). \end{aligned} \tag{3.243}$$

Higher order contributions are given by hypergeometric functions with larger numerical values of the parameters. They can be simplified by using Gaussian recurrence relations for hypergeometric functions (see e.g. Ref. [101]).

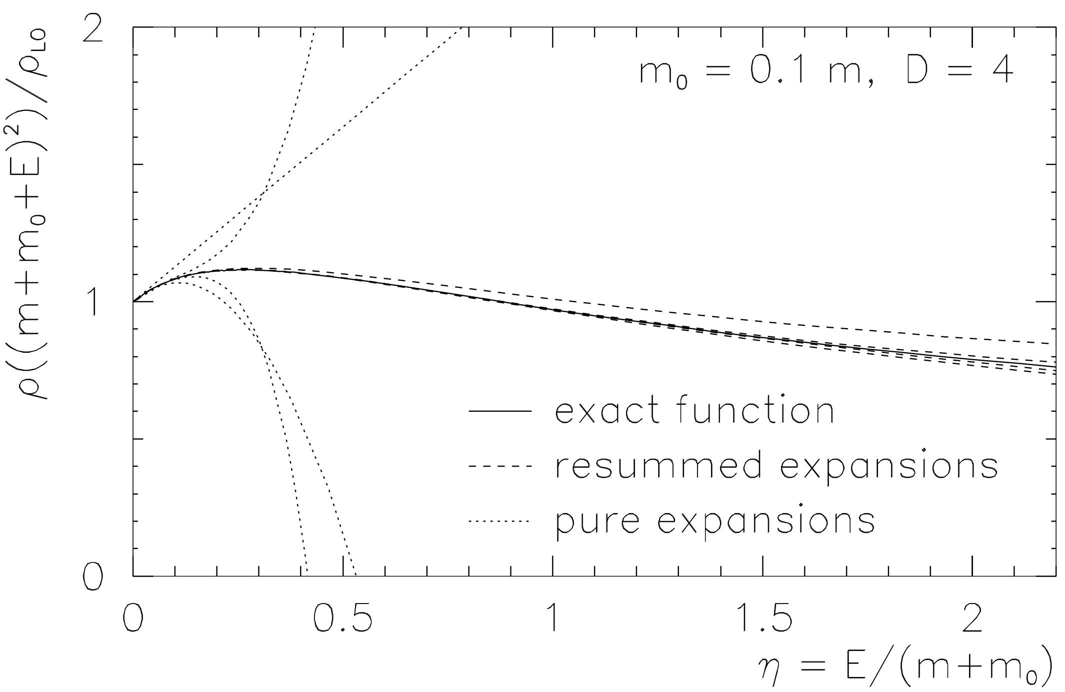


Figure 3.9: The same as Fig. 3.8 where the spectral density is normalized to the leading order expression of the pure threshold expansion.

The convergence of the expansion in Eq. (3.241) breaks down only at $E \sim M = m+m_0$. The resummation leads to an essential improvement of the convergence in comparison with the pure threshold expansion. In Fig. 3.9 the same curves are shown divided by the leading order term. This representation is more convenient for the diagrams which will be discussed in following subsections.

Note that Eq. (3.243) does not lead to the exact function in Eq. (3.235) because terms of order E^N stemming from the difference part $\Pi^{di}(p)$ of the correlator are missing. It simply corrects the behaviour of the coefficient functions by the small mass contributions.

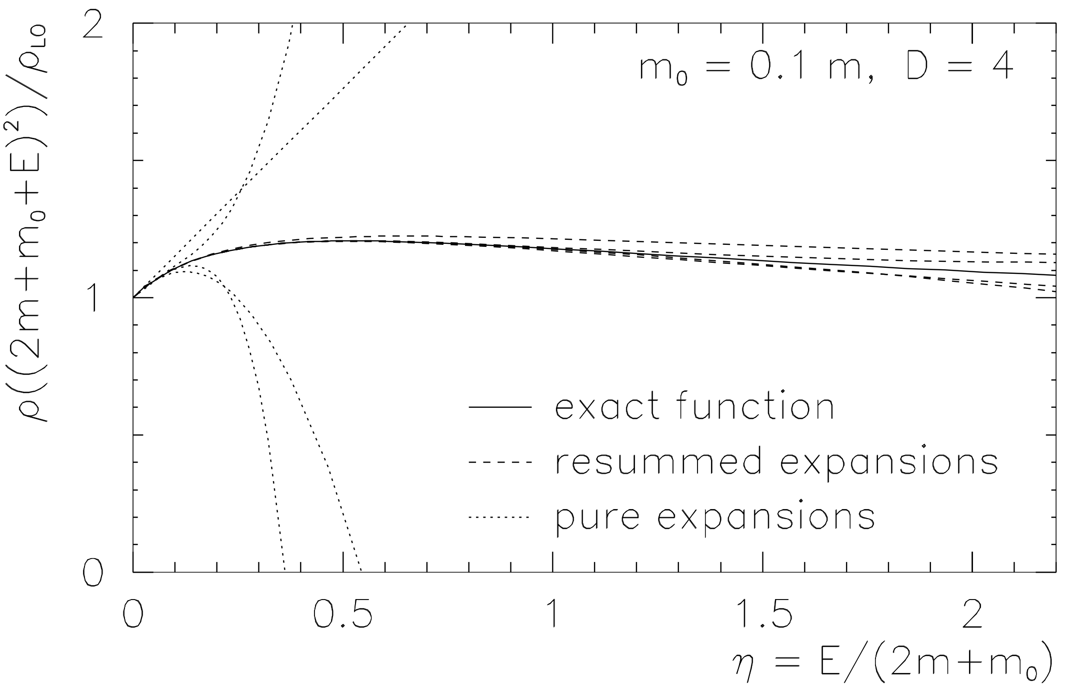


Figure 3.10: The spectral density for the sunset diagram in $D = 4$ space-time dimensions with a tiny mass m_0, normalized to the general power behaviour. Shown are the exact result obtained by using Eq. (3.217) (solid curve), the threshold expansion according to Eq. (3.244) (dotted curves), and the result for the resummation of the smallest mass contributions according to Eq. (3.245) (dashed curves).

3.6.5 The sunset diagram with a small mass

As next the sunset diagram with two equal masses m and a third mass $m_0 \ll m$ ($m_0 = m/10$) is analyzed. The exact result obtained by using Eq. (3.217) and normalized to the leading order term is shown in Fig. 3.10 as the solid curve. The pure expansion near the threshold reads

$$\begin{aligned}\rho^{das}\left((M+E)^2\right) &= \frac{mE^2\sqrt{m_0M}}{128\pi^3M^2}\Bigg\{1+\left(\frac{1}{m_0}+\frac{2}{m}-\frac{13}{M}\right)\frac{E}{8}+ \\ &\quad -\left(\frac{5}{m_0^2}+\frac{4}{m^2}+\frac{39}{m_0m}+\frac{153}{mM}-\frac{1115}{M^2}\right)\frac{E^2}{512}+O(E^3)\Bigg\}. \qquad (3.244)\end{aligned}$$

It is shown by the dotted curves in Fig. 3.10. In case of the resummation of the smallest mass contributions one obtains hypergeometric functions which do not obviously reduce to elementary functions in this case. The result for the spectral density within the asymptotic expansion up to the second order in Eq. (3.236) is given by

$$\begin{aligned}
\rho^{pas}\left((M+E)^2\right) &= \frac{mE^2(E+2m_0)^2}{512\pi^3(E+m_0)^{3/2}(E+M)^{3/2}}\bigg\{ {}_2F_1\left(\frac{3}{4},\frac{5}{4};3;1-\frac{m_0^2}{(E+m_0)^2}\right)+\\
&+\frac{E(E+2m_0)}{8m(E+M)}\left(1+\frac{3m}{2(E+m_0)}\right)\ {}_2F_1\left(\frac{5}{4},\frac{7}{4};4;1-\frac{m_0^2}{(E+m_0)^2}\right)+\\
&-\frac{E^2(E+2m_0)^2}{512m^2(E+M)^2}\left(1+\frac{5m}{2(E+m_0)}\right)\left(1+\frac{9m}{2(E+m_0)}\right)\times\\
&\quad {}_2F_1\left(\frac{7}{4},\frac{9}{4};5;1-\frac{m_0^2}{(E+m_0)^2}\right)\bigg\}. \qquad (3.245)
\end{aligned}$$

It can be seen that the dashed curves in Fig. 3.10 that represent the result in Eq. (3.245) approximate the exact curve much better than the dotted curves.

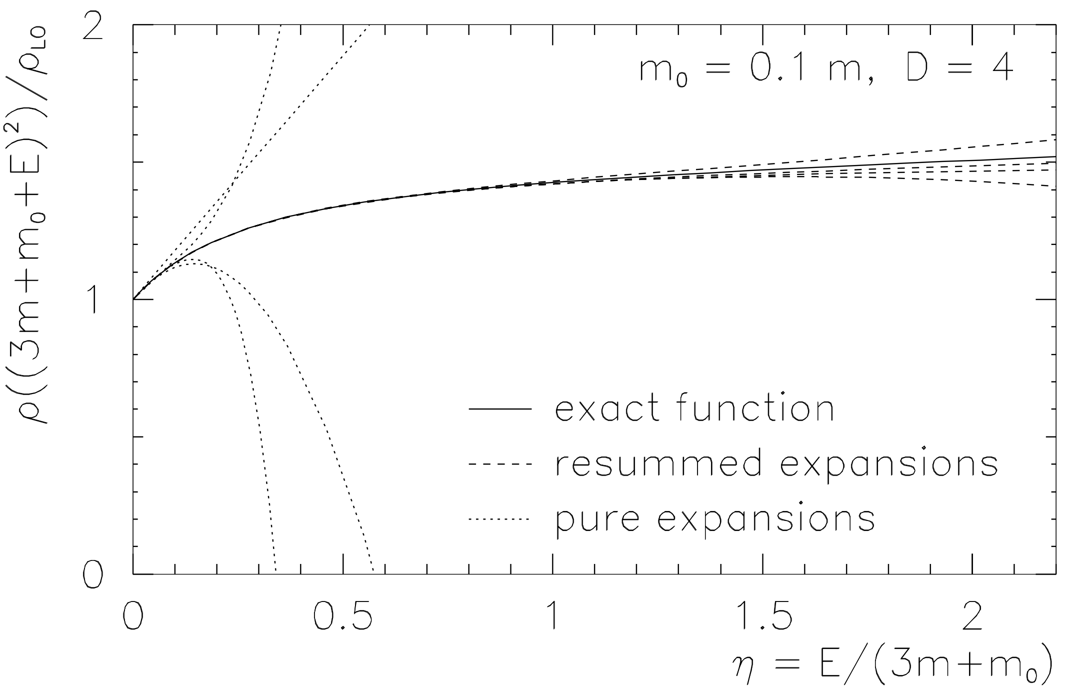


Figure 3.11: The spectral density for the four-line water melon diagram in $D = 4$ space-time dimensions with a tiny mass m_0, normalized to the general power behaviour. Shown are the exact result obtained by using Eq. (3.217) (solid curve), the threshold expansion according to Eq. (3.246) (dotted curves), and the result for the resummation of the smallest mass contributions according to Eq. (3.247) (dashed curves).

3.6.6 The four-line water melon with a small mass

With this example the consideration of the strongly asymmetric case is finished and at the same time the way to the multi-line water melon diagrams which can be treated in an analogous manner is shown. The result for the exact expression obtained by using Eq. (3.217) is shown in Fig. 3.11 as a solid line, normalized to the leading order term.

The dotted lines represent the results for the pure expansion near threshold which is given by

$$\begin{aligned}\rho^{das}\left((E+M)^2\right) &= \frac{m^{3/2}E^{7/2}\sqrt{2m_0}}{3360\pi^5 M^{3/2}}\left\{1+\left(\frac{1}{m_0}+\frac{3}{m}-\frac{19}{M}\right)\frac{E}{12}+\right. \\ &\quad \left.-\left(\frac{5}{m_0^2}-\frac{3}{m^2}+\frac{28}{m_0 m}+\frac{368}{mM}-\frac{2195}{M^2}\right)\frac{E^2}{1056}+O(E^3)\right\}.\end{aligned} \tag{3.246}$$

The asymptotic expansion to the second order in Eq. (3.236) gives

$$\begin{aligned}\rho^{pas}\left((M+E)^2\right) &= \frac{m^{3/2}E^{7/2}(E+2m_0)^{7/2}}{26880\pi^5(E+m_0)^3(E+M)^{3/2}}\left\{{}_2F_1\left(\frac{3}{2},2;\frac{9}{2};1-\frac{m_0^2}{(E+m_0)^2}\right)+\right. \\ &\quad +\frac{E(E+2m_0)}{8m(E+M)}\left(1+\frac{8m}{3(E+m_0)}\right)\,{}_2F_1\left(2,\frac{5}{2};\frac{11}{2};1-\frac{m_0^2}{(E+m_0)^2}\right)+ \\ &\quad \left.+\frac{E^2(E+2m_0)^2}{1408(E+M)^2}\left(1-\frac{32m^2}{3(E+m_0)^2}\right)\,{}_2F_1\left(\frac{5}{2},3;\frac{13}{2};1-\frac{m_0^2}{(E+m_0)^2}\right)\right\}.\end{aligned} \tag{3.247}$$

In Fig. 3.11 one can see how the expansion improves if the resummation of the smallest mass contributions (displayed as dashed lines) is performed.

The result of this section is quite general and applicable to all cases of one small mass. For some particular arrangement of masses one can obtain even simpler expressions as discussed in the next section.

3.6.7 The convolution with a small mass

In this section a result for the resummation of the smallest mass effects is obtained along a different route, namely, via the *convolution of spectral densities*. However, this method works in a narrower kinematic region than the method described in the previous section. In $D=4$ space-time dimensions, the convolution weight is given by

$$\rho(s;s_1;s_2)=\frac{1}{(4\pi)^2 s}\sqrt{(s-s_1-s_2)^2-4s_1s_2}. \tag{3.248}$$

The upper limit of the integration is determined by the requirement of positivity of the the square root argument. The zeros of the square root with respect to s_2 are given by $s_2^\pm=(\sqrt{s}\pm\sqrt{s}_1)^2$, and the demand $(s_2-s_2^+)(s_2-s_2^-)>0$ together with $s_2^+>s_2^-$ leads to $s_2>s_2^+$ or $s_2<s_2^-$. The physical region is the latter one. With $\rho_1(s)=\delta(s-m_0^2)$ for the spectral density of the single small mass line one obtains

$$\begin{aligned}\rho(s) &= \int_0^\infty ds_1\int_{M'^2}^{(\sqrt{s}-\sqrt{s_1})^2} ds_2\rho(s;s_1;s_2)\rho_1(s_1)\rho_2(s_2) = \\ &= \frac{1}{(4\pi)^2 s}\int_{M'^2}^{(\sqrt{s}-m_0)^2}\sqrt{(s-m_0^2-s_2)^2-4m_0^2s_2}\;\rho_2(s_2)ds_2\end{aligned} \tag{3.249}$$

where the low limit of integration is $M'=M-m_0$. Inserting $s=(M+E)^2$ and $s_2=(M'+E')^2$ one obtains

$$\begin{aligned}\hat\rho(E) &= \frac{1}{(4\pi)^2(M+E)^2}\int_0^E\sqrt{(E-E')(E+E'+2M)+m_0^2}\times \\ &\quad \times\sqrt{(E-E'+2m_0)(E+E'+2M')+m_0^2}\;\frac{\hat\rho'(E')dE'}{2(M'+E')}\end{aligned} \tag{3.250}$$

where $\hat{\rho}(E) = \rho_2((M+E)^2)$ and $\hat{\rho}'(E') = \rho_2((M'+E')^2)$. For the second of these functions one uses the threshold expansion in E'/M' as expansion parameter. For small $E < M'$ the threshold expansion inserted for $\hat{\rho}'(E')$ is valid because $E < M'$ implies $E' < M'$. The described procedure can be extended to the case of a very light sub-block of the diagram, e.g. a light fish diagram. In this case one has to replace $\rho_1(s)$ by the spectral density of the light sub-diagram which is well-known.

3.6.8 Recovering $\tilde{\Pi}(-p^2)$ through $\rho(s)$ near threshold

The analytic structure of water melon diagrams is completely fixed by the dispersion representation. Therefore, the focus of this presentation have been on the computation of the spectral density as the basic quantity important both for applications and the theoretical investigation of the diagram. However, with an analytical expression for the spectral density $\rho(s)$ at hand one can readily reconstruct the non-analytic piece of the polarization function in momentum space by using the dispersion relation

$$\tilde{\Pi}(-p^2) = \int \frac{\rho(s)ds}{s-p^2}. \tag{3.251}$$

One rewrites this equation in terms of threshold parameters according to $p = M - \Delta$ and $s = (M+E)^2$ using $\hat{\Pi}(\Delta) := \tilde{\Pi}(-p^2)$ and $\hat{\rho}(E) := \rho(s)$ and obtains

$$\hat{\Pi}(\Delta) = \int_0^\infty \frac{2(M+E)\hat{\rho}(E)dE}{(E+\Delta)(2M+E-\Delta)}, \qquad p = M - \Delta. \tag{3.252}$$

UV singularities can be removed by subtraction or by dimensional regularization. Again the unorthodox dimensional regularization prescription is used. For a general form of the threshold expansion $\hat{\rho}(E) = E^\gamma \sum a_k E^k$ one has to calculate integrals of the form

$$\hat{\Pi}^\sigma(\Delta) \ = \ \int_0^\infty \frac{E^\sigma dE}{(E+\Delta)(2M+E-\Delta)} = -\frac{\pi}{\sin(\pi\sigma)} \frac{\Delta^\sigma - (2M-\Delta)^\sigma}{2(M-\Delta)}. \tag{3.253}$$

Only the powers Δ^σ contribute to the singular part of the polarization function. Expressions like the one presented in Eq. (3.253) then allow one to restore that part of the correlation function $\tilde{\Pi}(-p^2)$ which has singularities near the threshold.

Chapter 4

The correlator of finite mass baryon currents

Baryons form a rich family of particles which has been experimentally studied with high accuracy [127]. A theoretical analysis of these experimental data gives a lot of information about the structure of QCD and the numerical values of its parameters. The hypothetical limit $N_c \to \infty$ for the number N_c of colours which is a very powerful tool for investigating the general properties of gauge interactions was especially successful for baryons [128]. The spectrum of baryons is contained in the correlator of two baryonic currents and the spectral density associated with it. To leading order the correlator is given by a product of N_c fermionic propagators. The diagrams of this topology, known as sunrise type diagrams, have been studied in the previous chapter in detail (cf. Refs. [107, 108, 112, 116, 129, 130, 131, 132]). They are rather frequently used in phenomenological applications [109, 84, 133]. With the advent of new accelerators and detectors many properties of baryons containing a heavy quark have been experimentally measured in recent years [127]. However, theoretical calculations beyond the leading order have not been done yet for many interesting cases.

In this chapter the α_s corrections to the correlator of two baryonic currents with one finite mass quark and two massless quarks are calculated in analytical form. The magnitude of the α_s corrections will be discussed and the massless and HQET limits are obtained as special cases. Analytical results for the moments of the spectral density associated with the correlator are presented as well. Note that the massless case has been known since long ago [134]. The mesonic analogue of the baryonic calculation was completed some time ago [135] and has subsequently provided a rich source of inspiration for many applications in meson physics.

The chapter is closed with considerations about the pole mass. The reason for these considerations is that in recent times the pole mass became somewhat "condemned" as being of no practical use for perturbation theory calculations. In the closing section it is argued why it is important to consider the energy region for which a mass concept is used and that the pole mass at its natural place (namely close to threshold) is still the best choice for perturbation theory calculations in this region.

4.1 Finite mass baryonic currents and correlators

The *three-quark current* considered here is of the form

$$j = \epsilon^{abc}(u_a^T C\Gamma d_b)\Gamma'\Psi_c \tag{4.1}$$

where Ψ is a finite mass quark with mass parameter m, u and d are massless quarks and C is the charge conjugation matrix. ϵ^{abc} is the totally antimetric tensor and a, b, c are colour indices for the $SU(3)$ colour group. Finally, Γ and Γ' are Dirac gamma matrices which correspond to the quantum numbers of the baryon. For instance, taking $\Gamma = 1$ and $\Gamma' = \gamma_5$ corresponds to a current describing a $J^P = 1/2^-$ baryon. In order to exhibit the general features of the calculation it suffices to take $\Gamma = \Gamma' = 1$. The correlator of two baryonic currents is expanded as

$$i\int \langle Tj(x)\bar{j}(0)\rangle e^{iqx}dx = \gamma_\nu q^\nu \Pi_q(-q^2) + m\Pi_m(-q^2). \tag{4.2}$$

The argument $(-q^2)$ is a convention to indicate that the correlator is considered in Minkowskian metric. $\Gamma' = \gamma_5$ leads to the trivial change $\Pi_q(-q^2) \to -\Pi_q(-q^2)$. Before going into further details, the diagrams to next-to-leading order are displayed in Fig. 4.1. The figure shows the general situation, even for different masses. In the case which is considered here, the massive line is taken out of the trace in all cases, therefore only diagrams with label ending with "1" are considered.

The leading and next-to-leading order contribution is given by two- and three-loop integrals, respectively. As will be explained later, the last integration will be replaced by a convolution of the spectral density with a weight (or convolution) function, similar to the treatment in the previous chapter. One therefore ends up with one- and two-loop integrals. At the beginning of this chapter, therefore, it is natural to introduce the standard integrals. A second step consists in explaining how to obtain the spectral densities for these integrals in a more general manner before starting with the leading order contribution.

4.1.1 Correlator functions to one- and two-loop order

The *massless standard one-loop integral* $G(n_1, n_2)$ in the Euclidean domain is given by

$$\frac{1}{(4\pi)^{D/2}}(p^2)^{D/2-n_1-n_2}G(n_1,n_2) := \int \frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2)^{n_1}((p-k)^2)^{n_2}}. \tag{4.3}$$

$G(n_1, n_2)$ can be expressed purely in terms of Euler's Gamma functions and does not depend on the moments. The expression is given by

$$G(n_1,n_2) = \frac{\Gamma(n_1+n_2-D/2)}{\Gamma(n_1)\Gamma(n_2)}\frac{\Gamma(D/2-n_1)\Gamma(D/2-n_2)}{\Gamma(D-n_1-n_2)} \tag{4.4}$$

which vanishes if n_1 or n_2 is zero or a negative integer. A basic element of the future results will be

$$G(1,1) = \frac{\Gamma(\varepsilon)}{\Gamma(1)^2}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} = \frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)^2}{\varepsilon(1-2\varepsilon)\Gamma(1-2\varepsilon)} =: \frac{G}{\varepsilon} \tag{4.5}$$

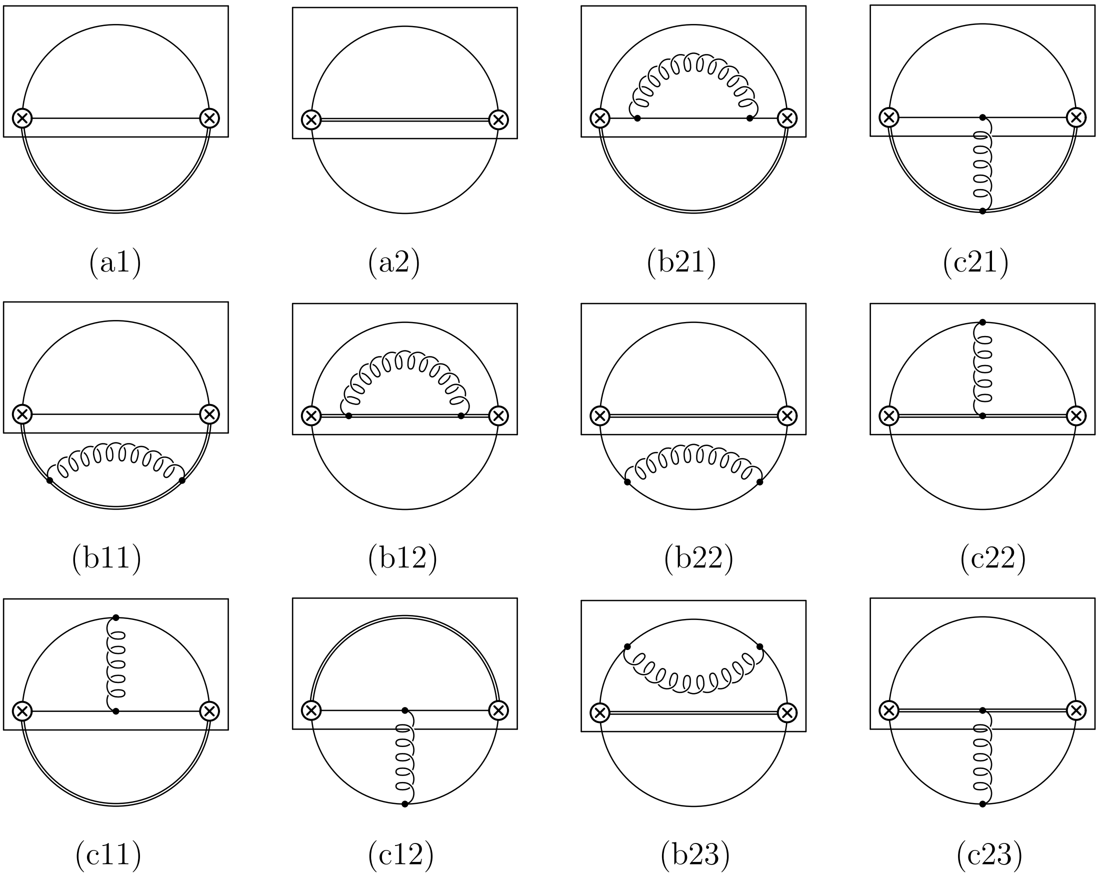


Figure 4.1: Leading order and next-to-leading order diagrams for the finite mass hadronic current correlator. Double lines indicate the finite mass quark, single lines the massless quarks. The springy lines stand for the gluons. The letters denote the leading order diagram (a), the self energy diagram (b), and the fish diagram (c). The first number for (b) and (c) assigns whether the massive line is involved in the radiative corrections (2) or not (1). The last number indicates which of the lines (massive line 1 or massless lines 2 and 3) is excluded from the trace, shown in the figure as a box drawn over the diagram.

which fixes the so-called *G-scheme* where powers in G are not expanded. In a similar manner one defines the *massless standard two-loop integral*

$$\begin{aligned}\frac{1}{(4\pi)^D}(p^2)^{D-n_1-n_2-n_3-n_4-n_5}G(n_1,n_2,n_3,n_4,n_5) &= \\ := \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(k^2)^{n_1}(l^2)^{n_2}((p-k)^2)^{n_3}((p-l)^2)^{n_4}((k-l)^2)^{n_5}}. \qquad (4.6)\end{aligned}$$

$G(n_1,n_2,n_3,n_4,n_5)$ can be reduced to one-loop integrals and thereby to Euler's Gamma functions by using the integration-by-parts technique. According to these almost traditional settings one defines one- and two-loop integrals in the finite mass case. The *single mass standard one-loop integral* $V(n_1,n_2;p^2/m^2)$ is defined as

$$\frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-n_1-n_2}V(n_1,n_2;p^2/m^2) := \int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2+m^2)^{n_1}((p-k)^2)^{n_2}}. \qquad (4.7)$$

Using Feynman parametrization one obtains

$$V(n_1,n_2;p^2/m^2)=\frac{\Gamma(n_1+n_2-D/2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1(1-x)^{D/2-n_2-1}x^{n_2-1}\left(1+\frac{xp^2}{m^2}\right)^{D/2-n_1-n_2}dx. \tag{4.8}$$

Special cases are

$$\begin{aligned}
V(n_1,n_2;p^2/m^2) &= 0 \qquad \text{for } n_1\le 0,\\
V(n_1,0;p^2/m^2) &= V(n_1,0;-1) \;=\; \frac{\Gamma(n_1-D/2)}{\Gamma(n_1)},\quad V(1,0;-1) \;=\; -\frac{G}{\varepsilon}+1+O(\varepsilon),\\
V(1,1;p^2/m^2) &= \Gamma(2-D/2)\int_0^1(1-x)^{D/2-2}\left(1+\frac{xp^2}{m^2}\right)^{D/2-2}dx,\\
V(1,1;-1) &= \Gamma(2-D/2)\int_0^1(1-x)^{D-4}dx \;=\; \frac{\Gamma(\varepsilon)}{1-2\varepsilon} \;=\; \frac{G}{\varepsilon}+O(\varepsilon).
\end{aligned} \tag{4.9}$$

The *standard single mass two-loop integral* $V(n_1,n_2,n_3,n_4,n_5;p^2/m^2)$ finally is given by

$$\begin{aligned}
&\frac{1}{(4\pi)^D}(m^2)^{D-n_1-n_2-n_3-n_4-n_5}V(n_1,n_2,n_3,n_4,n_5;p^2/m^2) \;=\\
&:= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(k^2+m^2)^{n_1}(l^2+m^2)^{n_2}((p-k)^2)^{n_3}((p-l)^2)^{n_4}((k-l)^2)^{n_5}}.
\end{aligned} \tag{4.10}$$

4.1.2 The integral representation of the basic spectral densities

To obtain the spectral density, one inverts the dispersion relation

$$\Pi(q^2)=\int\frac{\rho(s)ds}{q^2+s} \tag{4.11}$$

(the correlator function now in Euclidean metric) to obtain (cf. Chapter 2)

$$\rho(s)=\frac{1}{2\pi i}\operatorname{Disc}\Pi(s):=\frac{1}{2\pi i}\left(\Pi(se^{-i\pi})-\Pi(se^{+i\pi})\right) \tag{4.12}$$

This has to be applied to the standard one- and two-loop integrals V and leads to spectral densities ρ_V. One can calculate the discontinuity of the one-loop integral $V(n_1,n_2;p^2/m^2)$ even by keeping the integration in Eq. (4.8). In order to perform this operation it is useful to use the dimensional representation of the standard integrals (indicated by the tilde),

$$\begin{aligned}
V(n_1,n_2;p^2/m^2) &= (m^2)^{n_1+n_2-D/2}\tilde V(n_1,n_2;p^2),\\
V(n_1,n_2,n_3,n_4,n_5;p^2/m^2) &= (m^2)^{n_1+n_2+n_3+n_4+n_5-D}\tilde V(n_1,n_2,n_3,n_4,n_5;p^2),
\end{aligned} \tag{4.13}$$

Therefore, one starts with

$$\tilde V(n_1,n_2;p^2)=\frac{\Gamma(n_1+n_2-D/2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1(1-x)^{D/2-n_2-1}x^{n_2-1}\left(m^2+xp^2\right)^{D/2-n_1-n_2}dx. \tag{4.14}$$

The crucial point in the calculation of the discontinuity for the integrand is the non-integer power of $(m^2 + xp^2)$. In the case $p^2 < -m^2$, the integration encounters a cut for $x > -m^2/p^2$. The discontinuity is therefore given by

$$\begin{aligned}
\mathrm{Disc}(m^2 - xs)^{-n} &= (m^2 + xse^{-i\pi})^{-n} - (m^2 + xse^{i\pi})^{-n} = \\
&= \left[\left((xs - m^2)e^{-i\pi}\right)^{-n} - \left((xs - m^2)e^{i\pi}\right)^{-n}\right]\theta(xs - m^2) = \\
&= \left(e^{in\pi} - e^{-in\pi}\right)(xs - m^2)^{-n}\theta(xs - m^2) = 2i\sin(n\pi)(xs - m^2)^n\theta(xs - m^2)
\end{aligned} \tag{4.15}$$

where n is in general a non-integer number. Making use of

$$\sin(n\pi) = \frac{\pi}{\Gamma(n)\Gamma(1-n)} \tag{4.16}$$

one obtains

$$\begin{aligned}
\tilde{\rho}_V(n_1, n_2; s) &= \\
&= C_{n_1 n_2} \int_{m^2/s}^1 (1-x)^{D/2-n_2-1} x^{n_2-1} (xs - m^2)^{D/2-n_1-n_2} dx = \\
&= C_{n_1 n_2} (m^2)^{D/2-n_1-n_2} \int_{m^2/s}^1 (1-x)^{D/2-n_2-1} x^{n_2-1} \left(\frac{xs}{m^2} - 1\right)^{D/2-n_1-n_2} dx = \\
&= C_{n_1 n_2} s^{D/2-n_1-n_2} \int_{m^2/s}^1 (1-x)^{D/2-n_2-1} x^{n_2-1} \left(x - \frac{m^2}{s}\right)^{D/2-n_1-n_2} dx
\end{aligned} \tag{4.17}$$

where

$$C_{n_1 n_2} := \frac{1}{\Gamma(n_1)\Gamma(n_2)\Gamma(1 + D/2 - n_1 - n_2)}. \tag{4.18}$$

The last two expressions in Eq. (4.17) indicate the two possible dimensionless representations,

$$\begin{aligned}
\tilde{\rho}_V(n_1, n_2; s) &= (m^2)^{D/2-n_1-n_2} \rho_V(n_1, n_2; s/m^2), \\
\tilde{\rho}_V(n_1, n_2; s) &= s^{D/2-n_1-n_2} \hat{\rho}_V(n_1, n_2; m^2/s).
\end{aligned} \tag{4.19}$$

While the procedure of the calculations allows for a direct transition from V to ρ, the hatted quantities $\hat{\rho}$ are mainly used for the final result because of their explicit s-dependence. On the one-loop level they read

$$\hat{\rho}_V(n_1, n_2; z) = C_{n_1 n_2} \int_{m^2/s}^1 (1-x)^{D/2-n_2-1} x^{n_2-1} (x - z)^{D/2-n_1-n_2} dx \tag{4.20}$$

where the abbreviation $z = m^2/s$ is used. The general relations between hatted and unhatted spectral densities are given by

$$\rho_V(n_1, n_2; 1/z) = z^{n_1+n_2-D/2} \hat{\rho}_V(n_1, n_2; z) \qquad \text{and} \tag{4.21}$$

$$\rho_V(n_1, n_2, n_3, n_4, n_5; 1/z) = z^{n_1+n_2+n_3+n_4+n_5-D} \hat{\rho}_V(n_1, n_2, n_3, n_4, n_5; z). \tag{4.22}$$

Finally one uses the explicit formula for $\hat{\rho}_V(n_1,n_2;z)$ to obtain a second main ingredient for the final result. For $z\le 1$ one has

$$\begin{aligned}
\hat{\rho}_V(1,1;z) &= \frac{1}{\Gamma(1-\varepsilon)}\int_z^1 (1-x)^{-\varepsilon}(x-z)^{-\varepsilon}dx \;=\; \qquad (x':=x-z)\\
&= \frac{1}{\Gamma(1-\varepsilon)}\int_0^{1-z}(1-z-x')^{-\varepsilon}x'^{-\varepsilon}dx' \;=\; \frac{(1-z)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\int_0^{1-z}\left(1-\frac{x'}{1-z}\right)^{-\varepsilon}x'^{-\varepsilon}dx' \;=\\
&= \frac{(1-z)^{-\varepsilon}}{\Gamma(1-\varepsilon)}\int_0^1 (1-x'')^{-\varepsilon}(1-z)^{-\varepsilon}x''^{-\varepsilon}(1-z)dx'' \;=\; \qquad \left(x'':=\frac{x'}{1-z}\right)\\
&= \frac{(1-z)^{1-2\varepsilon}}{\Gamma(1-\varepsilon)}\int_0^1(1-x'')^{-\varepsilon}x''^{-\varepsilon}dx'' \;=\; \frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)\Gamma(1-\varepsilon)}(1-z)^{1-2\varepsilon} \;=\\
&= \frac{G}{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}(1-z)^{1-2\varepsilon}, \quad \text{using again} \quad G=\frac{\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)}. \qquad (4.23)
\end{aligned}$$

4.2 The mass part of the correlator

According to Eq. (4.2), two main ingredients have to be considered, namely the *momentum part* $\Pi_q(-q^2)$ and the *mass part* $\Pi_m(-q^2)$ of the correlator function. This section deals with the (easier) mass part, starting with the leading order diagram.

4.2.1 Contribution from the leading order diagram (a1)

The contribution of the leading order diagram (a1) is given by

$$\begin{aligned}
-iV_{a1}(-q^2) &= \int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\left(\frac{i}{\not q-\not p}\,\frac{i}{\not p-\not k}\right)\frac{i}{\not k-m} \;=\\
&= -i\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}((\not q-\not p)(\not p-\not k))}{(q-p)^2(p-k)^2}\,\frac{\not k+m}{k^2-m^2} \;=\\
&= -4i\int\frac{d^Dk}{(2\pi)^D}\frac{\not k+m}{k^2-m^2}\int\frac{d^Dp}{(2\pi)^D}\frac{(q-p)(p-k)}{(q-p)^2(p-k)^2} \;=\\
&= -2i\int\frac{d^Dk}{(2\pi)^D}\frac{\not k+m}{k^2-m^2}\int\frac{d^Dp}{(2\pi)^D}\frac{(q-k)^2-(q-p)^2-(p-k)^2}{(q-p)^2(p-k)^2} \;=\\
&= \frac{-2}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}\frac{\not k+m}{k^2-m^2}\left(-(q-k)^2\right)^{D/2-1}G(1,1). \qquad (4.24)
\end{aligned}$$

For the mass part one has

$$\begin{aligned}
V^m_{a1}(-q^2) &= \frac{-2i}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}\frac{G(1,1)}{(k^2-m^2)(-(q-k)^2)^{1-D/2}} \;=\\
&= \frac{-2}{(4\pi)^D}(m^2)^{D-2}G(1,1)V(1,1-D/2;-q^2/m^2). \qquad (4.25)
\end{aligned}$$

The corresponding spectral density is given by

$$\begin{aligned}
\rho^m_{a1}(s) &= \frac{-2}{(4\pi)^D}(m^2)^{2-2\varepsilon}G(1,1)\rho_V(1,\varepsilon-1;s/m^2) \;=\\
&= \frac{-2}{(4\pi)^D}G(1,1)\tilde{\rho}_V(1,\varepsilon-1;s) \;=\; -\frac{2G(1,1)}{(4\pi)^D}s^{2-2\varepsilon}\hat{\rho}_V(1,\varepsilon-1;m^2/s). \qquad (4.26)
\end{aligned}$$

4.2.2 The colour factors

The colour factor for the leading order diagram is simply given by

$$\epsilon_{ijk}\epsilon^{ijk} = N_c! = 6. \tag{4.27}$$

For a diagram containing a self energy loop one uses

$$\sum_a (T^a)^i{}_{i'}(T^a)^j{}_{j'} = \frac{1}{2}\left(\delta^i_{j'}\delta^j_{i'} - \frac{1}{N_c}\delta^i_{i'}\delta^j_{j'}\right) \tag{4.28}$$

to obtain

$$\begin{aligned} C_{\rm self} &= \epsilon_{ijk}\epsilon^{i''jk}(T^a)^i{}_{i'}(T^b)^{i'}{}_{i''}\delta_{ab} = \frac{1}{2}\epsilon_{ijk}\epsilon^{i''jk}\left(\delta^i_{i''}\delta^{i'}_{i'} - \frac{1}{N_c}\delta^i_{i'}\delta^{i'}_{i''}\right) = \\ &= \frac{1}{2}\left(N_c\epsilon_{ijk}\epsilon^{ijk} - \frac{1}{N_c}\epsilon_{ijk}\epsilon^{ijk}\right) = \frac{N_c^2-1}{2N_c}\epsilon_{ijk}\epsilon^{ijk} = C_F N_c!. \end{aligned} \tag{4.29}$$

For a diagram containing a fish one obtains

$$\begin{aligned} C_{\rm fish} &= \epsilon_{ijk}\epsilon^{i'j'k}(T^a)^i{}_{i'}(T^b)^j{}_{j'}\delta_{ab} = \frac{1}{2}\epsilon_{ijk}\epsilon^{i'j'k}\left(\delta^i_{j'}\delta^j_{i'} - \frac{1}{N_c}\delta^i_{i'}\delta^j_{j'}\right) = \\ &= \frac{1}{2}\left(\epsilon_{ijk}\epsilon^{jik} - \frac{1}{N_c}\epsilon_{ijk}\epsilon^{ijk}\right) = -\frac{N_c+1}{2N_c}\epsilon_{ijk}\epsilon^{ijk} = -C_B N_c!. \end{aligned} \tag{4.30}$$

4.2.3 The light results: self energy diagram (b21) and fish (c11)

Light diagrams are those where the gluon is exchanged between the massless quark lines. The results for the two diagrams (b21) and (c11) are taken from the output of the MATHEMATICA package developed for massless three-loop integrals, being

$$\rho^m_{b21}(s) = \frac{4G^2g_s^2}{(4\pi)^{3D/2}}s^{2-3\varepsilon}\left(\frac{1}{4\varepsilon^2}+\frac{1}{8\varepsilon}+\frac{11}{16}\right)\hat\rho_V(1,2\varepsilon-1;m^2/s), \tag{4.31}$$

$$\rho^m_{c11}(s) = \frac{4G^2g_s^2}{(4\pi)^{3D/2}}s^{2-3\varepsilon}\left(\frac{2}{\varepsilon^2}+\frac{3}{2\varepsilon}+\frac{27}{4}-6\zeta(3)\right)\hat\rho_V(1,2\varepsilon-1;m^2/s) \tag{4.32}$$

Both parts have to be combined, taking care also on the colour factors which are $N_c!$ for the leading order diagram (a1), $N_c!C_F$ ($C_F = 4/3$) for the self energy diagram (b21) and $-N_c!C_B$ ($C_B = 2/3$) for the fish diagram (c11). Dropping the general factor that normalizes to the leading order diagram for a moment, one obtains (note the additional factor 2 because the self energy diagram is occuring twice)

$$\begin{aligned} \rho^m_{\rm light}(s) &= 2N_c!C_F\rho^m_{b21}(s) - N_c!C_B\rho^m_{c11}(s) = \\ &= \frac{4G^2g_s^2N_c!}{(4\pi)^{3D/2}}s^{2-3\varepsilon} \times \\ &\times \left(2C_F\left(\frac{1}{4\varepsilon^2}+\frac{1}{8\varepsilon}+\frac{11}{16}\right) - C_B\left(\frac{2}{\varepsilon^2}+\frac{3}{2\varepsilon}+\frac{27}{4}-6\zeta(3)\right)\right)\hat\rho_V(1,2\varepsilon-1;m^2/s) = \\ &= \frac{2G^2g_s^2N_c!}{(4\pi)^{3D/2}}s^{2-3\varepsilon}C_F\left(\frac{1}{\varepsilon^2}+\frac{1}{2\varepsilon}+\frac{11}{4}-\frac{2}{\varepsilon^2}-\frac{3}{2\varepsilon}-\frac{27}{4}+6\zeta(3)\right)\hat\rho_V(1,2\varepsilon-1;m^2/s) = \\ &= \frac{2G^2g_s^2N_c!}{(4\pi)^{3D/2}}s^{2-3\varepsilon}C_F\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}-4+6\zeta(3)\right)\hat\rho_V(1,2\varepsilon-1;m^2/s) = \\ &= -\frac{2G^2g_s^2N_c!}{(4\pi)^{3D/2}}s^{2-3\varepsilon}C_F\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}+4-6\zeta(3)\right)\hat\rho_V(1,2\varepsilon-1;m^2/s). \end{aligned} \tag{4.33}$$

The light correction is generally represented as

$$\rho^m_{\rm light}(s) = -\frac{2N_c!}{(4\pi)^D}\frac{\alpha_s C_F}{4\pi}G(1,1)^2 s^{2-3\varepsilon}\left(B_0+B_1\varepsilon+B_2\varepsilon^2\right)\hat\rho_V(1,2\varepsilon-1;m^2/s). \tag{4.34}$$

4.2.4 Combining leading order and light corrections

The combination of the two parts is given by

$$\rho^m_{\rm leading}(s) = -\frac{2N_c!}{(4\pi)^D}G(1,1)s^{2-2\varepsilon}\hat\rho_V(1,\varepsilon-1;m^2/s) \tag{4.35}$$

and

$$\rho^m_{\rm light}(s) = -\frac{2N_c!}{(4\pi)^D}\frac{\alpha_s C_F}{4\pi}G(1,1)^2 s^{2-3\varepsilon}\left(B_0+B_1\varepsilon+B_2\varepsilon^2\right)\hat\rho_V(1,2\varepsilon-1;m^2/s). \tag{4.36}$$

After having obtained this result, the next step is to factor out common structures. Separating the pure integral from the common factor, one writes

$$\hat\rho_V(1,\varepsilon-1;z) =: C_1\varepsilon\hat g_1(z), \qquad \hat\rho_V(1,2\varepsilon-1;z) =: C_2\varepsilon\hat g_2(z) \tag{4.37}$$

with

$$\begin{aligned}\hat g_n(z) &:= \int_z^1 (1-x)^{2-(1+n)\varepsilon}x^{n\varepsilon-2}(x-z)^{2-(n+1)\varepsilon}dx,\\ C_n\varepsilon &:= \frac{1}{\Gamma(n\varepsilon-1)\Gamma(D/2-n\varepsilon+1)} = \frac{1}{\Gamma(n\varepsilon-1)\Gamma(3-(n+1)\varepsilon)}\end{aligned} \tag{4.38}$$

(i.e. $C_1=-1/2+O(\varepsilon)$ and $C_2=1+O(\varepsilon)$). Using this, one obtains

$$\begin{aligned}\rho^m_l(s) &:= \rho^m_{\rm leading}(s)+\rho^m_{\rm light}(s) =\\ &= -\frac{2N_c!}{(4\pi)^D}G(1,1)s^{2-2\varepsilon}C_1\varepsilon\Bigg(\hat g_1(m^2/s)\ +\\ &\qquad +\frac{\alpha_s C_F}{4\pi}s^{-\varepsilon}\frac{C_2}{C_1}G(1,1)(B_0+B_1\varepsilon+B_2\varepsilon^2)\hat g_2(m^2/s)\Bigg) =\\ &=: C^g\left(\hat g_1(m^2/s)+\frac{\alpha_s C_F}{4\pi}C^r\hat g_2(m^2/s)\right) =\\ &= C^g\left(\left(1+\frac{\alpha_s C_F}{4\pi}C^r\right)\hat g_1(m^2/s)+\frac{\alpha_s C_F}{4\pi}C^r\left(\hat g_2(m^2/s)-\hat g_1(m^2/s)\right)\right)\end{aligned} \tag{4.39}$$

with

$$C^g = -\frac{2N_c!}{(4\pi)^D}G(1,1)s^{2-2\varepsilon}C_1\varepsilon = -\frac{2N_c!G}{(4\pi)^D}s^{2-2\varepsilon}C_1 \to \frac{N_c!s^2}{(4\pi)^4}. \tag{4.40}$$

While the singularity contained in the first part is absorbed by the light current renormalization factor, the second part is finite, even though C^r is proportional to $1/\varepsilon$ and is expanded as

$$\begin{aligned}C^r &= \left(\frac{\mu^2}{s}\right)^\varepsilon\frac{C_2}{C_1}G(1,1)(B_0+B_1\varepsilon+B_2\varepsilon^2+O(\varepsilon^3)) =\\ &= \frac{G}{\varepsilon}\left(2B_0+\varepsilon\left(2B_0\ln\left(\frac{\mu^2}{s}\right)+2B_1+B_0\right)\right)+O(\varepsilon^2) = \frac{C^r_0}{\varepsilon}+C^r_1+O(\varepsilon).\end{aligned} \tag{4.41}$$

The reason is that the integral difference $\hat{g}_2(z) - \hat{g}_1(z)$ is of order ε,

$$\begin{aligned}
\hat{g}_2(z) - \hat{g}_1(z) &= \\
&= \int_z^1 (1-x)^{2-3\varepsilon} x^{2\varepsilon-2} (x-z)^{2-3\varepsilon} dx - \int_z^1 (1-x)^{2-2\varepsilon} x^{\varepsilon-2} (x-z)^{2-2\varepsilon} dx = \\
&= \varepsilon \int_z^1 \frac{(1-x)^2 (x-z)^2}{x^2} (\ln x - \ln(1-x) - \ln(x-z))\, dx + O(\varepsilon^2) = \\
&= \varepsilon \Big\{ 4z(1+z)(\mathrm{Li}_2(z) - \mathrm{Li}_2(1)) - 2\left(\frac{1}{3} + 3z - 3z^2 - \frac{1}{3}z^3\right) \ln(1-z) + \\
&\qquad - \left(z + 4z^2 + \frac{1}{3}z^3\right) \ln z + \frac{1}{9} + \frac{7}{3}z - \frac{7}{3}z^2 - \frac{1}{9}z^3 \Big\} + O(\varepsilon^2) = \\
&=: \varepsilon \hat{g}_{21}^1(z) + O(\varepsilon^2).
\end{aligned} \tag{4.42}$$

Therefore, one obtains

$$\rho_l^m(s) = \frac{N_c! s^2}{(4\pi)^4} \left(\left(1 + \frac{\alpha_s C_F}{4\pi\varepsilon} C_0^r\right) \hat{g}_1(m^2/s) + \frac{\alpha_s C_F}{4\pi} \left(C_1^r \hat{g}_1^0(m^2/s) + C_0^r \hat{g}_{21}^1(m^2/s) \right) \right) + O(\varepsilon). \tag{4.43}$$

The first part is absorbed into the light current renormalization factor while the second gives the finite correction.

4.2.5 The massive contribution (b11)

This subsection contains the calculations for the *self energy correction to the massive line*. It starts with the calculation of the self energy diagram, leads to the extraction of the singular parts which will renormalize the mass and the wave function, and will finally result in a rather compact form for the finite parts wich remain. The calculations in this subsection are done within Minkowskian metric (indicated by the negative arguments, in contrast to the fish).

Starting with the master bubble

Basic ingredient will be the master bubble diagram

$$\begin{aligned}
\Pi_B(-p^2) &= \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-m^2)(p-k)^2} = \frac{(m^2)^{D/2-2}}{(4\pi)^{D/2}} V(1,1;-p^2/m^2) = \int \frac{\rho_B(s) ds}{s-p^2}, \\
\rho_B(s) &= \frac{(m^2)^{D/2-2}}{(4\pi)^{D/2}} \rho_V(1,1;s/m^2) = \frac{1}{(4\pi)^{D/2}} \tilde{\rho}_V(1,1;s).
\end{aligned} \tag{4.44}$$

One starts with the self energy diagram, given by

$$\begin{aligned}
-i\Sigma(-k^2) &= \int \frac{d^D l}{(2\pi)^D} (-ig_s\gamma_\alpha) \frac{i}{\not{l} - m} (-ig_s\gamma^\alpha) \frac{-i}{(k-l)^2} = \\
&= -g_s^2 \int \frac{d^D l}{(2\pi)^D} \frac{\gamma_\alpha(\not{l} - m)\gamma^\alpha}{(l^2-m^2)(k-l)^2} = -i\not{k}\Sigma_p(-k^2) - im\Sigma_m(-k^2).
\end{aligned} \tag{4.45}$$

Therefore, the momentum part reads

$$\begin{aligned}
\Sigma_p(-k^2) &= -ig_s^2\int\frac{d^Dl}{(2\pi)^D}\frac{\frac14\mathrm{Tr}(\not k\gamma_\alpha(\not l+m)\gamma^\alpha)}{k^2(l^2-m^2)(k-l)^2} = \\
&= -i(2-D)g_s^2\int\frac{d^Dl}{(2\pi)^D}\frac{kl}{k^2(l^2-m^2)(k-l)^2} = \\
&= -i\frac{2-D}{2}g_s^2\int\frac{d^Dl}{(2\pi)^D}\frac{k^2+m^2+l^2-m^2-(k-l)^2}{k^2(l^2-m^2)(k-l)^2} = \\
&= \frac{(2-D)g_s^2}{2(4\pi)^{D/2}}\left(\left(1+\frac{m^2}{k^2}\right)\tilde V(1,1;-k^2)-\tilde V(1,0;-k^2)\right) \qquad \Rightarrow \\
\tilde\rho_p(s) &= \frac{(2-D)g_s^2}{2(4\pi)^{D/2}}\left(1+\frac{m^2}{s}\right)\tilde\rho_V(1,1;s) = \frac{2-D}{2}g_s^2\left(1+\frac{m^2}{s}\right)\rho_B(s). \qquad (4.46)
\end{aligned}$$

The correlator function is given by the dispersion relation

$$\tilde\Pi_p(-k^2) = \int\frac{\tilde\rho_p(s)ds}{s-k^2} = \frac{2-D}{2}g_s^2\int\frac{(1+m^2/s)\rho_B(s)}{s-k^2}ds. \qquad (4.47)$$

For the mass part one obtains

$$\begin{aligned}
\Sigma_m(-k^2) &= -ig_s^2\int\frac{d^Dl}{(2\pi)^D}\frac{\frac14\mathrm{Tr}(\gamma_\alpha(\not l+m)\gamma^\alpha)}{m(l^2-m^2)(k-l)^2} = \\
&= -iDg_s^2\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2-m^2)(k-l)^2} = \frac{Dg_s^2}{(4\pi)^{D/2}}\tilde V(1,1;-k^2) \qquad \Rightarrow \\
\tilde\rho_m(s) &= \frac{Dg_s^2}{(4\pi)^{D/2}}\tilde\rho_V(1,1;s) = Dg_s^2\rho_B(s) \qquad \Rightarrow \\
\tilde\Pi_m(-k^2) &= \int\frac{\tilde\rho_m(s)ds}{s-k^2} = Dg_s^2\int\frac{\rho_B(s)ds}{s-k^2}. \qquad (4.48)
\end{aligned}$$

Renormalization of mass and wave function

For the renormalization one has to decide which subtraction point to use. For this purpose one considers the two possibilities for the master bubble,

$$\begin{aligned}
\Pi_B(0) &= \frac{\Gamma(\varepsilon)m^{-2\varepsilon}}{(4\pi)^{D/2}}\int_0^1(1-x)^{-\varepsilon}1^{-\varepsilon}dx = \frac{\Gamma(\varepsilon)m^{-2\varepsilon}}{(4\pi)^{D/2}}\int_0^1 x^{-\varepsilon}dx = \frac{\Gamma(\varepsilon)m^{-2\varepsilon}}{(4\pi)^{D/2}(1-\varepsilon)} = \\
&= \frac{\Gamma(1+\varepsilon)m^{-2\varepsilon}}{(4\pi)^{D/2}\varepsilon(1-\varepsilon)} = \frac{Gm^{-2\varepsilon}}{(4\pi)^{D/2}\varepsilon}\Big(1-\varepsilon-(1-\zeta(2))\,\varepsilon^2+O(\varepsilon^3)\Big), \qquad (4.49) \\
\Pi_B(-m^2) &= \frac{\Gamma(\varepsilon)m^{-2\varepsilon}}{(4\pi)^{D/2}}\int_0^1(1-x)^{-\varepsilon}(1-x)^{-\varepsilon}dx = \frac{\Gamma(\varepsilon)m^{-2\varepsilon}}{(4\pi)^{D/2}}\int_0^1 x^{-2\varepsilon}dx = \qquad (4.50) \\
&= \frac{\Gamma(\varepsilon)m^{-2\varepsilon}}{(4\pi)^{D/2}(1-2\varepsilon)} = \frac{\Gamma(1+\varepsilon)m^{-2\varepsilon}}{(4\pi)^{D/2}\varepsilon(1-2\varepsilon)} = \frac{Gm^{-2\varepsilon}}{(4\pi)^{D/2}\varepsilon}\Big(1+\zeta(2)\varepsilon^2+O(\varepsilon^3)\Big).
\end{aligned}$$

It is obvious that $\Pi_B(-m^2)$ is much "closer" to G/ε than $\Pi_B(0)$. Using $\Pi_B(-m^2)$, therefore, has the advantage that one has not to calculate all orders in ε of the corresponding coefficient. It is more convenient, therefore, to take the subtraction at the point $s=m^2$.

The self energy contributions is associated with the wave function and the mass renormalization. To see this, one uses an ansatz for the renormalization and calculates

$$\begin{aligned}
\frac{i(1+a)}{\not{k}-m(1+b)} &= \frac{i(1+a)}{(\not{k}-m)(1-mb/(\not{k}-m))} \approx \frac{i(1+a)}{\not{k}-m}\left(1+\frac{mb}{\not{k}-m}\right) = \\
&= \frac{i(1+a)}{\not{k}-m}+\frac{i}{\not{k}-m}(1+a)mb\frac{1}{\not{k}-m} = \\
&\approx \frac{i}{\not{k}-m}+\frac{i}{\not{k}-m}(\not{k}-m)a\frac{1}{\not{k}-m}+\frac{i}{\not{k}-m}mb\frac{1}{\not{k}-m} = \\
&= \frac{i}{\not{k}-m}+\frac{i}{\not{k}-m}\left(-i\not{k}a+ima-imb\right)\frac{i}{\not{k}-m} = \\
&= \frac{i}{\not{k}-m}+\frac{i}{\not{k}-m}\left(-i\not{k}a-im(b-a)\right)\frac{i}{\not{k}-m}.
\end{aligned} \tag{4.51}$$

From this result one obtains $\tilde{\Pi}_p = a$, $\tilde{\Pi}_m = b-a$, so that

$$\begin{aligned}
a(-k^2) &= \tilde{\Pi}_p(-k^2) = \frac{2-D}{2}g_s^2\int\frac{\rho_B(s)ds}{s-k^2}\left(1+\frac{m^2}{s}\right) =: \int\frac{\rho_a(s)}{s-k^2}ds, \\
b(-k^2) &= \tilde{\Pi}_p(-k^2)+\tilde{\Pi}_m(-k^2) = \\
&= g_s^2\int\frac{\rho_B(s)ds}{s-k^2}\left(\frac{D+2}{2}-\frac{D-2}{2}\frac{m^2}{s}\right) =: \int\frac{\rho_b(s)}{s-k^2}ds.
\end{aligned} \tag{4.52}$$

Now one splits off the singular parts $a(-m^2)$ and $b(-m^2)$ resp. to obtain

$$a(-k^2) = \int\frac{\rho_a(s)}{s-m^2}ds+\int\left(\frac{1}{s-k^2}-\frac{1}{s-m^2}\right)\rho_a(s)ds = \tag{4.53}$$

$$\begin{aligned}
&= \int\frac{\rho_a(s)}{s-m^2}ds+(k^2-m^2)\int\frac{\rho_a(s)ds}{(s-m^2)(s-k^2)} =: a(-m^2)+a_f(-k^2), \\
b(-k^2) &= \int\frac{\rho_b(s)}{s-m^2}ds+(k^2-m^2)\int\frac{\rho_b(s)ds}{(s-m^2)(s-k^2)} =: b(-m^2)+b_f(-k^2).
\end{aligned}$$

These singular parts can be computed within dimensional regularization,

$$\begin{aligned}
a(-m^2) &= \int\frac{\rho_a(s)}{s-m^2}ds = \frac{2-D}{2}g_s^2\int\frac{\rho_B(s)ds}{s-m^2}\left(1+\frac{m^2}{s}\right) = \\
&= -\frac{(1-\varepsilon)g_s^2}{(4\pi)^{D/2}}\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\mu^{2\varepsilon}\int_{m^2}^\infty\frac{(s-m^2)^{-2\varepsilon}}{s^{1-\varepsilon}}\left(1+\frac{m^2}{s}\right)ds.
\end{aligned} \tag{4.54}$$

Using the substitution $z=m^2/s$, the integral results in

$$\begin{aligned}
(m^2)^\varepsilon\int_{m^2}^\infty\frac{(s-m^2)^{-2\varepsilon}}{s^{1-\varepsilon}}\left(1+\frac{m^2}{s}\right)ds &= (m^2)^\varepsilon\int_{m^2}^\infty\left(1-\frac{m^2}{s}\right)^{-2\varepsilon}s^{-1-\varepsilon}\left(1+\frac{m^2}{s}\right)ds = \\
&= \int_0^1(1-z)^{-2\varepsilon}z^{-1+\varepsilon}(1+z)dz = \int_0^1(1-z)^{-2\varepsilon}z^{-1+\varepsilon}dz+\int_0^1(1-z)^{-2\varepsilon}z^\varepsilon dz = \\
&= B(1-2\varepsilon,\varepsilon)+B(1-2\varepsilon,1+\varepsilon) = \frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-\varepsilon)}+\frac{\Gamma(1-2\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)} = \\
&= \frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)}(1-\varepsilon+\varepsilon) = \frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)}
\end{aligned} \tag{4.55}$$

so that

$$\begin{aligned}
a(-m^2) &= \frac{-g_s^2}{(4\pi)^{D/2}}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \frac{\Gamma(2-\varepsilon)}{\Gamma(2-2\varepsilon)}\,\frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)} = \\
&= \frac{-g_s^2}{(4\pi)^{D/2}}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} = \frac{-g_s^2}{(4\pi)^{D/2}}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \frac{G}{\varepsilon}\,\frac{\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)^2} = \\
&= \frac{-g_s^2}{(4\pi)^{D/2}}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \frac{G}{\varepsilon}\left(1+\zeta(2)\varepsilon^2+O(\varepsilon^3)\right). \qquad (4.56)
\end{aligned}$$

Similarly one obtains

$$\begin{aligned}
b(-m^2) &= \int\frac{\rho_b(s)ds}{s-m^2} = g_s^2\int\frac{\rho_B(s)ds}{s-m^2}\left(\frac{D+2}{2}-\frac{D-2}{2}\,\frac{m^2}{s}\right) = \\
&= \frac{g_s^2}{(4\pi)^{D/2}}\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\mu^{2\varepsilon}\int_{m^2}^\infty\frac{(s-m^2)^{-2\varepsilon}}{s^{1-\varepsilon}}\left(3-\varepsilon-(1-\varepsilon)\frac{m^2}{s}\right)ds = \dots \\
&= \frac{g_s^2}{(4\pi)^{D/2}}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\,\frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)}((1-\varepsilon)(3-\varepsilon)-\varepsilon(1-\varepsilon)) = \\
&= \frac{g_s^2}{(4\pi)^{D/2}}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}(3-2\varepsilon) = \dots \\
&= \frac{g_s^2}{(4\pi)^{D/2}}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \frac{G}{\varepsilon}\left(3-2\varepsilon+3\zeta(2)\varepsilon^2+O(\varepsilon^3)\right). \qquad (4.57)
\end{aligned}$$

These singular parts are absorbed into the wave function and the mass renormalization, respectively, while the finite parts are remaining.

The finite contribution from the massive self energy diagram (b11)

Having collected the contributions to an effective propagator, one expands this propagator again to obtain the finite corrections to the diagram itself,

$$\begin{aligned}
\frac{i(1+a_f)}{\not k-m(1+b_f)} &= \frac{i(1+a_f)(\not k+m(1+b_f))}{k^2-m^2(1+b_f)^2} = \\
&\approx \frac{i(\not k+m)(1+a_f)+imb_f}{k^2-m^2-2m^2b_f} = \frac{i(\not k+m)(1+a_f)+imb_f}{(k^2-m^2)(1-2m^2b_f/(k^2-m^2))} = \\
&\approx \frac{i(\not k+m)(1+a_f)+imb_f}{k^2-m^2}\left(1+\frac{2m^2b_f}{k^2-m^2}\right) = \\
&\approx \frac{i(\not k+m)(1+a_f)+imb_f}{k^2-m^2}+\frac{2im^2(\not k+m)b_f}{(k^2-m^2)^2} = \\
&= \frac{i\not k}{k^2-m^2}\left(1+a_f+\frac{2m^2b_f}{k^2-m^2}\right)+\frac{im}{k^2-m^2}\left(1+a_f+b_f+\frac{2m^2b_f}{k^2-m^2}\right) = \\
&=: \frac{i\not k}{k^2-m^2}(1+P(-k^2))+\frac{im}{k^2-m^2}(1+M(-k^2)). \qquad (4.58)
\end{aligned}$$

Because only the mass part of the diagram is considered, only the M part of this corrected propagator is needed. It consists of two contributions, the one called "$a+b$" and the final,

"b"-contribution. Defining $\rho_{a+b}:=\rho_a+\rho_b$, one obtains

$$\frac{1}{k^2-m^2}(a_f(-k^2)+b_f(-k^2))=\int\frac{\rho_{a+b}(s)ds}{(s-m^2)(s-k^2)} \tag{4.59}$$

and

$$\begin{aligned}\frac{2m^2}{(k^2-m^2)^2}b_f(-k^2) &= \frac{2m^2}{k^2-m^2}\int\frac{\rho_b(s)ds}{(s-m^2)(s-k^2)} = \\ &= 2m^2\int\frac{\rho_b(s)ds}{(s-m^2)^2}\left(\frac{1}{k^2-m^2}-\frac{1}{k^2-s}\right).\end{aligned} \tag{4.60}$$

Therefore, the mass part of the effective propagator is given by

$$\begin{aligned}D^m_{\text{eff}}(-k^2) &= \frac{i}{k^2-m^2}(1+M(-k^2)) = \\ &= \frac{i}{k^2-m^2}\left(1+a_f(-k^2)+b_f(-k^2)+\frac{2m^2b_f(-k^2)}{k^2-m^2}\right) = \\ &= \frac{i}{k^2-m^2}+i\int\frac{\rho_{a+b}(s)ds}{(s-m^2)(s-k^2)}+2im^2\int\frac{\rho_b(s)ds}{(s-m^2)^2}\left(\frac{1}{k^2-m^2}-\frac{1}{k^2-s}\right).\end{aligned} \tag{4.61}$$

To see how one has to insert this effective propagator into the diagram, one considers again the leading order diagram. It reads

$$\begin{aligned}-iV_{a1}(-q^2) &= \int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\left(\frac{i}{\not q-\not p}\,\frac{i}{\not p-\not k}\right)\frac{i}{\not k-m} = \\ &= -2\int\frac{d^Dk}{(2\pi)^D}\frac{i}{\not k-m}\int\frac{d^Dp}{(2\pi)^D}\frac{((q-k)^2-(q-p)^2-(p-k)^2)}{(q-p)^2(p-k)^2} = \\ &= 2\int\frac{d^Dk}{(2\pi)^D}\frac{i}{\not k-m}\left(-(q-k)^2\right)\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(q-p)^2(p-k)^2} = \\ &= \frac{2i}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}\frac{i}{\not k-m}\left(-(q-k)^2\right)^{D/2-1}G(1,1).\end{aligned} \tag{4.62}$$

Therefore, one concludes that

$$V_{a1}(-q^2)+V_{b11}(-q^2)=-\frac{2G(1,1)}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}\frac{D_{\text{eff}}(-k^2)}{(-(q-k)^2)^{1-D/2}} \tag{4.63}$$

and

$$\begin{aligned}V^m_{b11}(-q^2) &= \frac{2iG(1,1)}{(4\pi)^{D/2}}\int\frac{\rho_{a+b}(s)ds}{s-m^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-s)(-(q-k)^2)^{1-D/2}}+ \\ &\quad -\frac{4iG(1,1)}{(4\pi)^{D/2}}\int\frac{m^2\rho_b(s)ds}{(s-m^2)^2}\int\frac{d^Dk}{(2\pi)^D}\left(\frac{1}{k^2-m^2}-\frac{1}{k^2-s}\right)\frac{1}{(-(q-k)^2)^{1-D/2}} = \\ &= \frac{2G(1,1)}{(4\pi)^D}\int\frac{\rho_{a+b}(s)}{s-m^2}s^{D-2}V(1,1-D/2;-q^2/s)ds\; + \\ &\quad -\frac{4G(1,1)}{(4\pi)^D}\int\frac{m^2\rho_b(s)}{(s-m^2)^2}\Big((m^2)^{D-2}V(1,1-D/2;-q^2/m^2)\; + \\ &\qquad -s^{D-2}V(1,1-D/2;-q^2/s)\Big)ds.\end{aligned} \tag{4.64}$$

The spectral density is given by

$$\begin{aligned}
\rho^m_{b11}(s) &= \frac{2G(1,1)}{(4\pi)^D}\int_{m^2}^s \frac{\rho_{a+b}(s_1)}{s_1-m^2}s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)ds_1 + \\
&\quad -\frac{4G(1,1)}{(4\pi)^D}\int_{m^2}^\infty \frac{m^2\rho_b(s_1)}{(s_1-m^2)^2}\Big((m^2)^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/m^2) + \\
&\qquad\qquad -s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)\Big)ds_1 = \\
&= \frac{2G(1,1)}{(4\pi)^D}\int_{m^2}^s \frac{\rho_{a+b}(s_1)}{s_1-m^2}s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)ds_1 + \\
&\quad -\frac{4G(1,1)}{(4\pi)^D}(m^2)^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/m^2)\int_s^\infty \frac{m^2\rho_b(s_1)}{(s_1-m^2)^2}ds_1 + \\
&\quad -\frac{4G(1,1)}{(4\pi)^D}\int_{m^2}^s \frac{m^2\rho_b(s_1)}{(s_1-m^2)^2}\Big((m^2)^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/m^2) + \\
&\qquad\qquad -s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)\Big)ds_1.
\end{aligned} \tag{4.65}$$

Now one defines the quantity

$$L_b(s) := \int_s^\infty \frac{m^2\rho_b(s_1)}{(s_1-m^2)^2}ds_1, \qquad L_b'(s) = -\frac{m^2\rho_b(s)}{(s-m^2)^2} \tag{4.66}$$

which can also be used as derivative in the last integrand, and obtains

$$\begin{aligned}
\rho^m_{b11}(s) &= \frac{2G(1,1)}{(4\pi)^D}\int_{m^2}^s \frac{\rho_{a+b}(s_1)}{s_1-m^2}s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)ds_1 + \\
&\quad -\frac{4G(1,1)}{(4\pi)^D}L_b(s)(m^2)^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/m^2) + \\
&\quad +\frac{4G(1,1)}{(4\pi)^D}\int_{m^2}^s L_b'(s_1)\Big((m^2)^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/m^2) - s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)\Big)ds_1 = \\
&= \frac{2G(1,1)}{(4\pi)^D}\int_{m^2}^s \frac{\rho_{a+b}(s_1)}{s_1-m^2}s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)ds_1 + \\
&\quad -\frac{4G(1,1)}{(4\pi)^D}L_b(s)(m^2)^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/m^2) + \\
&\quad +\frac{4G(1,1)}{(4\pi)^D}\Big[L_b(s_1)\Big((m^2)^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/m^2) - s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)\Big)\Big]_{m^2}^s + \\
&\quad +\frac{4G(1,1)}{(4\pi)^D}\int_{m^2}^s L_b(s_1)\frac{d}{ds_1}\Big(s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)\Big)ds_1 = \\
&= \frac{2G(1,1)}{(4\pi)^D}\int_{m^2}^s \frac{\rho_{a+b}(s_1)}{s_1-m^2}s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)ds_1 + \\
&\quad -\frac{4G(1,1)}{(4\pi)^D}L_b(s)s^{2-2\varepsilon}\rho_V(1,\varepsilon-1;1) + \\
&\qquad +\frac{4G(1,1)}{(4\pi)^D}\int_{m^2}^s L_b(s_1)\frac{d}{ds_1}\Big(s_1^{2-2\varepsilon}\rho_V(1,\varepsilon-1;s/s_1)\Big)ds_1 = \\
&= \frac{2G}{(4\pi)^D}\int_{m^2}^s \frac{\rho_{a+b}(s_1)}{s_1-m^2}s_1^{2-2\varepsilon}\rho^1_V(1,\varepsilon-1;s/s_1)ds_1 + \\
&\quad +\frac{4G}{(4\pi)^D}\int_{m^2}^s L_b(s_1)\frac{d}{ds_1}\Big(s_1^{2-2\varepsilon}\rho^1_V(1,\varepsilon-1;s/s_1)\Big)ds_1.
\end{aligned} \tag{4.67}$$

(note that $\rho_V(1,\varepsilon-1;1)=0$). Finally one has to determine the different parts of this rather compact expression. These are

$$\begin{aligned}
\rho_{a+b}(s) &= \rho_a(s)+\rho_b(s) = g_s^2\left(\frac{2-D}{2}\left(1+\frac{m^2}{s}\right)+D+\frac{2-D}{2}\left(1+\frac{m^2}{s}\right)\right)\rho_B(s) = \\
&= g_s^2\left((2-D)\left(1+\frac{m^2}{s}\right)+D\right)\rho_B(s) = g_s^2\left(2\left(1+\frac{m^2}{s}\right)-D\frac{m^2}{s}\right)\rho_B(s) = \\
&= g_s^2\left(2-(D-2)\frac{m^2}{s}\right)\rho_B(s) = 2g_s^2\left(1-\frac{D-2}{2}\frac{m^2}{s}\right)\rho_B(s) = \qquad (4.68)\\
&= 2g_s^2\left(1-\frac{m^2}{s}\right)\rho_B(s) = \frac{2g_s^2}{(4\pi)^{D/2}}\left(1-\frac{m^2}{s}\right)^2\theta(s-m^2) =: \frac{g_s^2}{(4\pi)^{D/2}}\hat\rho_{a+b}(m^2/s),\\
L_b(s) &= \int_s^\infty\frac{m^2\rho_b(s_1)}{(s_1-m^2)^2}ds_1 = g_s^2\int_s^\infty\frac{m^2\rho_B(s_1)}{(s_1-m^2)^2}\left(\frac{D+2}{2}-\frac{D-2}{2}\frac{m^2}{s_1}\right)ds_1 = \\
&= \frac{g_s^2}{(4\pi)^{D/2}}\int_s^\infty\frac{m^2ds_1}{s_1(s_1-m^2)}\left(3-\frac{m^2}{s_1}\right) = \frac{g_s^2}{(4\pi)^{D/2}}\int_s^\infty\frac{m^2ds_1}{s_1^2(1-m^2/s_1)}\left(3-\frac{m^2}{s_1}\right) = \\
&= \frac{g_s^2}{(4\pi)^{D/2}}\int_{1-m^2/s}^1\frac{dx}{x}(2+x) = \frac{g_s^2}{(4\pi)^{D/2}}\Big[2\ln x+x\Big]_{1-m^2/s}^1 = \\
&= \frac{g_s^2}{(4\pi)^{D/2}}\left(-2\ln\left(1-\frac{m^2}{s}\right)+\frac{m^2}{s}\right) =: \frac{g_s^2}{(4\pi)^{D/2}}\hat L_b(m^2/s) \qquad (4.69)
\end{aligned}$$

(using $x=1-m^2/s_1$) where the last lines are taken at $D=4$, and finally

$$\rho_V^1(1,\varepsilon-1;1/z) = -\left(1+\frac{1}{z}\right)\ln z+\frac{3}{2}\left(1-\frac{1}{z}\right)+\frac{z}{6}\left(1-\frac{1}{z^3}\right) = \frac{1}{z^2}\hat\rho_V^1(1,\varepsilon-1;z). \qquad (4.70)$$

Using $\rho_V(1,\varepsilon-1;1/z)=z^{2\varepsilon-2}\hat\rho_V(1,\varepsilon-1;z)$, one gets rid of the factor in front of the spectral density. Substituing $z_1=m^2/s_1$ with

$$dz_1 = -\frac{m^2}{s_1^2}ds_1 \quad\Rightarrow\quad \frac{ds_1}{s_1} = -\frac{dz_1}{z_1}, \qquad \frac{d}{ds_1}(\ldots)\,ds_1 = \frac{d}{dz_1}(\ldots)\,dz_1 \qquad (4.71)$$

while the limits are changed to $[1,m^2/s]=-[m^2/s,1]$ in both parts, one can continue to

$$\begin{aligned}
\rho_{b11}^m(s) &= \frac{2g_s^2Gs^{2-2\varepsilon}}{(4\pi)^{3D/2}}\int_{m^2/s}^1\frac{\hat\rho_{a+b}(z_1)}{z_1(1-z_1)}\hat\rho_V^1(1,\varepsilon-1;m^2/sz_1)dz_1 + \\
&\qquad -\frac{4g_s^2Gs^{2-2\varepsilon}}{(4\pi)^{3D/2}}\int_{m^2/s}^1\hat L_b(z_1)\frac{d}{dz_1}\left(\hat\rho_V^1(1,\varepsilon-1;m^2/sz_1)\right)dz_1,\\
\rho_{b11}^m(m^2/z) &= \frac{4g_s^2Gs^{2-2\varepsilon}}{(4\pi)^{3D/2}}\left[\int_z^1\frac{\hat\rho_{a+b}(z_1)}{2z_1(1-z_1)}\hat\rho_V^1(1,\varepsilon-1;z/z_1)dz_1 + \right.\\
&\qquad \left.-\int_z^1\hat L_b(z_1)\frac{d}{dz_1}\left(\hat\rho_V^1(1,\varepsilon-1;z/z_1)\right)dz_1\right]. \qquad (4.72)
\end{aligned}$$

4.2.6 The fish contribution (c21)

This subsection contains the calculations for the *half-covered, semi-massive fish*. The diagram (c21) of Fig. 4.1 is calculated by keeping out the line which is not involved in the gluon exchange. The corresponding spectral density is written in terms of so-called prototypes and afterwards has to be convoluted with the remaining part. An algebra of convolution functions is developed to reduce the result to basic expressions.

The contribution of the semi-massive half-covered fish (c21)

The contribution of diagram (c21) is given by

$$\begin{aligned}
-iV_{c21}(-q^2) &= \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\times \\
&\times \operatorname{Tr}\left(\frac{i}{\not p-\not l}(-ig_s\gamma_\alpha)\frac{i}{\not p-\not k}\frac{i}{(\not q-\not p)}\right)\frac{i}{\not l-m}(-ig_s\gamma^\alpha)\frac{i}{\not k-m}\frac{-i}{(k-l)^2} = \\
&= -g_s^2\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{\operatorname{Tr}((\not p-\not l)\gamma_\alpha(\not p-\not k)(\not q-\not p))}{(p-k)^2(p-l)^2(q-p)^2(k-l)^2}\frac{(\not l+m)\gamma^\alpha(\not k+m)}{(l^2-m^2)(k^2-m^2)}. \qquad (4.73)
\end{aligned}$$

The massive part is considered by taking the trace divided by m,

$$\begin{aligned}
-iV^m_{c21}(-q^2) &= \\
&= -g_s^2\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{\operatorname{Tr}((\not p-\not l)\gamma_\alpha(\not p-\not k)(\not q-\not p))}{(p-k)^2(p-l)^2(q-p)^2(k-l)^2}\frac{(k+l)^\alpha}{(k^2-m^2)(l^2-m^2)} = \\
&= -g_s^2\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{\operatorname{Tr}((\not p-\not l)(\not k+\not l)(\not p-\not k)(\not q-\not p))}{(k^2-m^2)(l^2-m^2)(p-k)^2(p-l)^2(q-p)^2(k-l)^2} = \\
&= -g_s^2\int\frac{d^Dp}{(2\pi)^D}\frac{(q-p)_\mu}{(q-p)^2}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{\operatorname{Tr}((\not p-\not l)(\not k+\not l)(\not p-\not k)\gamma^\mu)}{(k^2-m^2)(l^2-m^2)(p-k)^2(p-l)^2(k-l)^2} = \\
&= -g_s^2\int\frac{d^Dp}{(2\pi)^D}\frac{qp-p^2}{(q-p)^2p^2}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{\operatorname{Tr}((\not p-\not l)(\not k+\not l)(\not p-\not k)\not p)}{(k^2-m^2)(l^2-m^2)(p-k)^2(p-l)^2(k-l)^2} = \\
&=: \int\frac{d^Dp}{(2\pi)^D}\frac{qp-p^2}{(q-p)^2p^2}\tilde V^m_{c21}(-p^2). \qquad (4.74)
\end{aligned}$$

where

$$\tilde V^m_{c21}(-p^2) := -g_s^2\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{\operatorname{Tr}((\not p-\not l)(\not k+\not l)(\not p-\not k)\not p)}{(k^2-m^2)(l^2-m^2)(p-k)^2(p-l)^2(k-l)^2} \qquad (4.75)$$

is calculated by a computer code in MATHEMATICA. In order to convert to spectral densities, the dispersion relation

$$\tilde V^m_{c21}(-p^2) = \int\frac{\tilde\rho^m_{c21}(s)ds}{s-p^2} \qquad (4.76)$$

is inserted. One obtains

$$\begin{aligned}
V^m_{c21}(-q^2) &= i\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{(q-p)^2p^2}\int\frac{\tilde\rho^m_{c21}(s)ds}{s-p^2} = \\
&= \int\left(i\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{(q-p)^2p^2(s-p^2)}\right)\tilde\rho^m_{c21}(s)ds \;=:\; \int\tilde\lambda(-q^2,s)\tilde\rho^m_{c21}(s)ds, \\
\rho^m_{c21}(s) &= \int\lambda(s,s_1)\tilde\rho^m_{c21}(s_1)ds_1. \qquad (4.77)
\end{aligned}$$

$\lambda(s,s_1)$ is the *convolution function*. The general procedure is to calculate the inner integral of Eq. (4.74), given in Eq. (4.75), reformulate it in terms of spectral densities and convolute it with the convolution function. The inner integral is given by

$$\begin{aligned}
\tilde{V}^m_{c21}(-p^2) &= \frac{4g_s^2}{(4\pi)^D}\Big[(p^2-m^2)^2\tilde{V}(1,1,1,1,1;-p^2)-p^2\tilde{V}(1,1,1,1,0;-p^2)\ +\\
&+\frac{1}{2}(p^2+m^2)\left(\tilde{V}(1,1,0,1,1;-p^2)+\tilde{V}(1,1,1,0,1;-p^2)\right)\ +\\
&+(p^2-m^2)\left(\tilde{V}(0,1,1,1,1;-p^2)+\tilde{V}(1,0,1,1,1;-p^2)\right)\ +\\
&-\frac{1}{2}\left(\tilde{V}(1,0,0,1,1;-p^2)+\tilde{V}(0,1,1,0,1;-p^2)\right)+\tilde{V}(0,0,1,1,1;-p^2)\Big]
\end{aligned}\tag{4.78}$$

and the corresponding spectral density reads

$$\begin{aligned}
\tilde{\rho}^m_{c21}(s) &= \frac{4g_s^2}{(4\pi)^D}\Big[(s-m^2)^2\tilde{\rho}_V(1,1,1,1,1;s)-s\tilde{\rho}_V(1,1,1,1,0;s)\ +\\
&+(s+m^2)\tilde{\rho}_V(1,1,0,1,1;s)+2(s-m^2)\tilde{\rho}_V(0,1,1,1,1;s)\ +\\
&+\tilde{\rho}_V(0,0,1,1,1;s)-\tilde{\rho}_V(0,1,1,0,1;s)\Big]\ =\\
&= \frac{4g_s^2s^{D-3}}{(4\pi)^D}\Big[(1-z)^2\hat{\rho}_V(1,1,1,1,1;z)-\hat{\rho}_V(1,1,1,1,0;z)\ +\\
&+(1+z)\hat{\rho}_V(1,1,0,1,1;z)+2(1-z)\hat{\rho}_V(0,1,1,1,1;z)\ +\\
&-\hat{\rho}_V(0,1,1,0,1;z)+\hat{\rho}_V(0,0,1,1,1;z)\Big]
\end{aligned}\tag{4.79}$$

where the short hand notation $z=m^2/s$ and the symmetry properties of the standard integrals are used. The different spectral densities in this expression are called *prototypes* and will be calculated in the next subsection. The convolution function will be dealt with next by constructing an *algebra of convolution functions*.

The algebra of convolution functions

The elements of the algebra are integrals $\lambda_n(s,s_1)$ of the type of the master bubble spectral density in Eq. (4.44). The algebra which turns out to be a simple reduction chain allows one to reduce all elements to a single basic element. On the level of correlator integrals within Minkowskian spacetime the algebra reads

$$\begin{aligned}
\tilde{\lambda}_n(-q^2,s) &:= i\int\frac{d^Dp}{(2\pi)^D}\frac{(-p^2)^n}{s^n(s-p^2)(q-p)^2}\ =\ i\int\frac{d^Dp}{(2\pi)^D}\frac{(s-p^2-s)(-p^2)^{n-1}}{s^n(s-p^2)(q-p)^2}\ =\\
&= i\int\frac{d^Dp}{(2\pi)^D}\frac{(p^2)^{n-1}}{s^n(q-p)^2}-i\int\frac{d^Dp}{(2\pi)^D}\frac{(-p^2)^{n-1}}{s^{n-1}(s-p^2)(q-p)^2}\ =\\
&= -\tilde{\lambda}_{n-1}(-q^2,s)\ =\ \ldots\ =\ (-1)^n\tilde{\lambda}_0(-q^2,s),\\
\tilde{\lambda}_0(-q^2,s) &= i\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(s-p^2)(q-p)^2}\ =\ -i\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(p^2-s)(q-p)^2}\ =\\
&= \frac{s^{D/2-2}}{(4\pi)^{D/2}}V(1,1;-q^2/s).
\end{aligned}\tag{4.80}$$

Within the Euclidean domain one has

$$\tilde\lambda_n(q^2,s) \;=\; \int\frac{d^Dp}{(2\pi)^D}\frac{(p^2)^n}{s^n(s+p^2)(q-p)^2} \;=\; (-1)^n\tilde\lambda_0(q^2,s), \tag{4.81}$$

$$\tilde\lambda_0(q^2,s) \;=\; \int\frac{d^Dp}{(2\pi)^D}\frac{1}{(s+p^2)(q-p)^2} \;=\; \frac{s^{D/2-2}}{(4\pi)^{D/2}}V(1,1;q^2/s). \tag{4.82}$$

Taking the discontinuity for both expressions, one obtains

$$\lambda_n(s,s_1)=(-1)^n\lambda_0(s,s_1),\quad \lambda_0(s,s_1)=\frac{s_1^{-\varepsilon}}{(4\pi)^{2-\varepsilon}}\rho_V(1,1;s/s_1)=\frac{s^{-\varepsilon}}{(4\pi)^{2-\varepsilon}}\hat\rho_V(1,1;s_1/s). \tag{4.83}$$

The only weight

Looking at the correlator representation of the convolution function in the present case,

$$\tilde\lambda(-q^2,s)=i\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{(q-p)^2p^2(s-p^2)} \tag{4.84}$$

one recognizes that there is a negative power of p^2. This would lead out of the algebra described above and produce an explicite infrared divergence. In order to avoid this one has to modify the fish contribution by a subtraction at $p^2=0$,

$$\begin{aligned}
V^m_{c21}(-q^2) &= i\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{(q-p)^2p^2}\int\frac{\tilde\rho^m_{c21}(s)ds}{s-p^2} = \\
&= i\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{(q-p)^2p^2}\left(\tilde V^m_{c21}(0)+\int\left(\frac{1}{s-p^2}-\frac{1}{s}\right)\tilde\rho^m_{c21}(s)ds\right) = \\
&= i\tilde V^m_{c21}(0)\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{(q-p)^2p^2}+i\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{(q-p)^2}\int\frac{\tilde\rho^m_{c21}(s)ds}{s(s-p^2)}.
\end{aligned} \tag{4.85}$$

It can be shown that the first term vanishes, one can therefore proceed with

$$V^m_{c21}(-q^2)=\int\left(i\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{s(q-p)^2(s-p^2)}\right)\tilde\rho^m_{c21}(s)ds=\int\tilde\lambda(-q^2,s)\tilde\rho^m_{c21}(s)ds \tag{4.86}$$

where $\tilde\lambda$ is now changed to

$$\begin{aligned}
\tilde\lambda(-q^2,s) &= i\int\frac{d^Dp}{(2\pi)^D}\frac{(qp-p^2)}{s(q-p)^2(s-p^2)} = \frac{i}{2}\int\frac{d^Dp}{(2\pi)^D}\frac{(q^2-p^2-(q-p)^2)}{s(q-p)^2(s-p^2)} = \\
&= \frac12\left(\frac{q^2}{s}\tilde\lambda_0(-q^2,s)+\tilde\lambda_1(-q^2,s)\right) = \frac12\left(\frac{q^2}{s}-1\right)\tilde\lambda_0(-q^2,s)
\end{aligned} \tag{4.87}$$

(which means that the algebra of convolution functions is extended to negative n) such that

$$\begin{aligned}
\lambda(s,s_1) &= \frac12\left(\frac{s}{s_1}-1\right)\lambda_0(s,s_1) = \frac{s^{-\varepsilon}}{2(4\pi)^{2-\varepsilon}}\left(\frac{s}{s_1}-1\right)\hat\rho_V(1,1;s_1/s) = \\
&= \frac{-1}{2(4\pi)^2}\left(1-\frac{s}{s_1}\right)\left(1-\frac{s_1}{s}\right)\theta(s-s_1)+O(\varepsilon).
\end{aligned} \tag{4.88}$$

4.3 The momentum part of the correlator

The same procedure as before will now be applied to the momentum part of the correlator function. The expressions are roughly twice the length of the expressions for the mass part. In addition there is some complication for the renormalization which will be mentioned at the very end.

4.3.1 The leading order diagram (a1)

Taking the contribution of the leading order diagram (a1) as given in Eq. (4.25), the momentum part is obtained by calculating one fourth of the trace with $\not{q}$ and dividing the expression by q^2. One obtains

$$\begin{aligned}
V^q_{a1}(-q^2) &= \frac{-2iG(1,1)}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}\frac{qk}{q^2(k^2-m^2)}\left(-(q-k)^2\right)^{D/2-1} = \\
&= \frac{-iG(1,1)}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}\frac{q^2+m^2+k^2-m^2-(q-k)^2}{q^2(k^2-m^2)}\left(-(q-k)^2\right)^{D/2-1} = \\
&= \frac{-iG(1,1)}{(4\pi)^{D/2}}\left(1+\frac{m^2}{q^2}\right)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-m^2)(-(q-k)^2)^{1-D/2}} + \\
&\qquad -\frac{iG(1,1)}{(4\pi)^{D/2}q^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-m^2)(-(q-k)^2)^{-D/2}} = \\
&= -\frac{G(1,1)}{(4\pi)^D}\left(1+\frac{m^2}{q^2}\right)(m^2)^{D/2-1-1-D/2}V(1,1-D/2;-q^2/m^2) + \\
&\qquad -\frac{G(1,1)}{(4\pi)^Dq^2}(m^2)^{D/2-1+D/2}V(1,-D/2;-q^2/m^2) = \\
&= -\frac{G(1,1)}{(4\pi)^D}(m^2)^{D-2}\left(\left(1+\frac{m^2}{q^2}\right)V(1,1-D/2;-q^2/m^2)+\frac{m^2}{q^2}V(1,-D/2;q^2/m^2)\right)
\end{aligned}\tag{4.89}$$

such that

$$\begin{aligned}
\rho^q_{a1}(s) &= -\frac{G(1,1)}{(4\pi)^D}(m^2)^{2-2\varepsilon}\left(\left(1+\frac{m^2}{s}\right)\rho_V(1,\varepsilon-1;s/m^2)+\frac{m^2}{s}\rho_V(1,\varepsilon-2;s/m^2)\right) = \\
&= -\frac{G(1,1)}{(4\pi)^D}s^{2-2\varepsilon}\left(\left(1+\frac{m^2}{s}\right)\hat\rho_V(1,\varepsilon-1;m^2/s)+\hat\rho_V(1,\varepsilon-2;m^2/s)\right) = \\
&= -\frac{G}{(4\pi)^D}s^{2-2\varepsilon}\frac{1}{\varepsilon}\left(\left(1+\frac{m^2}{s}\right)\hat\rho_V(1,\varepsilon-1;m^2/s)+\hat\rho_V(1,\varepsilon-2;m^2/s)\right).
\end{aligned}\tag{4.90}$$

The difference to the mass part is that $\hat\rho_V(1,\varepsilon-1;z)$ is replaced by

$$\frac{1}{2}\left((1+z)\hat\rho_V(1,\varepsilon-1;z)+\hat\rho_V(1,\varepsilon-2;z)\right).\tag{4.91}$$

Something similar happens to the contribution from the correction of the light part of the diagram as well as to the massive line correction as it will become obvious soon.

4.3.2 The light contributions (b21) and (c11)

The calculation of the contribution due to the correction of the light part is calculated by the MATHEMATICA package. The results are given by

$$\begin{aligned}
\rho^q_{b21}(s) &= \frac{4g_s^2G^2}{(4\pi)^{3D/2}}s^{2-3\varepsilon}\left(\frac{1}{8\varepsilon^2}+\frac{1}{16\varepsilon}+\frac{11}{32}\right)\times \qquad (4.92)\\
&\qquad\times\left(\left(1+\frac{m^2}{s}\right)\hat\rho_V(1,2\varepsilon-1;m^2/s)+\hat\rho_V(1,2\varepsilon-2;m^2/s)\right),\\
\rho^q_{c11}(s) &= \frac{4g_s^2G^2}{(4\pi)^{3D/2}}s^{2-3\varepsilon}\left(\frac{1}{\varepsilon^2}+\frac{3}{4\varepsilon}+\frac{27}{8}-3\zeta(3)\right)\times \qquad (4.93)\\
&\qquad\times\left(\left(1+\frac{m^2}{s}\right)\hat\rho_V(1,2\varepsilon-1;m^2/s)+\hat\rho_V(1,2\varepsilon-2;m^2/s)\right).
\end{aligned}$$

The colour factors for the leading order diagram are given by $N_c!$, for the massless line self energy diagram (b21) by $N_c!C_F$, and for the massless fish (c11) by $-N_c!C_B$. The massless line self energy correction is applied at each of the lines, such that the leading order term and the light contribution are given by

$$\begin{aligned}
\rho^q_{\rm leading}(s) &= -\frac{N_c!G}{(4\pi)^D}s^{2-2\varepsilon}\frac{1}{\varepsilon}\left(\left(1+\frac{m^2}{s}\right)\hat\rho_V(1,\varepsilon-1;m^2/s)+\hat\rho_V(1,\varepsilon-2;m^2/s)\right),\\
\rho^q_{\rm light}(s) &= 2N_c!C_F\rho^q_{b21}(s)-N_c!C_B\rho^q_{c11}(s) = \qquad (4.94)\\
&= -\frac{N_c!C_Fg_s^2G^2}{(4\pi)^{3D/2}}s^{2-3\varepsilon}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}+4-6\zeta(3)\right)\times\\
&\qquad\times\left(\left(1+\frac{m^2}{s}\right)\hat\rho_V(1,2\varepsilon-1;m^2/s)+\hat\rho_V(1,2\varepsilon-2;m^2/s)\right) =\\
&=: -\frac{N_c!C_Fg_s^2G(1,1)^2}{(4\pi)^{3D/2}}s^{2-3\varepsilon}\left(B_0+B_1\varepsilon+B_2\varepsilon^2\right)\times \qquad (4.95)\\
&\qquad\times\left(\left(1+\frac{m^2}{s}\right)\hat\rho_V(1,2\varepsilon-1;m^2/s)+\hat\rho_V(1,2\varepsilon-2;m^2/s)\right)
\end{aligned}$$

where $B_0=B_1=1$ and $B_2=4-6\zeta(3)$ as in the mass part calculation. Before combining these two expressions one rewrites them in terms of pure integrals, extracting the Γ structure,

$$\begin{aligned}
\hat\rho_V(1,n\varepsilon-1;z) &= \frac{1}{\Gamma(n\varepsilon-1)\Gamma(D/2-n\varepsilon+1)}\int_z^1(1-x)^{D/2-n\varepsilon}x^{n\varepsilon-2}(x-z)^{D/2-n\varepsilon}dx =\\
= \frac{1}{\Gamma(n\varepsilon-1)\Gamma(3-(n+1)\varepsilon)}&\int_z^1(1-x)^{2-(n+1)\varepsilon}x^{n\varepsilon-2}(x-z)^{2-(n+1)\varepsilon}dx, \qquad (4.96)\\
\hat\rho_V(1,n\varepsilon-2;z) &= \frac{1}{\Gamma(n\varepsilon-2)\Gamma(4-(n+1)\varepsilon)}\int_z^1(1-x)^{3-(n+1)\varepsilon}x^{n\varepsilon-3}(x-z)^{3-(n+1)\varepsilon}dx.
\end{aligned}$$

The relative factor is given by

$$C^r_{\rho n} := \frac{\Gamma(n\varepsilon-1)\Gamma(3-(n+1)\varepsilon)}{\Gamma(n\varepsilon-2)\Gamma(4-(n+1)\varepsilon)} = \frac{n\varepsilon-2}{3-(n+1)\varepsilon} = -\frac{2}{3}\left(1+\frac{2-n}{6}\varepsilon+O(\varepsilon^2)\right). \qquad (4.97)$$

With this one can define

$$(1+z)\hat\rho_V(1,\varepsilon-1;z)+\hat\rho_V(1,\varepsilon-2;z) \;=:\; 2C_1\varepsilon\hat g_1^q(z), \tag{4.98}$$

$$(1+z)\hat\rho_V(1,2\varepsilon-1;z)+\hat\rho_V(1,2\varepsilon-2;z) \;=:\; 2C_2\varepsilon\hat g_2^q(z) \tag{4.99}$$

with $C_n\varepsilon=(\Gamma(n\varepsilon-1)\Gamma(3-(n+1)\varepsilon))^{-1}$ as in the mass part calculation and

$$\begin{aligned} 2\hat g_n^q(z) &= (1+z)\int_z^1(1-x)^{2-(n+1)\varepsilon}x^{n\varepsilon-2}(x-z)^{2-(n+1)\varepsilon}dx\ + \\ &\quad +C_{\rho n}^r\int_z^1(1-x)^{3-(n+1)\varepsilon}x^{n\varepsilon-3}(x-z)^{3-(n+1)\varepsilon}dx. \end{aligned} \tag{4.100}$$

Now one can collect the terms to obtain

$$\begin{aligned} \rho_l^q(s) &:= \rho_{\rm leading}^q(s)+\rho_{\rm light}^q(s) = \\ &= -\frac{2N_c!}{(4\pi)^D}s^{2-2\varepsilon}G(1,1)C_1\varepsilon\bigg(\hat g_1^q(m^2/s)\ + \\ &\qquad +\frac{\alpha_sC_F}{4\pi}s^{-\varepsilon}\frac{C_2}{C_1}G(1,1)(B_0+B_1\varepsilon+B_2\varepsilon^2)\hat g_2^q(m^2/s)\bigg) = \\ &= \frac{N_c!s^2}{(4\pi)^4}\left(\left(1+\frac{\alpha_sC_F}{4\pi\varepsilon}C_0^r\right)\hat g_1^q(m^2/s)+\frac{\alpha_sC_F}{4\pi}\left(C_1^r\hat g_1^{q0}(m^2/s)+C_0^r\hat g_{21}^{q1}(m^2/s)\right)\right) \end{aligned} \tag{4.101}$$

in analogy to the mass part calculation, with

$$\hat g_1^q(z)=g_1^{q0}(z)+O(\varepsilon),\qquad \hat g_{21}^q(z):=\hat g_2^q(z)-\hat g_1^q(z)=\hat g_{21}^{q1}(z)\varepsilon+O(\varepsilon^2). \tag{4.102}$$

One obtains

$$\begin{aligned} \hat g_1^{q0}(z) &= \int_z^1\left((1+z)\frac{(1-x)^2(x-z)^2}{2x^2}-\frac{(1-x)^3(x-z)^3}{3x^3}\right)dx = \\ &= \frac{1}{12}-\frac{2}{3}z+\frac{2}{3}z^3-\frac{1}{12}z^4-z^2\ln z. \end{aligned} \tag{4.103}$$

A complete calculation up to $O(\varepsilon)$ shows that $\hat g_{21}^{q1}(z)$ is given by

$$\begin{aligned} \hat g_{21}^{q1}(z) &= \int_z^1\left((1+z)\frac{(1-x)^2(x-z)^2}{2x^2}-\frac{(1-x)^3(x-z)^3}{3x^3}\right)\ \times \\ &\qquad\times\ (\ln x-\ln(1-x)-\ln(x-z))\,dx\ + \\ &\quad +\frac{1}{18}\int_z^1\frac{(1-x)^3(x-z)^3}{x^3}dx\ = \\ &= \frac{7}{144}-\frac{1}{18}z+\frac{1}{18}z^3-\frac{7}{144}z^4-2z^2\left(\mathrm{Li}_2(z)-\mathrm{Li}_2(1)\right)\ + \\ &\qquad -\left(\frac{1}{6}-\frac{4}{3}z+\frac{4}{3}z^3-\frac{1}{6}z^4\right)\ln(1-z)+\left(\frac{3}{2}z^2+\frac{2}{3}z^3-\frac{1}{12}z^4\right)\ln z. \end{aligned} \tag{4.104}$$

The result in Eq. (4.101) represents the three contributions: The first one needs no expansion of $\hat g_1^q$ but is absorbed in the renormalization factor for the light current. The second term is the offspring of the singular part, and the third one the finite mass correction. One has

$$C_0^r=2B_0G,\qquad C_1^r=\left(2B_0\ln\left(\frac{\mu^2}{s}\right)+2B_1+B_0\right)G. \tag{4.105}$$

4.3.3 The massive contribution (b11)

According to the considerations for the mass part, the finite contribution of the momentum part of the effective massive propagator is given by

$$D^q_{\rm eff}(-k^2) = \frac{i}{k^2-m^2} - i\int\frac{\rho_a(s)ds}{(s-m^2)(k^2-s)} + 2im^2\int\frac{\rho_b(s)ds}{(s-m^2)^2}\left(\frac{1}{k^2-m^2}-\frac{1}{k^2-s}\right). \tag{4.106}$$

This term has to replace the expression $i/(k^2-m^2)$ in the leading order contribution

$$\begin{aligned} V^q_{a1}(-q^2) &= -\frac{G(1,1)}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-m^2}\times \qquad (4.107)\\ &\times\left(\left(1+\frac{m^2}{q^2}\right)\left(-(q-k)^2\right)^{D/2-1}+\frac{k^2-m^2}{q^2}\left(-(q-k)^2\right)^{D/2-1}+\frac{1}{q^2}\left(-(q-k)^2\right)^{D/2}\right). \end{aligned}$$

As one will see, in contrast to the leading order term, one has to take into account also the second term which would vanish for the leading order calculation itself. One obtains

$$\begin{aligned} V^q_{a1}(-q^2)+V^q_{b11}(-q^2) &= -\frac{G(1,1)}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}D^k_{\rm eff}(-k^2)\Bigg(\left(1+\frac{m^2}{q^2}\right)\left(-(q-k)^2\right)^{D/2-1}+\\ &\quad+\frac{k^2-m^2}{q^2}\left(-(q-k)^2\right)^{D/2-1}+\frac{1}{q^2}\left(-(q-k)^2\right)^{D/2}\Bigg), \qquad (4.108)\\ V^q_{b11}(-q^2) &= \frac{iG(1,1)}{(4\pi)^{D/2}}\int\frac{\rho_a(s)ds}{s-m^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-s}\Bigg(\left(1+\frac{m^2}{q^2}\right)\left(-(q-k)^2\right)^{D/2-1}+\\ &\quad+\left(\frac{k^2-s}{q^2}+\frac{s-m^2}{q^2}\right)\left(-(q-k)^2\right)^{D/2-1}+\frac{1}{q^2}\left(-(q-k)^2\right)^{D/2}\Bigg)+\\ &\quad-\frac{2iG(1,1)}{(4\pi)^{D/2}}\int\frac{m^2\rho_b(s)ds}{(s-m^2)^2}\int\frac{d^Dk}{(2\pi)^D}\left(\frac{1}{k^2-m^2}-\frac{1}{k^2-s}\right)\times\\ &\quad\times\Bigg(\left(1+\frac{m^2}{q^2}\right)\left(-(q-k)^2\right)^{D/2-1}+\\ &\quad+\frac{k^2-m^2}{q^2}\left(-(q-k)^2\right)^{D/2-1}+\frac{1}{q^2}\left(-(q-k)^2\right)^{D/2}\Bigg) =\\ &= \frac{G(1,1)}{(4\pi)^D}\int\frac{\rho_a(s)ds}{s-m^2}s^{D-2}\Bigg(\left(1+\frac{m^2}{q^2}\right)V(1,1-D/2;-q^2/s)+\\ &\quad+\frac{s-m^2}{q^2}V(1,-D/2;-q^2/s)+\frac{s}{q^2}V(1,-D/2;-q^2/s)\Bigg)+\\ &\quad-\frac{2G(1,1)}{(4\pi)^D}\int\frac{m^2\rho_b(s)ds}{(s-m^2)^2}(m^2)^{D-2}\times\\ &\quad\times\left(\left(1+\frac{m^2}{q^2}\right)V(1,1-D/2;-q^2/m^2)+\frac{m^2}{q^2}V(1,-D/2;-q^2/m^2)\right)+\\ &\quad+\frac{2G(1,1)}{(4\pi)^D}\int\frac{m^2\rho_b(s)ds}{(s-m^2)^2}s^{D-2}\Bigg(\left(1+\frac{m^2}{q^2}\right)V(1,1-D/2;-q^2/s)+\\ &\quad+\frac{s-m^2}{q^2}V(1,1-D/2;-q^2/s)+\frac{s}{q^2}V(1,-D/2;-q^2/s)\Bigg). \qquad (4.109) \end{aligned}$$

It is obvious that in the first and third line the first and second terms can be combined so that the explicit m^2 dependence vanishes. The spectral density is therefore given by

$$\begin{aligned}
\rho^q_{b11}(s) &= \frac{G(1,1)}{(4\pi)^D}\int\frac{\rho_a(s_1)ds_1}{s_1-m^2}s_1^{2-2\varepsilon}\ \times \\
&\qquad\times\ \left(\left(1+\frac{s_1}{s}\right)\rho_V(1,\varepsilon-1;s/s_1)+\frac{s_1}{s}\rho_V(1,\varepsilon-2;s/s_1)\right)\ + \\
&\quad -\frac{2G(1,1)}{(4\pi)^D}\int\frac{m^2\rho_b(s_1)ds_1}{(s_1-m^2)^2}(m^2)^{2-2\varepsilon}\ \times \\
&\qquad\times\ \left(\left(1+\frac{m^2}{s}\right)\rho_V(1,\varepsilon-1;s/m^2)+\frac{m^2}{s}\rho_V(1,\varepsilon-2;s/m^2)\right)\ + \\
&\quad +\frac{2G(1,1)}{(4\pi)^D}\int\frac{m^2\rho_b(s_1)ds_1}{(s_1-m^2)^2}s^{2-2\varepsilon}\ \times \\
&\qquad\times\ \left(\left(1+\frac{s_1}{s}\right)\rho_V(1,\varepsilon-1;s/s_1)+\frac{s_1}{s}\rho_V(1,\varepsilon-2;s/s_1)\right)\ =\ \ldots \\
&= \frac{G}{(4\pi)^D}\int_{m^2}^s\frac{\rho_a(s_1)}{s_1-m^2}\ \times \\
&\qquad\times\ s_1^{2-2\varepsilon}\left(\left(1+\frac{s_1}{s}\right)\rho^1_V(1,\varepsilon-1;s/s_1)+\frac{s_1}{s}\rho^1_V(1,\varepsilon-2;s/s_1)\right)ds_1\ + \\
&\quad +\frac{2G}{(4\pi)^D}\int_{m^2}^s L_b(s_1)\ \times \\
&\qquad\times\ \frac{d}{ds_1}\left(s_1^{2-2\varepsilon}\left(\left(1+\frac{s_1}{s}\right)\rho^1_V(1,\varepsilon-1;s/s_1)+\frac{s_1}{s}\rho^1_V(1,\varepsilon-2;s/s_1)\right)\right)ds_1.
\end{aligned} \tag{4.110}$$

Using

$$\begin{aligned}
\rho_a(s) &= \frac{2-D}{2}g_s^2\left(1+\frac{m^2}{s}\right)\rho_B(s)\ =\ \frac{-g_s^2}{(4\pi)^{D/2}}\left(1+\frac{m^2}{s}\right)\left(1-\frac{m^2}{s}\right)\theta(s-m^2)\ = \\
&=: \frac{g_s^2}{(4\pi)^{D/2}}\hat\rho_a(m^2/s),\qquad \hat\rho_a(z)\ =\ -(1+z)(1-z),
\end{aligned} \tag{4.111}$$

$$L_b(s) = \frac{g_s^2}{(4\pi)^{D/2}}\hat L_b(m^2/s),\qquad \hat L_b(z)\ =\ z-2\ln(1-z) \tag{4.112}$$

and the quantities $\hat\rho_V$, one ends up with

$$\begin{aligned}
\rho^q_{b11}(m^2/z) &= \frac{2g_s^2Gs^{2-2\varepsilon}}{(4\pi)^{3D/2}}\ \times \\
\times\ \Bigg[\int_z^1&\frac{\hat\rho_a(z_1)}{2z_1(1-z_1)}\left(\left(1+\frac{z}{z_1}\right)\hat\rho^1_V(1,\varepsilon-1;z/z_1)+\hat\rho^1_V(1,\varepsilon-2;z/z_1)\right)dz_1\ + \\
&-\int_z^1\hat L_b(z_1)\frac{d}{dz_1}\left(\left(1+\frac{z}{z_1}\right)\hat\rho^1_V(1,\varepsilon-1;z/z_1)+\hat\rho^1_V(1,\varepsilon-2;z/z_1)\right)dz_1\Bigg].
\end{aligned} \tag{4.113}$$

4.3.4 The fish contribution (c21)

Starting with Eq. (4.73), the momentum part is given by

$$\begin{aligned} V^q_{c21}(-q^2) &= -ig_s^2\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\times\\ &\times\frac{\mathrm{Tr}((\not q-\not p)(\not p-\not l)\gamma_\alpha(\not p-\not k))\frac{1}{4}\mathrm{Tr}(\not q(\not l+m)\gamma^\alpha(\not k+m))}{q^2(q-p)^2(k^2-m^2)(l^2-m^2)(p-k)^2(p-l)^2(k-l)^2} =\\ =:\ & i\int\frac{d^Dp}{(2\pi)^D}\frac{(q-p)_\mu q_\nu}{(q-p)^2q^2}\tilde V^{q\mu\nu}_{c21}(-p^2). \end{aligned} \tag{4.114}$$

The *tensor integral* $\tilde V^{q\mu\nu}_{c21}(-p^2)$ can be expressed in terms of *covariants*,

$$\tilde V^{q\mu\nu}_{c21}(-p^2) = g^{\mu\nu}p^2\tilde V^{q1}_{c21}(-p^2)+p^\mu p^\nu\tilde V^{q2}_{c21}(-p^2) \tag{4.115}$$

where the covariants can be determined by contracting with the *dual basis*,

$$Dp^2\tilde V^{q1}_{c21}(-p^2)+p^2\tilde V^{q2}_{c21}(-p^2) = g_{\mu\nu}\tilde V^{q\mu\nu}_{c21}(-p^2) =:\ p^2\hat V^{q1}_{c21}(-p^2), \tag{4.116}$$

$$p^4\tilde V^{q1}_{c21}(-p^2)+p^4\tilde V^{q2}_{c21}(-p^2) = p_\mu p_\nu\tilde V^{q\mu\nu}_{c21}(-p^2) =:\ p^4\hat V^{q2}_{c21}(-p^2) \tag{4.117}$$

and thus

$$\tilde V^{q1}_{c21}(-p^2)=\frac{\hat V^{q1}_{c21}(-p^2)-\hat V^{q2}_{c21}(-p^2)}{D-1},\quad \tilde V^{q2}_{c21}(-p^2)=\frac{D\hat V^{q2}_{c21}(-p^2)-\hat V^{q1}_{c21}(-p^2)}{D-1}. \tag{4.118}$$

Inserting this, one obtains

$$\begin{aligned} V^q_{c21}(-q^2) &= i\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(q-p)^2q^2}\Big(p^2(q^2-qp)\tilde V^{q1}_{c21}(-p^2)+(qp-p^2)(qp)\tilde V^{q2}_{c21}(-p^2)\Big) =\\ &= i\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(D-1)(q-p)^2q^2}\Bigg(p^2(q^2-qp)\Big(\hat V^{q1}_{c21}(-p^2)-\hat V^{q2}_{c21}(-p^2)\Big)+\\ &\qquad +(qp-p^2)(qp)\Big(D\hat V^{q2}_{c21}(-p^2)-\hat V^{q1}_{c21}(-p^2)\Big)\Bigg) =\\ &= i\int\frac{d^Dp}{(2\pi)^D}\Bigg[\frac{p^2(q^2-qp)-(qp-p^2)(qp)}{(D-1)(q-p)^2q^2}\hat V^{q1}_{c21}(-p^2)+\\ &\qquad +\frac{D(qp-p^2)(qp)-p^2(q^2-qp)}{(D-1)(q-p)^2q^2}\hat V^{q2}_{c21}(-p^2)\Bigg] =\\ &= i\int\frac{d^Dp}{(2\pi)^D}\Bigg[\frac{q^2p^2-(qp)^2}{(D-1)(q-p)^2q^2}\hat V^{q1}_{c21}(-p^2)+\frac{D(qp-p^2)(qp)-p^2(q^2-qp)}{(D-1)(q-p)^2q^2}\hat V^{q2}_{c21}(-p^2)\Bigg]. \end{aligned} \tag{4.119}$$

Expressing the correlator functions by using the dispersion relation

$$\hat V^{qi}_{c21}(-p^2)=\int\frac{\hat\rho^{qi}_{c21}(s_1)ds_1}{s_1-p^2} \tag{4.120}$$

and formally obtains

$$\rho^q_{c21}(s)=\int\Big(\lambda^1(s,s_1)\hat\rho^{q1}_{c21}(s_1)+\lambda^2(s,s_1)\hat\rho^{q2}_{c21}(s_1)\Big)ds_1. \tag{4.121}$$

All these parts have to be calculated.

The convolution functions

In the present case one has

$$\begin{aligned}4(q^2p^2-(qp)^2) &= 4q^2p^2-\left(q^2+p^2-(q-p)^2\right)^2 = \\ &= 2q^2p^2-q^4-p^4+2q^2(q-p)^2+2p^2(q-p)^2-(q-p)^4 \qquad (4.122)\end{aligned}$$

and therefore (all vanishing contributions are omitted)

$$\begin{aligned}\tilde\lambda^1(-q^2,s_1) &= \frac{i}{(D-1)q^2}\int\frac{d^Dp}{(2\pi)^D}\frac{q^2p^2-(qp)^2}{(q-p)^2(s_1-p^2)} = \\ &= \frac{i}{4(D-1)q^2}\int\frac{d^Dp}{(2\pi)^D}\frac{2q^2p^2-q^4-p^4}{(q-p)^2(s_1-p^2)} = \\ &= \frac{1}{4(D-1)q^2}\left(-2q^2s_1\tilde\lambda_1(-q^2,s_1)-q^4\tilde\lambda_0(-q^2,s_1)-s_1^2\tilde\lambda_2(-q^2,s_1)\right) = \\ &= \frac{1}{4(D-1)q^2}\left(2q^2s_1-q^4-s_1^2\right)\tilde\lambda_0(-q^2,s) = -\frac{(q^2-s_1)^2}{4(D-1)q^2}\tilde\lambda_0(-q^2,s_1), \qquad (4.123)\end{aligned}$$

or in terms of spectral densities

$$\lambda^1(s,s_1)=-\frac{(s-s_1)^2}{4(D-1)s}\lambda_0(s,s_1),\quad \lambda_0(s,s_1)=\frac{-1}{(4\pi)^2}\left(1-\frac{s_1}{s}\right)\theta(s-s_1)+O(\varepsilon). \quad (4.124)$$

For the second convolution function one calculates

$$\begin{aligned}&4\left(D(qp-p^2)(qp)-p^2(q^2-qp)\right) = \\ &= D\left(q^2-p^2-(q-p)^2\right)\left(q^2+p^2-(q-p)^2\right)-2p^2\left(q^2-p^2+(q-p)^2\right) = \\ &= D\left(q^4-2q^2(q-p)^2+(q-p)^4-p^4\right)-2p^2\left(q^2-p^2+(q-p)^2\right) = \\ &= Dq^4-(D-2)p^4-2p^2q^2-2Dq^2(q-p)^2-2p^2(q-p)^2+D(q-p)^4 \qquad (4.125)\end{aligned}$$

and therefore obtains

$$\begin{aligned}\tilde\lambda^2(-q^2,s_1) &= \frac{1}{4(D-1)q^2}\Big(Dq^4\tilde\lambda_0(-q^2,s_1)\ + \\ &\qquad -(D-2)s_1^2\tilde\lambda_2(-q^2,s_1)+2q^2s_1\tilde\lambda_1(-q^2,s_1)\Big) = \\ &= \frac{1}{4(D-1)q^2}\left(Dq^4-(D-2)s_1^2-2q^2s_1\right)\tilde\lambda_0(-q^2,s_1), \qquad (4.126)\end{aligned}$$

or in terms of spectral densities

$$\begin{aligned}\lambda^2(s,s_1) &= \frac{1}{4(D-1)s}\left(Ds^2-(D-2)s_1^2-2ss_1\right)\lambda_0(s,s_1) = \\ &= \frac{(s-s_1)}{4(D-1)s}(Ds+(D-2)s_1)\lambda_0(s,s_1). \qquad (4.127)\end{aligned}$$

The first projection

The first projection is given by

$$
\begin{aligned}
p^2\hat{V}^{q1}_{c21}(-p^2) = g_{\mu\nu}\tilde{V}^{q\mu\nu}_{c21}(-p^2) &= \\
= g_s^2 \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{\mathrm{Tr}(\gamma_\mu(\not{p}-\not{l})\gamma_\alpha(\not{p}-\not{k}))\frac{1}{4}\mathrm{Tr}(\gamma^\mu(\not{l}+m)\gamma^\alpha(\not{k}+m))}{(k^2-m^2)(l^2-m^2)(p-k)^2(p-l)^2(k-l)^2} &= \\
= \frac{g_s^2}{(4\pi)^D}\Big[4(p^2-m^2)^2\tilde{V}(1,1,1,1,1;-p^2) + 8(p^2-m^2)\tilde{V}(0,1,1,1,1;-p^2) &+ \\
+8p^2\tilde{V}(1,1,1,0,1;-p^2) - 4p^2\tilde{V}(1,1,1,1,0;-p^2) &+ \\
+4\tilde{V}(0,0,1,1,1;-p^2) + 4\tilde{V}(1,1,0,0,1;-p^2) &+ \\
-2(2-D)\left(\tilde{V}(0,1,0,1,1;-p^2) + \tilde{V}(0,1,1,0,1;-p^2)\right) &+ \\
+2(2-D)\left(\tilde{V}(0,1,1,1,0;-p^2) + \tilde{V}(1,1,1,0,0;-p^2)\right) &+ \\
-(2-D)\tilde{V}(1,1,1,1,-1;-p^2)\Big], & \qquad (4.128)
\end{aligned}
$$

and the spectral density reads ($z = m^2/s$)

$$
\begin{aligned}
\hat{\rho}^{q1}_{c21}(s) &= \frac{g_s^2}{(4\pi)^D s}\Big[4(s-m^2)^2\tilde{\rho}_V(1,1,1,1,1;s) + 8(s-m^2)\tilde{\rho}_V(0,1,1,1,1;s) + \\
&\quad +8s\tilde{\rho}_V(1,1,1,0,1;s) - 4s\tilde{\rho}_V(1,1,1,1,0;s) + \\
&\quad +4\tilde{\rho}_V(0,0,1,1,1;s) + 4\tilde{\rho}_V(1,1,0,0,1;s) + \\
&\quad -2(2-D)\left(\tilde{\rho}_V(0,1,0,1,1;s) + \tilde{\rho}_V(0,1,1,0,1;s)\right) + \\
&\quad +2(2-D)\left(\tilde{\rho}_V(0,1,1,1,0;s) + \tilde{\rho}_V(1,1,1,0,0;s)\right) + \\
&\quad -(2-D)\tilde{\rho}_V(1,1,1,1,-1;s)\Big] = \\
&= \frac{g_s^2 s^{D-4}}{(4\pi)^D}\Big[4\left(1-z\right)^2\hat{\rho}_V(1,1,1,1,1;z) + 8(1-z)\hat{\rho}_V(0,1,1,1,1;z) + \\
&\quad +8\hat{\rho}_V(1,1,1,0,1;z) - 4\hat{\rho}_V(1,1,1,1,0;z) + \\
&\quad +4\hat{\rho}_V(0,0,1,1,1;z) + 4\hat{\rho}_V(1,1,0,0,1;z) + \\
&\quad -2(2-D)\left(\hat{\rho}_V(0,1,0,1,1;z) + \hat{\rho}_V(0,1,1,0,1;z)\right) + \\
&\quad +2(2-D)\left(\hat{\rho}_V(0,1,1,1,0;z) + \hat{\rho}_V(1,1,1,0,0;z)\right) + \\
&\quad -(2-D)\hat{\rho}_V(1,1,1,1,-1;z)\Big]. \qquad (4.129)
\end{aligned}
$$

The second projection

The second projection is given by

$$
p^4\hat{V}^{q2}_{c21}(-p^2) = p_\mu p_\nu \tilde{V}^{q\mu\nu}_{c21}(-p^2) =
$$

$$
\begin{aligned}
&= g_s^2 \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{Tr}(\not{p}(\not{p}-\not{l})\gamma_\alpha(\not{p}-\not{k}))\frac{1}{4}\mathrm{Tr}(\not{p}(\not{l}+m)\gamma^\alpha(\not{k}+m))}{(k^2-m^2)(l^2-m^2)(p-k)^2(p-l)^2(k-l)^2} = \\
&= \frac{g_s^2}{(4\pi)^D}\Big[2(p^2+m^2)(p^2-m^2)^2\tilde{V}(1,1,1,1,1;-p^2)+4(p^4-m^4)\tilde{V}(0,1,1,1,1;-p^2)\;+ \\
&\qquad +4(p^4+m^4)\tilde{V}(1,1,1,0,1;-p^2)-(3p^4+m^4)\tilde{V}(1,1,1,1,0;-p^2)\;+ \\
&\qquad +2(p^2+m^2)\tilde{V}(0,0,1,1,1;-p^2)+2p^2\tilde{V}(1,1,0,0,1;-p^2)\;+ \\
&\qquad +2(p^2-m^2)\tilde{V}(0,1,0,1,1;-p^2)-2(p^2+m^2)\tilde{V}(0,1,1,0,1;-p^2)\;+ \\
&\qquad -2(p^2-m^2)\tilde{V}(0,1,1,1,0;-p^2)-2(p^2+m^2)\tilde{V}(1,1,1,0,0;-p^2)\;+ \\
&\qquad +2m^2\tilde{V}(1,1,-1,1,1;-p^2)+p^2\tilde{V}(1,1,1,1,-1;-p^2)-2\tilde{V}(-1,1,1,0,1;-p^2)\;+ \\
&\qquad +2\tilde{V}(0,0,0,1,1;-p^2)-\tilde{V}(0,0,1,1,0;-p^2)+2\tilde{V}(0,1,0,0,1;-p^2)\;+ \\
&\qquad -2\tilde{V}(0,1,1,-1,1;-p^2)+2\tilde{V}(0,1,1,0,0;-p^2)-\tilde{V}(1,1,0,0,0;-p^2)\Big]
\end{aligned}
\tag{4.130}
$$

and the spectral density reads ($z = m^2/s$)

$$
\begin{aligned}
\hat{\rho}^{q2}_{c21}(s) &= \frac{g_s^2}{(4\pi)^D s^2}\Big[2(s+m^2)(s-m^2)^2\tilde{\rho}_V(1,1,1,1,1;s)+4(s^2-m^4)\tilde{\rho}_V(0,1,1,1,1;s)\;+ \\
&\qquad +4(s^2+m^4)\tilde{\rho}_V(1,1,1,0,1;s)-(3s^2+m^4)\tilde{\rho}_V(1,1,1,1,0;s)\;+ \\
&\qquad +2(s+m^2)\tilde{\rho}_V(0,0,1,1,1;s)+2s\tilde{\rho}_V(1,1,0,0,1;s)\;+ \\
&\qquad +2(s-m^2)\tilde{\rho}_V(0,1,0,1,1;s)-2(s+m^2)\tilde{\rho}_V(0,1,1,0,1;s)\;+ \\
&\qquad -2(s-m^2)\tilde{\rho}_V(0,1,1,1,0;s)-2(s+m^2)\tilde{\rho}_V(1,1,1,0,0;s)\;+ \\
&\qquad +2m^2\tilde{\rho}_V(1,1,-1,1,1;s)+s\tilde{\rho}_V(1,1,1,1,-1;s)-2\tilde{\rho}_V(-1,1,1,0,1;s)\;+ \\
&\qquad +2\tilde{\rho}_V(0,0,0,1,1;s)-\tilde{\rho}_V(0,0,1,1,0;s)+2\tilde{\rho}_V(0,1,0,0,1;s)\;+ \\
&\qquad -2\tilde{\rho}_V(0,1,1,-1,1;s)+2\tilde{\rho}_V(0,1,1,0,0;s)-\tilde{\rho}_V(1,1,0,0,0;s)\Big] = \\
&= \frac{g_s^2 s^{D-4}}{(4\pi)^D}\Big[2(1+z)(1-z)^2\hat{\rho}_V(1,1,1,1,1;z)\;+ \\
&\qquad +4(1-z^2)\hat{\rho}_V(0,1,1,1,1;z)\;+ \\
&\qquad +4(1+z^2)\hat{\rho}_V(1,1,1,0,1;z)-(3+z^2)\hat{\rho}_V(1,1,1,1,0;z)\;+ \\
&\qquad +2(1+z)\hat{\rho}_V(0,0,1,1,1;z)+2\hat{\rho}_V(1,1,0,0,1;z)\;+ \\
&\qquad +2(1-z)\hat{\rho}_V(0,1,0,1,1;z)-2(1+z)\hat{\rho}_V(0,1,1,0,1;z)\;+ \\
&\qquad -2(1-z)\hat{\rho}_V(0,1,1,1,0;z)-2(1+z)\hat{\rho}_V(1,1,1,0,0;z)\;+ \\
&\qquad +2z\hat{\rho}_V(1,1,-1,1,1;z)+\hat{\rho}_V(1,1,1,1,-1;z)-2\hat{\rho}_V(-1,1,1,0,1;z)\;+ \\
&\qquad +2\hat{\rho}_V(0,0,0,1,1;z)-\hat{\rho}_V(0,0,1,1,0;z)+2\hat{\rho}_V(0,1,0,0,1;z)\;+ \\
&\qquad -2\hat{\rho}_V(0,1,1,-1,1;z)+2\hat{\rho}_V(0,1,1,0,0;z)-\hat{\rho}_V(1,1,0,0,0;z)\Big].
\end{aligned}
\tag{4.131}
$$

4.4 The fish prototypes

As one can see from the last expression of the previous section, there are many spectral functions of the standard two-loop integrals V, the so-called *prototypes*, necessary in order to built up the final results. These prototypes will be dealt with in this section. Before dealing with the calculations, however, the prototypes will be classified and two integrals will be calculated in advance which are of help in the following.

4.4.1 General simplification paths for the prototypes

Starting with the two-loop integrals V, also on this level there are relations which can be used in order to simplify the calculations. First of all, the symmetry property

$$V(n_1,n_2,n_3,n_4,n_5;p^2/m^2)=V(n_2,n_1,n_4,n_3,n_5;p^2/m^2) \tag{4.132}$$

can be used to collect the integrals. The symmetry property is of course valid on the level of spectral densities as well. For two vanishing entries the integrals can be split up into two one-loop standard integrals V. Because of $V(0,n_2;p^2/m^2)=0$, most of the integrals

$$\begin{aligned}
V(0,0,0,1,1;p^2/m^2) &= V(0,1;p^2/m^2)V(0,1;p^2/m^2) \;=\; 0,\\
V(0,0,1,0,1;p^2/m^2) &= V(0,1;p^2/m^2)V(0,1;p^2/m^2) \;=\; 0,\\
V(0,1,0,0,1;p^2/m^2) &= V(0,1;p^2/m^2)V(1,0;p^2/m^2) \;=\; 0,\\
V(1,0,0,0,1;p^2/m^2) &= V(1,0;p^2/m^2)V(0,1;p^2/m^2) \;=\; 0,\\
V(0,0,1,1,0;p^2/m^2) &= V(0,1;p^2/m^2)V(0,1;p^2/m^2) \;=\; 0,\\
V(0,1,0,1,0;p^2/m^2) &= V(0,0;p^2/m^2)V(1,1;p^2/m^2) \;=\; 0,\\
V(1,0,0,1,0;p^2/m^2) &= V(1,0;p^2/m^2)V(0,1;p^2/m^2) \;=\; 0,\\
V(0,1,1,0,0;p^2/m^2) &= V(0,1;p^2/m^2)V(1,0;p^2/m^2) \;=\; 0,\\
V(1,0,1,0,0;p^2/m^2) &= V(1,1;p^2/m^2)V(0,0;p^2/m^2) \;=\; 0,\\
V(1,1,0,0,0;p^2/m^2) &= V(1,0;p^2/m^2)V(1,0;p^2/m^2)
\end{aligned} \tag{4.133}$$

vanish. The last integral is the only non-vanishing one. Because of $V(n_1,0;p^2/m^2)=V(n_1,0;-1)$, however, the corresponding spectral density vanishes as well. But also in other cases other simplifications are possible. If the last entry vanishes, the integral can generally be written as a product of two massive one-loop integrals. If two other entries vanish or take negative values, one of the integrals can be performed first. If this integral it is a massless one-loop integral G, the result is again a product of two one-loop integrals where the massive one-loop integral has a non-integer entry. Cases which can be resolved in this kind are

$$\begin{aligned}
V(0,0,1,1,1:p^2/m^2) &= 0,\\
V(0,1,0,1,1;p^2/m^2) &= V(0,1;p^2/m^2)V(1,1;p^2/m^2) \;=\; 0,
\end{aligned}$$

$$\begin{aligned}
V(0,1,1,0,1;p^2/m^2) &= G(1,1)V(1,2-D/2;p^2/m^2),\\
V(0,1,1,1,0;p^2/m^2) &= V(0,1;p^2/m^2)V(1,1;p^2/m^2) \;=\; 0,\\
V(1,0,0,1,1;p^2/m^2) &= G(1,1)V(1,2-D/2;p^2/m^2),\\
V(1,0,1,0,1;p^2/m^2) &= V(1,1;p^2/m^2)V(0,1;p^2/m^2) \;=\; 0,\\
V(1,0,1,1,0;p^2/m^2) &= V(1,1;p^2/m^2)V(0,1;p^2/m^2) \;=\; 0 \qquad (4.134)
\end{aligned}$$

where for the first two

$$\begin{aligned}
&\frac{1}{(4\pi)^D}(m^2)^{D-3}V(0,0,1,1,1;p^2/m^2) \;=\\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(p-k)^2(p-l)^2(k-l)^2} \;=\; \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{k^2l^2(k-l)^2} = 0,\\
&\frac{1}{(4\pi)^D}(m^2)^{D-3}V(0,1,0,1,1;p^2/m^2) \;=\\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)(p-l)^2(k-l)^2} = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)(p-l)^2} \;=\\
&= \frac{1}{(4\pi)^D}(m^2)^{D-3}V(0,1;p^2/m^2)V(1,1;p^2/m^2), \qquad (4.135)
\end{aligned}$$

the first vanishing because of the absence of an outer scale to represent the integral (this rule will be called "*scaling rule*" in the following).

The nested integrals $V^\varepsilon(n_1,n_2;p^2/m^2)$

In some of the cases of taking one integration in advance, one is left with *nested integrals* of the type

$$\frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-n_1-n_2}V^\varepsilon(n_1,n_2;q^2/m^2) := \int\frac{d^Dp}{(2\pi)^D}\frac{V(1,1;p^2/m^2)}{(p^2+m^2)^{n_1}((q-p)^2)^{n_2}}. \qquad (4.136)$$

A way to obtain a value for the integral for $n_1 = 0$ is given by resolving the part $V(1,1;p^2/m^2)$,

$$\begin{aligned}
\frac{1}{(4\pi)^D}(m^2)^{D-n_2-2}V^\varepsilon(0,n_2;q^2/m^2) &= \frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-2}\int\frac{d^Dp}{(2\pi)^D}\frac{V(1,1;p^2/m^2)}{((q-p)^2)^{n_2}} \;=\\
&= \int\frac{d^Dp}{(2\pi)^D}\frac{1}{((q-p)^2)^{n_2}}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2+m^2)(p-k)^2} \;=\\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2+m^2)}\int\frac{d^Dp}{(2\pi)^D}\frac{1}{((q-p)^2)^{n_2}(p-k)^2} \;=\\
&= \frac{1}{(4\pi)^{D/2}}\int\frac{d^Dk}{(2\pi)^D}\frac{G(1,n_2)}{(k^2+m^2)((q-k)^2)^{n_2+1-D/2}} \;=\\
&= \frac{1}{(4\pi)^D}(m^2)^{D/2-1-n_2-1+D/2}G(1,n_2)V(1,1+n_2-D/2;q^2/m^2), \qquad (4.137)
\end{aligned}$$

therefore

$$\begin{aligned} V^\varepsilon(0,n_2;q^2/m^2) &= G(1,n_2)V(1,1+n_2-D/2;q^2/m^2),\\ \rho^\varepsilon(0,n_2;s/m^2) &= G(1,n_2)\rho_V(1,1+n_2-D/2;s/m^2). \end{aligned} \tag{4.138}$$

For $n_1 \neq 0$ a more involved method has been used, similar to the one used in the calculations for the massive contributions. However, the integrals will be calculated only for the cases which are of interest, namely for $n_1 = 0$ and $n_1 = 1$. For $n_1 = 0$ one obtains

$$\begin{aligned} \frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-n_2}V^\varepsilon(0,n_2;q^2/m^2) &= \\ = \int\frac{d^Dp}{(2\pi)^D}\frac{V(1,1;p^2/m^2)}{((q-p)^2)^{n_2}} &= \int\frac{d^Dp}{(2\pi)^D}\frac{1}{((q-p)^2)^{n_2}}\int\frac{\rho_V(1,1;s/m^2)ds}{s+p^2} = \\ = \int\rho_V(1,1;s/m^2)ds\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(p^2+s)((q-p)^2)^{n_2}} &= \\ = \frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-1-n_2}\int\rho_V(1,1;s/m^2)V(1,n_2;q^2/s)ds & \end{aligned} \tag{4.139}$$

while for $n_1 = 1$

$$\begin{aligned} \frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-1-n_2}V^\varepsilon(1,n_2;q^2/m^2) &= \int\frac{d^Dp}{(2\pi)^D}\frac{V(1,1;p^2/m^2)}{(p^2+m^2)((q-p)^2)^{n_2}} = \\ = \int\frac{d^Dp}{(2\pi)^D}\frac{1}{(p^2+m^2)((q-p)^2)^{n_2}}\int\frac{\rho_V(1,1;s/m^2)ds}{s+p^2} &= \\ = \int\frac{d^Dp}{(2\pi)^D}\frac{1}{((q-p)^2)^{n_2}}\int\frac{\rho_V(1,1;s/m^2)}{s-m^2}\left(\frac{1}{p^2+m^2}-\frac{1}{p^2+s}\right)ds &= \\ = \frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-1-n_2}\int\frac{\rho_V(1,1;s/m^2)}{s-m^2}\Big(V(1,n_2;q^2/m^2)-V(1,n_2;q^2/s)\Big)ds. & \end{aligned} \tag{4.140}$$

In this (second) case the first part results in the infinite constant

$$V(1,1;-1) = \int_{m^2}^\infty\frac{\rho_V(1,1;s/m^2)}{s-m^2}ds \tag{4.141}$$

which will be a basic element later on. Therefore, one obtains

$$V^\varepsilon(0,n_2;q^2/m^2) = \int\frac{\rho_V(1,1;s/m^2)}{m^2}V(1,n_2;q^2/s)ds, \tag{4.142}$$
$$V^\varepsilon(1,n_2;q^2/m^2) = V(1,1;-1)V(1,n_2;q^2/m^2) - \int_{m^2}\frac{\rho_V(1,1;s/m^2)}{s-m^2}V(1,n_2;q^2/s)ds.$$

The corresponding spectral densities are given by finite expressions,

$$\rho_V^\varepsilon(0,n_2;s/m^2) = \int_{m^2}^s\frac{\rho_V(1,1;s_1/m^2)}{m^2}\rho_V(1,n_2;s/s_1)ds_1, \tag{4.143}$$
$$\rho_V^\varepsilon(1,n_2;s/m^2) = V(1,1;-1)\rho_V(1,n_2;s/m^2) - \int_{m^2}^s\frac{\rho_V(1,1;s_1/m^2)}{s_1-m^2}\rho_V(1,n_2;s/s_1)ds_1$$

The integrals are limited to the interval $[m^2, s]$ by the two spectral densities involved. The identification of an infinite term in the last expression can be called *"passive subtraction"*. One obtains the same result in *"active subtraction"* by inserting

$$V(1,1;p^2/m^2) = \int \frac{\rho_V(1,1;s/m^2)}{s+p^2} ds = V(1,1;-1) - (p^2+m^2) \int \frac{\rho_V(1,1;s/m^2)}{(s-m^2)(s+p^2)} ds. \tag{4.144}$$

To conclude, one obtains

$$\begin{aligned}
\rho_V(0,n_2;1/z) &= \int_z^1 \frac{\rho_V(1,1;1/z_1)}{z_1^2} \rho_V(1,n_2;z_1/z) dz_1, \qquad (4.145)\\
\rho_V(1,n_2;1/z) &= V(1,1;-1)\rho_V(1,n_2;1/z) - \int_z^1 \frac{\rho_V(1,1;1/z_1)}{z_1(1-z_1)} \rho_V(1,n_2;z_1/z) dz_1.
\end{aligned}$$

For later convenience one changes to the hatted quantities

$$\hat\rho_V^\varepsilon(n_1,n_2;z) = z^{D/2-n_1-n_2-\varepsilon} \rho_V(n_1,n_2;1/z) \tag{4.146}$$

and obtains

$$\begin{aligned}
\hat\rho_V^\varepsilon(0,n_2;z) &= z^{1-\varepsilon} \int_z^1 \frac{\hat\rho_V(1,1;z_1)}{z_1^{n_2+1}} \hat\rho_V(1,n_2;z/z_1) dz_1, \qquad (4.147)\\
\hat\rho_V^\varepsilon(1,n_2;z) &= z^{-\varepsilon} \left(V(1,1;-1)\hat\rho_V(1,n_2;z) - \int_z^1 \frac{\hat\rho_V(1,1;z_1)}{z_1^{n_2}(1-z_1)} \hat\rho_V(1,n_2;z/z_1) dz_1 \right).
\end{aligned}$$

Finally one can look at two special cases which are used in the following. The spectral density $\hat\rho_V(1,0;z)$ vanishes, so that

$$\hat\rho_V^\varepsilon(0,0;z) = \hat\rho_V(1,0;z) = 0. \tag{4.148}$$

For $n_2 = 1$ one can use $\rho_V(1,1;z) = (1-z)\theta(1-z) + O(\varepsilon)$ to obtain

$$\begin{aligned}
\int_z^1 \frac{\hat\rho_V(1,1;z_1)}{z_1^2} \hat\rho_V(1,1;z/z_1) dz_1 &= \int_z^1 \frac{1-z_1}{z_1^2} \left(1 - \frac{z}{z_1}\right) dz_1 = \frac{1-z^2}{2z} + \ln z, \qquad (4.149)\\
\int_z^1 \frac{\hat\rho_V(1,1;z_1)}{z_1(1-z_1)} \hat\rho_V(1,1;z/z_1) dz_1 &= \int_z^1 \frac{1-z_1}{z_1(1-z_1)} \left(1 - \frac{z}{z_1}\right) dz_1 = z - 1 - \ln z
\end{aligned}$$

and therefore (with $V(1,1;-1) = G/\varepsilon$)

$$\begin{aligned}
\hat\rho_V^\varepsilon(0,1;z) &= z^{1-\varepsilon} \left(\frac{1-z^2}{2z} + \ln z \right) = \frac{1}{2}(1-z^2) + z\ln z,\\
\hat\rho_V^\varepsilon(1,1;z) &= z^{-\varepsilon} \left(V(1,1;-1)\hat\rho_V(1,1;z) + 1 - z + \ln z \right) =\\
&= \frac{G}{\varepsilon} z^{-\varepsilon} \hat\rho_V(1,1;z) + 1 - z + \ln z. \qquad (4.150)
\end{aligned}$$

The vector integral $V'(1,1;p^2/m^2)$

In cases of negative entries, one is left with a *vector integral* of the kind

$$\frac{1}{(4\pi)^{D/2}} (m^2)^{D/2-2} p^\mu V'(1,1;p^2/m^2) := \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu}{(k^2+m^2)(p-k)^2}. \tag{4.151}$$

These integrals will also be calculated in advance at this point. By contracting the integral with p_μ one obtains

$$\begin{aligned}
\frac{1}{(4\pi)^{D/2}}p^2(m^2)^{D/2-2}V'(1,1;p^2/m^2) &= \int\frac{d^Dk}{(2\pi)^D}\frac{pk}{(k^2+m^2)(p-k)^2} =\\
&= \frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{p^2+k^2-(p-k)^2}{(k^2+m^2)(p-k)^2} = \frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{p^2-m^2+k^2+m^2-(p-k)^2}{(k^2+m^2)(p-k)^2} =\\
&= \frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(p-k)^2}+\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{p^2-m^2}{(k^2+m^2)(p-k)^2}-\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2+m^2} =\\
&= \frac{1}{2(4\pi)^{D/2}}(m^2)^{D/2-2}\Big((p^2-m^2)V(1,1;p^2/m^2)-m^2V(1,0;-1)\Big), \qquad (4.152)
\end{aligned}$$

such that

$$V'(1,1;p^2/m^2) = \frac{1}{2}\left(1-\frac{m^2}{p^2}\right)V(1,1;p^2/m^2)-\frac{m^2}{2p^2}V(1,0;-1). \qquad (4.153)$$

The corresponding spectral density is given by

$$\rho'_V(1,1;s/m^2) = \frac{1}{2}\left(1+\frac{m^2}{s}\right)\rho_V(1,1;s/m^2), \qquad \rho'_V(1,1;1/z) = \frac{1}{2}(1+z)\hat\rho_V(1,1;1/z). \qquad (4.154)$$

Finally, one can restore the correlator by using the dispersion relation and an "active subtraction",

$$\begin{aligned}
V'(1,1;p^2/m^2) &= \int\frac{\rho'_V(1,1;s/m^2)}{s+p^2}ds =\\
&= \int\frac{\rho'_V(1,1;s/m^2)}{s-m^2}-(p^2+m^2)\int\frac{\rho'_V(1,1;s/m^2)}{(s+p^2)(s-m^2)}ds =\\
&= V'(1,1;-1)-(p^2+m^2)\int\frac{\rho'_V(1,1;s/m^2)}{(s+p^2)(s-m^2)}ds. \qquad (4.155)
\end{aligned}$$

One still has to calculate the singular part $V'(1,1;-1)$. Here one uses

$$\begin{aligned}
\rho_V(1,1;1/z) &= \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}z^\varepsilon(1-z)^{1-2\varepsilon}, \qquad \text{therefore}\\
\rho'_V(1,1;1/z) &= \frac{\Gamma(1-\varepsilon)}{2\Gamma(2-2\varepsilon)}(1+z)z^\varepsilon(1-z)^{1-2\varepsilon} \qquad (4.156)
\end{aligned}$$

and so $V(1,1;-1)=\Gamma(\varepsilon)/(1-2\varepsilon)$ as before and

$$\begin{aligned}
V'(1,1;-1) &= \int_{m^2}^\infty\frac{\rho'_V(1,1;s/m^2)}{s-m^2}ds = \int_0^1\frac{\rho'_V(1,1;1/z)}{m^2/z-m^2}\,\frac{m^2}{z^2}dz =\\
&= \int_0^1\frac{\rho'_V(1,1;1/z)}{z(1-z)}dz = \frac{\Gamma(1-\varepsilon)}{2\Gamma(2-2\varepsilon)}\int_0^1(1+z)z^{\varepsilon-1}(1-z)^{-2\varepsilon}dz =\\
&= \frac{\Gamma(1-\varepsilon)}{2\Gamma(2-2\varepsilon)}\left(\int_0^1 z^{\varepsilon-1}(1-z)^{-2\varepsilon}dz+\int_0^1 z^\varepsilon(1-z)^{-2\varepsilon}dz\right) =\\
&= \frac{\Gamma(1-\varepsilon)}{2\Gamma(2-2\varepsilon)}\left(\frac{\Gamma(\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)}+\frac{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(2-\varepsilon)}\right) =\\
&= \frac{\Gamma(\varepsilon)}{2(1-2\varepsilon)}\left(1+\frac{\varepsilon}{1-\varepsilon}\right) = \frac{\Gamma(\varepsilon)}{2(1-\varepsilon)(1-2\varepsilon)} = \frac{G}{2\varepsilon}(1+\varepsilon)+O(\varepsilon). \qquad (4.157)
\end{aligned}$$

4.4.2 Calculation of the prototypes

After having done some work in advance, most of the calculations for the prototypes are straightforward. The calculation starts with the two most complicated prototypes, the *proper fish* $\hat{\rho}_V(1,1,1,1,1;z)$ and the *spectacle* $\hat{\rho}_V(1,1,1,1,0;z)$ and leads to simpler cases. In the end all prototypes are listed in order to have them at hand when the fish diagrams are constructed.

The prototype $\hat{\rho}_V(1,1,1,1,1;z)$ (proper fish)

One starts with the integral which is provided in Refs. [135, 137]. In the notation of Ref. [135] it is the one called I, given by

$$I = F(1) + F(x_a x_b) - F(x_a) - F(x_b) + O(\varepsilon) \tag{4.158}$$

where

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^3}\left(2 + (2 - n\ln x)^2\right) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}\left(6 - 4\ln x + n^2 \ln^2 x\right) = \\ &= 6\,\mathrm{Li}_3(x) - 4\,\mathrm{Li}_2(x)\ln x + \mathrm{Li}_1(x)\ln^2 x. \end{aligned} \tag{4.159}$$

In the limit $m_b \to 0$, x_b tends to zero and x_a to $m^2/(m^2-q^2)$. After a thorough analysis of the limiting process one obtains

$$I = F(1) - F\left(\frac{m^2}{m^2-q^2}\right) + O(\varepsilon) = 6\zeta(3) - F\left(\frac{m^2}{m^2-q^2}\right) + O(\varepsilon). \tag{4.160}$$

The next step is to construct a connection to the integrals V (expressed in powers of m^2 so that no "external" discontinuity contributions from $-q^2$ can arise). One obtains

$$\begin{aligned} I &= -\frac{(-q^2 e^{\gamma_E})^{2\varepsilon}}{\pi^D}\int \frac{q^2 d^D k\, d^D l}{(k^2-m^2)(l^2-m^2)(q-k)^2(q-l)^2(k-l)^2} = \\ &= \frac{e^{2\gamma_E\varepsilon}(-q^2)^{1+2\varepsilon}}{\pi^D}(2\pi)^{2D} \times \\ &\qquad \times \int \frac{d^D k}{(2\pi)^D}\frac{d^D l}{(2\pi)^D}\frac{1}{(k^2-m^2)(l^2-m^2)(q-k)^2(q-l)^2(k-l)^2} = \\ &= \frac{e^{2\gamma_E\varepsilon}(-q^2)^{5-D}}{\pi^D}(2\pi)^{2D}\frac{1}{(4\pi)^D}(m^2)^{D-5}V(1,1,1,1,1;-q^2/m^2) = \\ &= e^{2\gamma_E\varepsilon}\left(\frac{-q^2}{m^2}\right)^{5-D} V(1,1,1,1,1;-q^2/m^2) \end{aligned} \tag{4.161}$$

and thus, turned to Euclidean space,

$$V(1,1,1,1,1;q^2/m^2) = e^{-2\gamma_E\varepsilon}\left(\frac{q^2}{m^2}\right)^{D-5}\left(F(1) - F\left(\frac{m^2}{q^2+m^2}\right)\right) + O(\varepsilon). \tag{4.162}$$

Fortunately, one can use $D=4$ because the quantities one has to calculate are finite. Using

$$\mathrm{Disc}\ln\left(\frac{m^2}{m^2-s}\right) = 2\pi i, \qquad \mathrm{Disc}\ln^2\left(\frac{m^2}{m^2-s}\right) = 4\pi i\ln\left(\frac{m^2}{s-m^2}\right) = -4\pi i\ln\left(\frac{s}{m^2}-1\right) \tag{4.163}$$

and (cf. Appendix E.3)

$$\mathrm{Li}_1\left(\frac{m^2}{m^2-s}\right) = -\ln\left(1-\frac{m^2}{m^2-s}\right) = -\ln\left(\frac{s}{s-m^2}\right) = \ln\left(1-\frac{m^2}{s}\right) \tag{4.164}$$

one obtains

$$\frac{1}{2\pi i}\,\mathrm{Disc}\,F\left(\frac{m^2}{m^2-s}\right) = -4\,\mathrm{Li}_2\left(\frac{m^2}{m^2-s}\right) - 2\ln\left(1-\frac{m^2}{s}\right)\ln\left(\frac{s}{m^2}-1\right). \tag{4.165}$$

Therefore, one ends up with

$$\begin{aligned}\rho_V(1,1,1,1,1;s/m^2) &= \\ &= \left(-\frac{s}{m^2}\right)^{-1}\left(4\,\mathrm{Li}_2\left(\frac{m^2}{m^2-s}\right) + 2\ln\left(1-\frac{m^2}{s}\right)\ln\left(\frac{s}{m^2}-1\right)\right) = \\ &= -\frac{m^2}{s}\left(4\,\mathrm{Li}_2\left(\frac{m^2}{m^2-s}\right) + 2\ln\left(1-\frac{m^2}{s}\right)\ln\left(\frac{s}{m^2}-1\right)\right).\end{aligned} \tag{4.166}$$

With $z = m^2/s$ one obtains

$$\rho_V(1,1,1,1,1;1/z) = -z\left(4\,\mathrm{Li}_2\left(\frac{z}{z-1}\right) + 2\ln(1-z)\ln\left(\frac{1}{z}-1\right)\right). \tag{4.167}$$

Now the dilogarithm identity

$$\mathrm{Li}_2\left(\frac{z}{z-1}\right) = -\,\mathrm{Li}_2(z) - \frac{1}{2}\ln^2(1-z) \tag{4.168}$$

can be used to obtain

$$\begin{aligned}\rho_V(1,1,1,1,1;1/z) &= z\Big(4\,\mathrm{Li}_2(z) + 2\ln^2(1-z) - 2\ln(1-z)\left(\ln(1-z)-\ln z\right)\Big) = \\ &= z\left(4\,\mathrm{Li}_2(z) + 2\ln(1-z)\ln z\right).\end{aligned} \tag{4.169}$$

In the final representation the prototype reads

$$\begin{aligned}\hat\rho_V(1,1,1,1,1;z) &= z^{D-5}\rho_V(1,1,1,1,1;1/z) = z^{-1-2\varepsilon}\rho_V(1,1,1,1,1;1/z) = \\ &= 4\left(\mathrm{Li}_2(z) + \frac{1}{2}\ln(1-z)\ln z\right) + O(\varepsilon).\end{aligned} \tag{4.170}$$

The prototype $\hat\rho_V(1,1,1,1,0;z)$ (spectacle)

Switching to the Euclidean domain, the first (naive) approach to this prototype on the correlator level is given by

$$\begin{aligned}&\frac{(m^2)^{1-\varepsilon}}{(4\pi)^{D/2}}V(1,1,1,1,0,1;q^2/m^2) = \\ &= \int\frac{d^Dp}{(2\pi)^D}\frac{1}{(q-p)^2}V(1,1;p^2/m^2)V(1,1;p^2/m^2) = \\ &= \int\frac{d^Dp}{(2\pi)^D}\frac{1}{(q-p)^2}\int\frac{\rho_V(1,1;s_1/m^2)}{s_1+p^2}ds_1\int\frac{\rho_V(1,1;s_2/m^2)}{s_2+p^2}ds_2 = \\ &= \int\tilde\lambda(q^2,s_1,s_2)\rho_V(1,1;s_1/m^2)\rho_V(1,1;s_2/m^2)ds_1ds_2\end{aligned} \tag{4.171}$$

where the entry 1 attached to the arguments indicates the explicit propagator. However, the convolution function will lead to an explicite infrared divergence. This cannot be seen in the two-fold integral but will emerge at the end. In order to avoid this one can use the subtraction at the point $p^2=-m^2$ by writing

$$V(1,1;p^2/m^2)=\int\frac{\rho_V(1,1;s/m^2)ds}{s+p^2}=V(1,1;-1)-\int\frac{(p^2+m^2)\rho_V(1,1;s/m^2)}{(s-m^2)(s+p^2)}ds. \tag{4.172}$$

Therefore, one ends up with three contributions,

$$\begin{aligned}
&\frac{(m^2)^{1-\varepsilon}}{(4\pi)^{D/2}}V(1,1,1,1,0,1;q^2/m^2) \ = \qquad\qquad (4.173)\\
&=\int\frac{d^Dp}{(2\pi)^D}\frac{V(1,1;-1)^2}{(q-p)^2}-2\int\frac{d^Dp}{(2\pi)^D}\frac{V(1,1;-1)}{(q-p)^2}\int\frac{(p^2+m^2)\rho_V(1,1;s_1/m^2)}{(s_1-m^2)(s_1+p^2)}ds_1+\\
&\quad+\int\frac{d^Dp}{(2\pi)^D}\frac{1}{(q-p)^2}\int\frac{(p^2+m^2)\rho_V(1,1;s_1/m^2)}{(s_1-m^2)(s_1+p^2)}ds_1\int\frac{(p^2+m^2)\rho_V(1,1;s_2/m^2)}{(s_2-m^2)(s_2+p^2)}ds_2 \ =\\
&=\int\tilde\lambda_a(q^2,s_1)\rho_V(1,1;s_1/m^2)ds_1+\int\tilde\lambda_b(q^2,s_1,s_2)\rho_V(1,1;s_1/m^2)\rho_V(1,1;s_2/m^2)ds_1ds_2
\end{aligned}$$

where the first term vanishes. One again has an *algebra of convolution functions* which can be constructed within the Euclidean domain as

$$\tilde\lambda_n(q^2,s) \ = \ \int\frac{d^Dp}{(2\pi)^D}\frac{(p^2+m^2)^n}{(s-m^2)^n(s+p^2)(q-p)^2} \ = \ (-1)^n\tilde\lambda_0(q^2,s), \tag{4.174}$$

$$\tilde\lambda_0(q^2,s) \ = \ \int\frac{d^Dp}{(2\pi)^D}\frac{1}{(s+p^2)(q-p)^2} \ = \ \frac{1}{(4\pi)^{D/2}}V(1,1;q^2/s). \tag{4.175}$$

The convolution function for the remaining first (mixed) part is then given by

$$\begin{aligned}
\tilde\lambda_a(q^2,s_1) &= -2V(1,1;-1)\int\frac{d^Dp}{(2\pi)^D}\frac{p^2+m^2}{(s_1-m^2)(s_1+p^2)(q-p)^2} \ =\\
&= -2V(1,1;-1)\tilde\lambda_1(q^2,s_1) \ = \ 2V(1,1;-1)\tilde\lambda_0(q^2,s_1), \qquad (4.176)\\
\lambda_a(s,s_1) &= 2V(1,1;-1)\lambda_0(s,s_1) \qquad (4.177)
\end{aligned}$$

while for the last (bilinear) part one needs more effort. First, one can perform a partial fractioning for the integrand to obtain

$$\begin{aligned}
\tilde\lambda_b(q^2,s_1,s_2) \ &= \ \int\frac{d^Dp}{(2\pi)^D}\frac{(p^2+m^2)^2}{(s_1-m^2)(s_2-m^2)(s_1+p^2)(s_2+p^2)(q-p)^2} \ =\\
&= \ \frac{1}{(s_1-m^2)(s_2-m^2)(s_2-s_1)}\int\frac{d^Dp}{(2\pi)^D}\left(\frac{(p^2+m^2)^2}{(s_1+p^2)(q-p)^2}-\frac{(p^2+m^2)^2}{(s_2+p^2)(q-p)^2}\right) \ =\\
&= \ \frac{(s_1-m^2)^2\tilde\lambda_2(q^2,s_1)-(s_2-m^2)^2\tilde\lambda_2(q^2,s_2)}{(s_1-m^2)(s_2-m^2)(s_2-s_1)} \ =\\
&= \ \frac{(s_1-m^2)^2\tilde\lambda_0(q^2,s_1)-(s_2-m^2)^2\tilde\lambda_0(q^2,s_2)}{(s_1-m^2)(s_2-m^2)(s_2-s_1)}.
\end{aligned} \tag{4.178}$$

Therefore, one has

$$\lambda_b(s,s_1,s_2)=\frac{(s_1-m^2)^2\lambda_0(s,s_1)-(s_2-m^2)^2\lambda_0(s,s_2)}{(s_1-m^2)(s_2-m^2)(s_2-s_1)}. \tag{4.179}$$

Second, one uses the symmetry of this expression with respect to the interchange of s_1 and s_2 and performs the integration over s_2 with the spectral density $\rho_V(1,1,s_2/m^2)$ explicitly, obtaining an effective convolution function

$$\begin{aligned}
\lambda_b(s,s_1) &= 2\lambda_0(s,s_1)\int_{m^2}^\infty \frac{s_1-m^2}{s_2-m^2}\,\mathrm{Prin}\left(\frac{1}{s_2-s_1}\right)\left(1-\frac{m^2}{s_2}\right)ds_2 = \\
&= 2\lambda_0(s,s_1)\,\mathrm{Prin}\int_{m^2}^\infty \frac{s_1-m^2}{s_2(s_2-s_1)}ds_2 = \\
&= 2\lambda_0(s,s_1)\frac{s_1-m^2}{s_1}\,\mathrm{Prin}\int_{m^2}^\infty\left(\frac{1}{s_2-s_1}-\frac{1}{s_2}\right)ds_2 = \\
&= 2\lambda_0(s,s_1)\frac{s_1-m^2}{s_1}\lim_{s_0\to\infty}\lim_{\epsilon\to 0}\left\{\int_{m^2}^{s_1-\epsilon}\frac{ds_2}{s_2-s_1}+\int_{s_1+\epsilon}^{s_0}\frac{ds_2}{s_2-s_1}-\int_{m^2}^{s_0}\frac{ds_2}{s_2}\right\} = \\
&= 2\lambda_0(s,s_1)\frac{s_1-m^2}{s_1}\lim_{s_0\to\infty}\left\{\ln\left(\frac{s_0}{s_1-m^2}\right)-\ln\left(\frac{s_0}{m^2}\right)\right\} = \\
&= -2\lambda_0(s,s_1)\frac{s_1-m^2}{s_1}\ln\left(\frac{s_1-m^2}{m^2}\right). \qquad (4.180)
\end{aligned}$$

The principal value integration necessary here works fine, in contrast to the unsubtracted case. One then can collect both contributions to obtain an effective convolution function

$$\lambda(s,s_1)=\lambda_a(s,s_1)+\lambda_b(s,s_1)=2\lambda_0(s,s_1)\left(V(1,1;-1)-\left(1-\frac{m^2}{s_1}\right)\ln\left(\frac{s_1}{m^2}-1\right)\right). \qquad (4.181)$$

If one inserts this into the integrand of Eq. (4.173) and takes the discontinuity, one obtains

$$\begin{aligned}
&\frac{(m^2)^{1-\varepsilon}}{(4\pi)^{D/2}}\rho_V(1,1,1,1,0,1;s/m^2) = \\
&= \int 2\lambda_0(s,s_1)\left(V(1,1;-1)-\left(1-\frac{m^2}{s_1}\right)\ln\left(\frac{s_1}{m^2}-1\right)\right)\rho_V(1,1;s_1/m^2)ds_1. \qquad (4.182)
\end{aligned}$$

Therefore, one can also extract a two-loop spectral density which is given by

$$\rho_V(1,1,1,1,0;1/z)=2\left(V(1,1;-1)-(1-z)\ln\left(\frac{1}{z}-1\right)\right)\rho_V(1,1;1/z), \qquad (4.183)$$

and which is transformed to

$$\begin{aligned}
\hat\rho_V(1,1,1,1,0;z) &= z^{D-4}\rho_V(1,1,1,1,0;1/z) = \\
&= 2z^{-2\varepsilon}\left(V(1,1;-1)-(1-z)\ln\left(\frac{1}{z}-1\right)\right)\rho_V(1,1;1/z) = \\
&= 2z^{-\varepsilon}\left(V(1,1;-1)-(1-z)\ln\left(\frac{1}{z}-1\right)\right)\hat\rho_V(1,1;z) = \\
&= 2\frac{G}{\varepsilon}z^{-\varepsilon}\hat\rho_V(1,1;z)+2(1-z)^2\left(\ln(1-z)-\ln z\right). \qquad (4.184)
\end{aligned}$$

The prototype $\hat\rho_V(1,1,0,1,1;z)$ and other prototypes of that kind

Prototypes with the entries $n_1=n_2=n_5=1$ and one of the other entries vanishing or negative reduce to the nested integrals or the vector integral. The first two are easily obtained using

$$\begin{aligned}
\frac{1}{(4\pi)^D}(m^2)^{D-3-n_4}V(1,1,0,n_4,1) &= \\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(k^2+m^2)(l^2+m^2)((p-l)^2)^{n_4}(k-l)^2} = \\
&= \int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)((p-l)^2)^{n_4}}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2+m^2)(l-k)^2} = \\
&= \frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-2}\int\frac{d^Dl}{(2\pi)^D}\frac{V(1,1;l^2/m^2)}{(l^2+m^2)((p-l)^2)^{n_4}} = \frac{(m^2)^{D-3-n_4}}{(4\pi)^D}V^\varepsilon(1,n_4;p^2/m^2)
\end{aligned}\tag{4.185}$$

so simply $\hat\rho_V(1,1,0,n_4,1;z)=\hat\rho_V^\varepsilon(1,n_4;z)$ and therefore

$$\begin{aligned}
\hat\rho_V(1,1,0,1,1;z) &= \hat\rho_V^\varepsilon(1,1;z) = z^{-\varepsilon}V(1,1;-1)\hat\rho_V(1,1;z)+1-z+\ln z,\\
\hat\rho_V(1,1,0,0,1;z) &= \hat\rho_V^\varepsilon(1,0;z) = 0.
\end{aligned}\tag{4.186}$$

For the last member of this family one has to calculate more,

$$\begin{aligned}
\frac{1}{(4\pi)^D}(m^2)^{D-3}V(1,1,-1,1,1;p^2/m^2) &= \\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{(p-k)^2}{(k^2+m^2)(l^2+m^2)(p-l)^2(k-l)^2} = \\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{(p^2-m^2)+(k^2+m^2)-2pk}{(k^2+m^2)(l^2+m^2)(p-l)^2(k-l)^2} = \\
&= (p^2-m^2)\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(k^2+m^2)(l^2+m^2)(p-l)^2(k-l)^2} + \\
&\qquad -2p_\mu\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{k^\mu}{(k^2+m^2)(l^2+m^2)(p-l)^2(k-l)^2} = \\
&= (p^2-m^2)\frac{1}{(4\pi)^D}(m^2)^{D-4}V(1,1,0,1,1;p^2/m^2) + \\
&\qquad -\frac{2}{(4\pi)^{D/2}}(m^2)^{D/2-2}p_\mu\int\frac{d^Dl}{(2\pi)^D}\frac{l^\mu V'(1,1;l^2/m^2)}{(l^2+m^2)(p-l)^2}
\end{aligned}\tag{4.187}$$

where the vector integral came in. In using Eq. (4.155) one continues with

$$\begin{aligned}
\frac{1}{(4\pi)^D}(m^2)^{D-3}\left(V(1,1,-1,1,1;p^2/m^2)+\left(1-\frac{p^2}{m^2}\right)V(1,1,0,1,1;p^2/m^2)\right) &= \\
= -\frac{2}{(4\pi)^{D/2}}(m^2)^{D/2-2}V'(1,1;-1)p_\mu\int\frac{d^Dl}{(2\pi)^D}\frac{l^\mu}{(l^2+m^2)(p-l)^2} + \\
+\frac{2}{(4\pi)^{D/2}}(m^2)^{D/2-2}\int\frac{d^Dl}{(2\pi)^D}\frac{pl}{(p-l)^2}\int\frac{\rho_V'(1,1;s/m^2)}{(s+l^2)(s-m^2)}ds &= \\
= -\frac{2}{(4\pi)^D}(m^2)^{D/2-4}V'(1,1;-1)p^2V'(1,1;p^2/m^2) + \\
+\frac{2}{(4\pi)^{D/2}}(m^2)^{D/2-2}\int_{m^2}^\infty\frac{\rho_V'(1,1;s/m^2)}{s-m^2}ds\int\frac{d^Dl}{(2\pi)^D}\frac{pl}{(s+l^2)(p-l)^2} &=
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{(4\pi)^D}(m^2)^{D/2-4}p^2V'(1,1;-1)V'(1,1;p^2/m^2) + \\
&\qquad +\frac{2}{(4\pi)^D}(m^2)^{D-4}p^2\int_{m^2}^{\infty}\frac{\rho_V'(1,1;s/m^2)}{s-m^2}V'(1,1;p^2/s)ds
\end{aligned} \tag{4.188}
$$

and thus

$$
\begin{aligned}
&V(1,1,-1,1,1;p^2/m^2) + \left(1-\frac{p^2}{m^2}\right)V(1,1,0,1,1;p^2/m^2) = \\
&= -2\frac{p^2}{m^2}V'(1,1;-1)V'(1,1;p^2/m^2) + 2\frac{p^2}{m^2}\int_{m^2}^{\infty}\frac{\rho_V'(1,1;s/m^2)}{s-m^2}V'(1,1;p^2/s)ds.
\end{aligned} \tag{4.189}
$$

For the spectral density one therefore obtains

$$
\begin{aligned}
&\rho_V(1,1,-1,1,1;s/m^2) + \left(1+\frac{s}{m^2}\right)\rho_V(1,1,0,1,1;s/m^2) = \\
&= \frac{2s}{m^2}V'(1,1;-1)\rho_V'(1,1;s/m^2) - \frac{2s}{m^2}\int_{m^2}^{s}\frac{\rho_V'(1,1;s_1/m^2)}{s_1-m^2}\rho_V'(1,1;s/s_1)ds_1
\end{aligned} \tag{4.190}
$$

or

$$
\begin{aligned}
&\rho_V(1,1,-1,1,1;1/z) + \left(1+\frac{1}{z}\right)\rho_V(1,1,0,1,1;1/z) = \\
&= \frac{2}{z}V'(1,1;-1)\rho_V'(1,1;1/z) - \frac{2}{z}\int_{m^2}^{m^2/z}\frac{\rho_V'(1,1;s_1/m^2)}{s_1-m^2}\rho_V'(1,1;m^2/s_1z)ds_1 = \\
&= \frac{1}{z}(1+z)V'(1,1;-1)\rho_V(1,1;1/z) - \frac{2}{z}\int_z^1\frac{\rho_V'(1,1;1/z_1)}{z_1(1-z_1)}\rho_V'(1,1;z_1/z)dz_1.
\end{aligned} \tag{4.191}
$$

Finally, in using the hat notation,

$$
\begin{aligned}
&\hat{\rho}_V(1,1,-1,1,1;z) + (1+z)\hat{\rho}_V(1,1,0,1,1;z) = \\
&= (1+z)V'(1,1;-1)z^{-\varepsilon}\hat{\rho}_V(1,1;z) - 2\int_z^1\frac{\hat{\rho}_V'(1,1;z_1)}{z_1(1-z_1)}\hat{\rho}_V'(1,1;z/z_1)dz_1.
\end{aligned} \tag{4.192}
$$

The integral in the last expression can be calculated, one obtains

$$
\begin{aligned}
\int_z^1\frac{\hat{\rho}_V'(1,1;z_1)}{z_1(1-z_1)}\hat{\rho}_V'(1,1;z/z_1)dz_1 &= \frac{1}{4}\int_z^1\frac{(1-z_1^2)}{z_1(1-z_1)}\left(1-\frac{z^2}{z_1^2}\right)dz_1 = \\
&= \frac{1}{4}\int_z^1\frac{1+z}{z}\left(1-\frac{z^2}{z_1^2}\right)dz_1 = \frac{1}{8}\left(1-4z+3z^2-2\ln z\right)
\end{aligned} \tag{4.193}
$$

and therefore

$$
\begin{aligned}
\hat{\rho}_V(1,1,-1,1,1;z) &= -(1+z)\rho_V(1,1,0,1,1;z) + \\
&\qquad +(1+z)V'(1,1;-1)z^{-\varepsilon}\hat{\rho}_V(1,1;z) - \frac{1}{4} + z - \frac{3}{4}z^2 + \frac{1}{2}\ln z = \\
&= -(1+z)\rho_V(1,1,0,1,1;z) + \\
&\qquad +\frac{1}{2}(1+z)\frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z) + \frac{1}{2}(1+z)(1-z) - \frac{1}{4} + z - \frac{3}{4}z^2 + \frac{1}{2}\ln z = \\
&= -(1+z)\rho_V(1,1,0,1,1;z) + \frac{1}{2}(1+z)\frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z) + \frac{1}{4} + z - \frac{5}{4}z^2 + \frac{1}{2}\ln z = \\
&= -\frac{1}{2}(1+z)\frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z) - \frac{3}{4} + z - \frac{1}{4}z^2 - \left(\frac{1}{2}+z\right)\ln z.
\end{aligned} \tag{4.194}
$$

The prototype $\hat\rho_V(1,1,1,1,-1;z)$

For this prototype one obtains

$$\begin{aligned}
\lefteqn{\frac{1}{(4\pi)^D}(m^2)^{D-3}V(1,1,1,1,-1;p^2/m^2) \ =}\\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{(k-l)^2}{(k^2+m^2)(l^2+m^2)(p-k)^2(p-l)^2} \ =\\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{(k^2+m^2)+(l^2+m^2)-2kl-2m^2}{(k^2+m^2)(l^2+m^2)(p-k)^2(p-l)^2} \ =\\
&= -2m^2\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(k^2+m^2)(l^2+m^2)(p-k)^2(p-l)^2} \ +\\
&\qquad -2\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{kl}{(k^2+m^2)(l^2+m^2)(p-k)^2(p-l)^2} \ =\\
&= -2m^2\frac{1}{(4\pi)^D}(m^2)^{D-4}V(1,1,1,1,0;p^2/m^2) \ +\\
&\qquad -2\frac{1}{(4\pi)^D}(m^2)^{D-4}p_\mu p^\mu V'(1,1;p^2/m^2)V'(1,1;p^2/m^2),
\end{aligned} \tag{4.195}$$

so

$$V(1,1,1,1,-1;p^2/m^2) = -2V(1,1,1,1,0;p^2/m^2) - 2\frac{p^2}{m^2}V'(1,1;p^2/m^2)^2. \tag{4.196}$$

Next one subtracts the divergence by taking

$$V'(1,1;p^2/m^2) = V'(1,1;-1) + \Delta V'(1,1;p^2/m^2), \tag{4.197}$$

where

$$\begin{aligned}
\Delta V'(1,1;p^2/m^2) \ &= \ V'(1,1;p^2/m^2) - V'(1,1;-1) \ = \qquad (4.198)\\
= -(p^2+m^2)\int_{m^2}^\infty\frac{\rho'_V(1,1;s/m^2)}{(s+p^2)(s-m^2)}ds &= -\frac12(p^2+m^2)\int_{m^2}^\infty\frac{(1-m^4/s^2)ds}{(s+p^2)(s-m^2)} \ =\\
= -\frac12(p^2+m^2)\int_0^1\frac{(1-z^2)dz}{(m^2+p^2z)(1-z)} &= -\frac12\left(1+\frac{m^2}{p^2}\right)\left(1+\left(1-\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2}{m^2}\right)\right).
\end{aligned}$$

Here the result for $\rho'_V(1,1;s/m^2)$ in Eq. (4.154) in $D=4$ space-time dimensions has been used. The squared expression is

$$\begin{aligned}
&\left(\Delta V'(1,1;p^2/m^2)\right)^2 \ = \qquad (4.199)\\
&= \frac14\left(1+\frac{m^2}{p^2}\right)^2\left(1+2\left(1-\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2}{m^2}\right)+\left(1-\frac{m^2}{p^2}\right)^2\ln^2\left(1-\frac{p^2}{m^2}\right)\right).
\end{aligned}$$

With

$$\frac{1}{2\pi i}\,\mathrm{Disc}\,\ln\left(1-\frac{s}{m^2}\right) = -1, \qquad \frac{1}{2\pi i}\,\mathrm{Disc}\,\ln^2\left(1-\frac{s}{m^2}\right) = -2\ln\left(\frac{s}{m^2}-1\right) \tag{4.200}$$

one obtains

$$
\begin{aligned}
&\frac{1}{2\pi i}\,\mathrm{Disc}\;\Delta V'(1,1;-s/m^2) \;=\\
&=\;\frac{1}{2}\left(1-\frac{m^2}{s}\right)\left(1+\frac{m^2}{s}\right)\;=\;\frac{1}{2}\left(1-\frac{m^4}{s^2}\right)=\rho_V'(1,1;s/m^2),\\
&\frac{1}{2\pi i}\,\mathrm{Disc}\left(\Delta V'(1,1;-s/m^2)\right)^2\;=\\
&=\;\frac{1}{4}\left(1-\frac{m^2}{s}\right)^2\left(-2\left(1+\frac{m^2}{s}\right)-2\left(1+\frac{m^2}{s}\right)^2\ln\left(\frac{s}{m^2}-1\right)\right)\;=\\
&=\;-\frac{1}{2}\left(1-\frac{m^2}{s}\right)^2\left(1+\frac{m^2}{s}\right)\left(1+\left(1+\frac{m^2}{s}\right)\ln\left(\frac{s}{m^2}-1\right)\right).
\end{aligned}
\tag{4.201}
$$

The first result was expected. But because one multiplies this quantity with the divergence $V'(1,1;-1)$, one has to keep the exact expression $\rho_V'(1,1;s/m^2)$. Now one has

$$
\begin{aligned}
&V(1,1,1,1,-1;p^2/m^2)+2V(1,1,1,1,0;p^2/m^2)\;=\\
&=\;-2\frac{p^2}{m^2}V'(1,1;-1)^2-4\frac{p^2}{m^2}V'(1,1;-1)\Delta V'(1,1;p^2/m^2)-2\frac{p^2}{m^2}\left(\Delta V'(1,1;p^2/m^2)\right)^2
\end{aligned}
\tag{4.202}
$$

and therefore

$$
\begin{aligned}
&\rho_V(1,1,1,1,-1;s/m^2)+2\rho_V(1,1,1,1,0;s/m^2)\;=\\
&=\;\frac{4s}{m^2}V'(1,1;-1)\rho_V'(1,1;s/m^2)\;+\\
&\qquad-\frac{s}{m^2}\left(1-\frac{m^2}{s}\right)^2\left(1+\frac{m^2}{s}\right)\left(1+\left(1+\frac{m^2}{s}\right)\ln\left(\frac{s}{m^2}-1\right)\right).
\end{aligned}
\tag{4.203}
$$

or (using $\rho_V'(1,1;1/z)=(1+z)\rho_V(1,1;1/z)/2$)

$$
\begin{aligned}
\rho_V(1,1,1,1,-1;1/z)\;&=\;-2\rho_V(1,1,1,1,0;1/z)+\\
&+\frac{2}{z}(1+z)V'(1,1;-1)\rho_V(1,1;1/z)-\frac{1}{z}(1-z)^2(1+z)\left(1+(1+z)\ln\left(\frac{1}{z}-1\right)\right).
\end{aligned}
\tag{4.204}
$$

Finally, in the hat notation, one arrives at

$$
\begin{aligned}
\hat\rho_V(1,1,1,1,-1;z)\;&=\;-2z\hat\rho_V(1,1,1,1,0;z)+\\
&+2(1+z)V'(1,1;-1)z^{-\varepsilon}\hat\rho_V(1,1;z)-(1-z)^2(1+z)\left(1+(1+z)\ln\left(\frac{1}{z}-1\right)\right)\;=\\
&=\;-2z\hat\rho_V(1,1,1,1,0;z)+(1+z)\frac{G}{\varepsilon}(1+\varepsilon)z^{-\varepsilon}\hat\rho_V(1,1;z)\;+\\
&\qquad-1+z+z^2-z^3-(1-2z^2+z^4)\left(\ln(1-z)-\ln z\right)\;=\\
&=\;(1-3z)\frac{G}{\varepsilon}z^{-\varepsilon}\hat\rho_V(1,1;z)+z-z^3-(1-4z+6z^2-4z^3+z^4)\left(\ln(1-z)-\ln z\right)\;=
\end{aligned}
\tag{4.205}
$$

$$
=\;(1-3z)\frac{G}{\varepsilon}z^{-\varepsilon}\hat\rho_V(1,1;z)+(1-z^2)z-(1-z)^4\left(\ln(1-z)-\ln z\right).
\tag{4.206}
$$

The prototype $\hat\rho_V(0,1,1,1,1;z)$ and other prototypes of that kind

If one of the first two entries vanishes, the massless contribution can be integrated first. In general one has

$$\begin{aligned}
&\frac{1}{(4\pi)^D}(m^2)^{D-n_2-n_3-n_4-n_5}V(0,n_2,n_3,n_4,n_5;p^2/m^2) \;=\\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)^{n_2}((p-k)^2)^{n_3}((p-l)^2)^{n_4}((k-l)^2)^{n_5}} \;=\\
&= \int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)^{n_2}((p-l)^2)^{n_4}}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{((p-k)^2)^{n_3}((k-l)^2)^{n_5}} \;=\\
&= \frac{1}{(4\pi)^{D/2}}G(n_3,n_5)\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)^{n_2}((p+l)^2)^{n_3+n_4+n_5-D/2}} \;=\\
&= \frac{1}{(4\pi)^D}(m^2)^{D/2-n_2-n_3-n_4-n_5+D/2}G(n_3,n_5)V(n_2,n_3+n_4+n_5-D/2;p^2/m^2) \qquad (4.207)
\end{aligned}$$

and therefore

$$\rho_V(0,n_2,n_3,n_4,n_5;s/m^2) \;=\; G(n_3,n_5)\rho_V(n_2,n_3+n_4+n_5-D/2;s/m^2), \quad (4.208)$$

$$\hat\rho_V(0,n_2,n_3,n_4,n_5;z) \;=\; G(n_3,n_5)\hat\rho_V(n_2,n_3+n_4+n_5-D/2;z). \qquad (4.209)$$

For the prototypes in question one obtains

$$\begin{aligned}
\hat\rho_V(0,1,1,1,1;z) &= G(1,1)\hat\rho_V(1,1+\varepsilon;z),\\
\hat\rho_V(0,1,1,0,1;z) &= G(1,1)\hat\rho_V(1,\varepsilon;z),\\
\hat\rho_V(0,1,1,-1,1;z) &= G(1,1)\hat\rho_V(1,\varepsilon-1;z). \qquad (4.210)
\end{aligned}$$

The last member of this family, $\hat\rho_V(-1,1,1,0,1;z)$ is more complicated again,

$$\begin{aligned}
\frac{1}{(4\pi)^D}(m^2)^{D-2}V(-1,1,1,0,1;p^2/m^2) \;&=\; \int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)}\int\frac{d^Dk}{(2\pi)^D}\frac{k^2+m^2}{(p-k)^2(k-l)^2} \;=\\
&= \int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)}\int\frac{d^Dk}{(2\pi)^D}\frac{(p-k)^2+m^2}{k^2(p-l-k)^2} \;=\\
&= (p^2+m^2)\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(p-l-k)^2} \;+\\
&\qquad -2p_\mu\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)}\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu}{k^2(p-l-k)^2} \;+\\
&\qquad +\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(p-l-k)^2}. \qquad (4.211)
\end{aligned}$$

The inner integral of the last part vanishes because of the scaling rule, the one in the first part is proportional to $G(1,1)$. The vector integral in the second part can be expanded in the outer momentum $(p-l)$. Therefore, one proceeds with

$$\frac{1}{(4\pi)^D}(m^2)^{D-2}V(-1,1,1,0,1;p^2/m^2) \;=$$

$$
\begin{aligned}
= \;& \frac{1}{(4\pi)^{D/2}}(p^2+m^2)G(1,1)\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+m^2)(p-l)^{2-D/2}} \;+ \\
& -\int\frac{d^Dl}{(2\pi)^D}\frac{p^2-pl}{(l^2+m^2)(p-l)^2}\int\frac{d^Dk}{(2\pi)^D}\frac{2(p-l)k}{k^2(p-l-k)^2} \;= \\
= \;& \frac{1}{(4\pi)^D}(m^2)^{D/2-3+D/2}(p^2+m^2)G(1,1)V(1,2-D/2;p^2/m^2) \;+ \\
& -\int\frac{d^Dl}{(2\pi)^D}\frac{p^2-pl}{(l^2+m^2)(p-l)^2}\int\frac{d^Dk}{(2\pi)^D}\frac{(p-l)^2+k^2-(p-l-k)^2}{k^2(p-l-k)^2} \;= \\
= \;& \frac{1}{(4\pi)^D}(m^2)^{D-2}\left(1+\frac{p^2}{m^2}\right)G(1,1)V(1,2-D/2;p^2/m^2) \;+ \\
& -\int\frac{d^Dl}{(2\pi)^D}\frac{p^2-pl}{(l^2+m^2)}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(p-l-k)^2} \;+ \\
& -\int\frac{d^Dl}{(2\pi)^D}\frac{p^2-pl}{(l^2+m^2)(p-l)^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2} \;+ \\
& +\int\frac{d^Dl}{(2\pi)^D}\frac{p^2-pl}{(l^2+m^2)(p-l)^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(p-l-k)^2}
\end{aligned}
\tag{4.212}
$$

where the last two parts vanish again because of the scaling rule. Again one proceeds with the first two integrals to obtain

$$
\begin{aligned}
& \frac{1}{(4\pi)^D}(m^2)^{D-2}V(-1,1,1,0,1;p^2/m^2) \;= \\
= \;& \frac{1}{(4\pi)^D}(m^2)^{D-2}\left(1+\frac{p^2}{m^2}\right)G(1,1)V(1,2-D/2;p^2/m^2) \;+ \\
& -\frac{1}{2}\int\frac{d^Dl}{(2\pi)^D}\frac{p^2+m^2-(l^2+m^2)+(p-l)^2}{(l^2+m^2)}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(p-l-k)^2} \;= \\
= \;& \frac{1}{(4\pi)^D}(m^2)^{D-2}\left(1+\frac{p^2}{m^2}\right)G(1,1)V(1,2-D/2;p^2/m^2) \;+ \\
& -\frac{1}{2(4\pi)^{D/2}}G(1,1)\int\frac{d^Dl}{(2\pi)^D}\frac{p^2+m^2-(l^2+m^2)+(p-l)^2}{(l^2+m^2)(p-l)^{2-D/2}} \;= \\
= \;& \frac{1}{(4\pi)^D}(m^2)^{D-2}\left(1+\frac{p^2}{m^2}\right)G(1,1)V(1,2-D/2;p^2/m^2) \;+ \\
& -\frac{1}{2(4\pi)^D}G(1,1)\Big[(m^2)^{D/2-3+D/2}(p^2+m^2)V(1,2-D/2;p^2/m^2) \;+ \\
& \qquad -(m^2)^{D/2-2+D/2}V(0,2-D/2;p^2/m^2) \;+ \\
& \qquad +(m^2)^{D/2-2+D/2}V(1,1-D/2;p^2/m^2)\Big] \;= \\
= \;& \frac{1}{(4\pi)^D}(m^2)^{D-2}\left(1+\frac{p^2}{m^2}\right)G(1,1)V(1,2-D/2;p^2/m^2) \;+ \\
& -\frac{1}{2(4\pi)^D}(m^2)^{D-2}G(1,1)\Big[\left(1+\frac{p^2}{m^2}\right)V(1,2-D/2;p^2/m^2) \;+ \\
& \qquad -V(0,2-D/2;p^2/m^2)+V(1,1-D/2;p^2/m^2)\Big]
\end{aligned}
\tag{4.213}
$$

such that finally

$$\begin{aligned} V(-1,1,1,0,1;p^2/m^2) &= G(1,1)\Bigg[\left(1+\frac{p^2}{m^2}\right)V(1,2-D/2;p^2/m^2)+ \\ &\quad -\frac{1}{2}\left(1+\frac{p^2}{m^2}\right)V(1,2-D/2;p^2/m^2)-\frac{1}{2}V(1,1-D/2;p^2/m^2)\Bigg] = \\ &= \frac{1}{2}G(1,1)\Bigg[\left(1+\frac{p^2}{m^2}\right)V(1,2-D/2;p^2/m^2)-V(1,1-D/2;p^2/m^2)\Bigg]. \end{aligned} \tag{4.214}$$

The spectral density of this integral is given by

$$\rho_V(-1,1,1,0,1;1/z)=\frac{1}{2}G(1,1)\left(\left(1-\frac{1}{z}\right)\rho_V(1,\varepsilon;1/z)-\rho_V(1,\varepsilon-1;1/z)\right) \tag{4.215}$$

or

$$\hat\rho_V(-1,1,1,0,1;z)=-\frac{1}{2}G(1,1)\left((1-z)\hat\rho_V(1,\varepsilon;z)+\hat\rho_V(1,\varepsilon-1;z)\right) \tag{4.216}$$

The different members of the family can be calculated by using

$$\begin{aligned} \hat\rho_V(1,\varepsilon-1;z) &= \frac{1}{\Gamma(\varepsilon-1)\Gamma(3-2\varepsilon)}\int_z^1(1-x)^{2-2\varepsilon}x^{\varepsilon-2}(x-z)^{2-2\varepsilon}dx, \\ \hat\rho_V(1,\varepsilon;z) &= \frac{1}{\Gamma(\varepsilon)\Gamma(2-2\varepsilon)}\int_z^1(1-x)^{1-2\varepsilon}x^{\varepsilon-1}(x-z)^{1-2\varepsilon}dx, \\ \hat\rho_V(1,\varepsilon+1;z) &= \frac{1}{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)}\int_z^1(1-x)^{-2\varepsilon}x^{\varepsilon}(x-z)^{-2\varepsilon}dx. \end{aligned} \tag{4.217}$$

Only the first member of this family contains a singularity. This singularity can be extracted by a subtraction method. Using $G(1,1)=G/\varepsilon$, one models the singular part in the same way as in the previous cases, namely one gets it to be proportional to $z^{-\varepsilon}$,

$$G(1,1)\hat\rho_V(1,\varepsilon+1;z)=\frac{G}{\varepsilon}z^{-\varepsilon}\hat\rho_V(1,1;z)+\frac{G}{\varepsilon}\left(\hat\rho_V(1,\varepsilon+1;z)-z^{-\varepsilon}\hat\rho_V(1,1;z)\right) \tag{4.218}$$

With

$$\hat\rho_V(1,1;z)=\frac{1}{\Gamma(1-\varepsilon)}\int_z^1(1-x)^{-\varepsilon}(x-z)^{-\varepsilon}dx \tag{4.219}$$

one has (using $\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)=\Gamma(1-\varepsilon)+O(\varepsilon^2)$)

$$\begin{aligned} &\frac{1}{\varepsilon}\left(\hat\rho_V(1,1+\varepsilon;z)-z^{-\varepsilon}\rho_V(1,1;z)\right) = \\ &\quad = \frac{1}{\varepsilon}\int_z^1(1-x)^{-\varepsilon}(x-z)^{-\varepsilon}\left((1-x)^{-\varepsilon}x^{\varepsilon}(x-z)^{-\varepsilon}-z^{-\varepsilon}\right)dx+O(\varepsilon) = \\ &\quad = \int_z^1\left(\ln z-\ln(1-x)+\ln x-\ln(x-z)\right)dx+O(\varepsilon) = \\ &\quad = 1-z+(1-2z)\ln z-2(1-z)\ln(1-z)+O(\varepsilon), \end{aligned} \tag{4.220}$$

such that

$$\hat\rho_V(0,1,1,1,1;z)=\frac{G}{\varepsilon}z^{-\varepsilon}\hat\rho_V(1,1;z)+1-z+(1-2z)\ln z-2(1-z)\ln(1-z)+O(\varepsilon). \tag{4.221}$$

The complete list of prototypes of this paragraph is given by

$$
\begin{aligned}
\hat{\rho}_V(0,1,1,-1,1;z) &= G(1,1)\hat{\rho}_V(1,\varepsilon-1;z) \;=\; -\frac{1}{2}\int_z^1 \frac{(1-x)^2(x-z)^2}{x^2}dx \;= \\
&= -\frac{1}{2}\left(\frac{1}{3}+3z-3z^2-\frac{1}{3}z^3+2z(1+z)\ln z\right) \;=:\; -\frac{1}{2}\hat{f}_1(z), \qquad (4.222)\\
\hat{\rho}_V(0,1,1,0,1;z) &= G(1,1)\hat{\rho}_V(1,\varepsilon;z) \;= \\
&= \int_z^1 \frac{(1-x)(x-z)}{x}dx \;=\; \frac{1-z^2}{2}+z\ln z \;=:\; \hat{f}_2(z), \qquad (4.223)\\
\hat{\rho}_V(0,1,1,1,1;z) &= \frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z)+1-z+(1-2z)\ln z-2(1-z)\ln(1-z), \quad (4.224)\\
\hat{\rho}_V(-1,1,1,0,1;z) &= -\frac{1}{2}(1-z)\hat{f}_1(z)+\frac{1}{4}\hat{f}_2(z) \;=\; -\frac{1}{6}+z-\frac{1}{2}z^2-\frac{1}{3}z^3+z^2\ln z \qquad (4.225)
\end{aligned}
$$

where the finite parts are given up to $O(\varepsilon^0)$.

4.4.3 Tabulating all prototypes for the fish

Before they will be combined, all prototypes are listed in this subsection, starting from the most complicated one, the proper fish prototype, to the vanishing ones. One has

$$
\begin{aligned}
\hat{\rho}_V(1,1,1,1,1;z) &= 4\left(\mathrm{Li}_2(z)+\frac{1}{2}\ln(1-z)\ln z\right),\\
\hat{\rho}_V(1,1,1,1,0;z) &= 2\frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z)-2(1-z)^2\left(\ln(1-z)-\ln z\right),\\
\hat{\rho}_V(1,1,1,1,-1;z) &= (1-3z)\frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z)+(1-z^2)z-(1-z)^4\left(\ln(1-z)-\ln z\right),\\
\hat{\rho}_V(1,1,1,0,0;z) &= V(1,0;-1)z^{1-\varepsilon}\hat{\rho}(1,1;z) \;=\; -\frac{G}{\varepsilon}z^{1-\varepsilon}\rho_V(1,1;z)+(1-z)z,\\
\hat{\rho}_V(1,1,-1,1,1;z) &= -\frac{1}{2}(1+z)\frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z)-\frac{3}{4}+z-\frac{1}{4}z^2-\left(\frac{1}{2}+z\right)\ln z,\\
\hat{\rho}_V(1,1,0,1,1;z) &= \frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z)+1-z+\ln z,\\
\hat{\rho}_V(0,1,1,1,1;z) &= \frac{G}{\varepsilon}z^{-\varepsilon}\hat{\rho}_V(1,1;z)+1-z+(1-2z)\ln z-2(1-z)\ln(1-z),\\
\hat{\rho}_V(0,1,1,0,1;z) &= \frac{1-z^2}{2}+z\ln z,\\
\hat{\rho}_V(0,1,1,-1,1;z) &= -\frac{1}{2}\left(\frac{1}{3}+3z-3z^2-\frac{1}{3}z^3+2z(1+z)\ln z\right),\\
\hat{\rho}_V(-1,1,1,0,1;z) &= -\frac{1}{6}+z-\frac{1}{2}z^2-\frac{1}{3}z^3+z^2\ln z \qquad (4.226)
\end{aligned}
$$

while

$$
\begin{aligned}
\hat{\rho}_V(1,1,0,0,1;z) &= \hat{\rho}_V(0,1,1,1,0;z) \;=\; \hat{\rho}_V(0,1,0,1,1;z) \;=\; \hat{\rho}_V(0,0,1,1,1;z) \;=\; 0,\\
\hat{\rho}_V(1,1,0,0,0;z) &= \hat{\rho}_V(0,1,1,0,0;z) \;=\; \hat{\rho}_V(0,0,1,1,0;z) \;=\; \hat{\rho}_V(0,0,0,1,1;z) \;=\; 0,\\
\hat{\rho}_V(0,1,0,0,1;z) &= \hat{\rho}_V(0,1,0,1,0;z) \;=\; 0. \qquad (4.227)
\end{aligned}
$$

The list in terms of dimensional spectral densities

A second representation of the prototypes shall be given in terms of the dimensional spectral densities $\tilde{\rho}_V(n_1, n_2, n_3, n_4, n_5; s)$. Here one has $(z = m^2/s)$

$$\begin{aligned}
\tilde{\rho}_V(1,1,1,1,1;s) &= \frac{4}{s}\left(\mathrm{Li}_2(z) + \frac{1}{2}\ln(1-z)\ln z\right), \\
\tilde{\rho}_V(1,1,1,1,-1;s) &= -2m^2\tilde{\rho}_V(1,1,1,1,0;s) + \\
&\quad +(s+m^2)\Pi_B(m^2)\rho_B(s) + s(1-z^2)\left(z - (1-z^2)\ln\left(\frac{z}{1-z}\right)\right), \\
\tilde{\rho}_V(1,1,1,1,0;s) &= 2\Pi_B(m^2)\rho_B(s) + 2(1-z)^2\ln\left(\frac{z}{1-z}\right), \\
\tilde{\rho}_V(1,1,1,0,0;s) &= -m^2\Pi_B(m^2)\rho_B(s) + m^2(1-z), \\
\tilde{\rho}_V(1,1,0,1,1;s) &= \Pi_B(m^2)\rho_B(s) + 1 - z + \ln z, \\
\tilde{\rho}_V(1,1,-1,1,1;s) &= -(s+m^2)\tilde{\rho}_V(1,1,0,1,1;s) + \\
&\quad +\frac{1}{2}(s+m^2)\Pi_B(m^2)\rho_B(s) + s\left(\frac{1}{4} + z - \frac{5}{4}z^2 + \frac{1}{2}\ln z\right), \\
\tilde{\rho}_V(0,1,1,1,1;s) &= \Pi_B(m^2)\rho_B(s) + 1 - z + (1-2z)\ln z - 2(1-z)\ln(1-z), \\
\tilde{\rho}_V(0,1,1,0,1;s) &= s\left(\frac{1}{2} - \frac{1}{2}z^2 + z\ln z\right), \\
\tilde{\rho}_V(0,1,1,-1,1;s) &= -\frac{1}{2}s^2\left(\frac{1}{3} + 3z - 3z^2 - \frac{1}{3}z^3 + 2z(1+z)\ln z\right), \\
\tilde{\rho}_V(-1,1,1,0,1;s) &= s^2\left(-\frac{1}{6} + z - \frac{1}{2}z^2 - \frac{1}{3}z^3 + z^2\ln z\right)
\end{aligned} \tag{4.228}$$

and

$$\begin{aligned}
\tilde{\rho}_V(1,1,0,0,1;s) &= \tilde{\rho}_V(0,1,1,1,0;s) = \tilde{\rho}_V(0,1,0,1,1;s) = \tilde{\rho}_V(0,0,1,1,1;s) = 0, \\
\tilde{\rho}_V(1,1,0,0,0;s) &= \tilde{\rho}_V(0,1,1,0,0;s) = \tilde{\rho}_V(0,0,1,1,0;s) = \tilde{\rho}_V(0,0,0,1,1;s) = 0, \\
\tilde{\rho}_V(0,1,0,1,0;s) &= \tilde{\rho}_V(0,1,0,0,1;s) = 0.
\end{aligned} \tag{4.229}$$

4.5 The general renormalization procedure

It turns out that the method developed for the light contribution and the massive self energy contribution to extract and combine the singularities in a renormalization factor also works for the fish contribution. Therefore, it can be used for the whole contribution. The following considerations will give a general outline of this procedure. One starts with the assumption that before the convolution is done, the spectral density is given by

$$\tilde{\rho} = \Pi_B(-m^2)\tilde{\rho}^s + \tilde{\rho}^f \tag{4.230}$$

where the singularity is parametrized by $\Pi_B(-m^2)$, and $\tilde{\rho}^s$ is called the singular and $\tilde{\rho}^f$ the finite part. Note that one has $\Pi_B(-m^2)\varepsilon = 1 + O(\varepsilon^2)$. If the convolution function λ

is split up into a zeroth and first order term according to $\lambda = \lambda^0 + \varepsilon\lambda^1$, the product of both results in

$$\lambda\tilde{\rho} = \Pi_B(-m^2)\lambda^0\tilde{\rho}^s + \Pi_B(-m^2)\varepsilon\lambda^1\tilde{\rho}^s + \lambda^0\tilde{\rho}^f = \Pi_B(-m^2)\lambda^0\tilde{\rho}^s + \lambda^1\tilde{\rho}^s + \lambda^0\tilde{\rho}^f. \quad (4.231)$$

The integration (convolution) therefore results in

$$\rho = \int \lambda\tilde{\rho} = \Pi_B(-m^2)\int \lambda^0\tilde{\rho}^s + \int(\lambda^1\tilde{\rho}^s + \lambda^0\tilde{\rho}^f) = \Pi_B(-m^2)\rho^s + \rho^f \quad (4.232)$$

so that

$$\rho^s = \int \lambda^0\tilde{\rho}^s, \qquad \rho^f = \int(\lambda^1\tilde{\rho}^s + \lambda^0\tilde{\rho}^f). \quad (4.233)$$

If $\rho_0 = \rho_0^0 + \varepsilon\rho_0^1$ is the leading order spectral density, the bare spectral density to first order is given by $\rho^B = \rho_0 + \rho$. This quantity has to be renormalized. This is done by multiplying it with the inverse of the renormalization factor $Z = 1 + \Pi_B(-m^2)Z_1$ to obtain

$$\begin{aligned}
\rho^R &= Z^{-1}\rho^B = \left(1 - \Pi_B(-m^2)Z_1\right)\left(\rho_0^0 + \varepsilon\rho_0^1 + \Pi_B(-m^2)\rho^s + \rho^f\right) = \\
&= \rho_0^0 - \Pi_B(-m^2)Z_1\rho_0^0 - \Pi_B(-m^2)Z_1\varepsilon\rho_0^1 + \Pi_B(-m^2)\rho^s + \rho^f = \\
&= \rho_0^0 + \Pi_B(-m^2)\left(\rho^s - Z_1\rho_0^0\right) + \rho^f - Z_1\rho_0^1. \qquad (4.234)
\end{aligned}$$

If the renormalization is multiplicative, the coefficient Z_1 is such that $Z_1\rho_0^0 = \rho^s$, and with this Z_1 one then obtains $\rho^R = \rho_0^0 + \rho^f - Z_1\rho_0^1 = \rho_0 + \rho_1$. The renormalization factor can even be a part of the whole factor (and the coefficient Z_1 a part of the sum) if ρ is a part of the first order correction (e.g. the fish contribution) only.

A more elegant method is possible if the singular part $\tilde{\rho}^s$ of the first order correction is proportional to $\tilde{\rho}_0$ where $\rho_0 = \int \lambda\tilde{\rho}_0$. In this case one can collect for a renormalization factor,

$$\tilde{\rho}_0 + \Pi_B(-m^2)\tilde{\rho}^s = Z\tilde{\rho}_0. \quad (4.235)$$

Then the renormalization is already possible before the convolution takes place, the leading order term need not to be expanded, and the finite first order contribution will not be changed,

$$\tilde{\rho}^B = Z\tilde{\rho}_0 + \tilde{\rho}^f, \qquad \tilde{\rho}^R = Z^{-1}\tilde{\rho}^B = \tilde{\rho}_0 + \tilde{\rho}^f. \quad (4.236)$$

Both methods can be compared if one convolutes the latter result with λ. One then has

$$\begin{aligned}
\rho^B &= \int \lambda(Z\tilde{\rho}_0 + \tilde{\rho}^f) = Z\rho_0 + \int \lambda\tilde{\rho}^f = Z\rho_0 + \int \lambda^0\tilde{\rho}^f, \qquad (4.237)\\
\rho^R &= \rho_0 + \int \lambda^0\tilde{\rho}^f \stackrel{!}{=} \rho_0 + \int(\lambda^1\tilde{\rho}^s + \lambda^0\tilde{\rho}^f) - Z_1\rho_0^1 = \rho_0 + \rho^f - Z_1\rho_0^1.
\end{aligned}$$

Together with the condition for the vanishing of the singular part one obtains

$$Z_1\rho_0^0 = \rho^s = \int \lambda^0\tilde{\rho}^s, \quad Z_1\rho_0^1 = \rho^f - \int \lambda^0\tilde{\rho}^f = \int \lambda^1\tilde{\rho}^s \quad \Rightarrow \quad Z_1\rho_0 = \int \lambda\tilde{\rho}^s. \quad (4.238)$$

This equality can be checked for the mass part and turns out to be valid. This will be shown here in the exact ε dependence. The coefficient of the divergence is given by

$$\tilde{\rho}^s(m^2/z_1) = 4(1-z_1)\times z_1^{-\varepsilon}\hat{\rho}_V(1,1;z_1)\times(m^2/z_1)^{1-2\varepsilon} = 4(m^2)^{1-2\varepsilon}G\frac{(1-z_1)^{2-2\varepsilon}z_1^{\varepsilon-1}}{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)} \quad (4.239)$$

where the different factor (separated by "×") come from (1) the coefficient of the divergent part, (2) the divergence itself (up to $\Pi_B(-m^2)$), and (3) the common factor $s_1^{D-3}=s_1^{1-2\varepsilon}$ of $\tilde\rho^q_{c21}(s_1)$. As next, the convolution function is given by (cf. Eq. (4.88))

$$\begin{aligned}\lambda(s,s_1) &= \frac{s^{-\varepsilon}}{2(4\pi)^{2-\varepsilon}}\left(\frac{s}{s_1}-1\right)\hat\rho_V(1,1;s_1/s) \quad\Rightarrow\\ \lambda\left(\frac{m^2}{z},\frac{m^2}{z_1}\right) &= \frac{(m^2)^{-\varepsilon}z^\varepsilon}{2(4\pi)^{2-\varepsilon}}\left(\frac{z_1}{z}-1\right)\hat\rho_V(1,1;z/z_1)\ =\\ &= \frac{(m^2)^{-\varepsilon}z^\varepsilon}{2(4\pi)^{2-\varepsilon}}\left(\frac{z_1}{z}-1\right)\frac{G}{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}\left(1-\frac{z}{z_1}\right)^{1-2\varepsilon}\ =\\ &= \frac{(m^2)^{-\varepsilon}G}{2(4\pi)^{2-\varepsilon}}\frac{z^{\varepsilon-1}(z_1-z)^{2-2\varepsilon}z_1^{2\varepsilon-1}}{\Gamma(1-\varepsilon)(1+\varepsilon)}.\end{aligned}\tag{4.240}$$

For the convolution one therefore obtains

$$\begin{aligned}\rho(m^2/z) &= \int\lambda(m^2/z,s_1)\tilde\rho^s(s_1)ds_1 \ =\ \int_z^1\lambda\left(\frac{m^2}{z},\frac{m^2}{z_1}\right)\tilde\rho^s\left(\frac{m^2}{z_1}\right)\frac{dz_1}{z_1^2}\ =\\ &= \frac{2(m^2)^{2-3\varepsilon}z^{\varepsilon-1}G^2}{(4\pi)^{2-\varepsilon}\Gamma(1+\varepsilon)^2\Gamma(1-\varepsilon)^2}\int_z^1(1-z_1)^{2-2\varepsilon}z_1^{3\varepsilon-4}(z_1-z)^{2-2\varepsilon}dz_1.\end{aligned}\tag{4.241}$$

This appears not to be proportional to ρ_0. However, one can perform a conformal transformation $z_1\to z/z_1$ and obtains

$$\begin{aligned}\rho(m^2/z) &= \frac{2(m^2)^{2-3\varepsilon}z^{\varepsilon-1}G^2}{(4\pi)^{2-\varepsilon}\Gamma(1-\varepsilon)^2\Gamma(1-\varepsilon)^2}\int_z^1\left(1-\frac{z}{z_1}\right)^{2-2\varepsilon}\left(\frac{z}{z_1}\right)^{3\varepsilon-4}\left(\frac{z}{z_1}-z\right)^{2-2\varepsilon}z\frac{dz_1}{z_1^2}\ =\\ &= \frac{2(m^2)^{2-3\varepsilon}z^{\varepsilon-1+3\varepsilon-4+2-2\varepsilon+1}G^2}{(4\pi)^{2-2\varepsilon}\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)^2}\int_z^1(z_1-z)^{2-2\varepsilon}z_1^{-2+2\varepsilon+4-3\varepsilon-2+2\varepsilon-2}(1-z_1)^{2-2\varepsilon}dz_1\ =\\ &= \frac{2(m^2)^{2-3\varepsilon}z^{2\varepsilon-2}G^2}{(4\pi)^{2-2\varepsilon}\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)^2}\int_z^1(z_1-z)^{2-2\varepsilon}z_1^{\varepsilon-2}(1-z_1)^{2-2\varepsilon}dz_1.\end{aligned}\tag{4.242}$$

This has to be compared with $\rho_0(s)=\rho^m_{a1}(s)$,

$$\rho_0(m^2/z)=-\frac{2(m^2/z)^{2-2\varepsilon}G}{(4\pi)^{2-\varepsilon}}C_1\hat g_1(z)\tag{4.243}$$

where $\hat g_n(z)$ and C_n are defined in Eq. (4.38). Now one has

$$\begin{aligned}C_1^{-1} &= \varepsilon\Gamma(\varepsilon-1)\Gamma(3-2\varepsilon)\ =\ \varepsilon\Gamma(\varepsilon-1)(2-2\varepsilon)\Gamma(2-2\varepsilon)\ =\\ &= -2\varepsilon(\varepsilon-1)\Gamma(\varepsilon-1)\Gamma(2-2\varepsilon)\ =\ -2\Gamma(1+\varepsilon)\Gamma(2-2\varepsilon)\ =\\ &= -2\Gamma(1+\varepsilon)^2\Gamma(1-\varepsilon)^2/G\end{aligned}\tag{4.244}$$

(cf. Eq. (4.5)) and therefore

$$\rho_0(m^2/z)=\frac{(m^2/z)^{2-2\varepsilon}G^2}{(4\pi)^{2-2\varepsilon}\Gamma(1+\varepsilon)^2\Gamma(1-\varepsilon)^2}\int_z^1(1-z_1)^{2-2\varepsilon}z_1^{\varepsilon-2}(z_1-z)^{2-2\varepsilon}dz_1.\tag{4.245}$$

Therefore, one finally obtains

$$\rho(m^2/z)=2(m^2)^{-\varepsilon}\rho_0(m^2/z).\tag{4.246}$$

4.5.1 The final result for the mass part

The final result for the mass part reads

$$\rho_m(s) = \frac{1}{128\pi^4}\rho^m(s), \qquad \rho^m(s) = s^2\left\{\rho_0^m(s)\left(1+\frac{\alpha_s}{\pi}\ln\left(\frac{\mu^2}{m^2}\right)\right)+\frac{\alpha_s}{\pi}\rho_1^m(s)\right\} \quad (4.247)$$

where

$$\rho_0^m(m^2/z) = 1+9z-9z^2-z^3+6z(1+z)\ln z, \quad (4.248)$$

$$\begin{aligned}\rho_1^m(m^2/z) &= 9+\frac{665}{9}z-\frac{665}{9}z^2-9z^3-\left(\frac{58}{9}+42z-42z^2-\frac{58}{9}z^3\right)\ln(1-z)+\\ &+\left(2+\frac{154}{3}z-\frac{22}{3}z^2-\frac{58}{9}z^3\right)\ln z+4\left(\frac{1}{3}+3z-3z^2-\frac{1}{3}z^3\right)\ln(1-z)\ln z\ +\\ &+12z\left(2+3z+\frac{1}{9}z^2\right)\left(\frac{1}{2}\ln^2 z-\zeta(2)\right)+4\left(\frac{2}{3}+12z+3z^2-\frac{1}{3}z^3\right)\mathrm{Li}_2(z)\ +\\ &+24z(1+z)\left(\mathrm{Li}_3(z)-\zeta(3)-\frac{1}{3}\mathrm{Li}_2(z)\ln z\right). \end{aligned} \quad (4.249)$$

4.5.2 The final result for the momentum part

For the momentum part one has to work harder to come to a final result. The reason is three-fold. First, the leading order term consists of a combination of two (or three) fundamental spectral densities. Besides this, however, the light and massive contribution can be worked on like in the mass part calculation because the unintegrated singular parts are proportional to the unintegrated leading order contribution. The two other reasons are related to the fish contributions. First, the vector structure is replaced by the tensor structure, therefore there are two instead of one contribution which have to be convoluted. Second and more serious, the unintegrated singular part of the fish is *not* proportional to the unintegrated leading order contribution. But one can now profit from the thorough considerations at the beginning of this section and use the integrated subtraction which works because the integrated singular part is proportional to the leading order term.

To conclude, the result for the momentum part is given by the same general structure as for the mass part,

$$\rho_q(s) = \frac{1}{128\pi^4}\rho^q(s), \qquad \rho^q(s) = s^2\left\{\rho_0^q(s)\left(1+\frac{\alpha_s}{\pi}\ln\left(\frac{\mu^2}{m^2}\right)\right)+\frac{\alpha_s}{\pi}\rho_1^q(s)\right\} \quad (4.250)$$

where

$$\rho_0^q(s) = \frac{1}{4}-2z+2z^3-\frac{1}{4}z^4-3z^2\ln z, \quad (4.251)$$

$$\begin{aligned}\rho_1^q(s) &= \frac{1}{4}\left\{\frac{71}{12}-\frac{565}{9}z-\frac{7}{2}z^2+\frac{625}{9}z^3-\frac{109}{12}z^4+\right.\\ &-\frac{1}{9}\left(49-464z+464z^3-49z^4\right)\ln(1-z)+\left(1-\frac{68}{3}z-44z^2+\frac{452}{9}z^3-\frac{49}{9}z^4\right)\ln z\ +\\ &+\frac{4}{3}\left(1-8z+8z^3-z^4\right)\ln(1-z)\ln z+z^2\left(54+8z-z^2\right)\left(\frac{2}{9}\pi^2-\frac{2}{3}\ln^2 z\right)+\\ &\left.+\frac{4}{3}\left(2-16z-54z^2+8z^3-z^4\right)\mathrm{Li}_2(z)-16z^2\left(3\,\mathrm{Li}_3(z)-3\zeta(3)-\mathrm{Li}_2(z)\ln(z)\right)\right\}. \end{aligned} \quad (4.252)$$

4.6 Comparison with different QCD limits

The results given in Eqs. (4.249) and (4.252) represent the full next-to-leading order solution. Since the anomalous dimension of the current in Eq. (4.1) is known up to two-loop order [138], the results shown in Eqs. (4.249) and (4.252) complete the ingredients necessary for an analysis of the correlator in Eq. (4.2) within an operator product expansion at the next-to-leading order level.

The next step is the analysis Eqs. (4.249) and (4.252). Two limiting cases of general interest are the near-threshold and the high energy asymptotics. With the result given in Eqs. (4.249) and (4.252) both limits can be taken explicitly. The asymptotic expressions can be also obtained in the framework of effective theories which can be viewed as special devices for such calculations.

4.6.1 The high energy limit

In the high energy (or, equivalently, small mass) limit $z \to 0$ the corrections read

$$\begin{aligned}\rho_1^m(s) &= 9+83z-4\pi^2 z+2\ln z+50z\ln z+12z\ln^2 z-24z\zeta(3)+O(z^2),\\ \rho_1^q(s) &= \frac{71}{48}+\frac{1}{4}\ln z-\frac{41}{3}z-6z\ln z+O(z^2).\end{aligned}\tag{4.253}$$

Therefore one obtains

$$\begin{aligned}\rho^q(s) &= \rho^q_{\rm massless}(s) = \qquad (4.254)\\ &= \frac{s^2}{4}\left\{1+\frac{\alpha_s}{\pi}\left(\ln\left(\frac{\mu^2}{s}\right)+\frac{71}{12}\right)\right\}-2m^2_{\overline{\rm MS}}(\mu)s\left\{1+\frac{\alpha_s}{\pi}\left(3\ln\left(\frac{\mu^2}{s}\right)+\frac{19}{2}\right)\right\},\\ m\rho^m(s) &= m_{\overline{\rm MS}}(\mu)\rho^m_{\rm massless}(s) = m_{\overline{\rm MS}}(\mu)s^2\left\{1+\frac{\alpha_s}{\pi}\left(2\ln\left(\frac{\mu^2}{s}\right)+\frac{31}{3}\right)\right\}\qquad (4.255)\end{aligned}$$

where $\rho^\alpha_{\rm massless}(s)$ $(\alpha=m,q)$ is the result of calculating the correlator in the effective theory of massless quarks. For the momentum part $\rho^q(s)$ the $O(m^2)$ correction is retained. The relation between the pole mass m and the $\overline{\rm MS}$ mass $m_{\overline{\rm MS}}(\mu)$ used here reads

$$m=m_{\overline{\rm MS}}(\mu)\left\{1+\frac{\alpha_s}{\pi}\left(\ln\left(\frac{\mu^2}{m^2}\right)+\frac{4}{3}\right)\right\}.\tag{4.256}$$

Note that the massless effective theory cannot reproduce the mass singularities (terms like $z\ln(z)$ in Eq. (4.253)). These singularities can be parametrized by condensates of local operators. The first m^2 correction in the leading order parts of Eqs. (4.249) and (4.252) as well as in Eq. (4.253) (or, equivalently, the m^3 correction to the expression in Eq. (4.255)) can be found if the perturbative value of the heavy quark condensate $\langle 0|\bar\Psi\Psi|0\rangle$ taken from the full theory is added [139]. The composite operator $(\bar\Psi\Psi)$ should be understood within a mass independent renormalization scheme such as the $\overline{\rm MS}$ scheme. This value (perturbatively, $\langle 0|\bar\Psi\Psi|0\rangle \sim m^3\ln(\mu^2/m^2)$) cannot be computed within the effective theory of massless quarks. It provides the proper matching between the perturbative expressions for the correlators of the full (massive) and effective (massless) theories. This matching procedure allows one to restore higher order terms of the mass expansion in the full theory from the effective massless theory with the mass term treated as a perturbation [140].

Accounting for the mass term as a perturbation in a massless theory is justified at high energies and greatly simplifies the calculations (see e.g. Ref. [141]). Note that the correction of order m^2/s to $\rho^m(s)$ can actually be found in this manner because it depends only on one local operator ($\bar{\Psi}\Psi$) and, therefore, the calculation is technically feasible. In case of the function $\rho^q(s)$ the situation is different because there is no gauge invariant operator of dimension two in the effective massless theory. Therefore, the mass singularities of the form $m^2\log(m^2/s)$ should not appear in the expansion for $\rho^q(s)$ at large energies. The result in Eq. (4.254) shows this explicitly. Note that the absence of such singularities is one of the checks for the correctness of the calculation.

4.6.2 The near-threshold limit

In the near-threshold limit $E \to 0$ with $s = (m+E)^2$ one explicitly obtains

$$\begin{aligned}\rho^m_{\rm thr}(m,E) &= \frac{16E^5}{5m}\Bigg\{1+\frac{\alpha_s}{\pi}\ln\left(\frac{\mu^2}{m^2}\right)+ \\ &\qquad +\frac{\alpha_s}{\pi}\left(\frac{54}{5}+\frac{4\pi^2}{9}+4\ln\left(\frac{m}{2E}\right)\right)\Bigg\}+O\left(\frac{E^6}{m^2}\right).\end{aligned} \tag{4.257}$$

The invariant function $\rho^m(s)$ suffices to determine the complete leading HQET behaviour since one has $\not{q}\rho^q + m\rho^m \to (\not{v}+1)\rho_{\rm HQET}$ for the leading term. This relation was explictly checked. In this region the appropriate device to compute the limit of the correlator is HQET (see e.g. Refs. [142, 143]). Writing

$$m\rho^m_{\rm thr}(m,E) = C(m/\mu,\alpha_s)^2\rho_{\rm HQET}(E,\mu) \tag{4.258}$$

one obtains the known result for $\rho_{\rm HQET}(E,\mu)$ [144]

$$\rho_{\rm HQET}(E,\mu) = \frac{16E^5}{5}\left\{1+\frac{\alpha_s}{\pi}\left(\frac{182}{15}+\frac{4\pi^2}{9}+4\ln\frac{\mu}{2E}\right)\right\}+O(E^6) \tag{4.259}$$

with the matching coefficient $C(m/\mu,\alpha_s)$ given by [145]

$$C(m/\mu,\alpha_s) = 1+\frac{\alpha_s}{\pi}\left(\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)-\frac{2}{3}\right). \tag{4.260}$$

The matching procedure allows one to restore the near-threshold limit of the full correlator starting from the simpler effective theory near the threshold [146]. Note that the higher order corrections in E/m to Eq. (4.257) can easily be obtained from the explicit result given in Eqs. (4.249) and (4.252). Indeed, the next-to-leading order corrections in the low energy expansion read

$$\Delta\rho^m_{\rm thr}(m,E) = -\frac{24E^6}{5m^2}\left\{1+\frac{\alpha_s}{\pi}\left(\ln\left(\frac{\mu^2}{m^2}\right)+\frac{584}{45}+\frac{4\pi^2}{9}+\frac{44}{9}\ln\left(\frac{m}{2E}\right)\right)\right\}, \tag{4.261}$$

$$\Delta\rho^q_{\rm thr}(m,E) = -\frac{8E^6}{m^2}\left\{1+\frac{\alpha_s}{\pi}\left(\ln\left(\frac{\mu^2}{m^2}\right)+\frac{908}{75}+\frac{4\pi^2}{9}+\frac{68}{15}\ln\left(\frac{m}{2E}\right)\right)\right\} \tag{4.262}$$

To obtain this result starting from HQET is a more difficult task requiring the analysis of contributions of higher dimension operators.

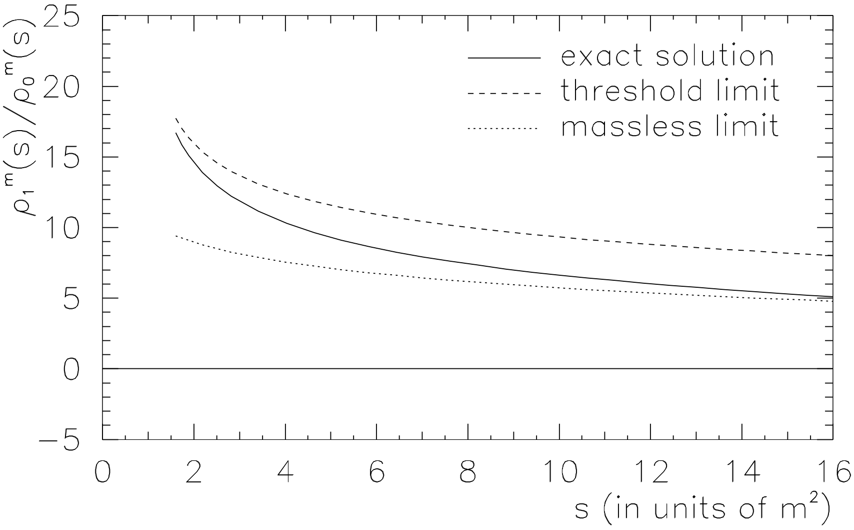


Figure 4.2: The ratio $\rho_1^m(s)/\rho_0^m(s)$ of the next-to-leading correction and the leading order term in dependence of the energy square s

4.6.3 Discussion of quantitative features

Some quantitative features of the correction given in Eqs. (4.249) and (4.252) will be discussed as next. Of interest is whether the two limiting expressions (the massless limit expression as given in Eqs. (4.254) and (4.255) and the HQET limit expression in Eqs. (4.257) and (4.258)) can be used to characterise the full functionional dependence for all energies. For this discussion one compares components of the baryonic spectral function in leading and next-to-leading order. In Figs. 4.2 and 4.3 the ratio $\rho_1^\alpha(s)/\rho_0^\alpha(s)$ is shown for $\alpha = m$ and $\alpha = q$, respectively. In the following the specific renormalization scale value $\mu = m$ is used always if it is not written explicitly. One can see that a simple interpolation between the two limits can give a rather good approximation for the next-to-leading order correction in the complete region of s.

4.7 Moments of the spectral density

An informative set of observables are moments of the spectral density

$$\mathcal{M}_n^\alpha = \int_{m^2}^\infty \frac{\rho^\alpha(s)ds}{s^n} = m^{2(3-n)} M_n^\alpha \tag{4.263}$$

where M_n^α are dimensionless quantities. One finds

$$M_n^\alpha = M_n^{\alpha(0)}\left\{1 + \frac{\alpha_s}{\pi}\left(\ln\left(\frac{\mu^2}{m^2}\right) + \delta_n^\alpha\right)\right\} \tag{4.264}$$

where

$$M_n^{q(0)} = \frac{12}{(n+1)n(n-1)^2(n-2)(n-3)}, \tag{4.265}$$

$$M_n^{m(0)} = \frac{12}{n(n-1)^2(n-2)^2(n-3)}, \tag{4.266}$$

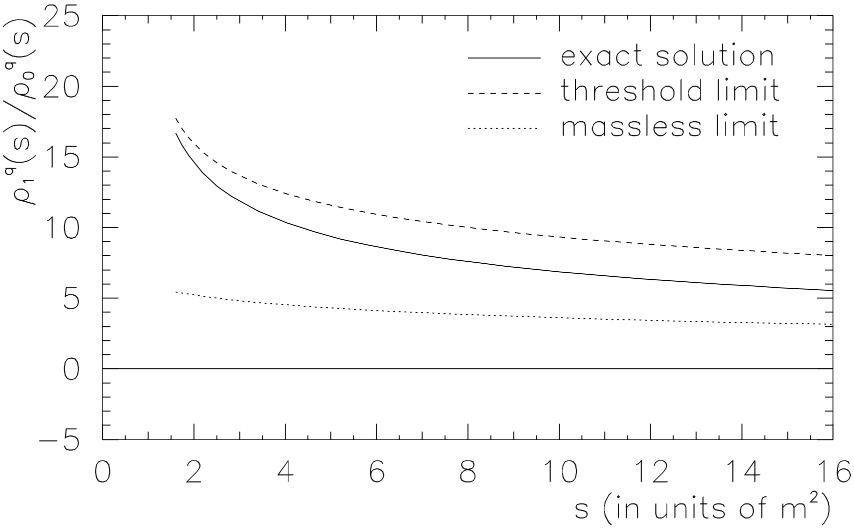


Figure 4.3: The ratio $\rho_1^q(s)/\rho_0^q(s)$ of the next-to-leading correction and the leading order term in dependence of the energy square s

and

$$\delta_n^\alpha = A_n^\alpha + \frac{2\pi^2}{9}. \tag{4.267}$$

The coefficients A_n^α are rational numbers, and the closed form expressions for δ_n^α are found in Appendix D.5. Only the first values for A_n^m and A_n^q are shown in the second column of Table 4.1.

n	A_n^m	$\delta_n^m - \delta_{n-1}^m$	A_q^m	$\delta_q^m - \delta_{q-1}^m$
4	3		9/2	
5	13/2	3.500000	22/3	2.833333
6	17/2	2.000000	109/12	1.750000
7	535/54	1.407407	5593/540	1.274074
8	1187/108	1.083333	6133/540	1.000000
9	64093/5400	0.878333	460351/37800	0.821190
10	22691/1800	0.737037	40553/3150	0.695370
11	1167767/88200	0.633878	148574/11025	0.602132
12	2433499/176400	0.555357	2470739/176400	0.530357

Table 4.1: Values for the rational part A_n^α of the first moments δ_n^α and their relative difference $\delta_n^\alpha - \delta_{n-1}^\alpha$

By representing the moments through

$$\frac{M_n^\alpha}{M_n^{\alpha(0)}} = \frac{M_N^\alpha}{M_N^{\alpha(0)}}\left\{1 + \frac{\alpha_s}{\pi}(\delta_n^\alpha - \delta_N^\alpha)\right\} \tag{4.268}$$

all corrections can be normalized to the moment M_N^α of fixed order N. Note that the difference $\delta_n^\alpha - \delta_N^\alpha$ is scheme-independent. This feature can be used in the high precision analysis of heavy quark properties within NRQCD (see e.g. Refs. [45, 117, 118, 119, 120, 121, 122]). With Eq. (4.268) one can find the actual (invariant or scheme-independent) magnitude of the correction. Indeed, for any given N one can find a set of perturbatively commensurate moments M_n^α with $n \sim N$ for which the requirement of the chosen precision is satisfied. In the third column of Table 4.1, therefore, numerical values are presented only for the differences of the δ_n^α for subsequent orders.

Note that the moments represent massive vacuum bubbles, i.e. diagrams with massive lines without external momenta. These diagrams have been comprehensively analyzed in Refs. [111, 110]. The analytical results for the first few moments at three-loop level can be checked independently with existing computer programs for symbolic calculations in high energy physics (see e.g. Ref. [147]).

4.7.1 Phenomenological consequences

The presented results have phenomenological applications within the sum rule analysis of baryon properties (see e.g. Refs. [148, 149, 150] and Chapter 7). As an example the integral of $\rho^q(s)$ is calculated up to the energy cut $\sqrt{s_0}$,

$$\mathcal{M}_0^q(s_0) = \int_{m^2}^{s_0} \rho^q(s) ds \tag{4.269}$$

which is related to the coupling constant (residue) of a baryon to the current in Eq. (4.1). In NLO the integral is represented by

$$\mathcal{M}_0^q(s_0) = \mathcal{M}_0^{q(0)}(s_0) \left(1 + \frac{\alpha_s}{\pi}\left(\ln\left(\frac{\mu^2}{m^2}\right) + \Delta(s_0)\right)\right) \tag{4.270}$$

which leads to the renormalization of the LO result for the residue in the form

$$Z_R^{01} = \frac{\mathcal{M}_0^q(s_0)}{\mathcal{M}_0^{q(0)}(s_0)} = 1 + \frac{\alpha_s}{\pi}\left(\ln\left(\frac{\mu^2}{m^2}\right) + \Delta(s_0)\right) + O(\alpha_s^2). \tag{4.271}$$

For e.g. $s_0 = 2m^2$, $\mu^2 = m^2$ one finds numerically

$$Z_R^{01} = 1 + \frac{\alpha_s}{\pi}\Delta(2m^2) = 1 + \frac{\alpha_s}{\pi}15.4117\ldots \tag{4.272}$$

It is obvious that the NLO correction to the residue in the $\overline{\rm MS}$ scheme is rather large. For the numerical value of the coupling constant $\alpha_s \approx 0.3$ which is a typical value for baryons containing a c-quark, the NLO correction in the $\overline{\rm MS}$ scheme reaches the 100% level.

One can see that the corrections to the moments basically reflect the form of the correction to the spectrum. Even the massless approximation is reasonably good for relative corrections for the first few moments despite of the unfavourable shape of the weight function $1/s^n$. It can be improved by changing the subtraction point μ, i.e. by switching from the $\overline{\rm MS}$ scheme to some other renormalization scheme, or by resumming the integrand [81] which lies beyond the scope of finite order perturbation theory though.

4.7.2 A word about why the pole mass is used here

As the difference $\delta^\alpha_{n+1}-\delta^\alpha_n$ vanishes like $1/n$ for high values of n (see Appendix D.5), the contributions δ^α_n themselves behave like $\ln n$. One may consider the possibility to reduce or even eliminate this logarithmic increase by the choice of another renormalization scheme, for instance the $\overline{\rm MS}$ scheme. As seen earlier, the pole mass m can be expressed by the $\overline{\rm MS}$ mass $m_{\overline{\rm MS}}$ through

$$m=m_{\overline{\rm MS}}\left(1+\frac{\alpha_s}{\pi}A_m\right). \tag{4.273}$$

Inserting this into Eq. (4.263) and using Eq. (4.264) one obtains

$$\mathcal{M}^\alpha_n=m_{\overline{\rm MS}}^{2(3-n)}M_n^{\alpha(0)}\left\{1+\frac{\alpha_s}{\pi}\left(\ln\left(\frac{\mu^2}{m^2_{\overline{\rm MS}}}\right)+\delta^\alpha_n+2(3-n)A_m\right)\right\}. \tag{4.274}$$

However, while the logarithm is now combined with a linear function, the situation does not improve but actually becomes worse. At the same time it becomes obvious that the pole mass chosen here in some natural way is the only way for which the difference $\delta^\alpha_{n+1}-\delta^\alpha_n$ actually vanishes for $n\to\infty$. Therefore, the pole mass is highly preferable here.

4.8 General considerations about the pole mass

In order to prepare the field for the sowing the seed for the last two chapters, the advantage of the *pole mass* will be outlined in this section. At least one advantage was already mentioned just before, namely the improved convergence of the moments. The section starts with the calculation of contributions to the quark self energy due to the *"large β_0 approximation"*. This approximation allows for an exact resummation of perturbative contributions. Even though the approximation contains the resummation of only a small portion of diagrams, the results obtained by using this approximation are normally used to argue against the use of the pole mass. In this section these arguments are shown to be relative, especially depending on the energy region (Euclidean, close to threshold, etc.) in which the perturbative results are calculated. Weakening the arguments against its use, the pole mass enters back into the game.

4.8.1 The "large β_0 approximation"

β_0 is the leading coefficient in the renormalization group equation. In the "large β_0 approximation" the self energy correction to arbitrary high order is approximated by the replacement of the gluon propagator by a chain of gluon propagators and one-loop gluon self energy diagrams. A single gluon self energy diagram can be expressed in dimensional regularization as the transverse propagator (see Eq. (H.60)),

$$\begin{aligned}\Pi_{\mu\nu}(-k^2)&=\frac{-ik^2C}{(-k^2)^\varepsilon}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right),\\ C&=\frac{g_s^2\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}\left\{\frac{1}{\varepsilon}\left(\frac{5}{3}C_A-\frac{2}{3}N_L\right)+\frac{31}{9}C_A-\frac{10}{9}N_L\right\}\end{aligned} \tag{4.275}$$

where $C_A=3$ and N_L is the number of light degrees. The divergence in C is absorbed into the renormalization of the coupling. Because $(g_{\mu\nu}-k_\mu k_\nu/k^2)$ is a projector onto the transverse components, all longitudinal components of the gluon propagators

$$D_{\mu\nu}(-k^2)=\frac{-i}{k^2}\left(g_{\mu\nu}-(1-\alpha_g)\frac{k_\mu k_\nu}{k^2}\right) \tag{4.276}$$

vanish. For an n-fold chain one obtains

$$(D\Pi D\cdots\Pi D)_{\mu\nu}(-k^2)=\frac{-i}{k^2}\left(\frac{C}{(-k^2)^\varepsilon}\right)^n\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)=:D^{\perp(n)}(-k^2)\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right). \tag{4.277}$$

Using this short form, one calculates the correlator for the complete quark self energy diagram,

$$\begin{aligned}
-i\Sigma^{(n)}(q^2) &= \\
&= \int\frac{d^Dk}{(2\pi)^D}(-ig_s\gamma^\mu)\frac{i}{\not{k}-m}(-ig_s\gamma^\nu)D^{\perp(n)}\left(-(q-k)^2\right)\left(g_{\mu\nu}-\frac{(q-k)_\mu(q-k)_\nu}{(q-k)^2}\right) = \\
&= g_s^2C^n\int\frac{d^Dk}{(2\pi)^D}\frac{\gamma^\mu(\not{k}+m)\gamma^\nu}{(k^2-m^2)(-(q-k)^2)^{1+n\varepsilon}}\left(g_{\mu\nu}-\frac{(q-k)_\mu(q-k)_\nu}{(q-k)^2}\right) = \\
&= g_s^2C^n\int\frac{d^Dk}{(2\pi)^D}\left(\frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(k^2-m^2)(-(q-k)^2)^{1+n\varepsilon}}+\frac{(\not{q}-\not{k})(\not{k}+m)(\not{q}-\not{k})}{(k^2-m^2)(-(q-k)^2)^{2+n\varepsilon}}\right).
\end{aligned} \tag{4.278}$$

With

$$\begin{aligned}
\gamma^\mu(\not{k}+m)\gamma_\mu &= (2-D)\not{k}+Dm \quad\Rightarrow \\
\frac{1}{4}\mathrm{Tr}\left(\gamma^\mu(\not{k}+m)\gamma_\mu\right) &= Dm, \\
\frac{1}{4}\mathrm{Tr}\left(\not{q}\gamma^\mu(\not{k}+m)\gamma_\mu\right) &= (2-D)(qk) \;=\; \frac{2-D}{2}\left(q^2+k^2-(q-k)^2\right), \\
(\not{q}-\not{k})(\not{k}+m)(\not{q}-\not{k}) &= (\not{q}-\not{k})\not{k}(\not{q}-\not{k})+m(q-k)^2 \quad\Rightarrow \\
\frac{1}{4}\mathrm{Tr}\left((\not{q}-\not{k})(\not{k}+m)(\not{q}-\not{k})\right) &= m(q-k)^2, \\
\frac{1}{4}\mathrm{Tr}\left(\not{q}(\not{q}-\not{k})(\not{k}+m)(\not{q}-\not{k})\right) &= 2(q^2-qk)(qk-k^2)-(qk)(q-k)^2 \;= \\
&= \frac{1}{2}(q^2-k^2)^2-\frac{1}{2}(q^2+k^2)(q-k)^2
\end{aligned} \tag{4.279}$$

one obtains

$$\Sigma^{(n)}(q^2)=\not{q}\Sigma_q^{(n)}(q^2)+m\Sigma_m^{(n)}(q^2) \tag{4.280}$$

where

$$\begin{aligned}
m\Sigma_m^{(n)}(q^2) &= ig_s^2C^n(D-1)m\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-m^2)(-(q-k)^2)^{1+n\varepsilon}} = \\
&= ig_s^2C^n(D-1)m\frac{-i}{(4\pi)^{D/2}}(m^2)^{D/2-2-n\varepsilon}V(1,1+n\varepsilon;-q^2/m^2) \quad\Rightarrow \\
\Sigma_m^{(n)}(q^2) &= \frac{g_s^2}{(4\pi)^{2-\varepsilon}}C^n(m^2)^{-(n+1)\varepsilon}(3-2\varepsilon)V(1,1+n\varepsilon;-q^2/m^2)
\end{aligned} \tag{4.281}$$

and

$$
\begin{aligned}
q^2\Sigma_q^{(n)}(q^2) = ig_s^2C^n\int\frac{d^Dk}{(2\pi)^D}\Bigg(&\frac{(2-D)(q^2+m^2)}{2(k^2-m^2)(-(q-k)^2)^{1+n\varepsilon}}+\frac{2-D}{2(k^2-m^2)(-(q-k)^2)^{n\varepsilon}}+\\
&+\frac{(q^2-m^2)^2}{2(k^2-m^2)(-(q-k)^2)^{2+n\varepsilon}}+\frac{q^2+m^2}{2(k^2-m^2)(-(q-k)^2)^{1+n\varepsilon}}\Bigg)=\\
=\frac{g_s^2}{(4\pi)^{D/2}}C^n\Bigg(&\frac{1}{2}(q^2-m^2)^2(m^2)^{D/2-3-n\varepsilon}V(1,2+n\varepsilon;-q^2/m^2)\ +\\
&+\frac{3-D}{2}(q^2+m^2)(m^2)^{D/2-2-n\varepsilon}V(1,1+n\varepsilon;-q^2/m^2)\ +\\
&+\frac{2-D}{2}(m^2)^{D/2-1-n\varepsilon}V(1,n\varepsilon;-q^2/m^2)\Bigg)\quad\Rightarrow\\
\Sigma_q^{(n)}(q^2) = \frac{g_s^2}{(4\pi)^{2-\varepsilon}}&C^n(m^2)^{-(n+1)\varepsilon}\Bigg(\frac{1}{2}\left(1-\frac{m^2}{q^2}\right)^2\frac{q^2}{m^2}V(1,2+n\varepsilon;-q^2/m^2)+\\
-\left(\frac{1}{2}-\varepsilon\right)&\left(1+\frac{m^2}{q^2}\right)V(1,1+n\varepsilon;-q^2/m^2)-(1-\varepsilon)\frac{m^2}{q^2}V(1,n\varepsilon;-q^2/m^2)\Bigg).
\end{aligned}
\tag{4.282}
$$

Note that one can effectively set $k^2=m^2$ because (k^2-m^2) and powers of it will cancel the first numerator factor and therefore will give no contribution (cf. the scaling rule). Because of

$$
\begin{aligned}
V(n_1,n_2;-1) &= \frac{\Gamma(n_1+n_2-D/2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1(1-x)^{D/2-n_2-1}x^{n_2-1}(1-x)^{D/2-n_1-n_2}dx =\\
&= \frac{\Gamma(n_1+n_2-D/2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1(1-x)^{D-n_1-2n_2-1}x^{n_2-1}dx = \\
&= \frac{\Gamma(n_1+n_2-D/2)\Gamma(D-n_1-2n_2)\Gamma(n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(D-n_1-n_2)} = \frac{\Gamma(n_1+n_2-D/2)\Gamma(D-n_1-2n_2)}{\Gamma(n_1)\Gamma(D-n_1-n_2)}
\end{aligned}
\tag{4.283}
$$

one obtains at $q^2=m^2$

$$
V(1,n_2+n\varepsilon;-1)=\frac{\Gamma(n_2-1+(n+1)\varepsilon)\Gamma(3-2n_2-2(n+1)\varepsilon)}{\Gamma(3-n_2-(n+2)\varepsilon)}
\tag{4.284}
$$

and therefore

$$
\begin{aligned}
\Sigma_m^{(n)}(m^2) &= C_n\left(\frac{3}{\varepsilon}+3n+4+(3n^2+10n+8)\varepsilon+O(\varepsilon^2)\right),\\
\Sigma_q^{(n)}(m^2) &= C_n\left(-\frac{3}{2}n-\frac{1}{4}(9n+14)n\varepsilon+O(\varepsilon^2)\right)
\end{aligned}
\tag{4.285}
$$

where

$$
C_n:=g_s^2C^n\frac{(m^2)^{-(n+1)\varepsilon}}{(4\pi)^{2-\varepsilon}}\ \frac{\Gamma(1+(n+1)\varepsilon)\Gamma(1-2(n+1)\varepsilon)}{(n+1)\Gamma(1-(n+2)\varepsilon)}.
\tag{4.286}
$$

4.8.2 The pole mass defined by a scheme mass

The definition of the pole mass is deeply related to the calculation of the quark self energy. The (unrenormalized) self energy can be written as

$$
\Sigma^B(\not q)=\not q\Sigma_q^B(q^2)+m_B\Sigma_m^B(q^2).
\tag{4.287}
$$

The singularities contained in Σ_q and Σ_m are then absorbed in renormalization factors of the propagator and the mass. If one calculates the bare propagator by taking the inverse,

$$S^B = \frac{i}{\not{q} - m_B - \Sigma^B(\not{q})} = \frac{i}{\not{q}(1 - \Sigma_q^B(q^2)) - m_B(1 + \Sigma_m^B(q^2))} = \frac{iZ_2}{\not{q} - Z_m^{-1} m_B}, \tag{4.288}$$

one can identify

$$Z_2^{-1} = 1 - \Sigma_q^B(q^2), \qquad Z_m^{-1} = \frac{1 + \Sigma_m^B(q^2)}{1 - \Sigma_q^B(q^2)} = 1 + \Sigma_m^B(q^2) + \Sigma_q^B(q^2) + \ldots \tag{4.289}$$

where the ellipses stand for terms of second order in the coupling. However, the renormalization factors cannot depend on the energy square q^2. Instead one defines

$$(Z_2^S)^{-1} = 1 - \Sigma_q^{BS}, \qquad (Z_m^S)^{-1} = \frac{1 + \Sigma_m^{BS}}{1 - \Sigma_q^{BS}} = 1 + \Sigma_m^{BS} + \Sigma_q^{BS} + \ldots \tag{4.290}$$

where Σ_i^{BS} are the singular parts of the bare quantities $\Sigma_i^B(q^2)$ together with a (constant) finite part, depending on the chosen scheme (the additional index "S" stands for the particular renormalization scheme). One is therefore left with finite (and generally q dependent) parts,

$$S_S = \frac{i}{\not{q}(1 - \Sigma_q^S(q^2)) - m_S(1 + \Sigma_m^S(q^2))} = \frac{i(\not{q}(1 - \Sigma_q^S(q^2)) + m_S(1 + \Sigma_m^S(q^2)))}{q^2(1 - \Sigma_q^S(q^2))^2 - m_S^2(1 + \Sigma_m^S(q^2))}. \tag{4.291}$$

Now the *pole mass* is defined as pole of this function with respect to $\sqrt{q^2}$, so implicitly

$$m = m_S \frac{1 + \Sigma_m^S(q^2)}{1 - \Sigma_q^S(q^2)}\bigg|_{q^2 = m^2} = m_S\left(1 + \Sigma_m^S(m^2) + \Sigma_q^S(m^2) + \ldots\right) \tag{4.292}$$

This implicit equation is highly depending on the renormalization scheme S. For a choice not adequate to the problem the perturbation series on the right hand side can be divergent. Taking for instance the "large β_0 approximation", i.e. the results just calculated within the $\overline{\text{MS}}$ scheme, one obtains

$$m = m_S\left\{1 - C_n\left(\frac{3}{\varepsilon} + \frac{3}{2}n + 4 + \frac{1}{4}(3n^2 + 26n + 32)\varepsilon + O(\varepsilon^2)\right)\right\}. \tag{4.293}$$

However, one might raise the two following questions:

1. Is the "large β_0 approximation" sufficient for an estimate of convergence?

2. Is the divergence of the series a relative item, namely that a mass at threshold (the pole mass concept) and a renormalization scheme developed for Euclidean regions (the $\overline{\text{MS}}$ scheme) are not compatible with each other? Could this phenomenon also occur for quantities defined at *any* pair of values for q^2 far away from each other?

4.8.3 The pole mass renormalization

Especially the last question shall be answered in the following by the construction of a scheme independent renormalization for the pole mass. Even if not yet renormalized, the pole mass can be expressed by

$$m = m_B \frac{1 + \Sigma_m(q^2)}{1 - \Sigma_q(q^2)}\bigg|_{q^2=m^2}. \tag{4.294}$$

$\Sigma_m(q^2)$ and $\Sigma_q(q^2)$ are both parametrized by q^2 as well as by the bare mass m_B. While the bare mass can be replaced by the pole mass in the first order approximation, Eq. (4.294) is in general an implicit equation which can be solved iteratively for m, even if the divergences are not yet being subtracted. The solution is given by

$$m = Z_m^{-1} m_B \tag{4.295}$$

where Z_m is the renormalization factor for the pole mass. Therefore, a method has been found to determine the mass renormalization factor totally independent of specific schemes. This renormalization factor of course contains logarithms of masses. If one relates masses for different schemes by equating the bare masses,

$$Z_m^S m_S = m_B = Z_m m \quad \Leftrightarrow \quad m = \frac{Z_m^S}{Z_m} m_S, \tag{4.296}$$

for example for $S = \overline{\mathrm{MS}}$, the ratio as expanded in a series does not converge in general because of these logarithms if the scales are quite different. This is the case if one uses a scheme far from threshold. But for the definition of the pole mass itself this does not matter at all. Therefore, the argument against the pole mass is shown to be irrelevant.

Chapter 5

New developments in HQET

The *heavy quark effective theory (HQET)* [151] allows one to study the properties of heavy hadrons in a systematic $1/m_Q$ expansion where m_Q is the mass of the heavy quark included in this hadron (see e.g. Ref. [152]). The leading term of this expansion gives rise to the spin-flavour symmetry, also known as *heavy quark symmetry (HQS)* (for a detailed introduction to HQS see e.g. Ref. [143]).

Among the well-known predictions of HQS are for instance the relations between different heavy hadron transition form factors. As an example, the six form factors describing the transition $\Lambda_b \to \Lambda_c$ are reduced to one universal Isgur-Wise function in the HQS limit [142, 153, 154]. Even then one still remains with many non-perturbative parameters characterizing the process and the heavy baryons participating in it. These concern the functional behaviour of the Isgur-Wise function itself, the masses and residues of the heavy baryons, and, at next-to-leading order in the heavy mass expansion, the average kinetic and chromomagnetic energy of the heavy quark in the heavy baryon. All these non-perturbative parameters can be determined by using non-perturbative methods as e.g. lattice calculations (see Chapter 6), QCD sum rule methods [155] (see Chapter 7) or, in a less fundamental approach, by using potential models. While the first part of this chapter deals with the HQET itself, HQET is applied in the second part by calculating the one-loop corrections to the baryonic Isgur–Wise function. This work is not yet completed, as the recurrence algebra for the necessary integrals has not been solved to get to a final result. Therefore, only the concepts and the calculational steps are presented.

In order to be able to use HQET also to higher orders in $1/m_Q$, the coefficients of the expansion have to be matched to QCD. An two-loop adjustment of the so-called electric and magnetic form factors is done in the first part of this chapter. For this purpose a large amount of QCD diagrams correcting the interaction vertex of the heavy quark and an external gluon had to be calculated and assembled. This calculation extends the known results by including one more loop order.

HQET and the nonrelativistic QCD (NRQCD) [156, 157] are two effective theories that describe the interactions of hadrons containing almost on-shell heavy quarks. The HQET is mainly applied to hadrons containing a single heavy quark, such as a B meson, while the NRQCD describes the interactions between nonrelativistic quarks and is typically applied to $\bar{Q}Q$ bounded states like Υ. Applications of NRQCD can be found for instance in lattice QCD, as dealt with in Chapter 6.

5.1 HQET coefficients and their matching

Starting from the QCD Lagrangian for the heavy quark,

$$\mathcal{L}_h = \bar{q}_h(i\not{D} - m_Q)q_h, \qquad D_\mu = \partial_\mu + igA^a_\mu T_a, \tag{5.1}$$

the effective Lagrangian can be constructed in a straightforward manner. The first step is to partition the covariant derivative into a parallel and an orthogonal part,

$$D^\mu_\parallel = v^\mu(v\cdot D), \qquad D^\mu_\perp = D^\mu - D^\mu_\perp \tag{5.2}$$

with $v^2 = 1$. Considering the rest frame of the heavy hadron where $v = (1;0,0,0)$, the parallel component is combined with γ_0 in $\not{D}_\parallel$ which is a block diagonal matrix for the high energy representation of the Dirac matrices. A separation of the spinor q_h into a "large" and a "small" component decouples, if the other part $\not{D}_\perp$ is absent. This is of course not the case but can be accomblished by a transformation of the spinors using unitary matrices. This transformation is known from QED and is called *Foldy–Wouthuysen transformation*.

5.1.1 The Foldy–Wouthuysen transformation

The application of the Foldy–Wouthuysen transformation to the QCD is introduced in Ref. [158] and will be explained here in detail. The first step consists in the replacement of q_h by

$$q'_h = Uq_h, \qquad q'^\dagger_h = q^\dagger_h U^{-1} \quad\Rightarrow\quad \bar{q}'_h = \bar{q}_h\gamma_0 U^{-1}\gamma_0 \tag{5.3}$$

where the unitary transformation matrix is given by

$$U = \exp\left(-\frac{i\not{D}_\perp}{2m_Q}\right) \tag{5.4}$$

with

$$\gamma_0 U\gamma_0 = \exp\left(-\frac{i\gamma_0\not{D}_\perp\gamma_0}{2m_Q}\right) = \exp\left(\frac{i\gamma_0^2\not{D}_\perp}{2m_Q}\right) = \exp\left(\frac{i\not{D}_\perp}{2m_Q}\right) = U^{-1} \tag{5.5}$$

(this identity, in this form valid only in the heavy hadron rest frame, can be generalized to a moving frame). The Lagrangian can be rewritten as

$$\mathcal{L} = \bar{q}_h(i\not{D}_\parallel + i\not{D}_\perp - m_Q)q_h = \bar{q}'_h U^{-1}(i\not{D}_\parallel + i\not{D}_\perp - m_Q)U^{-1}q'_h. \tag{5.6}$$

An expansion in $1/m_Q$ of the unitary matrix results in

$$\begin{aligned}
U^{-1}(i\not{D}_\parallel + i\not{D}_\perp - m_Q)U^{-1} \;&=\\
\approx\;& \left(1+\frac{i\not{D}_\perp}{2m_Q} - \frac{\not{D}_\perp^2}{8m_Q^2}\right)(i\not{D}_\parallel + i\not{D}_\perp - m_Q)\left(1-\frac{i\not{D}_\perp}{2m_Q} - \frac{\not{D}_\perp^2}{8m_Q^2}\right) \;=\\
=\;& i\not{D}_\parallel + i\not{D}_\perp - m_Q - \frac{1}{2m_Q}\not{D}_\perp\not{D}_\parallel - \frac{1}{2m_Q}\not{D}_\perp^2 - \frac{i}{2}\not{D}_\perp + \frac{1}{8m_Q}\not{D}_\perp^2 \;+\\
& -\frac{1}{2m_Q}\not{D}_\parallel\not{D}_\perp - \frac{1}{2m_Q}\not{D}_\perp^2 - \frac{i}{2}\not{D}_\perp + \frac{1}{8m_Q}\not{D}_\perp^2 + \frac{1}{4m_Q}\not{D}_\perp^2 + O\left(\frac{1}{m_Q^2}\right) \;=\\
=\;& i\not{D}_\perp - m_Q - \frac{1}{2m_Q}\left(\not{D}_\perp\not{D}_\parallel + \not{D}_\parallel\not{D}_\perp\right) - \frac{1}{2m_Q}\not{D}_\perp^2 + O\left(\frac{1}{m_Q^2}\right) \;=:\; i\not{D}'_\parallel + i\not{D}'_\perp - m_Q
\end{aligned} \tag{5.7}$$

where

$$i\not{D}'_{\|} = i\not{D}_{\|} - \frac{1}{2m_Q}\not{D}_{\perp}^2, \qquad i\not{D}'_{\perp} = \frac{1}{2m_Q}\left(\not{D}_{\perp}\not{D}_{\|} + \not{D}_{\|}\not{D}_{\perp}\right). \tag{5.8}$$

$\not{D}'_{\|}$ is again block-diagonal while $\not{D}'_{\perp}$ is off-diagonal. Note that the leading term $\not{D}_{\perp}$ vanishes. The remaining expression is proportional to $1/m_Q$. The next step is the analogous replacement

$$q''_h = U'q'_h \quad \text{with} \quad U' = \exp\left(-\frac{i\not{D}'_{\perp}}{2m_Q}\right). \tag{5.9}$$

The iterative transformation performed by repeating this procedure will be completed by a final transformation. After n steps for instance this final transformation is given by

$$q_h^{(n)} = e^{-im_Q v\cdot x}q_v \tag{5.10}$$

therefore

$$i\not{D}_{\|}q_h^{(n)} = e^{-im_Q v\cdot x}(m_Q\not{v} + i\not{D}_{\|})q_v \tag{5.11}$$

removing the *pivotal point* given by m_Q. The whole procedure was done do decouple the so-called "big" and "small" components of the spinor. Using the high energy representation of the Dirac matrices and considering the heavy hadron rest frame, these components are given by the upper and lower part of the spinor. This goal is reached up to the order $1/m_Q^n$ because in the heavy hadron rest frame, the operator $\not{D}_{\|}$ is block-diagonal, while the (off-diagonal) operator $\not{D}_{\perp}$ is of the order $1/m_Q^n$. In a more general frame the two spinor components can be obtained by applying the projectors P^+ and P^-, defined by

$$P^{\pm} := \frac{1}{2}(1 \pm \not{v}), \qquad q_v^{\pm} := P^{\pm}q_v, \quad \not{v}\,q_v^{\pm} = \pm q_v^{\pm}. \tag{5.12}$$

Using these two spinors, the Lagrangian can be reformulated according to [143]

$$\begin{aligned} \mathcal{L} &= \bar{q}_h^{(n)}\left(i\not{D}_{\|}^{(n)} + i\not{D}_{\perp}^{(n)} - m_Q\right)q_h^{(n)} = \\ &= \bar{q}_v^{+}i\not{D}_{\|}^{(n)}q_v^{+} + \bar{q}_v^{-}\left(i\not{D}_{\|}^{(n)} - 2m_Q\right)q_v^{-} + \bar{q}_v^{+}i\not{D}_{\perp}^{(n)}q_v^{-} + \bar{q}_v^{-}i\not{D}_{\perp}^{(n)}q_v^{+} \end{aligned} \tag{5.13}$$

(note that $\not{D}_{\|}$ occurs in all $\not{D}_{\|}{}'$, $\not{D}_{\|}{}''$, $\ldots$, $\not{D}_{\|}{}^{(n)}$. Therefore, Eq. (5.11) can be applied). The two last (nondiagonal) parts in the Lagrangian are of order $1/m_Q^n$ and can be neglected for the consideration of lower order contributions. The second part describes the heavy, highly fluctuating degrees of freedom which will be eliminated. Only the first part of the Lagrangian is of interest, and using $Q = q_v^+$ this part constitutes the Lagragian of HQET,

$$\begin{aligned} \mathcal{L}_{\text{HQET}} &= \bar{Q}i\not{D}_{\|}^{(n)}Q = \\ &= \bar{Q}\Bigg\{i\not{D}_{\|} - \frac{1}{2m_Q}\not{D}_{\perp}^2 - \frac{i}{4m_Q^2}\left(\frac{1}{2}\not{D}_{\|}\not{D}_{\perp}^2 + \not{D}_{\perp}\not{D}_{\|}\not{D}_{\perp} + \frac{1}{2}\not{D}_{\perp}^2\not{D}_{\|}\right) + \\ &\quad + \frac{1}{8m_Q^3}\left(\not{D}_{\|}\not{D}_{\perp}\not{D}_{\|}\not{D}_{\perp} + \not{D}_{\|}\not{D}_{\perp}^2\not{D}_{\|} + \not{D}_{\perp}\not{D}_{\|}^2\not{D}_{\perp} + \not{D}_{\perp}\not{D}_{\|}\not{D}_{\perp}\not{D}_{\|} + \not{D}_{\perp}^4\right) + \ldots\Bigg\}Q, \end{aligned} \tag{5.14}$$

where

$$Q = P^+e^{im_Q v\cdot x}q_h^{(n)} = P^+e^{im_Q v\cdot x}U^{(n-1)}\cdots U'Uq_h, \qquad \not{v}Q = Q. \tag{5.15}$$

This calculation is done automatically and published in Ref. [158] up to order $1/m_Q^{12}$. There is a second which avoids the use of the Fouldy–Wouthuysen method, but this approach needs field redefinitions [162]. A final note is in order here about the structure of the higher order terms. Calculated in the heavy hadron rest frame, one obtains

$$\begin{aligned}\{\not{D}_\perp,\not{D}_\parallel\} &= \left\{\sum_{i=1}^3\gamma^iD_i,\gamma^0D_0\right\} = \sum_{i=1}^3\left(\gamma^i\gamma^0D_iD_0+\gamma^0\gamma^iD_0D_i\right) = \\ &= \sum_{i=1}^3\gamma^i\gamma^0(D_iD_0-D_0D_i) = \sum_{i=1}^3\gamma^i\gamma^0[D_i,D_0] = -igG_{i0}\gamma^i\gamma^0 \qquad (5.16)\end{aligned}$$

($[D_\mu,D_\nu]=ig_sG_{\mu\nu}$), in general

$$\{\not{D}_\perp,\not{D}_\parallel\}=ig_sG_{\mu\nu}\gamma^\mu v^\nu. \qquad (5.17)$$

With this result the expression

$$\frac{1}{2}\not{D}_\parallel\not{D}_\perp^2+\not{D}_\perp\not{D}_\parallel\not{D}_\perp+\frac{1}{2}\not{D}_\perp^2\not{D}_\parallel=\frac{1}{2}\left(\{\not{D}_\parallel,\not{D}_\perp\}\not{D}_\perp+\not{D}_\perp\{\not{D}_\parallel,\not{D}_\perp\}\right) \qquad (5.18)$$

like all higher orders does not contain any explicit $\not{D}_\parallel$, i.e. no time derivative in the heavy hadron rest frame. The consequence is that a later mixing of orders $1/m_Q$ is excluded, the result presented here is therefore the appropiate power expansion in $1/m_Q$.

5.1.2 The kinetic term and the Fermi term

Even though the representation chosen in Eq. (5.14) is the shortest one, for practical calculations this short form has to be resolved again. Looking at the first order term, one obtains

$$\not{D}_\perp^2=(\not{D}-\not{D}_\parallel)^2=\not{D}^2-\not{D}\not{D}_\parallel-\not{D}_\parallel\not{D}+\not{D}_\parallel\not{D}_\parallel. \qquad (5.19)$$

In the following a distinction between *operators of the first and second class* is of use here. The application of operators of the second class to the spinor results in zero, the simplest of these operators therefore result in the equations of motion. In the present case one has $\not{D}_\parallel Q=0$. The first and last term in Eq. (5.19) are operators of the first class, the remaining ones are operators of the second class. In Ref. [159] a systematic procedure has been developed to remove operators of the second class. An operator of the second class of order $1/m_Q^n$ can be generally written as

$$O_{\rm II}=\frac{1}{m_Q^n}\bar{Q}(i\not{D}_\parallel A+\bar{A}i\not{D}_\parallel)Q \qquad (5.20)$$

where A is a general operator, $\bar{A}=\gamma_0A^\dagger\gamma_0$. Using the replacement

$$Q\to\left(1-\frac{1}{m_Q^n}P^+A\right)Q \quad\Rightarrow\quad \bar{Q}\to\bar{Q}\left(1-\frac{1}{m_Q^n}\bar{A}P^+\right) \qquad (5.21)$$

with $\bar{Q}i\not{D}_\parallel Q$ as pivotal point, one obtains

$$\bar{Q}i\not{D}_\parallel Q\to\bar{Q}i\not{D}_\parallel Q-\frac{1}{m_Q^n}\bar{Q}(i\not{D}_\parallel P^+A+\bar{A}P^+i\not{D}_\parallel)Q. \qquad (5.22)$$

The projector P^+ is used in order to make sure that $\not{v}Q = Q$ is still valid for the redefined spinor. Because P^+ commutes with $i\not{D}_\parallel$, the subtraction term for the operator $O_{\rm II}$ can be found. Applying this to $A = \not{D}$ results in the removal of the operators of the second class in Eq. (5.19). For the remaining operators of the first class the general relation $\not{a}\not{a} = a^2 - i\sigma^{\mu\nu}[a_\mu, a_\nu]/2$ is used,

$$\begin{aligned}\not{D}^2 &= D^2 - \frac{i}{2}\sigma^{\mu\nu}[D_\mu, D_\nu] = D^2 + \frac{1}{2}g_s\sigma^{\mu\nu}G_{\mu\nu},\\ \not{D}_\parallel^2 &= D_\parallel^2, \quad D^2 = D_\parallel^2 + D_\perp^2, \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu],\end{aligned} \tag{5.23}$$

the result reads

$$\mathcal{L}_{\rm HQET} = \bar{Q}\left\{iD_\parallel - \frac{1}{2m_Q}D_\perp^2 - \frac{g_s}{4m_Q}\sigma^{\mu\nu}G_{\mu\nu} + O\left(\frac{1}{m_Q^2}\right)\right\}Q. \tag{5.24}$$

The two terms of order $1/m_Q$ are called the *kinetic term* (proportional to $D_\perp^2$) and the *Fermi term*. Note in this context that the Lagrangians of HQET and NRQCD look similar at this stage. However, the order of terms is counted differently. For HQET the expansion parameter is p/m_Q where the momentum transfer p is at the same order as the QCD scale, $p \sim \Lambda_{\rm QCD}$. Therefore, the first term is of order $\Lambda_{\rm QCD}$ while the second (kinetic) term is of order $\Lambda_{\rm QCD}^2/m_Q$. For NRQCD, the expansion parameter is the three-velocity $\vec{v}$. Then both terms are of the same order $m_Q\vec{v}^{\,2}$.

5.1.3 Loop corrections

The expressions in Eqs. (5.14) and (5.24) are pure tree-level results. If loop corrections are included, all the different terms are accompanied by a corresponding coefficient, the effective Lagrangian up to order $1/m_Q^3$ reads [160]

$$\begin{aligned}\mathcal{L}_{\rm HQET} = \bar{Q}\Big\{ & iD_\parallel - \frac{c_k}{2m_Q}D_\perp^2 - \frac{c_f g}{4m_Q}\sigma_{\alpha\beta}G^{\alpha\beta} - \frac{c_d g}{8m_Q^2}v^\alpha[D_\perp^\beta G_{\alpha\beta}] + \\ & + \frac{ic_s g}{8m_Q^2}v_\lambda\sigma_{\alpha\beta}\{D_\perp^\alpha, G^{\lambda\beta}\} + \frac{c_{k2}}{8m_Q^3}D_\perp^4 + \frac{c_{w1}g}{16m_Q^3}\{D_\perp^2, \sigma_{\mu\nu}G^{\mu\nu}\} + \\ & - \frac{c_{w2}g}{8m_Q^3}D_{\perp\rho}\sigma_{\mu\nu}G^{\mu\nu}D_\perp^\rho + \frac{c_{p'p}g}{8m_Q^3}\left(\sigma_{\mu\nu}(D_{\perp\rho}G^{\rho\mu}D_\perp^\nu + D_\perp^\nu G^{\rho\mu}D_{\perp\rho} - D_{\perp\rho}G^{\mu\nu}D_\perp^\rho)\right) + \\ & - \frac{ic_m g}{8m_Q^3}\left(D_{\perp\mu}[D_{\perp\nu}G^{\mu\nu}] + [D_{\perp\nu}G^{\mu\nu}]D_{\perp\mu}\right) + \frac{c_{a1}g^2}{16m_Q^3}G_{\mu\nu}G^{\mu\nu} + \frac{c_{a2}g^2}{16m_Q^3}v_\mu v_\nu G_\rho{}^\mu G^{\rho\nu} + \\ & + \frac{c_{a3}g^2}{16m_Q^3}{\rm Tr}(G_{\mu\nu}G^{\mu\nu}) + \frac{c_{a4}g^2}{16m_Q^3}{\rm Tr}(v_\mu v_\nu G_\rho{}^\mu G^{\rho\nu}) + \\ & - \frac{ic_{b1}g^2}{16m_Q^3}\sigma_{\mu\nu}[G_\rho{}^\mu, G^{\rho\nu}] - \frac{ic_{b2}g^2}{16m_Q^3}v_\mu v_\nu\sigma_{\rho\lambda}[G^{\rho\mu}, G^{\lambda\nu}]\Big\}Q\end{aligned} \tag{5.25}$$

where square brackets indicate that these expressions do not act on Q. The indices k, f, d and s stand for the *kinetic*, *Fermi*, *Darwin* and *spin-orbit term*, resp. There are of course also interactions between the heavy and a light quark. These will be discussed later.

5.1.4 The matching procedure

Before starting the *matching procedure* to match the coefficients to the full QCD result, an ambiguity has to be mentioned that relates the coefficients to all orders of perturbation theory. This ambiguity is known as *reparametrization invariance* [161]. Although the effective theories are of non-relativistic character, there is a freedom concerning the velocity v (see e.g. Ref. [143]). If this velocity is changed by an amount $\delta v \sim \Lambda_{\rm QCD}/m_Q$, the Lagrangian has to be the same. This symmetry leads to relations between coefficients of different orders in the $1/m_Q$ expansion which are valid to all orders of perturbation theory. The first two of them are

$$c_k = 1, \qquad c_s = 2c_f + 1. \tag{5.26}$$

For this reason not all of the coefficients have to be matched perturbatively. In order to match the coefficients, the effective vertex of a heavy QCD quark and a single gluon is considered,

$$\Gamma^\mu(p,q) = ig_s\bar u(p+q)\left\{F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m_Q}\right\}u(p) \tag{5.27}$$

where $p = m_Qv + k$ and $p+q = m_Qv + k + q$ are the momenta of the incoming and outgoing quark, and q is the momentum transfer, carried by the incoming gluon. k and $k+q$, the quark momenta up to a general momentum m_Qv, are known as *residual momenta*. The situation in terms of the effective vertex diagram is depicted in Fig. 5.1. By using

$$\begin{aligned}\bar u(p+q)i\sigma^{\mu\nu}q_\nu u(p) &= \frac{1}{2}\bar u(p+q)[\not q,\gamma^\mu]u(p) = \\ &= \bar u(p+q)(2m_Q\gamma^\mu - (2p+q)^\mu)u(p),\end{aligned} \tag{5.28}$$

this effective vertex can be written alternatively as

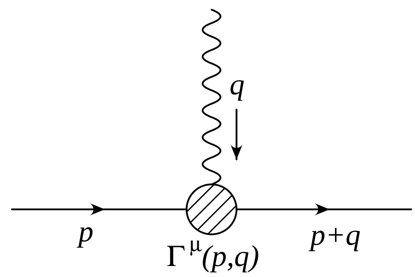


Figure 5.1: effective vertex diagram for the adjustment of the electric and magnetic form factor

$$\Gamma^\mu(p,q) = ig_s\bar u(p+q)\left\{\varepsilon(q^2)\frac{(2p+q)^\mu}{2m_Q} + \mu(q^2)\frac{[\not q,\gamma^\mu]}{4m_Q}\right\}u(p) \tag{5.29}$$

where $\varepsilon(q^2) = F_1(q^2)$ is the *electric* and $\mu(q^2) = F_1(q^2) + F_2(q^2)$ is the *magnetic form factor*. The connection between the QCD spinor $u(p)$ and the HQET spinor Q can easily be worked out. Starting from the effective vertex, all transformations via unitary matrices performed in the first part of the Foldy–Wouthuysen transformation are trivial. The only non-trivial transformation is the projection with P^+. Therefore, $Q = P^+u(p)$. Using the Dirac equation $(\not p - m_Q)u(p) = 0$, one can write

$$\begin{aligned}2m_Qu(p) &= m_Qu(p) + (m_Q\not v + \not k)u(p) = (m_Q(1+\not v) + \not k)\,u(p) = \\ &= 2m_QP^+u(p) + \not ku(p) = 2m_QQ + \not ku(p)\end{aligned} \tag{5.30}$$

and therefore

$$u(p) = \left(1 - \frac{\not k}{2m_Q}\right)^{-1}Q = \left(1 + \frac{\not k}{2m_Q} + \frac{k^2}{4m_Q^2} + \dots\right)Q. \tag{5.31}$$

Inserting this expansion into the effective vertex leads to

$$\begin{aligned}\Gamma^\mu(p,q) &= ig_s\bar{Q}\Bigg\{\varepsilon(q^2)\left(v^\mu+\frac{(2k+q)^\mu}{2m_Q}-\frac{q^2+[\not{k},\not{q}]}{8m_Q^2}v^\mu+\ldots\right)+\\ &\qquad +\mu(q^2)\left(\frac{[\not{q},\gamma^\mu]}{4m_Q}+\frac{q^2+[\not{k},\not{q}]}{4m_Q^2}+\ldots\right)\Bigg\}Q.\end{aligned}\tag{5.32}$$

Finally, the expansion

$$\varepsilon(q^2)=\varepsilon(0)+\varepsilon'(0)\frac{q^2}{m_Q^2}+\ldots,\qquad \mu(q^2)=\mu(0)+\mu'(0)\frac{q^2}{m_Q^2}+\ldots\tag{5.33}$$

of the electric and magnetic form factor leads to a result that has to be compared with the expression coming from the HQET Lagrangian,

$$ig_s\bar{Q}\left(v^\mu+c_k\frac{(2k+q)^\mu}{2m_Q}+c_f\frac{[\not{q},\gamma^\mu]}{4m_Q}+c_d\frac{q^2}{8m_Q^2}v^\mu+c_s\frac{[\not{k},\not{q}]}{8m_Q^2}v^\mu+\ldots\right)Q.\tag{5.34}$$

The comparison results in

$$\begin{aligned}c_k&=\varepsilon(0), & c_d&=8\varepsilon'(0)+2\mu(0)-\varepsilon(0),\\ c_f&=\mu(0), & c_s&=2\mu(0)-\varepsilon(0).\end{aligned}\tag{5.35}$$

The equality $\varepsilon(0)=1$ is a consequence of the reparametrization invariance and hence is valid to all orders of perturbation theory. It expresses the conservation of the chromoelectric charge, while the chromomagnetic charge need not be conserved.

5.1.5 Regularization and matching

As shown in Ref. [160], using dimensional regularization, for the HQET expression there are no finite parts for loop diagrams while the infrared (IR) and ultraviolet (UV) divergences cancel each other. Therefore, the only finite part that can occur is the matching coefficient $C_{\rm HQET}$ of the full QCD calculation (like c_k, c_d, c_f, and c_s). The HQET contribution for the radiative correction of a specific term of the Lagrangian is given by

$$F_{\rm HQET}=A_{\rm HQET}\left(\frac{1}{\varepsilon_{\rm UV}}-\frac{1}{\varepsilon_{\rm IR}}\right)+C_{\rm HQET}.\tag{5.36}$$

On the other hand, the QCD result will have the general structure

$$F_{\rm QCD}=A\left(\frac{1}{\varepsilon_{\rm UV}}+\ln\left(\frac{\mu}{m_Q}\right)\right)+B\left(\frac{1}{\varepsilon_{\rm IR}}+\ln\left(\frac{\mu}{m_Q}\right)\right)+C.\tag{5.37}$$

While the UV divergences are absorbed in the renormalization factors of HQET and QCD, resp., the remaining parts are used to write down the matching condition

$$-A_{\rm HQET}\frac{1}{\varepsilon_{\rm IR}}+C_{\rm HQET}=B\frac{1}{\varepsilon_{\rm IR}}+(A+B)\ln\left(\frac{\mu}{m_Q}\right)+C,\tag{5.38}$$

therefore $A_{\rm HQET}=-B$ and

$$C_{\rm HQET}=(A+B)\ln\left(\frac{\mu}{m_Q}\right)+C.\tag{5.39}$$

Here one only has to know the sum $A+B$ and thus does not have to distinguish between UV and IR divergences at all. Therefore, dimensional regularization is an appropiate tool.

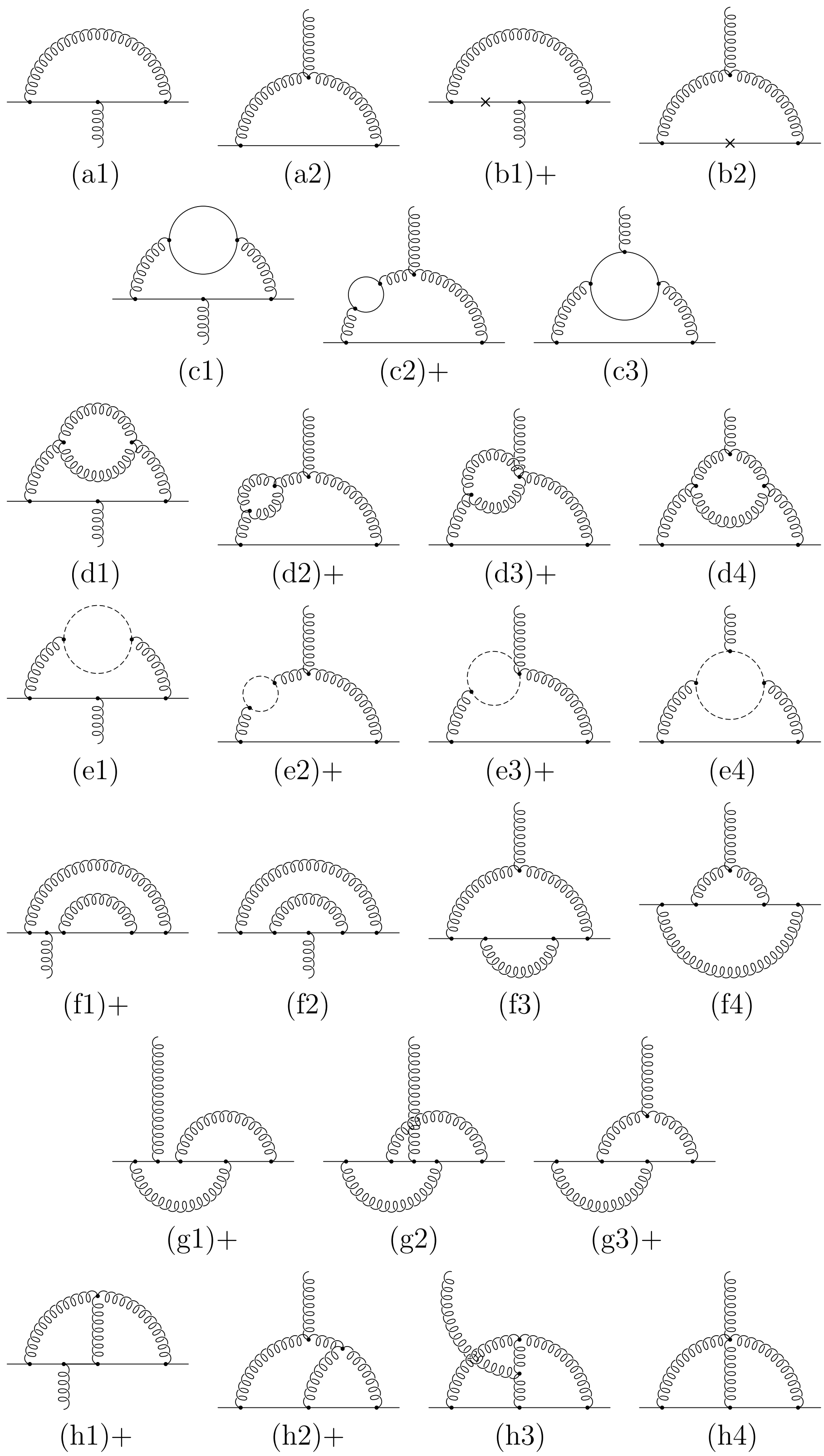


Figure 5.2: vertex diagrams. The "+" indicates an additional mirrored diagram

5.2 Two-loop matching of the second order terms

While the coefficients c_k and c_s are fixed due to reparametrization invariance, several attempts have been made to match the other coefficients. The one-loop matching of the Fermi coefficient c_f has been done in Refs. [163, 164]. In Refs. [165, 166] the anomalous dimension of the corresponding operator was calculated, the two-loop matching was done in Ref. [166], a resummation of all orders perturbation theory in the large β_0 limit was presented in Ref. [167]. All coefficients up to order $1/m_Q^3$ were matched in Ref. [160] up to one-loop order. In the work presented here the electric and magnetic form factor are determined up to two-loop order and to order q^2/m^2 in order to match the $1/m^2$ coefficients c_d and c_s (see Eq. (5.35)).

The diagrams which have to be calculated are shown in Fig. 5.2. The calculation was done using a general covariant gauge and the background field method [168] and made use of the package `recursor` written by David Broadhurst and Andrey Grozin [169] to handle the enormeous amount of two-loop integrals. In order to give an estimate about the complexity of the calculations one should note that each expansion in q to second order replaces the propagator containing this momentum by six propagators. The general covariant gauge replaces each of the gluon propagators by two terms. And the background field method applied to a three-gluon vertex also enlarges the number of terms by a factor between 1 and 2. The calculation was done in independent packages (in `reduce` and MATHEMATICA, resp.), where full agreement was obtained in the end.

5.2.1 The background field method

An off-shell gluon like the one considered in the vertex diagrams is given by a gauge dependent operator. Therefore, the calculated radiative corrections are not expected to be gauge independent as well. But there is a way out. On the level of one-loop integrals Bryce DeWitt developed the *background field method* in 1967 [170]. This method was later on extended to multiloop calculations by 't Hooft, DeWitt, Boulware and Abbott (for an overview see the paper of Laurence Abbott, Ref. [168]). The idea of this method is based on the introduction of an additional external gauge field. The starting point on the level of generating functionals and actions can already be understood for non-gauge theories and will be shown here. The generating functional is given by

$$Z[J] = \int [dA] \exp\{i(S_{\rm cl}[A] + J \cdot A)\} \tag{5.40}$$

where $S_{\rm cl}[A]$ is the classical action and $J \cdot A = \int J_\mu A^\mu d^4x$. In addition to the field A a *background field* B is introduced. The modified generating functional reads

$$Z[J,B] = \int [dA] \exp\{i(S_{\rm cl}[A+B] + J \cdot A)\}\,. \tag{5.41}$$

For the moment being, B is only an additional parameter. Therefore, all calculations in the functional calculus are the same as in the case where B is absent,

$$W[J,B] = -i \ln Z[J,B] \quad \Rightarrow \quad \bar{A} = \frac{\delta}{\delta J} W[J,B] \quad \Rightarrow \quad S_{\rm eff}[\bar{A},B] = W[J,B] - J \cdot \bar{A}. \tag{5.42}$$

It can easily be seen that $S_{\text{eff}}[\bar{A}, B] = S_{\text{eff}}[\bar{A} + B]$ where the latter expression is the non-modified effective action at $\bar{A} + B$, therefore one can set $\bar{A} = 0$ and obtains [168]

$$S_{\text{eff}}[0, B] = S_{\text{eff}}[B]. \tag{5.43}$$

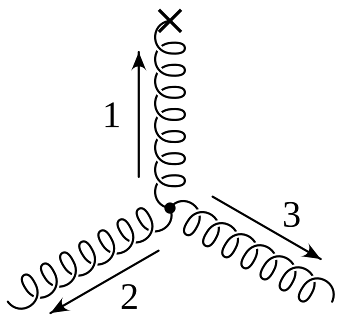


Figure 5.3: modification of the three-gluon vertex for the background field method. The cross represents the background field, the arrows show the momentum directions, and the numbers label the indices at the three ends of this Feynman diagram element.

The procedure explained here can be generalized to gauge theories like QCD with the same result. The great advantage of the background field method is that it retains explicit gauge invariance. There exists a choice for the gauge fixing term $-G_a G^a/2\alpha_g$ occuring in the generating functional for which the effective action $S_{\text{eff}}[0, B]$ is a gauge invariant functional of B. This gauge choice is given by the QCD field strength tensor

$$\bar{G}_a = \partial_\mu A_a^\mu + igf_{abc}A^{\mu b}B_\mu^c. \tag{5.44}$$

This choice has consequences for the Feynman rules. Relevant in the actual application are only Feynman rules including up to one background field. While the four-gluon vertex remains unchanged, the three-gluon vertex is changed. The three-gluon vertex between internal gluon lines reads

$$g_s f_{a_1a_2a_3}\left\{(k_2 - k_3)_{\mu_1} g_{\mu_2\mu_3} + (k_3 - k_1)_{\mu_2} g_{\mu_3\mu_1} + (k_1 - k_2)_{\mu_3} g_{\mu_1\mu_2}\right\} \tag{5.45}$$

while for the first gluon coupled to a background field as for the situation shown in Fig. 5.3, the three-gluon vertex is modified to

$$g_s f_{a_1a_2a_3}\left\{(k_2 - k_3)_{\mu_1} g_{\mu_2\mu_3} + \left(k_3 - k_1 + \frac{k_2}{\alpha_g}\right)_{\mu_2} g_{\mu_3\mu_1} + \left(k_1 - k_2 - \frac{k_3}{\alpha_g}\right)_{\mu_3} g_{\mu_1\mu_2}\right\}. \tag{5.46}$$

A further consequence of the background field method is that the relation between the renormalization factors of coupling (Z_g) and gauge field (Z_{1g}) reads $Z_{1g}Z_g = 1$. This is a consequence of the requirement that the renormalization of the field strength tensor

$$\begin{aligned} G_{\mu\nu}^b &= \left(\partial_\mu A_\nu^b - \partial_\nu A_\mu^b - ig^b[A_\mu^b, A_\nu^b]\right) = \\ &= \left(\partial_\mu Z_{1g}A_\nu - \partial_\nu Z_{1g}A_\mu - iZ_g g[Z_{1g}A_\mu, Z_{1g}A_\nu]\right) = \\ &= Z_{1g}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - iZ_g Z_{1g}[A_\mu, A_\nu]\right) \stackrel{!}{=} Z_{1g}G_{\mu\nu} \end{aligned} \tag{5.47}$$

is gauge invariant. Because of this relation, the *Slavnov-Taylor identities* are replaced by *Ward identities*. Therefore, all divergences along quark lines can be removed by a renormalization procedure using renormalization factors (multiplicative renormalization).

5.2.2 Projectors for the electric and magnetic form factor

In order to distinguish between the electric and the magnetic form factor in the result obtained for the vertex correction $\Gamma^\mu(p, q)$ from the calculation of the set of diagrams presented in the beginning of this section, one has to find projectors on these components. The general ansatz for this is given by

$$\begin{aligned}
&\mathrm{Tr}\,(B_\mu(\not p+\not q+m_Q)\Gamma^\mu(p,q)(\not p+m_Q)) = \\
&= \mathrm{Tr}\left(B_\mu(\not p+\not q+m_Q)\left(\varepsilon(q^2)\frac{(2p+q)_\mu}{2m_Q}+\mu(q^2)\frac{[\not q,\gamma^\mu]}{4m_Q}\right)(\not p+m_Q)\right). \quad (5.48)
\end{aligned}$$

where Γ^μ is the result obtained by calculating the diagrams, and $(p+q)^2 = p^2 = m_Q^2$ (therefore, $2pq = -q^2$). In order that the Ward identity is valid, B_μ must satisfy

$$\mathrm{Tr}\,(B_\mu(\not p+\not q+m_Q)q^\mu(\not p+m_Q)) = 0. \quad (5.49)$$

The two linearly independent possibilities to satisfy this equation are given by $B_\mu = \gamma_\mu$ and $B_\mu = 2p_\mu + q_\mu$. Therefore, B_μ have to be built up by these two elements. But on the other hand, the problem to solve Eq. (5.48) for ε and μ is unique. One obtains the system of equations

$$\begin{aligned}
&\mathrm{Tr}\,(\gamma_\mu(\not p+\not q+m_Q)\Gamma^\mu(p,q)(\not p+m_Q)) = \\
&= \mathrm{Tr}\left(\gamma_\mu(\not p+\not q+m_Q)\left(\varepsilon(q^2)\frac{(2p+q)_\mu}{2m_Q}+\mu(q^2)\frac{[\not q,\gamma^\mu]}{4m_Q}\right)(\not p+m_Q)\right) = \\
&= -2\varepsilon(q^2)(q^2-4m_Q^2)+2(D-1)\mu(q^2)q^2, \\
&2m_Q\mathrm{Tr}\,((2p+q)_\mu(\not p+\not q+m_Q)\Gamma^\mu(p,q)(\not p+m_Q)) = \\
&= 2m_Q\mathrm{Tr}\left((2p+q)_\mu(\not p+\not q+m_Q)\left(\varepsilon(q^2)\frac{(2p+q)_\mu}{2m_Q}+\mu(q^2)\frac{[\not q,\gamma^\mu]}{4m_Q}\right)(\not p+m_Q)\right) = \\
&= 2\varepsilon(q^2)(q^2-4m_Q^2)^2-2\mu(q^2)(q^2-4m_Q^2)q^2 \quad (5.50)
\end{aligned}$$

which can be solved for $\varepsilon(q^2)$ and $\mu(q^2)$,

$$\begin{aligned}
2(D-2)q^2\mu(q^2) &= \mathrm{Tr}\left(\left(\gamma_\mu+\frac{2m_Q(2p+q)_\mu}{q^2-4m_Q^2}\right)(\not p+\not q+m_Q)\Gamma^\mu(p,q)(\not p+m_Q)\right), \\
2(D-2)(q^2-4m_Q^2)\varepsilon(q^2) &= \\
&= \mathrm{Tr}\left(\left(\gamma_\mu+\frac{2m_Q(D-1)(2p+q)_\mu}{q^2-4m_Q^2}\right)(\not p+\not q+m_Q)\Gamma^\mu(p,q)(\not p+m_Q)\right). \quad (5.51)
\end{aligned}$$

The projections are given by the prescription to take the contaction and trace with

$$\begin{aligned}
P_\mu^E(p,q) &= \frac{\not p+m_Q}{2(D-2)(q^2-4m_Q^2)}\left(\gamma_\mu+\frac{2m_Q(D-1)(2p+q)_\mu}{q^2-4m_Q^2}\right)(\not p+\not q+m_Q), \\
P_\mu^M(p,q) &= \frac{\not p+m_Q}{2(D-2)q^2}\left(\gamma_\mu+\frac{2m_Q(2p+q)_\mu}{q^2+m_Q^2}\right)(\not p+\not q+m_Q). \quad (5.52)
\end{aligned}$$

Because the quark is on-shell (a condition that has already been used), one can replace $p = m_Qv$ and write $\Gamma^\mu(q) = \Gamma^\mu(p,q)$. The only remaining momentum is the gluon momentum q, and the expansion of the vertex in this momentum up to second order is given by

$$\Gamma^\mu(q) = \Gamma^\mu + \Gamma^\mu_\alpha q^\alpha + \Gamma^\mu_{\alpha\beta}q^\alpha q^\beta + O(q^3). \quad (5.53)$$

Expanding also the projectors in powers of q, one obtains

$$
\begin{aligned}
P^E_\mu(q) &= \frac{m_Q(1+v)}{2(D-2)(q^2-4m_Q^2)}\left\{\gamma_\mu + \frac{2m_Q(D-1)(2m_Q v+q)_\mu}{q^2-4m_Q^2}\right\}(m_Q(1+\not{v})+\not{q}) = \\
&= \frac{-(1+\not{v})}{8(D-2)m_Q}\left(1+\frac{q^2}{4m_Q^2}\right) \times \\
&\qquad \times \left\{\gamma_\mu - \frac{D-1}{2m_Q}\left(1+\frac{q^2}{4m_Q^2}\right)(2m_Q v+q)_\mu\right\}(m_Q(1+\not{v})+\not{q}) + O(q^3) = \\
&= \frac{-(1+\not{v})}{8(D-2)m_Q}\Big\{(\gamma_\mu-(D-1)v_\mu)\, m_Q(1+\not{v}) + \\
&\qquad -\frac{D-1}{2m_Q}q_\mu m_Q(1+\not{v}) + (\gamma_\mu-(D-1)v_\mu)\,\not{q} + \\
&\qquad -\frac{D-1}{2m_Q}q_\mu\not{q} + \frac{q^2}{4m_Q^2}(\gamma_\mu - 2(D-1)v_\mu)\, m_Q(1+\not{v}) + O(q^3)\Big\}, \qquad (5.54)\\
P^M_\mu(q) &= \frac{m_Q(1+\not{v})}{2(D-2)q^2}\left\{\gamma_\mu + \frac{2m_Q(2m_Q v+q)_\mu}{q^2-4m_Q^2}\right\}(m_Q(1+\not{v})+\not{q}) = \\
&= \frac{m_Q(1+\not{v})}{2(D-2)q^2}\left\{\gamma_\mu - \left(v_\mu+\frac{q_\mu}{2m_Q}\right)\left(1+\frac{q^2}{4m_Q^2}\right)\right\}(m_Q(1+\not{v})+\not{q}) + O(q^3) = \\
&= \frac{m_Q(1+\not{v})}{2(D-2)q^2}\Big\{(\gamma_\mu-v_\mu)m_Q(1+\not{v}) - \frac{q_\mu}{2m_Q}m_Q(1+\not{v}) + (\gamma_\mu-v_\mu)\not{q} + \\
&\qquad -\frac{q_\mu}{2m_Q}\not{q} - \frac{q^2}{4m_Q^2}v_\mu m_Q(1+\not{v}) + O(q^3)\Big\}. \qquad (5.55)
\end{aligned}
$$

The projectors which project out the different q dependencies of $\Gamma^\mu(q)$ can be constructed by a comparison using the Passarino–Veltman method (the factor 1/4 is by convention),

$$
\begin{aligned}
\frac{1}{4}\mathrm{Tr}(P^i_\mu\Gamma^\mu) &:= \mathrm{Tr}(P^i_\mu(q)\Gamma^\mu) = A^i + B^i_\rho q^\rho + C^i_{\rho\sigma}q^\rho q^\sigma + O(q^3),\\
\frac{1}{4}\mathrm{Tr}(P^i_{\mu\alpha}\Gamma^{\mu\alpha}) &:= \mathrm{Tr}(P^i_\mu(q)\Gamma^\mu_\alpha q^\alpha) = D^i_\rho q^\rho + E^i_{\rho\sigma}q^\rho q^\sigma + O(q^3),\\
\frac{1}{4}\mathrm{Tr}(P^i_{\mu\alpha\beta}\Gamma^{\mu\alpha\beta}) &:= \mathrm{Tr}(P^i_\mu(q)\Gamma^\mu_{\alpha\beta} q^\alpha q^\beta) = F^i_{\rho\sigma}q^\rho q^\sigma + O(q^3) \qquad (i=E,M) \qquad (5.56)
\end{aligned}
$$

(where $B^i_\rho = B^i v_\rho$, $C^i_{\rho\sigma} = C^i g_{\rho\sigma} + \tilde{C}^i v_\rho v_\sigma$ etc.) which results in

$$
\begin{aligned}
P^E_\mu(q^2) &= (1+\not{v})v_\mu + O(q^4),\\
P^E_{\mu\alpha}(q^2) &= \frac{q^2}{2m_Q}(1+\not{v})\left\{\frac{g_{\mu\alpha}+v_\mu\gamma_\alpha}{D-2} - \frac{\gamma_\mu(\gamma_\alpha+v_\alpha)}{(D-1)(D-2)} - \frac{(D+1)v_\mu v_\alpha}{D-1}\right\} + O(q^4),\\
P^E_{\mu\alpha\beta}(q^2) &= q^2(1+\not{v})v_\mu\frac{g_{\alpha\beta}-v_\alpha v_\beta}{D-1} + O(q^4),\\
P^M_\mu(q^2) &= (1+\not{v})\left\{\frac{\gamma_\mu - v_\mu}{D-1}\right\} + O(q^2),\\
P^M_{\mu\alpha}(q^2) &= \frac{-2m_Q}{(D-1)(D-2)}(1+\not{v})\left\{g_{\mu\alpha} - (\gamma_\mu-v_\mu)\gamma_\alpha - \gamma_\mu v_\alpha\right\} + O(q^2),\\
P^M_{\mu\alpha\beta}(q^2) &= O(q^2). \qquad (5.57)
\end{aligned}
$$

Using these definitions, one can find $\varepsilon(0)$, $\varepsilon'(0)$, and $\mu(0)$ by comparing powers of q^2,

$$\begin{aligned}
\varepsilon(q^2) &= \mathrm{Tr}\left(P^E_\mu(q)\Gamma^\mu(q)\right) = \\
&= \mathrm{Tr}\left(P^E_\mu(q)\Gamma^\mu\right) + \mathrm{Tr}\left(P^E_\mu(q)\Gamma^\mu_\alpha q^\alpha\right) + \mathrm{Tr}\left(P^E_\mu(q)\Gamma^\mu_{\alpha\beta}q^\alpha q^\beta\right) + O(q^4) = \\
&= \frac{1}{4}\mathrm{Tr}\left(P^E_\mu(q^2)\Gamma^\mu\right) + \frac{1}{4}\mathrm{Tr}\left(P^E_{\mu\alpha}(q^2)\Gamma^{\mu\alpha}\right) + \frac{1}{4}\mathrm{Tr}\left(P^E_{\mu\alpha\beta}(q^2)\Gamma^{\mu\alpha\beta}\right) + O(q^4) = \\
&= \varepsilon(0) + \varepsilon'(0)\frac{q^2}{m_Q^2} + O(q^4/m_Q^4),
\end{aligned} \tag{5.58}$$

$$\begin{aligned}
\mu(q^2) &= \mathrm{Tr}\left(P^M_\mu(q)\Gamma^\mu(q)\right) = \\
&= \mathrm{Tr}\left(P^M_\mu(q)\Gamma^\mu\right) + \mathrm{Tr}\left(P^M_\mu(q)\Gamma^\mu_\alpha q^\alpha\right) + \mathrm{Tr}\left(P^M_\mu(q)\Gamma^\mu_{\alpha\beta}q^\alpha q^\beta\right) + O(q^2) = \\
&= \frac{1}{4}\mathrm{Tr}\left(P^M_\mu(q^2)\Gamma^\mu\right) + \frac{1}{4}\mathrm{Tr}\left(P^M_{\mu\alpha}(q^2)\Gamma^{\mu\alpha}\right) + \frac{1}{4}\mathrm{Tr}\left(P^M_{\mu\alpha\beta}(q^2)\Gamma^{\mu\alpha\beta}\right) + O(q^2) = \\
&= \mu(0) + O(q^2/m_Q^2).
\end{aligned} \tag{5.59}$$

5.2.3 One-loop results

At this moment only some of the results will be presented here. As being work in progress, subsets of contributing diagrams have been completed but not the complete set. Only partial (though gauge independent) results for the two-loop diagrams will be given in the following subsections while the results for the one-loop diagrams (a1) and (a2) in Fig. 5.2 are subject of this subsection. The result is written in an intermediate step in terms of dimensionless integrals $M(n_1, n_2; 1/z)$ which are defined by

$$\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left(-k^2+m_Q^2\right)^{n_1}\left(-(p-k)^2\right)^{n_2}} =: \frac{i}{(4\pi)^{D/2}}(m_Q^2)^{D/2-n_1-n_2}M(n_1,n_2;-p^2/m_Q^2). \tag{5.60}$$

The final results for diagram (a1) (in case of $m_Q = \mu$) read

$$\begin{aligned}
\frac{1}{4}\mathrm{Tr}\left(P^E_\mu(q^2)\Gamma^\mu_{a1}\right) &= \frac{\alpha_s}{4\pi N_c}\Gamma(1+\varepsilon)\frac{D-1}{(D-4)(D-3)}, \\
\frac{1}{4}\mathrm{Tr}\left(P^E_{\mu\alpha}(q^2)\Gamma^{\mu\alpha}_{a1}\right) &= \frac{\alpha_s}{4\pi N_c}\left(\frac{q^2}{m_Q^2}\right)\Gamma(1+\varepsilon)\frac{3D^2-27D+52+4\alpha_g}{4(D-5)(D-4)(D-3)}, \\
\frac{1}{4}\mathrm{Tr}\left(P^E_{\mu\alpha\beta}(q^2)\Gamma^{\mu\alpha\beta}_{a1}\right) &= -\frac{\alpha_s}{4\pi N_c}\left(\frac{q^2}{m_Q^2}\right)\Gamma(1+\varepsilon)\frac{3D^3-25D^2+50D-24+12\alpha_g}{12(D-5)(D-4)(D-3)}, \\
\frac{1}{4}\mathrm{Tr}\left(P^M_\mu(q^2)\Gamma^\mu_{a1}\right) &= \frac{\alpha_s}{4\pi N_c}\Gamma(1+\varepsilon)\frac{\alpha_g}{(D-4)(D-3)}, \\
\frac{1}{4}\mathrm{Tr}\left(P^M_{\mu\alpha}(q^2)\Gamma^{\mu\alpha}_{a1}\right) &= \frac{\alpha_s}{4\pi N_c}\Gamma(1+\varepsilon)\frac{D^2-8D+19-\alpha_g}{(D-4)(D-3)}
\end{aligned} \tag{5.61}$$

With this one obtains the gauge independent results

$$\varepsilon_{a1}(0) = \frac{\alpha_s}{4\pi N_c}\Gamma(1+\varepsilon)\frac{D-1}{(D-4)(D-3)} = \frac{\alpha_s}{4\pi N_c}\left(-\frac{3}{2\varepsilon}-2\right),$$

$$\begin{aligned}
\varepsilon'_{a1}(0) &= -\frac{\alpha_s}{4\pi N_c}\Gamma(1+\varepsilon)\frac{3D^2+19D+36}{12(D-4)(D-3)} = \frac{\alpha_s}{4\pi N_c}\left(\frac{1}{3\varepsilon}+\frac{1}{4}\right), \\
\mu_{a1}(0) &= \frac{\alpha_s}{4\pi N_c}\Gamma(1+\varepsilon)\frac{D^2-8D+19}{(D-4)(D-3)} = \frac{\alpha_s}{4\pi N_c}\left(-\frac{3}{2\varepsilon}-3\right).
\end{aligned} \tag{5.62}$$

For diagram (a2) one obtains

$$\begin{aligned}
\frac{1}{4}\mathrm{Tr}\left(P^E_\mu(q^2)\Gamma^\mu_{a2}\right) &= -\frac{\alpha_s N_c}{4\pi}\Gamma(1+\varepsilon)\frac{D-1}{(D-4)(D-3)}, \\
\frac{1}{4}\mathrm{Tr}\left(P^E_{\mu\alpha}(q^2)\Gamma^{\mu\alpha}_{a2}\right) &= -\frac{\alpha_s N_c}{4\pi}\left(\frac{q^2}{m_Q^2}\right)\Gamma(1+\varepsilon)\frac{D^2-8D+15+2\alpha_g}{2(D-5)(D-4)(D-3)}, \\
\frac{1}{4}\mathrm{Tr}\left(P^E_{\mu\alpha\beta}(q^2)\Gamma^{\mu\alpha\beta}_{a2}\right) &= \frac{\alpha_s N_c}{4\pi}\left(\frac{q^2}{m_Q^2}\right)\Gamma(1+\varepsilon)\frac{D^2-7D+8+2\alpha_g}{2(D-5)(D-4)(D-3)}, \\
\frac{1}{4}\mathrm{Tr}\left(P^M_\mu(q^2)\Gamma^\mu_{a2}\right) &= -\frac{\alpha_s N_c}{4\pi}\Gamma(1+\varepsilon)\frac{\alpha_g}{(D-4)(D-3)}, \\
\frac{1}{4}\mathrm{Tr}\left(P^M_{\mu\alpha}(q^2)\Gamma^{\mu\alpha}_{a2}\right) &= -\frac{\alpha_s N_c}{4\pi}\Gamma(1+\varepsilon)\frac{5-\alpha_g}{(D-4)(D-3)},
\end{aligned} \tag{5.63}$$

therefore

$$\begin{aligned}
\varepsilon_{a2}(0) &= -\frac{\alpha_s N_c}{4\pi}\Gamma(1+\varepsilon)\frac{D-1}{(D-4)(D-3)} = \frac{\alpha_s N_c}{4\pi}\left(\frac{3}{2\varepsilon}+2\right), \\
\varepsilon'_{a2}(0) &= \frac{\alpha_s N_c}{4\pi}\Gamma(1+\varepsilon)\frac{D-7}{2(D-5)(D-4)(D-3)} = \frac{\alpha_s N_c}{4\pi}\left(-\frac{3}{4\varepsilon}-\frac{1}{2}\right), \\
\mu_{a2}(0) &= -\frac{\alpha_s N_c}{4\pi}\Gamma(1+\varepsilon)\frac{5}{(D-4)(D-3)} = \frac{\alpha_s N_c}{4\pi}\left(\frac{5}{2\varepsilon}+5\right).
\end{aligned} \tag{5.64}$$

Both results can be combined using the colour factors

$$C_F = \frac{N_c^2-1}{2N_c}, \qquad C_A = N_c \quad \Rightarrow \quad C_F - C_A = \frac{-1}{2N_c} \tag{5.65}$$

which gives the corrections to $\varepsilon(0)$, $\varepsilon'(0)$ and $\mu(0)$ up to one-loop order,

$$\begin{aligned}
\varepsilon(0) &= 1+\frac{\alpha_s}{4\pi}(C_F-C_A)\left(\frac{3}{\varepsilon}+4\right)+\frac{\alpha_s}{4\pi}C_A\left(\frac{3}{2\varepsilon}+2\right)+O(\alpha_s^2) = \\
&= 1+\frac{\alpha_s}{4\pi}\left(C_F-\frac{1}{2}C_A\right)\left(\frac{3}{\varepsilon}+4\right)+O(\alpha_s^2), \\
\varepsilon'(0) &= -\frac{\alpha_s}{4\pi}(C_F-C_A)\left(\frac{2}{3\varepsilon}+\frac{1}{2}\right)-\frac{\alpha_s}{4\pi}C_A\left(\frac{3}{4\varepsilon}+\frac{1}{2}\right)+O(\alpha_s^2) = \\
&= -\frac{\alpha_s}{4\pi}C_F\left(\frac{2}{3\varepsilon}+\frac{1}{2}\right)-\frac{\alpha_s}{4\pi}C_A\left(\frac{1}{12\varepsilon}\right)+O(\alpha_s^2), \\
\mu(0) &= 1+\frac{\alpha_s}{4\pi}(C_F-C_A)\left(\frac{3}{\varepsilon}+6\right)-\frac{\alpha_s}{4\pi}C_A\left(\frac{5}{2\varepsilon}+5\right)+O(\alpha_s^2) = \\
&= 1+\frac{\alpha_s}{4\pi}\left(3C_F-\frac{1}{2}C_A\right)\left(\frac{1}{\varepsilon}+2\right)+O(\alpha_s^2).
\end{aligned} \tag{5.66}$$

The contribution to $\varepsilon(0)$ is absorbed by the renormalization factor, so that the renormalized coefficient is $\varepsilon(0)=1$. The results presented here coincide with those obtained in Ref. [160] (note that in Ref. [160] $D=4-\varepsilon$ is used instead of $D=4-2\varepsilon$).

5.2.4 Details on the two-loop calculation

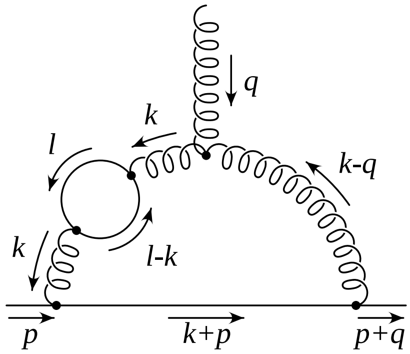


Figure 5.4: The diagram (c1) with momenta indicated

The diagram (c2) as shown in Fig. 5.4 can serve as an example to explain the necessary steps in order to calculate two-loop diagrams of this form in full QCD, as it is done by the MATHEMATICA package written for this purpose in a complete automatic way. As input for the package only the specification of the propagators is necessary and the choice for the form factor and the order in q. A typical element is the massive quark line symbol `massiline[a,k+p,b]` where `a` and `b` represent the two end points of the propagator and `k+p` is the momentum of the line. The horizontal line in Fig. 5.4 is called the base line of the diagram. The fermion lines are closed by final elements `massiend[a]`. The variable `a`, changed internally to a variable according to the programming systematics, is used for all "characters" of this vertex, i.e. the Lorentz index, the Dirac structure element, and the colour index. The propagators and associated vertex factors are combined by the package according to the usual QCD Feynman rules (see e.g. Ref. [171]). In case of diagram (c2) one obtains

$$\begin{aligned}\int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D} ig(T_b)_i{}^j\gamma_\beta\left(\frac{i}{\not k+\not p-m_Q}\right) ig(T_a)_j{}^k\gamma_\alpha \mathrm{Tr}\left(igT_s\gamma_\sigma\frac{i}{\not l-\not k}igT_r\gamma_\rho\frac{i}{\not l}\right)\times\\ \times\frac{-i\delta^{bl}}{(k-q)^2}\left(g^{\beta\lambda}-\frac{(k-q)^\beta(k-q)^\lambda}{(k-q)^2}\right)\frac{-i\delta^{ns}}{k^2}\left(g^{\nu\sigma}-\frac{k^\nu k^\sigma}{k^2}\right)\frac{-i\delta^{ra}}{k^2}\left(g^{\rho\alpha}-\frac{k^\rho k^\alpha}{k^2}\right)\times\\ \times\, gf_{mnl}\left(g_{\mu\nu}(k+q-\frac{1}{\alpha_g}(k-q))+g_{\nu\lambda}(q-2k)_\mu+g_{\lambda\mu}(k-2q-\frac{1}{\alpha_g}k)_\nu\right)\end{aligned} \tag{5.67}$$

The structure of this expression is already rather complicated. The structure becomes even more complicated when expanded in powers in q and projected. But the expression can be decomposed into different structural units which are dealt with separately in the package.

The colour structure

The colour structure is expressed by the Gell–Mann matrices T_a, the structure constants f_{abc}, and the Kronecker symbols δ_{ab}. Only when the four-gluon vertex appears, the factorization of the colour structure is no longer possible and one has to split the calculation into three separate blocks. The calculation of the colour structure leads to the *colour factors*

$$C_A=N_c,\quad C_F=\frac{N_c^2-1}{2N_c}\quad\text{and}\quad C_B=\frac{N_c-1}{2N_c} \tag{5.68}$$

where N_c is the number of colours.

The Dirac structure

The Dirac structure of diagram (c2) is given by the massive quark line and the massless quark loop (in the final result this massless quark loop is amended by the massive quark loop where the mass of this loop is the same as the mass of the base line). The Dirac structure for (c2) is given by

$$\gamma_\beta(\not{k}+\not{p}+m_Q)\gamma_\alpha \ \times \ \mathrm{Tr}(\gamma_\sigma(\not{l}-\not{k})\gamma_\rho\not{l}), \tag{5.69}$$

including the Dirac factors from the projector. The Dirac structure is an intermediate object because the trace is taken in the end. The package, however, gives the option to keep the structure up to the end. This is of interest especially in those cases where general vertex factors are considered. In the actual case, however, the Dirac structure will be attached to the momentum structure which is explained next.

The momentum structure

In order to extract the momentum structure, a common denominator is chosen for the propagators. In case that the gluon momentum q appears, this factor has to be expanded. The maximum degree of this expansion depends on the choice for the projector. The expansion is a geometric series expansion,

$$\frac{1}{(k-q)^2}=\frac{1}{k^2}\left(1+\frac{2kq-q^2}{k^2}+\left(\frac{2kq-q^2}{k^2}\right)^2+\dots\right). \tag{5.70}$$

A common denominator is chosen again, and only those parts are used which have the selected power of q. After the projection the momentum structure is resolved for this scalar integral to scalar products of the inner and outer momenta.

The initial integral

The initial integral is the integral over scalar propagators with the maximally necessary power of denominator factors. It can be identified by two different kinds of integral types (denoted by capital M and N) [169],

$$\begin{aligned}&\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\times\\&\quad\times\frac{1}{(-k^2)^{n_1}(-l^2)^{n_2}(-(k+p)^2+m_Q^2)^{n_3}(-(l+p)^2+m_Q^2)^{n_4}(-(k-l)^2)^{n_5}}=\\&\qquad=:\ \frac{-1}{(4\pi)^D}(m_Q^2)^{D-n_1-n_2-n_3-n_4-n_5}M(n_1,n_2,n_3,n_4,n_5;-p^2/m_Q^2),\end{aligned} \tag{5.71}$$

$$\begin{aligned}&\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\times\\&\quad\times\frac{1}{(-k^2)^{n_1}(-l^2)^{n_2}(-(k+p)^2+m_Q^2)^{n_3}(-(l+p)^2+m_Q^2)^{n_4}(-(k+l+p)^2-m_Q^2)^{n_5}}=\\&\qquad=:\ \frac{-1}{(4\pi)^D}(m_Q^2)^{D-n_1-n_2-n_3-n_4-n_5}N(n_1,n_2,n_3,n_4,n_5;-p^2/m_Q^2).\end{aligned} \tag{5.72}$$

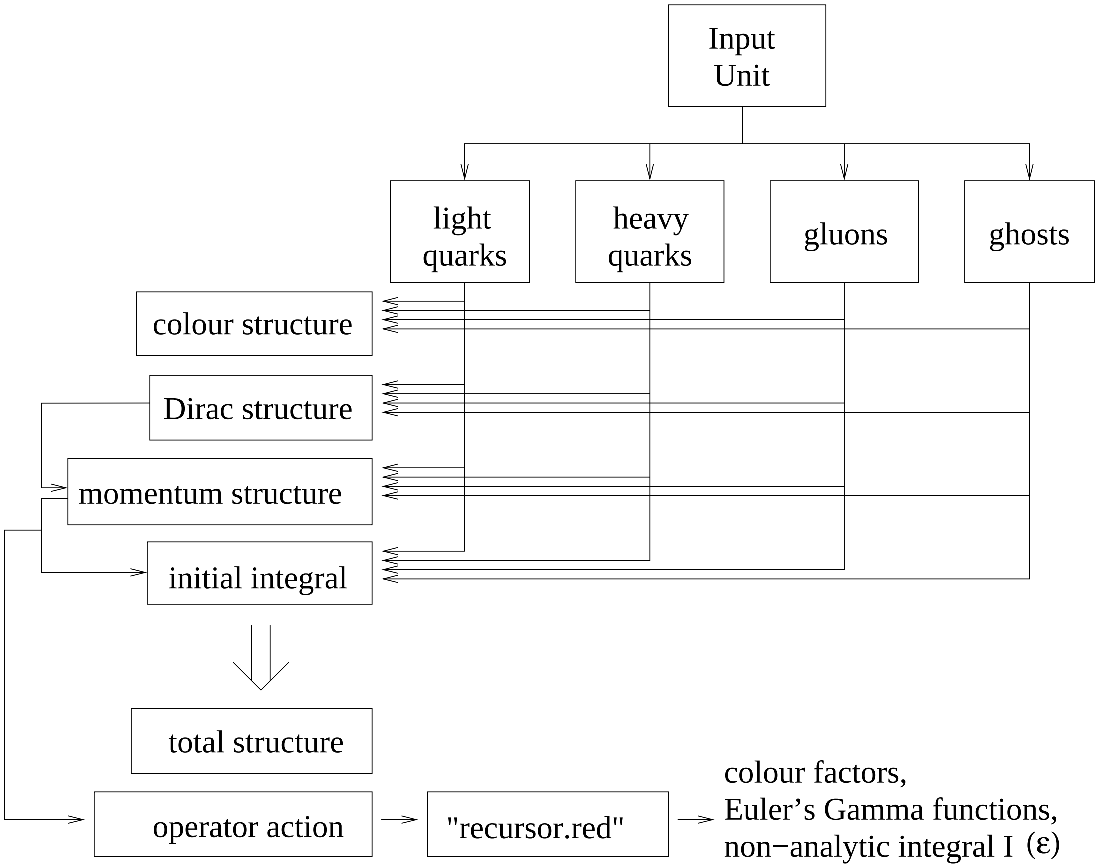


Figure 5.5: algorithmical structure of the MATHEMATICA package

First assembly

Looking at Fig. 5.5, one can see that after the treatment of these five structure elements an assembly is possible. An intermediate step is the assembly of expressions like

$$\{<\mathit{factor}>, <\mathit{momenta}>, <\mathit{products}>, <\mathit{Dirac\ structure}>, <\mathit{initial\ integral}>\} \tag{5.73}$$

where in this case the second and fourth entry remains empty because there is no Dirac structure left and no momenta coupled to it.

Operation on the entries

The scalar products occuring in the third entry can reduce the powers of the denominator factors and therefore change the entries n_i of the initial integrals according to rules as

$$2k\cdot l = k^2 + l^2 - (k-l)^2 \sim \mathbf{1}^- + \mathbf{2}^- - \mathbf{5}^-. \tag{5.74}$$

For this reason the momentum structure is changed to an operator structure which acts on the initial integral and decreases its entries n_i (see later on). The operator action does not lead out of the set of integral types M or N. Therefore, the diagrams (cx) with massive loop (called (mx)) and the diagrams (gx) result in integrals of type N. The remaining diagrams result in diagrams of type M.

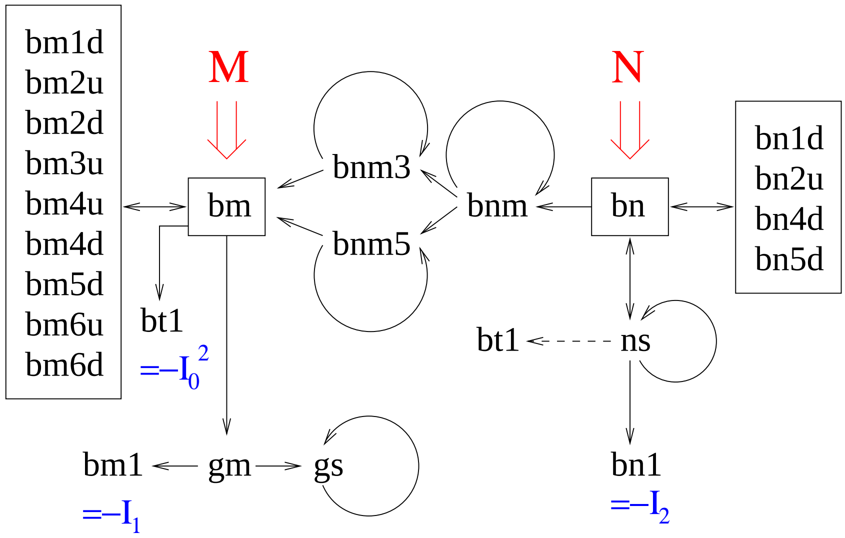


Figure 5.6: algorithmical scheme for the module `recursor.red`

The recursion

The integrals of the two types can be reduced to three basic integrals I_0^2, I_1 and I_2 which again are expressible in term of Euler's gamma functions and a single non-analytic integral $I(\varepsilon)$,

$$\begin{aligned}
I_0 &= i m_Q^{-2\varepsilon} M(0,1) \;= \\
&= \frac{1}{\pi^{D/2}} \int \frac{d^D k}{-(k+p)^2 + m_Q^2} \;=\; \frac{2\Gamma(\varepsilon)}{D-2}, \\
&\quad \text{where } M(0,0,0,1,1) = M(0,1)^2, \\[2ex]
I_1 &= -m_Q^{-4\varepsilon} M(1,0,0,1,1) \;= \\
&= \frac{1}{\pi^D} \int \frac{d^D k \, d^D l}{(-k^2)(-(k-l)^2)(-(l+p)^2 + m_Q^2)} \;= \\
&= \frac{4(2D-7)}{(D-3)(3D-8)(3D-10)} \, \frac{\Gamma^2(1-\varepsilon)\Gamma(1+2\varepsilon)\Gamma(1-4\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1-3\varepsilon)} \Gamma^2(\varepsilon), \\[2ex]
I_2 &= -m^{-4\varepsilon} N(0,0,1,1,1) \;= \\
&= \frac{1}{\pi^D} \int \frac{d^D k \, d^D l}{(-(k+p)^2 + m_Q^2)(-(l+p)^2 + m_Q^2)(-(k+l+p)^2 + m_Q^2)} \;= \\
&= \frac{3(D-2)^2(5D-18)}{2(D-3)(3D-8)(3D-10)} I_0^2 - \frac{2(D-4)}{2D-7} I_1 - \frac{16(D-4)^2}{(3D-8)(3D-10)} I(\varepsilon).
\end{aligned} \tag{5.75}$$

For the concise calculation the package `recursor.red` for the recursive calculation of general three-loop integrals [83] running under `reduce` has been translated to MATHEMATICA by the author and used in the own computer package. The algorithmic structure of this package is shown in Fig. 5.6 without going into detail.

5.2.5 Results for the two-loop calculation

At this point a preliminary results can be shown, as it is given by the contribution for the diagram (c2). The result is given by

$$\begin{aligned}
\varepsilon_{c2}(0) &= -\frac{g_s^4 C_A N_l T_F}{(4\pi)^4}\left[\frac{2(D-3)(D-2)}{(2D-7)}\right] I_1, \\
\varepsilon'_{c2}(0) &= -\frac{g_s^4 C_A N_l T_F}{(4\pi)^4}\left[\frac{(D-2)(8D^3-79D^2+240D-212+2(3D-8)\alpha_g)}{4(D-1)(2D-7)(2D-9)} + \right. \\
&\quad \left. -\frac{2(D-2)(3D^3-27D^2+69D-44+(3D-8)\alpha_g)}{4(D-1)(2D-7)(2D-9)}\right] I_1, \qquad (5.76) \\
\mu_{c2}(0) &= \frac{g_s^4 C_A N_l T_F}{(4\pi)^4}\left[\frac{(D-2)(D-4-(3D-8)\alpha_g)}{2(D-1)(2D-7)} - \frac{(D-2)(3D-8)(5-\alpha_g)}{2(D-1)(2D-7)}\right] I_1
\end{aligned}$$

where the two contributions in the square brackets come from the two projections for the resp. order in q^2/m_Q^2. It is easy to see that the gauge dependence cancels out. The contribution is gauge invariant. This is a good test for the reliability of the calculations already at this preliminary level because (c2) is the only diagram that has this specific colour factor. Therefore, a possible gauge dependence cannot be changed by another contribution.

5.2.6 Feynman rules for HQET

At this point the Feynman rules for HQET up to $O(1/m_Q)$ shall be listed [159],

$$\begin{aligned}
\text{heavy quark propagator:} &\quad \frac{1+\not{v}}{2}\,\frac{i}{p\cdot v+i0} \\
\text{heavy quark one-gluon vertex:} &\quad i g_s T_a v^\mu \\
\text{heavy quark no-gluon kinetic vertex:} &\quad \frac{i}{2m_Q} p^2 \\
\text{heavy quark one-gluon kinetic vertex:} &\quad \frac{i g_s}{2m_Q}(2p-k)^\mu T_a \\
\text{heavy quark two-gluon kinetic vertex:} &\quad \frac{i g_s^2}{2m_Q}\{T_a,T_b\} g^{\mu\nu} \\
\text{heavy quark one-gluon Fermi vertex:} &\quad \frac{-g_s}{2m_Q}\sigma^{\mu\nu} k_\nu T_a \\
\text{heavy quark two-gluon Fermi vertex:} &\quad \frac{i g_s^2}{2m_Q}\sigma^{\mu\nu} f_{abd} T_d \qquad (5.77)
\end{aligned}$$

where v is the velocity of the hadron, k is the incoming momentum of the gluon, and p is the outgoing heavy quark residual momentum at the corresponding vertex. These Feynman rules are given here for the use in the following sections.

5.3 A threshold mass definition

The masses of fermions and bosons are input parameters of the Standard Model. In case of heavy hadrons, an exact determination of the heavy quark mass, therefore, is mandatory for precision tests of the Standard Model. Although it is widely accepted that the quark masses are generated due to the Higgs mechanism, the value of the mass cannot be calculated from the Standard Model itself. Instead, quark masses have to be determined from the comparison of theoretical predictions and experimental data.

It is important to stress that there is no unique definition of the quark mass, as it might be suggested by the notation m_Q used up to now. As in solid state physics, the quark mass can be screened and modified by the neighbourhood of the quark and its state. If the quark is very energetic, the pole mass $m_{\rm pole}$, defined as the pole of the quark propagator, is an appropiate parameter to describe the mass properties of the quark. However, if the quark is in rest relative to the hadron, the spectator quarks have to be taken into account insofar as they build up an effective potential $V(r)$ for the heavy quark. Only the pole mass in combination with this potential turns out to be a sensible quantity to be considered. This is expressed by the fact that only the combination $2m_{\rm pole}+V(r)$ is free of infrared ambiguities which describe long-distant interactions [172, 173] (see also Refs. [174, 175]).

The combination $2m_{\rm pole}+V(r)$, however, includes the light degrees of freedom. In order to find a definition for the heavy quark mass independent of the light spectator quarks, different concepts for the so-called *threshold masses* have been introduced. Among these are the low scale (LS) mass [173], the potential subtracted (PS) mass [174], ore one half of the perturbative mass of a fictious 1^3S_1 ground state (called $1S$ mass) [121] (for a review see e.g. Ref. [117]). In the work shown here a further mass concept is presented, the *modified potential subtracted ($\overline{PS}$) mass* [176]. Even though developed by employing different concepts, the results of Ref. [174] are recovered in the static limit. The definition of the $\overline{\rm PS}$ is given by

$$m_{\overline{\rm PS}}=m_{\rm pole}-\delta m_{\overline{\rm PS}}\quad\text{with}\quad\delta m_{\overline{\rm PS}}=\Sigma_{\rm soft}(\not p)\Big|_{\not p=m}\tag{5.78}$$

where $\Sigma_{\rm soft}$ is the soft part of the heavy quark self energy. This definition is "natural" in the sense that the regular change of the mass due to the resummation of self energy contributions is modified insofar, as only the soft part of the radiative corrections are taken into account which contain the infrared ambiguity. In order to introduce the concept of soft regions, the next subsection deals with consequences from taking one of the gluons as soft, while a more general concept is developed thereafter.

5.3.1 Soft regions and effective potentials

The bulk of diagrams to be considered for the two-loop one-particle irreducible radiative corrections to the quark propagator are shown in Fig. 5.7. The consideration of the soft part of these diagrams means that at least one of the gluons should be soft, and therefore close to on-shell. For this reason, the diagrams (a) and (c) in Fig. 5.7 can be considered as one-loop diagrams where one of the vertices is given by an effective vertex. The effective vertex then includes the one-loop radiative corrections for particles which are close to

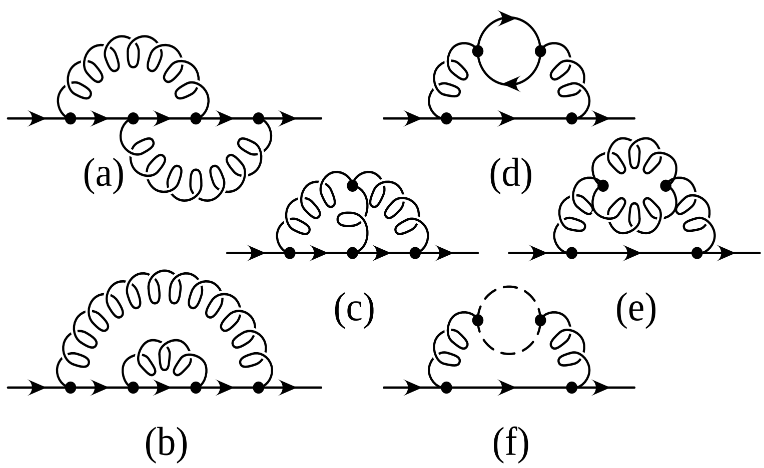


Figure 5.7: Two-loop contributions to the quark self energy

on-shell. A first approach to this calculation is found in Appendix H. However, a more general concept for the calculation is found which will be presented here.

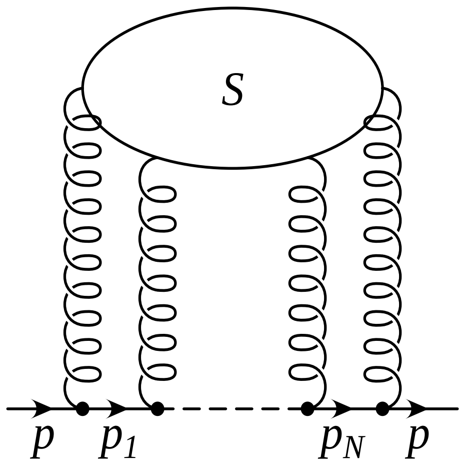


Figure 5.8: The general structure of the self energy diagram of a quark

The starting point of the considerations is an on-shell quark with mass m and momentum p (i.e. $p^2 = m^2$) which is considered to be at rest, $p = (m, \vec{0})$. This quark interacts with a number of gluons. The subdiagram S displayed in Fig. 5.8 describes the interaction between the gluons. In general the quark lines between the interaction points represent virtual quark states. However, if the virtual quark comes very close to the mass shell and the total momentum of the cloud of virtual gluons becomes soft, this situation gives rise to long-distance nonperturbative QCD interactions. The described (virtual) contributions result in the soft part of the self energy, $\Sigma_{\rm soft}$. For a precise definition one starts with a general self energy diagram as shown in Fig. 5.8,

$$-i\Sigma(\not{p}) = \int \prod_{m=1}^{M} \frac{d^4 l_m}{(2\pi)^4} S^{\{a_n\}}_{\{\alpha_n\}}(\{l_m\}) \left(-ig_s \gamma^{\alpha_{N+1}} T_{a_{N+1}}\right) \prod_{n=N}^{1} \frac{i}{\not{p}_n - m} (-ig_s \gamma^{\alpha_n} T_{a_n}) \quad (5.79)$$

where the last factor is a non-commutative product with decreasing index n. The *line* *momenta* k_n are linear combinations of the gluon loop momenta l_m. The particular representation is specified by the structure S. The symbol $\{l_m\}$ means the set of all these loop momenta. The same symbol is used for the Lorentz and colour indices. In general one has $M \leq N$ which means that line momenta can be correlated. The momenta of the virtual quark states are given by $p_n = p + k_n$. Taking this as the starting point one defines

$$-i\Sigma_{\rm soft}(\not p) = \sum_{i=1}^{N} \int \prod_{m=1}^{M} \frac{d^4 l_m}{(2\pi)^4} S^{\{a_n\}}_{\{\alpha_n\}}(\{l_m\}) \left(-ig_s\gamma^{\alpha_{N+1}}T_{a_{N+1}}\right) \prod_{n=N}^{i+1} \frac{i}{\not p_n - m} (-ig_s\gamma^{\alpha_n}T_{a_n}) \times$$
$$\times\, i(\not p_i + m)\left(-i\pi\delta(p_i^2 - m^2)\right)(-ig_s\gamma^{\alpha_i}T_{a_i}) \prod_{n=i-1}^{1} \frac{i}{\not p_n - m}(-ig_s\gamma^{\alpha_n}T_{a_n})\,. \qquad (5.80)$$

This equation is the definition of the soft part of the quark self energy. One can derive this expression from Eq. (5.79) by using the identity

$$\frac{1}{p^2 - m^2 + i\epsilon} = -i\pi\delta(p^2 - m^2) + \mathrm{Prin}\left(\frac{1}{p^2 - m^2}\right) \qquad (5.81)$$

and the fact that the principal value integral does not give any infrared sensitive contribution. The delta function can be used to remove the integration over the zero component of k_i. In order to parametrize the softness of the gluon cloud one imposes a cutoff on the spatial component, $|\vec{k}_i| < \mu_f$, and indicates this by a label μ_f written at the upper limit of the three-dimensional integral. This cutoff μ_f is also known as *factorization scale*. Therefore, one can rewrite Eq. (5.80) as

$$\Sigma_{\rm soft}(\not p, \mu_f) = -\frac{1}{2}\sum_{i=1}^{N} \int^{\mu_f} \frac{d^3 k_i}{(2\pi)^3} V(\vec{k}_i, p) \qquad (5.82)$$

where

$$V(\vec{k}_i, p) \;:=\; -\int \prod_{m=1}^{M-1} \frac{d^4 l_m}{(2\pi)^4} S^{\{a_n\}}_{\{\alpha_n\}}(\{l_m\}) \left(-ig_s\gamma^{\alpha_{N+1}}T_{a_{N+1}}\right) \prod_{n=N}^{i+1} \frac{i}{\not p_n - m}(-ig_s\gamma^{\alpha_n}T_{a_n}) \times$$
$$\times\, \frac{\not p_i + m}{2p_i^0}(-ig_s\gamma^{\alpha_i}T_{a_i}) \prod_{n=i-1}^{1} \frac{i}{\not p_n - m}(-ig_s\gamma^{\alpha_n}T_{a_n})\,. \qquad (5.83)$$

The range of the index m is reduced by one which indicates that one of the loop momenta is extracted as line momentum of the i-th line. In the following the different implemetations of this compact expression are dealt with. As one will see explicitly, the function $V(\vec{k}, p)$ occurring as integrand can be considered as *quark-antiquark potential* where one has summed over the spin of the tensor product of the spinors of a final state and an initial state. Because the static quark-antiquark potential is used in a similar way in Ref. [174], one recovers the result of Ref. [174] in the static limit. However, there is no kind of hierarchical order between both concepts because both solve the afore mentioned problems with non-perturbative uncertainties of the order $O(\Lambda_{\rm QCD})$. The role of non-perturbative effects of the order $1/m$ have to be studied at a later time.

5.3.2 The leading order perturbative contribution

The leading order contribution to the self energy of the quark is given by

$$\Sigma(\not p) = i\int \frac{d^4 k}{(2\pi)^4}(-ig_s\gamma_\alpha T_a)\frac{i}{\not p + \not k - m}(-ig_s\gamma^\alpha T_a)\frac{-i}{k^2} \qquad (5.84)$$

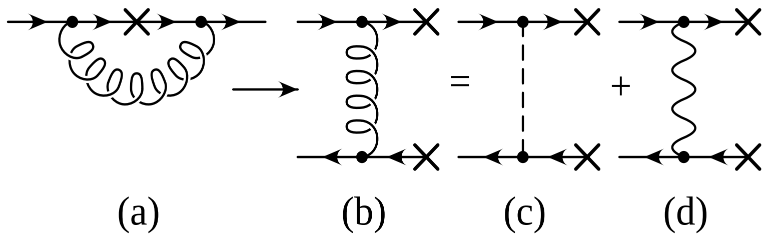


Figure 5.9: Leading order contribution to the quark self energy (a) and to the quark-antiquark potential (b). The cross indicates the point where the quark line is cut by imposing an on-shell condition to the virtual quark state. The gluon propagator can be decomposed in a Coulomb propagator (c) and a transverse propagator (d).

where the Feynman gauge is used for the gluon. The soft contribution thus reads

$$\begin{aligned}\Sigma_{\rm soft}(\not p) &= -ig_s^2C_F\int\frac{d^4k}{(2\pi)^4k^2}\gamma_\alpha(\not p+\not k+m)\gamma^\alpha\left(-i\pi\delta\left((p+k)^2-m^2\right)\right) = \\ &= -\pi g_s^2C_F\int\frac{d^4k}{(2\pi)^4k^2}\left(-2(\not p+\not k)+4m\right)\delta\left((p+k)^2-m^2\right).\end{aligned}\tag{5.85}$$

The fact that the self energy correction is located between on-shell quark states leads to simplifications which remove the Dirac structure of the integrand, because

$$\begin{aligned}\bar u(p)\not p u(p) &= \bar u(p)mu(p) \quad\Rightarrow\quad \not p\to m,\\ \bar u(p)m\not k u(p) &= \frac{1}{2}\bar u(p)(\not p\not k+\not k\not p)u(p) \;=\; (kp)\bar u(p)u(p) \quad\Rightarrow\quad \not k\to\frac{kp}{m}=k_0\end{aligned}\tag{5.86}$$

The last identity is valid in the quark rest frame. One obtains

$$\Sigma_{\rm soft}(\not p) = -\pi g_s^2C_F\int\frac{d^4k}{(2\pi)^4k^2}2(m-k_0)\delta\left((p+k)^2-m^2\right).\tag{5.87}$$

In terms of the zero component of the momentum k the Dirac delta function has two zeros $k_0=k_+$ and $k_0=k_-$ with

$$k_\pm := \pm\sqrt{\kappa^2+m^2}-m\tag{5.88}$$

where $\kappa=|\vec k|$. The delta function is therefore written as

$$\delta\left((p+k)^2-m^2\right)=\frac{1}{2\sqrt{\kappa^2+m^2}}\left(\delta(k_0-k_+)+\delta(k_0-k_-)\right).\tag{5.89}$$

The procedure which is done here is illustrated in Fig. 5.9(a–b). The cross indicates that one cuts the line at this point by imposing the on-shell condition to the corresponding (virtual) momentum. The diagram then proceeds to a quark-antiquark interaction diagram where the crosses have been kept to indicate the position of the cut line. This line carries the momentum $p+k$ while the other two external lines carry the momentum p. Accordingly one obtains

$$\Sigma_{\rm soft}(\not p,\mu_f) = -\frac{1}{2}\int^{\mu_f}\frac{d^3k}{(2\pi)^3}V(\vec k,p),\qquad V(\vec k,p)=V_+(\vec k,p)+V_-(\vec k,p)\tag{5.90}$$

where

$$V_\pm(\vec{k},p) = g_s^2 C_F \frac{m-k_\pm}{\sqrt{m^2+\kappa^2}(k_\pm^2-\kappa^2)} = \frac{g_s^2 C_F(\sqrt{m^2+\kappa^2}\mp 2m)}{2m\sqrt{m^2+\kappa^2}(\sqrt{m^2+\kappa^2}\mp m)}. \tag{5.91}$$

These are radially symmetric potentials. For $m \ll \mu_f$ the restriction of the three-dimensional integral by the radial bound μ_f allows for an expansion in κ/m. One obtains

$$\begin{aligned} V_+(\vec{k},p) &= -4\pi\alpha_s C_F \left\{\frac{1}{\kappa^2} - \frac{3}{4m^2} + O\left(\frac{\kappa^2}{m^4}\right)\right\}, \\ V_-(\vec{k},p) &= -4\pi\alpha_s C_F \left\{\quad - \frac{3}{4m^2} + O\left(\frac{\kappa^2}{m^4}\right)\right\} \end{aligned} \tag{5.92}$$

and therefore

$$V(\vec{k},p) = -4\pi\alpha_s C_F \left\{\frac{1}{\kappa^2} - \frac{3}{2m^2} + O\left(\frac{\kappa^2}{m^4}\right)\right\}. \tag{5.93}$$

The first term in Eq. (5.93) is the *Coulomb potential* for a quark-antiquark interaction. The second term can be related to the *Breit-Fermi potential of the quark-antiquark interaction* [177] by summing over the spin states of the tensor product of the final quark and the final antiquark spinor and using the same kinematic constraints. Moreover, one can identify $V_+(\vec{k},p)$ with the *scattering potential* and $V_-(\vec{k},p)$ with the *annihilation potential*.

Employing $(\sqrt{m^2+\kappa^2}\pm m)(\sqrt{m^2+\kappa^2}\mp m)=\kappa^2$ and the substitution

$$\kappa = \frac{m}{2t}(t^2-1) \quad\Rightarrow\quad \sqrt{m^2+\kappa^2} = \frac{m}{2t}(1+t^2), \quad d\kappa = \frac{m}{2t^2}(1+t^2)dt, \tag{5.94}$$

the potentials given in Eq. (5.91) can be integrated exactly up to the cut μ_f,

$$\begin{aligned} \int^{\mu_f}\frac{d^3k}{(2\pi)^3}V_\pm(\vec{k},p) &= \frac{g_s^2C_F}{2\pi^2}\int_0^{\mu_f}\frac{(\sqrt{m^2+\kappa^2}\pm m)(\sqrt{m^2+\kappa^2}\mp 2m)}{2m\sqrt{m^2+\kappa^2}}d\kappa = \\ = \frac{g_s^2C_F}{2\pi^2}\int_0^{\mu_f}\frac{\kappa^2-m^2\mp m\sqrt{m^2+\kappa^2}}{2m\sqrt{m^2+\kappa^2}}d\kappa &= \frac{g_s^2C_F}{2\pi^2}\int_0^{\mu_f}\left(\frac{\kappa^2-m^2}{2m\sqrt{m^2+\kappa^2}}\mp\frac{1}{2}\right)d\kappa = \\ = \frac{g_s^2C_Fm}{16\pi^2}\int_1^{\tau}\frac{t^4-6t^2+1}{8t^3}dt \mp \frac{g_s^2C_F\mu_f}{4\pi^2} &= \frac{g_s^2C_Fm}{16\pi^2}\left(\frac{\tau^2}{2}-\frac{1}{2\tau^2}-6\ln\tau\right)\mp\frac{g_s^2C_F\mu_f}{4\pi^2} \end{aligned} \tag{5.95}$$

where $\tau = (\mu_f + \sqrt{m^2+\mu_f^2})/m$ (see also Appendix I). One obtains

$$\Sigma_{\rm soft}(\mu_f) = \frac{\alpha_sC_F}{2\pi}m\left\{3\ln\left(\frac{\mu_f}{m}+\sqrt{\frac{\mu_f^2}{m^2}+1}\right)-\frac{\mu_f}{m}\sqrt{\frac{\mu_f^2}{m^2}+1}\right\}. \tag{5.96}$$

The expansion of this expression in small values of μ_f/m results in

$$\Sigma_{\rm soft}(\mu_f) = \frac{\alpha_sC_F}{\pi}\mu_f\left\{1-\frac{\mu_f^2}{2m^2}\right\}. \tag{5.97}$$

The first term reproduces the result given in Ref. [174] to leading order in α_s while the second term is the recoil correction to the static limit in this order of perturbation theory. This second term is related to the Breit-Fermi potential but does not coincide with it.

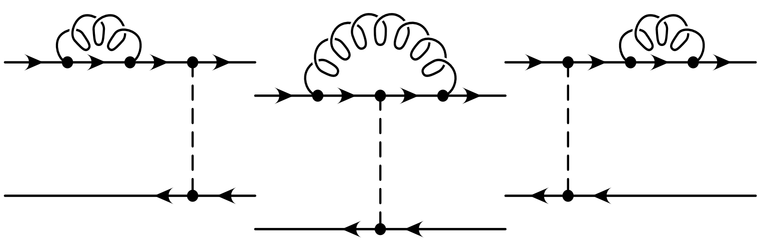


Figure 5.10: Diagrams which cancel due to the classical Ward identity

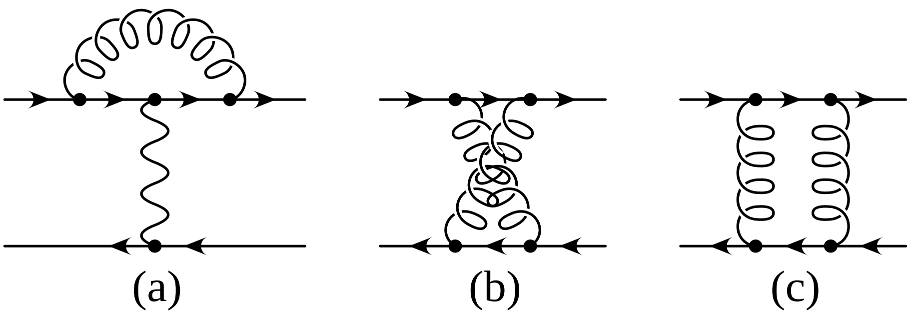


Figure 5.11: Diagrams which contribute to order $O(\mu_f^2/m^2)$

5.3.3 Two-loop contributions

To take a step beyond the leading order perturbation theory, the two-loop diagrams for the heavy quark self energy in Fig. 5.7 are considered. They are calculated in Coulomb gauge, even though it is shown that the final result is gauge invariant (cf. Appendix I.3). The gluon propagator in Coulomb gauge is given by

$$G_{00}^{ab}(k) = \frac{i\delta^{ab}}{\vec{k}^{\,2}}, \qquad G_{ij}^{ab}(k) = \frac{i\delta^{ab}}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^{\,2}}\right), \qquad i,j = 1,2,3. \tag{5.98}$$

The use of Coulomb gauge splits up the gluon propagators into a Coulomb term (Coulomb gluon) and a transverse term (transverse gluon) where the first one couples to the quark via the time components only. This splitting is shown in Fig. 5.9(b–d).

5.3.4 The abelian diagrams

The analysis of two-loop diagrams starts with the abelian diagrams shown in Figs. 5.7(a) and (b). In cutting the quark line in all possible ways one obtains many diagrams. However, it is found that the final contribution of these diagrams to the soft part of the self energy are suppressed by μ_f^2/m^2. There are different arguments for this suppression. First, in applying the classical Ward identity, the QED diagrams shown in Fig. 5.10 cancel exactly at $|\vec{k}| \to 0$. The remaining contribution is of order $O(\vec{k}^2/m^2)$. One should note that the Ward identity for the interaction vertex of a Coulomb gluon with the quark holds even in non-abelian theories [178]. A second argument is that the interaction between a transverse gluon and a non-relativistic quark as shown in Fig. 5.11(a) is suppressed by μ_f/m, leading to an overall μ_f^2/m^2 suppression. In addition, the box diagrams in Figs. 5.11(b) and (c) are either suppressed by a factor μ_f^2/m^2 or give an iteration of the leading order potential. To summarize, the diagrams in Figs. 5.7(a) and (b) give contributions only of the order $g_s^4\mu_f^2/m^2$.

5.3.5 The vacuum polarization of the gluon

The only abelian diagrams which can give a non-suppressed contribution to the soft part of the quark self energy are the diagrams containing the vacuum polarization of the gluon as shown in Fig. 5.7(d–f). The simple calculation of these diagrams within the $\overline{\mathrm{MS}}$ scheme, accounting only for N_f light fermion loops, gluon loop (and ghost loop if Feynman gauge is used) results after renormalization (cf. Appendix H) in

$$\begin{aligned}
\Sigma^A_{\rm soft} &= -\frac{1}{2}\int^{\mu}\frac{d^3k}{(2\pi)^3}\left(-\frac{4\pi\alpha_s(\mu)C_F}{|\vec{k}|^2}\right)\times \\
&\quad\times\left\{1+\frac{\alpha_s(\mu)}{4\pi}\left(\frac{31C_A}{9}-\frac{20T_FN_f}{9}-\left(\frac{11C_A}{3}-\frac{4T_FN_f}{3}\right)\ln\left(\frac{|\vec{k}|^2}{\mu^2}\right)\right)\right\} \\
&= \frac{\alpha_S(\mu)C_F}{\pi}\mu_f\left\{1+\frac{\alpha_s(\mu)}{4\pi}\left(a_1-\beta_0\ln\left(\frac{\mu_f^2}{\mu^2}\right)\right)\right\}. \qquad (5.99)
\end{aligned}$$

This result has been anticipated because the expression in the curly brackets of the integrand reproduces the next-to-leading order correction to the QCD Coulomb potential.

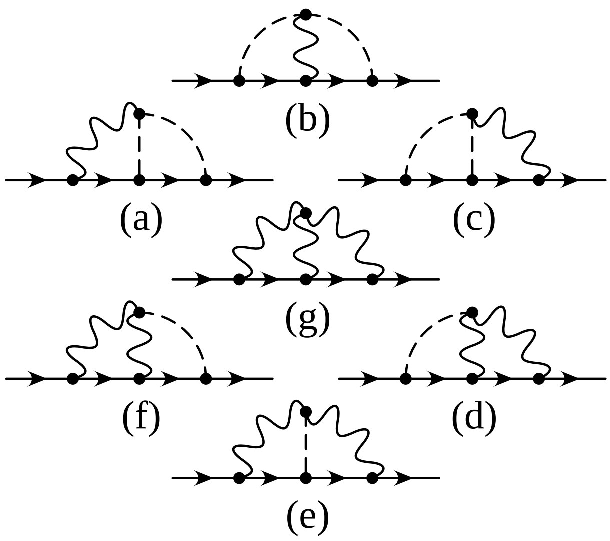


Figure 5.12: The non-abelian self energy diagram in Coulomb gauge; displayed are Coulomb gluons (wiggles) and transverse gluons (dashed lines)

5.3.6 The non-abelian diagrams

In this subsection the non-abelian diagram shown in Fig. 5.7(c) is calculated. In Coulomb gauge this diagram gives rise to seven two-loop diagrams which are shown in Fig. 5.12. Direct calculations show that only the diagram in Fig. 5.12(b) gives a contribution of order $g_s^4\mu_f/m$ while the other diagrams are of order $g_s^4\mu_f^2/m^2$ or vanish to this order in the coupling after applying the renormalization procedure (see e.g. Ref. [179] and Appendix J). The calculation of the diagram in Fig. 5.12(b) is simple and will be shown in detail. The contribution of this diagram to the self energy is given by

$$\begin{aligned}-i\Sigma^{6b}(\not p) &= \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{\vec k_1^2}\frac{i}{\vec k_2^2}\frac{i}{(k_1-k_2)^2}\left(\delta_{ij}-\frac{(k_1-k_2)_i(k_1-k_2)_j}{(\vec k_1-\vec k_2)^2}\right)\times \\ \times\ g_sf_{abc}(k_1+k_2)^jg^{00}&(-ig_s\gamma^0T_a)\frac{i}{\not p+\not k_2-m}(-ig_s\gamma^iT_b)\frac{i}{\not p+\not k_1-m}(-ig_s\gamma^0T_c). \end{aligned}\tag{5.100}$$

Here the Lorentz structure of the three-gluon vertex is reduced to k_1+k_2. If considered in the quark rest frame between on-shell states, represented by projectors $P^+=(1+\not v)/2$ on the left and the right, the Dirac structure of the integrand can be simplified to

$$P^+\left\{-(2m+k_{20})\gamma^i\vec k_1\vec\gamma-\vec k_2\vec\gamma\gamma^i(2m+k_{10})\right\}. \tag{5.101}$$

The remaining diagrams are symmetric with respect to the interchange of the two line momenta k_1 and k_2. This provides a further simplification of the Dirac structure. The self energy reduces to

$$\Sigma^{6b}(\not p)=g_s^4C_FC_A\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{(4m+k_{10}+k_{20})\left(\vec k_1^2\vec k_2^2-(\vec k_1\vec k_2)^2\right)}{\vec k_1^2\vec k_2^2(\vec k_1-\vec k_2)^2(k_1-k_2)^2((p+k_1)^2-m^2)((p+k_2)^2-m^2)} \tag{5.102}$$

where $T_aT_bT_cf_{abc}=iC_FC_A/2$ has been used. One now employs the substitutions

$$\frac{1}{(p+k_1)^2-m^2}\to -i\pi\delta((p+k_1)^2-m^2),\qquad \frac{1}{(p+k_2)^2-m^2}\to -i\pi\delta((p+k_2)^2-m^2) \tag{5.103}$$

i.e. one cuts the two intermediate quark lines separately to obtain the two parts $\Sigma^{6b}_{\rm soft1}$ and $\Sigma^{6b}_{\rm soft2}$ of the soft contribution $\Sigma^{6b}_{\rm soft}$ of the self energy, as shown in Fig. 5.12. Taking the first cut, the delta function removes the integration over k_{10}. At the same time the definition of the soft contribution imposes a restriction $|\vec k_1|<\mu_f$ on the space components of the first line momentum. The delta function

$$\delta\left((p+k_1)^2-m^2\right)=\frac{1}{2\sqrt{m^2+\vec k_1^2}}\left(\delta(k_{10}-k_{1+})+\delta(k_{10}-k_{1-})\right) \tag{5.104}$$

with

$$k_{1\pm}=\pm\sqrt{m^2+\vec k_1^2}-m \tag{5.105}$$

results in two contributions which are known as scattering and annihilation amplitude (according to k_{1+} and k_{1-}, resp.). The integration over k_{20} is done by using the residue theorem. Actually there are only two denominator factors which can contribute to poles of the integrand, namely $(k_1-k_2)^2$ and $((p+k_2)^2-m^2)$ (for the details of the calculation see Appendix J). In summing up the four contributions from the integration over k_{20} and k_{10} one ends up with two three-dimensional integrals over the space components of the two line momenta where the first integral is restricted by $|\vec k_1|<\mu_f$ as mentioned earlier. One now imposes the restriction $|\vec k_1|<\mu_f$ on the integrand to simplify it and obtains

$$\Sigma^{6b}_{\rm soft1}=\frac{g_s^4C_FC_A}{4m}\int^{\mu_f}\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\frac{\vec k_1^2\vec k_2^2-(\vec k_1\vec k_2)^2}{\vec k_1^2(\vec k_2^2)^2(\vec k_1-\vec k_2)^2}. \tag{5.106}$$

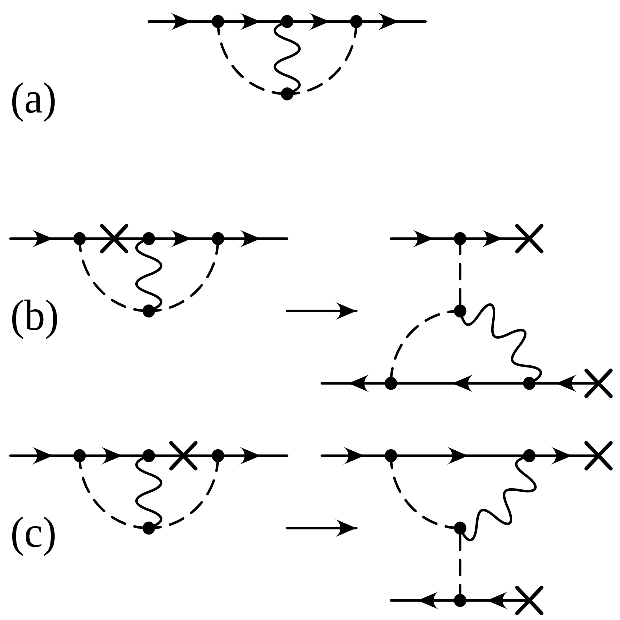


Figure 5.13: The soft part of the non-abelian diagram under consideration

The integral over the space components of k_2 can be easily done by performing the angular integration followed by the radial integration. One obtains

$$\int \frac{d^3k_2}{(2\pi)^3} \frac{\vec{k}_1^2\vec{k}_2^2 - (\vec{k}_1\vec{k}_2)^2}{\vec{k}_1^2(\vec{k}_2^2)^2(\vec{k}_1 - \vec{k}_2)^2} = \frac{1}{16|\vec{k}_1|} \tag{5.107}$$

and therefore, finally

$$\Sigma^{6b}_{\text{soft}1} = \frac{\alpha_s^2 C_F C_A}{16m}\mu_f^2. \tag{5.108}$$

Symmetry considerations show that $\Sigma^{6b}_{\text{soft}2}$ gives exactly the same contribution. As mentioned before, there are no other non-abelian contributions, therefore one ends up with

$$\Sigma^{NA}_{\text{soft}} = \frac{\alpha_s^2 C_F C_A}{8m}\mu_f^2. \tag{5.109}$$

This result has been anticipated, too, to be minus one half of the non-abelian correction to the QCD Coulomb potential, which is known in the literature (see for example Refs. [180, 181]),

$$\Sigma^{NA}_{\text{soft}} = -\frac{1}{2}\int^{\mu_f} \frac{d^3k}{(2\pi)^3}\left\{-\frac{\pi^2\alpha_s^2 C_F C_A}{m|\vec{k}|}\right\} = \frac{\alpha_s^2 C_F C_A}{8m}\mu_f^2. \tag{5.110}$$

This calculation concludes the considerations of the two-loop diagrams shown in Fig. 5.7.

5.3.7 The final result

Summarizing all contribution up to NNLO accuracy, one obtains

$$m_{\overline{\text{PS}}}(\mu_f) - m = -\frac{\alpha_s C_F}{\pi}\mu_f\left\{1 + C_0'\frac{\mu_f}{m} + C_0''\frac{\mu_f^2}{m^2} + \frac{\alpha_s}{4\pi}\left(C_1 + C_1'\frac{\mu_f}{m}\right) + C_2\left(\frac{\alpha_s}{4\pi}\right)^2\right\} \tag{5.111}$$

where m is the pole mass, μ is the renormalization scale, and $\alpha_s = \alpha_s(\mu)$. This result is of the order $O(\alpha_s^2)$ because one presumes that the ratio μ_f/m is typically of the order of α_s or smaller. The scale μ_f is the factorization scale, and

$$\begin{aligned}
C_0 &= 1, \qquad C_0' \;=\; 0, \qquad C_0'' \;=\; -\frac{1}{2}, \\
C_1 &= a_1 - 2\beta_0 \ln\left(\frac{\mu_f}{\mu}\right), \qquad C_1' \;=\; C_A\frac{\pi^2}{2}, \\
C_2 &= a_2 - 2(2a_1\beta_0 + \beta_1)\left(\ln\left(\frac{\mu_f}{\mu}\right) - 1\right) + 4\beta_0^2\left(\ln^2\left(\frac{\mu_f}{\mu}\right) - 2\ln\left(\frac{\mu_f}{\mu}\right) + 2\right). \qquad (5.112)
\end{aligned}$$

While β_0 and β_1 are the two first coefficients of the QCD β-function,

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F N_f, \quad \beta_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F N_f - 4C_F T_F N_f \qquad (5.113)$$

(cf. Eq. (2.20) in Chapter 2), the coefficients a_1 and a_2 were calculated in Refs. [182] and [183, 184], respectively, and are given by

$$\begin{aligned}
a_1 &= \frac{31}{9}C_A - \frac{20}{9}T_F N_f, \\
a_2 &= \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta_3\right)C_A^2 - \left(\frac{1798}{81} + \frac{56}{3}\zeta_3\right)C_A T_F N_f \\
&\quad + \left(\frac{20}{9}T_F N_f\right)^2 - \left(\frac{55}{3} - 16\zeta_3\right)C_F T_F N_f. \qquad (5.114)
\end{aligned}$$

The coefficients C_1 and C_2 have been derived in Ref. [174] by using known corrections to the QCD potential. In the work presented here the new coefficients C_0', C_0'', and C_1' have been derived [176]. Note that the result can be represented in a condensed form as

$$m_{\overline{\mathrm{PS}}}(\mu_f) - m = -\frac{1}{2}\int^{\mu_f}\frac{d^3k}{(2\pi)^3}\left(V_C(|\vec{k}|) + V_R(|\vec{k}|) + V_{NA}(|\vec{k}|)\right) \qquad (5.115)$$

where the first term V_C is the static Coulomb potential, V_R is the relativistic correction (which is related to the Breit-Fermi potential but does not coincide with it), and V_{NA} is the non-abelian correction.

5.3.8 Application of the result

In Ref. [176] the results obtained in this section are applied to the determination of the top quark production near the production threshold. In order to determine the relative cross section $R_{e^+e^-}$ in this threshold region, the non-relativistic Schrödinger equation has to be solved. It turns out that the use of threshold masses stabilizes the determination of the threshold in dependence on the renormalization scale as well as in dependence on the order of the perturbative expansion for the effective potential used in the Schrödinger equation [119]. The details of these arguments can be found in the literature [117, 176]. Note that the use of a one-scale running of the coupling is a second large step in stabilizing the threshold region (see e.g. Refs. [185, 186]). It remains as work for the future to test the consequences of the use of the $\overline{\mathrm{PS}}$ mass in one-scale running.

5.4 The baryonic Isgur–Wise function

After having presented calculations in connection to the HQET in the previous sections, starting with this section the HQET is used – especially the Heavy Quark Symmetry (HQS) as limiting case. In this limit of infinite heavy quark masses, the form factors for the weak decay describing semileptonic transitions $H(v) \to H'(v') + \ell + \nu_\ell$ between ground-state hadrons including one heavy quark are described by an universal form factor $\xi(v\cdot v')$, known as *Isgur–Wise function* [151]. The surprising fact that such transitions can be described by a single function of one variable, namely the *velocity transfer* $y = v \cdot v'$, is a consequence of the HQS [187]. At $y = 1$, the *zero recoil point* where the initial and final hadron have the same velocity, this function is normalized to $\xi(1) = 1$.

Being a nonperturbative quantity describing the cloud of light quarks and gluons surrounding the heavy quark, this function can only be determined by nonperturbative methods, as for instance in QCD sum rules [155] (see also Chapter 7). The precision of the QCD sum rule method can be improved by taking radiative corrections for the perturbative parts into account. After the incorporation of the leading logarithmic approximation [188], the full next-to-leading order corrections to the Isgur–Wise function for mesons was obtained in Ref. [189]. However, corresponding calculations for the case of baryons are still missing. The starting steps presented here for the calculation of two-loop corrections for the baryonic Isgur–Wise function shall fill this gap.

The first step is the calculation of the next-to-leading order corrections to the HQET three-point diagrams. This will be done in this section, employing configurations space methods which come in via integral transforms. However, the integrals turn out not to be analytically calculable. In order to obtain analytic results, in the section that follows the limit $y \approx 1$ is considered. The integrals can then be decomposed into standard integrals which are reducible to known functions by using the integration-by-parts technique [169].

5.4.1 The leading order diagram

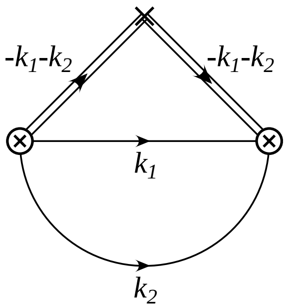


Figure 5.14: Leading order diagram where only the residual momenta are shown

To explain the main features of the method, one starts with the leading order diagram shown in Fig. 5.14. The heavy line which is denoted by the double line carries a momentum $-k_1-k_2$ while the light lines carry the momenta k_1 and k_2. The cross on the top of the diagram indicates the place where the velocity is changed from v (on the left) to v' (on the right). The first step consists in performing a Wick rotation. The next step is to replace the light propagators according to

$$\frac{1}{(k^2)^\alpha} = \frac{\Gamma(D/2-\alpha)}{4^\alpha \pi^{D/2}\Gamma(\alpha)} \int \frac{e^{ikx} d^D x}{(x^2)^{D/2-\alpha}}. \tag{5.116}$$

One thus obtains

$$\begin{aligned}
\Pi_0 &= \int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D} \mathrm{Tr}\left(\bar\Gamma \frac{i}{\not k_1}\Gamma\frac{i}{\not k_2}\right)\left(\frac{i}{\omega-(k_1+k_2)v}\right)\left(\frac{i}{\omega'-(k_1+k_2)v'}\right) = \\
&= \mathrm{Tr}(\bar\Gamma\gamma_\mu\Gamma_\nu)\int \frac{id^D k_1^E}{(2\pi)^D}\frac{id^D k_2^E}{(2\pi)^D}\frac{(k_1^E)^\mu (k_2^E)^\nu}{(k_1^E)^2(k_2^E)^2(\omega-i(k_1^E+k_2^E)v)(\omega'-i(k_1^E+k_2^E)v')} =
\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{Tr}(\bar{\Gamma}\gamma_\mu\Gamma\gamma_\nu)\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_1x_1}\frac{x_1^\mu d^Dx_1}{(x_1^2)^{D/2}}\right)\times \qquad (5.117)\\
&\quad\times\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_2x_2}\frac{x_2^\nu d^Dx_2}{(x_2^2)^{D/2}}\right)\frac{1}{(\omega-i(k_1+k_2)v)(\omega'-i(k_1+k_2)v')}
\end{aligned}
$$

where the index E indicating the Euclidean metric have been omitted. In addition

$$
\frac{k^\mu}{(k^2)^{\alpha+1}} = -\frac{1}{2\alpha}\frac{\partial}{\partial k_\mu}\frac{1}{(k^2)^\alpha} = -\frac{i\Gamma(D/2-\alpha)}{4^\alpha 2\pi^{D/2}\Gamma(\alpha+1)}\int\frac{x^\mu e^{ikx}d^Dx}{(x^2)^{D/2-\alpha}}. \qquad (5.118)
$$

has been used. The heavy quark propagators can be transformed into an exponential function by employing

$$
\frac{1}{A^\alpha} = \frac{(-1)^\alpha}{\Gamma(\alpha)}\int_0^\infty\lambda^{\alpha-1}e^{\lambda A}d\lambda \qquad \text{for } \mathrm{Re}\,A<0. \qquad (5.119)
$$

Applying this to $A=\omega-i(k_1+k_2)v$ and the primed version of it, one ends up with

$$
\begin{aligned}
\Pi_0 &= \mathrm{Tr}(\bar{\Gamma}\gamma_\mu\Gamma\gamma_\nu)\frac{\Gamma^2(D/2)}{4\pi^D}\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{x_1^\mu x_2^\nu d^Dx_1d^Dx_2}{(x_1^2)^{D/2}(x_2^2)^{D/2}}d\lambda\,d\lambda'\times\\
&\quad\times\exp(ik_1x_1+ik_2x_2+\lambda(\omega-i(k_1+k_2)v)+\lambda'(\omega'-i(k_1+k_2)v')). \qquad (5.120)
\end{aligned}
$$

In this situation one can perform the integrations over the inner momenta k_1 and k_2. These will result in $x_1=\lambda v+\lambda' v'=x_2$. Thus the x_i-integrations vanish. One obtains

$$
\Pi_0 = \mathrm{Tr}(\bar{\Gamma}\gamma_\mu\Gamma\gamma_\nu)\frac{\Gamma^2(D/2)}{4\pi^D}\int\frac{x^\mu x^\nu d\lambda\,d\lambda'}{(x^2)^D}e^{\lambda\omega+\lambda'\omega'} \qquad (5.121)
$$

where $x=v\lambda+v'\lambda'$. Finally, the quantity needed for the sum rule analysis is the one which is Borel transformed both with respect to ω and ω' [188]. The direct application of the Borel transformation to the final expression is somewhat difficult. Therefore one makes a comparison in applying the Borel transformation both to the left and to the right hand side of Eq. (5.119), obtaining

$$
e^{-ikv/T} = -\hat{B}_T^{(\omega)}\left(\frac{1}{\omega-ikv}\right) = \hat{B}_T^{(\omega)}\int_0^\infty e^{\lambda(\omega-ikv)}d\lambda = \int_0^\infty\hat{B}_T^{(\omega)}(e^{\lambda\omega})e^{-ikv\lambda}d\lambda \qquad (5.122)
$$

and states that $\hat{B}_T^{(\omega)}(e^{\lambda\omega})=\delta(\lambda-1/T)$. The direct proof using the definition of the *Borel transformation* is given by

$$
\begin{aligned}
\hat{B}_T^{(\omega)}(e^{\lambda\omega}) &= \lim_{-\omega,n\to\infty}\frac{(-\omega)^{n+1}}{n!}\frac{d^n}{d\omega^n}e^{\lambda\omega} = \lim_{-\omega,n\to\infty}\frac{(-\omega)^{n+1}}{n!}\lambda^ne^{\lambda\omega} =\\
&= \lim_{n\to\infty}\frac{(nT)^{n+1}}{n!}\lambda^ne^{-n\lambda T} = \qquad (\text{where } \omega=-nT,\ T \text{ fixed})\\
&= \lim_{n\to\infty}\frac{nT}{\sqrt{2\pi n}}\left(\frac{enT}{n}\right)^n\lambda^ne^{-n\lambda T} = \qquad \left(\text{using } n!\approx\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)\\
&= \lim_{n\to\infty}\frac{nT}{\sqrt{2\pi n}}\left(eTe^{-\lambda T}\lambda\right)^n = \frac{T}{\sqrt{2\pi}}\lim_{n\to\infty}\sqrt{n}\left(\lambda Te^{1-\lambda T}\right)^n. \qquad (5.123)
\end{aligned}
$$

One knows that

$$
|xe^{1-x}|\le 1, \qquad xe^{1-x}=1 \quad \text{for } x=1. \qquad (5.124)
$$

Therefore, the power takes the limit value 1 only in the case $\lambda T = 1$ while it gives 0 in all other cases. Because of the additional factor of $\sqrt{n}$ it is probable that the expression $\hat{B}_T^{(\omega)}(e^{\lambda\omega})$ turns out to be a Dirac delta function. This is checked numerically. Instead of the Borel parameter one takes $t = 1/T$ and therefore obtains

$$\Pi_0^B := \hat{B}_{1/t}^{(\omega)}\hat{B}_{1/t'}^{(\omega)}\Pi_0 = \mathrm{Tr}(\bar\Gamma\gamma_\mu\Gamma\gamma_\nu)\frac{\Gamma^2(D/2)}{4\pi^D}\frac{a^\mu a^\nu}{(a^2)^D} = \frac{\mathrm{Tr}(\bar\Gamma\not{a}\Gamma\not{a})}{4\pi^D(a^2)^D}\Gamma^2(D/2) \tag{5.125}$$

where $a = tv + t'v'$ and $a^2 = t^2 + 2tt'(v\cdot v') + t'^2$.

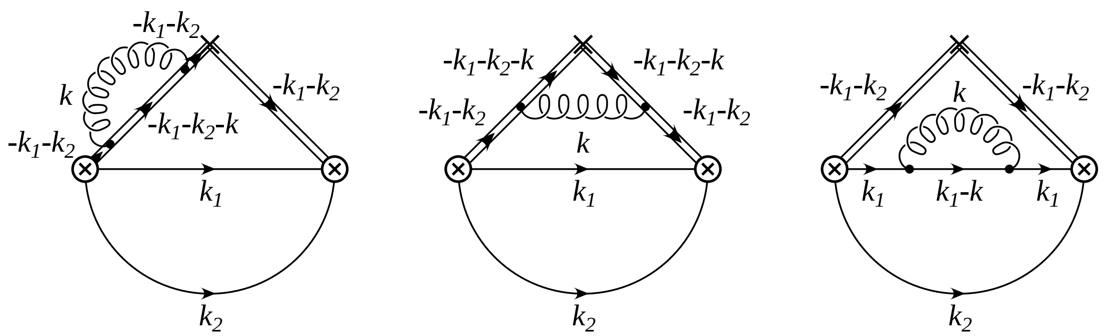


Figure 5.15: next-to-leading order self-energy diagrams

5.4.2 The massive self energy diagram

One now can proceed to the calculation of the first next-to-leading order diagrams shown in Fig. 5.15. The diagram on the left hand side is the massive self energy diagram which shall be calculated first. It is given by

$$\begin{aligned}
\Pi_1 &= \int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\left(\bar\Gamma\frac{i}{\not{k}_1}\Gamma\frac{i}{\not{k}_2}\right)\frac{i}{\omega'-(k_1+k_2)v'}\quad\times\\
&\times\ \frac{i}{\omega-(k_1+k_2)v}(-ig_sv_\alpha)\frac{i}{\omega-(k_1+k_2+k)v}(-ig_sv^\alpha)\frac{i}{\omega-(k_1+k_2)v}\left(\frac{-i}{k^2}\right)\ =\\
&= -ig_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\Gamma\gamma_\nu)\int\frac{id^Dk_1^E}{(2\pi)^D}\frac{id^Dk_2^E}{(2\pi)^D}\frac{id^Dk^E}{(2\pi)^D}\frac{(k_1^E)^\mu(k_2^E)^\nu}{(k_1^E)^2(k_2^E)^2(k^E)^2}\quad\times\\
&\quad\times\ \frac{1}{\omega'-i(k_1^E+k_2^E)v'}\left(\frac{1}{\omega-i(k_1^E+k_2^E)v}\right)^2\frac{1}{\omega-i(k_1^E+k_2^E+k^E)v}
\end{aligned} \tag{5.126}$$

where again the Wick rotation to Euclidean metric was performed. Skipping this index E and inserting the configuration space representations for the different propagators, one obtains

$$\begin{aligned}
\Pi_1 &= -g_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\Gamma\gamma_\nu)\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_1x_1}\frac{x_1^\mu d^Dx_1}{(x_1^2)^{D/2}}\right)\quad\times\\
&\quad\times\ \left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_2x_2}\frac{x_2^\nu d^Dx_2}{(x_2^2)^{D/2}}\right)\left(\frac{\Gamma(D/2-1)}{4\pi^{D/2}}e^{ikx_3}\frac{d^Dx_3}{(x_3^2)^{D/2-1}}\right)\quad\times\\
&\quad\times\ \left(-e^{(\omega-i(k_1+k_2+k)v)\lambda_1}d\lambda_1\right)\left(\lambda_2e^{(\omega-i(k_1+k_2)v)\lambda_2}d\lambda_2\right)\left(-e^{(\omega'-i(k_1+k_2)v')\lambda_3}d\lambda_3\right).
\end{aligned} \tag{5.127}$$

The first step to simplify this integral is to perform the integrations over k_1, k_2 and k. These give rise to

$$x_1 = v(\lambda_1 + \lambda_2) + v'\lambda_3 = x_2, \qquad x_3 = v\lambda_1. \tag{5.128}$$

Finally, the Borel transformations for ω and ω' are performed, resulting in $\lambda_2 = t - \lambda_1$, $\lambda_3 = t'$ which leaves one free parameter $\lambda_1 \in [0, t]$. With

$$x_1 = x_2 = vt + v't' = a, \qquad x_3 = v\lambda_1, \tag{5.129}$$

one obtains

$$\begin{aligned}
\Pi_1^B &= g_s^2 \mathrm{Tr}(\bar\Gamma\gamma_\mu\Gamma\gamma_\nu)\frac{\Gamma^2(D/2)\Gamma(D/2-1)}{16\pi^{3D/2}}\frac{a^\mu}{(a^2)^{D/2}}\frac{a^\nu}{(a^2)^{D/2}}\int_0^t \frac{(t-\lambda_1)d\lambda_1}{((v\lambda_1)^2)^{D/2-1}} = \\
&= \frac{g_s^2\mathrm{Tr}(\bar\Gamma \not a \Gamma \not a)}{4\pi^{3D/2}(a^2)^D}\Gamma^2(D/2)\Gamma(D/2-1)\int_0^t \frac{(t-\lambda_1)d\lambda_1}{\lambda_1^{D-2}} \qquad (\text{because } v^2 = 1).
\end{aligned} \tag{5.130}$$

The integral can be calculated,

$$\int_0^t \frac{(t-\lambda_1)d\lambda_1}{\lambda_1^{D-2}} = \frac{-t^{2\varepsilon}}{2\varepsilon(1-2\varepsilon)} = \frac{-t^{2\varepsilon}}{2\varepsilon}(1+2\varepsilon) + O(\varepsilon) = -\frac{1}{2\varepsilon} - 1 - \ln t + O(\varepsilon), \tag{5.131}$$

the result is therefore given by

$$\Pi_1^B = \frac{-g_s^2\mathrm{Tr}(\bar\Gamma\not a\Gamma\not a)}{16\pi^{6-3\varepsilon}(a^2)^{4-2\varepsilon}}\Gamma^2(2-\varepsilon)\Gamma(1-\varepsilon)\left(\frac{1}{2\varepsilon} + 1 + \ln t\right). \tag{5.132}$$

A corresponding diagram is given by the exchange $(v, t) \leftrightarrow (v', t')$.

5.4.3 The broken massive self energy diagram

As next the diagram in the middle of Fig. 5.15 is considered. Performing all the steps described above, one obtains

$$\begin{aligned}
\Pi_2 &= \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\left(\bar\Gamma\frac{i}{\not k_1}\Gamma\frac{i}{\not k_2}\right)\frac{i}{\omega-(k_1+k_2)v}(-ig_sv_\alpha)\frac{i}{\omega-(k_1+k_2+k)v} \times \\
&\qquad \times \frac{i}{\omega'-(k_1+k_2+k)v'}(-ig_sv^\alpha)\frac{i}{\omega'-(k_1+k_2)v'}\left(\frac{-i}{k^2}\right) = \\
&= -ig_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\Gamma\gamma_\nu)\int\frac{id^Dk_1^E}{(2\pi)^D}\frac{id^Dk_2^E}{(2\pi)^D}\frac{id^Dk^E}{(2\pi)^D}\frac{(k_1^E)^\mu(k_2^E)^\nu}{(k_1^E)^2(k_2^E)^2(k^E)^2}\left(\frac{1}{\omega-i(k_1^E+k_2^E)v}\right) \times \\
&\qquad \times \left(\frac{1}{\omega-i(k_1^E+k_2^E+k^E)v}\right)\left(\frac{1}{\omega'-i(k_1^E+k_2^E+k^E)v'}\right)\left(\frac{1}{\omega'-i(k_1^E+k_2^E)v'}\right) = \\
&= -g_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\Gamma\gamma_\nu)\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_1x_1}\frac{x_1^\mu d^Dx_1}{(x_1^2)^{D/2}}\right) \times \\
&\qquad \times \left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_2x_2}\frac{x_2^\nu d^Dx_2}{(x_2^2)^{D/2}}\right)\left(\frac{\Gamma(D/2-1)}{4\pi^{D/2}}e^{ikx_3}\frac{d^Dx_3}{(x_3^2)^{D/2-1}}\right) \times \\
&\qquad \times \left(-e^{(\omega-i(k_1+k_2)v)\lambda_1}d\lambda_1\right)\left(-e^{(\omega-i(k_1+k_2+k)v)\lambda_2}d\lambda_2\right) \times \\
&\qquad \times \left(-e^{(\omega'-i(k_1+k_2+k)v')\lambda_3}d\lambda_3\right)\left(-e^{(\omega'-i(k_1+k_2)v')\lambda_4}d\lambda_4\right).
\end{aligned} \tag{5.133}$$

The integration over k_1, k_2 and k results in

$$x_1 = v(\lambda_1 + \lambda_2) + v'(\lambda_3 + \lambda_4) = x_2, \qquad x_3 = v\lambda_2 + v'\lambda_3, \tag{5.134}$$

The Borel transformation leads to $\lambda_1 + \lambda_3 = t$ and $\lambda_3 + \lambda_4 = t'$. Therefore, $\lambda_1 = t - \lambda_2$ and $\lambda_4 = t' - \lambda_3$ can be eliminated and one obtains $x_1 = vt + v't' = x_2 = a$ and

$$\Pi_2 = \frac{g_s^2 \mathrm{Tr}(\bar\Gamma \not{a} \Gamma \not{a})}{16\pi^{3D/2}(a^2)^D}\Gamma^2(D/2)\Gamma(D/2-1)\int_0^t d\lambda_2 \int_0^t d\lambda_3 \frac{1}{((v\lambda_2 + v'\lambda_3)^2)^{D/2-1}}. \tag{5.135}$$

5.4.4 The light self energy diagram

For the light self energy diagram (right hand side of Fig. 5.15) one starts with

$$\begin{aligned}
\Pi_3 &= \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\left(\bar\Gamma\frac{i}{\not{k}_1}(-ig_s\gamma_\alpha)\frac{i}{\not{k}_1-\not{k}}(-ig_s\gamma^\alpha)\frac{i}{\not{k}_1}\Gamma\frac{i}{\not{k}_2}\right) \times \\
&\qquad \times \frac{i}{\omega-(k_1+k_2)v}\left(\frac{-i}{k^2}\right)\frac{i}{\omega'-(k_1+k_2)v'} = \\
&= -ig_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\gamma_\alpha\gamma_\nu\gamma^\alpha\gamma_\rho\Gamma\gamma_\sigma)\int\frac{id^Dk_1^E}{(2\pi)^D}\frac{id^Dk_2^E}{(2\pi)^D}\frac{id^Dk}{(2\pi)^D} \times \\
&\quad \times \frac{(k_1^E)^\mu(k_1^E-k^E)^\nu(k_1^E)^\rho(k_2^E)^\sigma}{(k_1^E)^2(k_1^E-k^E)^2(k_1^E)^2(k_2^E)^2}\left(\frac{1}{\omega-(k_1^E+k_2^E)v}\right)\frac{1}{(k^2)^E}\left(\frac{1}{\omega'-(k_1^E+k_2^E)v'}\right) = \\
&= -g_s^2(2-D)\mathrm{Tr}(\bar\Gamma\gamma_\mu\gamma_\nu\gamma_\rho\Gamma\gamma_\sigma)\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D} \times \\
&\qquad \times \left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_1x_1}\frac{x_1^\mu d^Dx_1}{(x_1^2)^{D/2}}\right)\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{i(k_1-k)x_2}\frac{x_2^\nu d^Dx_2}{(x_2^2)^{D/2}}\right) \times \\
&\qquad \times \left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_1x_3}\frac{x_3^\rho d^Dx_3}{(x_3^2)^{D/2}}\right)\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_2x_4}\frac{x_4^\sigma d^Dx_4}{(x_4^2)^{D/2}}\right) \times \\
&\quad \times \left(\frac{\Gamma(D/2-1)}{4\pi^{D/2}}e^{ikx_5}\frac{d^Dx_5}{(x_5^2)^{D/2-1}}\right)\left(-e^{(\omega-(k_1+k_2)v)\lambda_1}d\lambda_1\right)\left(-e^{(\omega'-(k_1+k_2)v')\lambda_2}d\lambda_2\right).
\end{aligned} \tag{5.136}$$

After integration over k_1, k_2 and k one obtains

$$x_1 + x_2 + x_3 = x_4 = v\lambda_1 + v'\lambda_2, \qquad x_2 - x_5 = 0. \tag{5.137}$$

while the Borel transformation leads to $\lambda_1 = t$ and $\lambda_2 = t'$. Thus the result is given by

$$\begin{aligned}
\Pi_3 &= \frac{-g_s^2(2-D)}{64\pi^{5D/2}(a^2)^{D/2}}\mathrm{Tr}(\bar\Gamma\gamma_\mu\gamma_\nu\gamma_\rho\Gamma\not{a})\Gamma^4(D/2)\Gamma(D/2-1) \times \\
&\qquad \times \int \frac{x_1^\mu(a-x_1-x_3)^\nu x_3^\rho d^Dx_1 d^Dx_3}{(x_1^2)^{D/2}((a-x_1-x_3)^2)^{D-1}(x_3^2)^{D/2}}.
\end{aligned} \tag{5.138}$$

Note that the tensor integral is symmetric in μ, ν, and ρ, the only outer momentum is a. Therefore it can be expressed by $Aa^\mu a^\nu a^\rho + B(a^\mu g^{\nu\rho} + a^\nu g^{\mu\rho} + a^\rho g^{\mu\nu})$.

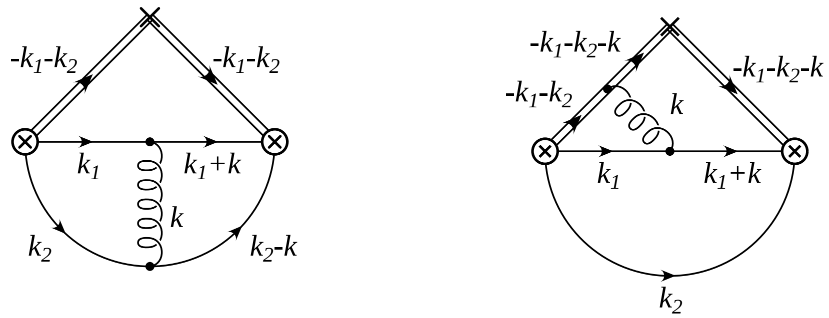


Figure 5.16: next-to-leading order fish diagrams

5.4.5 The light fish diagram

As in Chapter 4, *fish diagrams* are diagrams with a gluon propagator connecting two lines. The light fish diagram is shown in Fig. 5.16 on the left hand side. The contribution of this diagram is given by

$$
\begin{aligned}
\Pi_4 &= \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\left(\bar\Gamma\frac{i}{\not k_1+\not k}(-ig_s\gamma_\alpha)\frac{i}{\not k_1}\Gamma\frac{i}{\not k_2}(-ig_s\gamma^\alpha)\frac{i}{\not k_2-\not k}\right)\times\\
&\qquad\times\left(\frac{i}{\omega-(k_1+k_2)v}\right)\frac{-i}{k^2}\left(\frac{i}{\omega'-(k_1+k_2)v'}\right) =\\
&= -ig_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\gamma_\alpha\gamma_\nu\Gamma\gamma_\rho\gamma^\alpha\gamma_\sigma)\int\frac{id^Dk_1^E}{(2\pi)^D}\frac{id^Dk_2^E}{(2\pi)^D}\frac{id^Dk^E}{(2\pi)^D}\times\\
&\times\frac{(k_1^E+k^E)^\mu(k_1^E)^\nu(k_2^E)^\rho(k_2^E-k^E)^\sigma}{(k_1^E+k_2^E)^2(k_1^E)^2(k_2^E)^2(k_2^E-k^E)}\frac{i}{\omega-i(k_1^E+k_2^E)v}\left(\frac{-i}{(k^E)^2}\right)\frac{i}{\omega'-i(k_1^E+k_2^E)v'} =\\
&= -g_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\gamma_\alpha\gamma_\nu\Gamma\gamma_\rho\gamma^\alpha\gamma_\sigma)\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\times\\
&\qquad\times\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{i(k_1+k)x_1}\frac{x_1^\mu d^Dx_1}{(x_1^2)^{D/2}}\right)\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_1x_2}\frac{x_2^\nu d^Dx_2}{(x_2^2)^{D/2}}\right)\times\\
&\qquad\times\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_2x_3}\frac{x_3^\rho d^Dx_3}{(x_3^2)^{D/2}}\right)\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{i(k_2-k)x_4}\frac{x_4^\sigma d^Dx_4}{(x_4^2)^{D/2}}\right)\times\\
&\quad\times\left(\frac{\Gamma(D/2-1)}{4\pi^{D/2}}e^{ikx_5}\frac{d^Dx_5}{(x_1^2)^{D/2-1}}\right)\left(-e^{(\omega-i(k_1+k_2)v)\lambda_1}d\lambda_1\right)\left(-e^{(\omega'-i(k_1+k_2)v')\lambda_2}d\lambda_2\right).
\end{aligned}
\tag{5.139}
$$

The integration over k_1, k_2 and k results in

$$x_1+x_2=x_3+x_4=v\lambda_1+v'\lambda_2,\qquad x_1-x_4+x_5=0,\tag{5.140}$$

the Borel transformation gives $\lambda_1=t$ and $\lambda_2=t'$. One eliminates $x_1=a-x_3$, $x_4=a-x_2$ and $x_5=x_4-x_1=x_2-x_3$ and obtains

$$
\begin{aligned}
\Pi_4 &= -g_s^2\mathrm{Tr}(\bar\Gamma\gamma_\nu\gamma_\alpha\gamma_\nu\Gamma\gamma_\rho\gamma^\alpha\gamma_\sigma)\frac{\Gamma^4(D/2)\Gamma(D/2-1)}{64\pi^{5D/2}}\times\\
&\times\int\frac{(a-x_2)^\mu x_2^\nu x_3^\rho(a-x_3)^\sigma d^Dx_2d^Dx_3}{((a-x_2)^2)^{D/2}(x_2^2)^{D/2}(x_3^2)^{D/2}((a-x_3)^2)^{D/2}((x_2-x_3)^2)^{D/2-1}}.
\end{aligned}
\tag{5.141}
$$

Note that the integral is symmetric under simultaneous pair exchange of μ, ν, ρ, and σ.

5.4.6 The semi-massive fish diagram

The contribution of the diagram on the right hand side of Fig. 5.16 is given by

$$\begin{aligned}
\Pi_5 &= \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\left(\bar\Gamma\frac{i}{\not k_1+\not k}(-ig_s\gamma_\alpha)\frac{i}{\not k_1}\Gamma\frac{i}{\not k_2}\right)\left(\frac{-i}{k^2}\right) \times \\
&\quad \times \frac{i}{\omega'-(k_1+k_2+k)v'}\,\frac{i}{\omega-(k_1+k_2+k)v}(-ig_sv^\alpha)\frac{i}{\omega-(k_1+k_2)v} = \\
&= -ig_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\not v\gamma_\nu\Gamma\gamma_\rho)\int\frac{id^Dk_1^E}{(2\pi)^D}\frac{id^Dk_2^E}{(2\pi)^D}\frac{id^Dk^E}{(2\pi)^D}\frac{(k_1^E+k)^\mu(k_1^E)^\nu(k_2^E)^\rho}{(k_1^E+k^E)^2(k_1^E)^2(k_2^E)^2(k^E)^2} \times \\
&\quad \times \frac{1}{\omega'-i(k_1^E+k_2^E+k^E)v'}\,\frac{1}{\omega-i(k_1^E+k_2^E+k^E)v}\,\frac{1}{\omega-i(k_1^E+k_2^E)v} = \\
&= -g_s^2\mathrm{Tr}(\bar\Gamma\gamma_\mu\not v\gamma_\nu\Gamma\gamma_\rho)\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{i(k_1+k)x_4}\frac{x_4^\mu d^Dx_4}{(x_4^2)^{D/2}}\right) \times \\
&\quad \times \left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_1x_1}\frac{x_1^\nu d^Dx_1}{(x_1^2)^{D/2}}\right)\left(-\frac{i\Gamma(D/2)}{2\pi^{D/2}}e^{ik_2x_2}\frac{x_2^\rho d^Dx_1}{(x_2^2)^{D/2}}\right) \times \\
&\quad \times \left(\frac{\Gamma(D/2-1)}{4\pi^{D/2}}e^{ikx_3}\frac{d^Dx_3}{(x_3^2)^{D/2-1}}\right)\left(-e^{(\omega-i(k_1+k_2)v)\lambda_1}d\lambda_1\right) \times \\
&\quad \times \left(-e^{(\omega-i(k_1+k_2+k)v)\lambda_2}d\lambda_2\right)\left(-e^{(\omega'-i(k_1+k_2+k)v')\lambda_3}d\lambda_3\right). \qquad (5.142)
\end{aligned}$$

Note that there is one more variable x_4. It will remain as integration variable, too. The integration over the three inner momenta result in

$$x_1+x_4 = x_2 = v(\lambda_1+\lambda_2)+v'\lambda_3, \qquad x_3+x_4 = v\lambda_2+v'\lambda_3, \qquad (5.143)$$

The Borel transformation gives $\lambda_2 = t-\lambda_1$, $\lambda_3 = t'$ such that

$$x_1+x_4 = x_2 = vt+v't' = a, \qquad x_3+x_4 = vt+v't'-\lambda_1v = a-\lambda_1v. \qquad (5.144)$$

Therefore, one ends up with the Borel transformed integral

$$\begin{aligned}
\Pi_5^B &= \frac{-ig_s^2}{32\pi^{2D}}\mathrm{Tr}(\bar\Gamma\gamma_\mu\not v\gamma_\nu\Gamma\gamma_\rho)\Gamma^3(D/2)\Gamma(D/2-1) \times \\
&\qquad \times \int\frac{x_4^\mu(a-x_4)^\nu a^\rho d^Dx_4}{(x_4^2(a-x_4)^2)^{D/2}}\int_0^t\frac{d\lambda_1}{((a-\lambda_1v-x_4)^2)^{D/2-1}} = \qquad (5.145)\\
&= \frac{g_s^2}{32\pi^{2D}}\Gamma^3(D/2)\Gamma(D/2-1)\int\frac{\mathrm{Tr}(\bar\Gamma\not x_4\not v(\not a-\not x_4)\Gamma\not a)}{(x_4^2(a-x_4)^2)^{D/2}}d^Dx_4\int_0^t\frac{d\lambda_1}{((a-\lambda_1v-x_4)^2)^{D/2-1}}.
\end{aligned}$$

The solution of the integral in Eq. (5.145) is pending. It is extremely difficult to imagine that by standard techniques this integral could be reduced to an analytical expression. Probably the angular dependence between a, v and x_4 can be integrated out. But the two-fold integral over λ_1 and $|x_4|$ can then only be treated numerically. For this reason, the calculations are started again in the following section for the case where the Isgur–Wise function is considered close to the zero recoil point – a situation which is used in Ref. [189] as well.

5.5 The Isgur–Wise function close to zero recoil

As in the previous section, the baryonic three-point function is given by

$$\tilde{\Gamma}(\omega,\omega',v\cdot v')(v+v')^\mu = i^2\int\langle 0|T\{J_{\bar{\Gamma}}(x)V^\mu(0)J_\Gamma(y)\}|0\rangle e^{ipx-ip'y}d^4x\,d^4y \tag{5.146}$$

where p and p' are the residual momenta of the ingoing and outgoing heavy quarks, resp., v and v' are the velocities of the baryons, $\omega = p\cdot v$, $\omega' = p'\cdot v'$ with $v^2 = v'^2 = 1$, and the currents are given by

$$J_\Gamma = [(q_1)^{iT}C\Gamma\tau(q_2)^j]\Gamma'(h_v)^k\varepsilon_{ijk}, \qquad V^\mu = \bar{h}_v\gamma^\mu h_{v'}, \tag{5.147}$$

q_1, q_2 and h_v being the light and heavy quark fields, respectively. The calculational tools will be demonstrated in the following by using a special case, namely the heavy-to-light gluon exchange diagram.

5.5.1 The semimassive fish diagram

The diagram to be calculated is the heavy-to-light gluon exchange diagram, known from the last section as semimassive fish diagram (cf. Fig. 5.16 (right)) which is shown with new momentum conventions in Fig. 5.17 on the right. It is a three-loop diagram, so it contains three internal momenta k, l and p. The last momentum is attached to the light line not connected to the others by a gluon. As usual one can express the integral in terms of light and heavy propagators

$$\frac{i}{\not k} \qquad \frac{i}{\omega^{(\prime)}+k\cdot v^{(\prime)}} \tag{5.148}$$

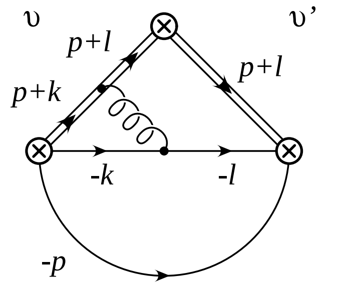


Figure 5.17: semimassive fish diagram

with internal (residual) momentum k, $\omega^{(\prime)} = v^{(\prime)}p$, and gluon vertices to the light and heavy lines, given by $-ig_s\gamma^\alpha$ and $-ig_sv^\alpha$ (cf. Eq. (5.77)). Leaving out the colour factor for the moment, the contribution of this diagram is given by

$$\begin{aligned} I &= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}\left(\Gamma'\frac{i}{-\not p}\Gamma\frac{i}{-\not k}(-ig_s\gamma_\alpha)\frac{i}{-\not l}\right)\left(\frac{-i}{(k-l)^2}\right)\times\\ &\quad\times\left(\frac{i}{\omega+pv+kv}\right)(-ig_sv^\alpha)\left(\frac{i}{\omega+pv+lv}\right)\left(\frac{i}{\omega'+pv'+lv'}\right) =\\ &= ig_s^2\mathrm{Tr}(\Gamma'\gamma_\mu\Gamma\gamma_\nu\not v\gamma_\rho)\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}p^\mu k^\nu l^\rho\left(\frac{-1}{p^2}\right)\left(\frac{-1}{k^2}\right)\left(\frac{-1}{l^2}\right)\left(\frac{-1}{(k-l)^2}\right)\times\\ &\quad\times\left(\frac{1}{\omega+pv+kv}\right)\left(\frac{1}{\omega+pv+lv}\right)\left(\frac{1}{\omega'+pv'+lv'}\right). \end{aligned}\tag{5.149}$$

While the colour factor is given by

$$\varepsilon_{ijk}(T^a)^i_{i'}(T^a)^j_{j'}\varepsilon^{i'j'k'} = \ldots = -\frac{N_c+1}{2N_c}\varepsilon_{ijk}\varepsilon^{ijk} = -C_BN_c!, \tag{5.150}$$

there are several steps necessary to calculate this integral.

Iterating the integrals

The integrals can be "iterated" which means that there are two integrations (namely those over k and l) that can be performed before performing the integration over p. The inner integral can be defined as

$$\begin{aligned} I_\mu(p,\omega,\omega') &= ig_s^2\mathrm{Tr}(\Gamma'\gamma_\mu\Gamma\gamma_\nu\not{v}\gamma_\rho)\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}k^\nu l^\rho\left(\frac{-1}{k^2}\right)\left(\frac{-1}{l^2}\right)\left(\frac{-1}{(k-l)^2}\right)\times \\ &\quad\times\left(\frac{1}{\omega+pv+kv}\right)\left(\frac{1}{\omega+pv+lv}\right)\left(\frac{1}{\omega'+pv'+lv'}\right) = \\ &= I_\mu(0,\omega+pv,\omega'+pv'). \end{aligned} \tag{5.151}$$

The last equality shows that this integral does not depend explicitly on the momentum vector p but is implicitly shifted by it. This integral I_μ is a vector integral. This integral may therefore be expressed in terms of the outer vectors. The only one which can be used are the velocities v and v' or linear combinations of them. For reasons that will be seen later on the expansion is not done in terms v and v' itself but in the orthonormal basis

$$v_\pm := \frac{v\pm v'}{2c_\pm} \qquad \Leftrightarrow \qquad v = c_+v_+ + c_-v_-, \quad v' = c_+v_+ - c_-v_- \tag{5.152}$$

where

$$c_\pm := \sqrt{\frac{y\pm 1}{2}}, \quad y := v\cdot v'. \tag{5.153}$$

This basis is (pseudo)orthonormal because of $v_+^2 = -v_-^2 = 1$ and $v_+v_- = 0$. The expansion of $I_\mu(0,\omega,\omega')$ in this basis is given by

$$I_\mu(0,\omega,\omega') = v_+\tilde{I}^+(\omega,\omega') + v_-\tilde{I}^-(\omega,\omega') \tag{5.154}$$

and leads to

$$\begin{aligned} I &= \int\frac{d^Dp}{(2\pi)^D}p^\mu\left(\frac{-1}{p^2}\right)I_\mu(p,\omega,\omega') = \int\frac{d^Dp}{(2\pi)^D}p^\mu\left(\frac{-1}{p^2}\right)I_\mu(0,\omega+pv,\omega'+pv') = \\ &= \int\frac{d^Dp}{(2\pi)^D}\left(\frac{-1}{p^2}\right)\Big((pv_+)\tilde{I}^+(\omega+pv,\omega'+pv') + (pv_-)\tilde{I}^-(\omega+pv,\omega'+pv')\Big) \end{aligned} \tag{5.155}$$

for the original expression.

Separating the Dirac structure

As the next step the Dirac structure of the integrals $\tilde{I}^\pm(\omega,\omega')$ given by the trace factor is separated, and the remaining tensor integrals are expanded in terms of covariants. Here the integral $\tilde{I}^+(\omega,\omega')$ is considered which can be written as

$$\tilde{I}^+(\omega,\omega') = ig_s^2\mathrm{Tr}\left(\Gamma'\not{v}_+\Gamma\gamma_\mu(c_+\not{v}_+ + c_-\not{v}_-)\gamma_\nu\right)I^{\mu\nu}(\omega,\omega'). \tag{5.156}$$

The expansion is given by

$$I^{\mu\nu}(\omega,\omega') = A_g g^{\mu\nu} + A_{++}v_+^\mu v_+^\nu + A_{+-}v_+^\mu v_-^\nu + A_{-+}v_-^\mu v_+^\nu + A_{--}v_-^\mu v_-^\nu \tag{5.157}$$

and the integral $\tilde{I}^+(\omega,\omega')$ thus reads

$$\begin{aligned}\tilde{I}^+(\omega,\omega') &= ig_s^2\Big[A_g\mathrm{Tr}(\Gamma'\not{v}_+\Gamma\gamma_\mu(c_+\not{v}_+ + c_-\not{v}_-)\gamma^\mu)+\\ &\quad +A_{++}\mathrm{Tr}(\Gamma'\not{v}_+\Gamma\not{v}_+(c_+\not{v}_+ + c_-\not{v}_-)\not{v}_+) + A_{+-}\mathrm{Tr}(\Gamma'\not{v}_+\Gamma\not{v}_+(c_+\not{v}_+ + c_-\not{v}_-)\not{v}_-) +\\ &\quad +A_{-+}\mathrm{Tr}(\Gamma'\not{v}_+\Gamma\not{v}_-(c_+\not{v}_+ + c_-\not{v}_-)\not{v}_+) + A_{--}\mathrm{Tr}(\Gamma'\not{v}_+\Gamma\not{v}_-(c_+\not{v}_+ + c_-\not{v}_-)\not{v}_-)\Big] =\\ &= ig_s^2\Big[(c_+((2-D)A_g + A_{++} - A_{--}) + c_-(A_{+-}+A_{-+}))\mathrm{Tr}(\Gamma'\not{v}_+\Gamma\not{v}_+) +\\ &\qquad +(c_+(A_{+-}+A_{-+}) + c_-((2-D)A_g - A_{++} + A_{--}))\mathrm{Tr}(\Gamma'\not{v}_+\Gamma\not{v}_-)\Big] \qquad (5.158)\end{aligned}$$

where one uses

$$\begin{aligned}\gamma_\mu(c_+\not{v}_+ + c_-\not{v}_-)\gamma^\mu &= (2-D)(c_+\not{v}_+ + c_-\not{v}_-),\\ \not{v}_+(c_+\not{v}_+ + c_-\not{v}_-)\not{v}_+ &= c_+\not{v}_+ + c_-\not{v}_+\not{v}_-\not{v}_+ \;=\; c_+\not{v}_+ - c_-\not{v}_-,\\ \not{v}_+(c_+\not{v}_+ + c_-\not{v}_-)\not{v}_- &= c_+\not{v}_- - c_-\not{v}_+,\\ \not{v}_-(c_+\not{v}_+ + c_-\not{v}_-)\not{v}_+ &= c_+\not{v}_- - c_-\not{v}_+,\\ \not{v}_-(c_+\not{v}_+ + c_-\not{v}_-)\not{v}_- &= c_+\not{v}_-\not{v}_+\not{v}_- + c_-\not{v}_- \;=\; c_+\not{v}_+ + c_-\not{v}_-. \qquad (5.159)\end{aligned}$$

By contracting with the dual basis given $g_{\mu\nu}$ and products of $v_\pm^\mu$ and $v_\pm^\nu$ one obtains

$$\begin{aligned}DA_g + A_{++} - A_{--} &= I_g(\omega,\omega') \;=\; g_{\mu\nu}I^{\mu\nu}(\omega,\omega'),\\ A_g + A_{++} \qquad &= I_{++}(\omega,\omega') \;=\; (v_+)_\mu(v_+)_\nu I^{\mu\nu}(\omega,\omega'),\\ -A_g \qquad + A_{--} &= I_{--}(\omega,\omega') \;=\; (v_-)_\mu(v_-)_\nu I^{\mu\nu}(\omega,\omega'), \qquad (5.160)\\ -A_{+-} &= I_{+-}(\omega,\omega') \;=\; (v_+)_\mu(v_-)_\nu I^{\mu\nu}(\omega,\omega'),\\ -A_{-+} &= I_{-+}(\omega,\omega') \;=\; (v_-)_\mu(v_+)_\nu I^{\mu\nu}(\omega,\omega').\end{aligned}$$

While the two last equations are already solved, the first three need to be inverted, the result reads

$$A_g = \frac{I_g - I_{++} + I_{--}}{D-2}, \qquad A_{++} = I_{++} - A_g, \quad A_{--} = I_{--} + A_g. \qquad (5.161)$$

In the following one concentrates again on one of these integrals, e.g. $I_g(\omega,\omega')$.

Expanding the heavy propagators

The integral one looks at is given by

$$I_g(\omega,\omega') = \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}(kl)\left(\frac{-1}{k^2}\right)\left(\frac{-1}{l^2}\right)\left(\frac{-1}{(k-l)^2}\right)\left(\frac{1}{\omega+kv}\right)\left(\frac{1}{\omega+lv}\right)\left(\frac{1}{\omega'+lv'}\right). \qquad (5.162)$$

Some word about the momentum factor (kl) are in order here. This factor and other factors resulting from the covariant expansion of the tensor integral can be expanded in terms of scalar products occuring in the denominator of the integrand and therefore cancel some of the propagator factors. This procedure can also be expressed in terms of operators on the powers of propagators. This technique is considered in a later step. Here the term is left as it is. Care is taken instead on the heavy quark propagators which give the integral a non-integrable form. By assuming that the recoil parameter $y = v \cdot v'$ is not very different from 1 one can express v and v' in the denominators by v_+ and v_- and expand the resulting expressions in the small parameter

$$c := \frac{c_-}{c_+} = \sqrt{\frac{y-1}{y+1}} = \sqrt{r} \qquad \text{where } r := \frac{y-1}{y+1}. \tag{5.163}$$

But before doing so, the momenta k and l are rescaled by c_+,

$$c_+ k \to k, \quad c_+ l \to l \quad \Rightarrow \quad \frac{d^D k}{(2\pi)^D} \to c_+^{-D} \frac{d^D k}{(2\pi)^D}, \quad \frac{d^D l}{(2\pi)^D} \to c_+^{-D} \frac{d^D l}{(2\pi)^D}. \tag{5.164}$$

Then one has

$$\begin{aligned}
\frac{1}{\omega + kv} &= \frac{1}{\omega + c_+ k v_+ + c_- k v_-} \quad \to \quad \frac{1}{\omega + k v_+ + c k v_-} = \sum_{n=0}^{\infty} \frac{n!(-c k v_-)^n}{(\omega + k v_+)^{n+1}}, \\
\frac{1}{\omega + lv} &= \frac{1}{\omega + c_+ l v_+ + c_- l v_-} \quad \to \quad \frac{1}{\omega + l v_+ + c l v_-} = \sum_{n=0}^{\infty} \frac{n!(-c l v_-)^n}{(\omega + l v_+)^{n+1}}, \\
\frac{1}{\omega' + lv'} &= \frac{1}{\omega' + c_+ l v_+ - c_- l v_-} \quad \to \quad \frac{1}{\omega' + l v_+ - c l v_-} = \sum_{n=0}^{\infty} \frac{n!(c l v_-)^n}{(\omega' + l v_+)^{n+1}}.
\end{aligned} \tag{5.165}$$

Therefore, one obtains

$$\begin{aligned}
I_g(\omega, \omega') &= c_+^{4-2D} \int \frac{d^D k}{(2\pi)^D} \frac{d^D l}{(2\pi)^D} (kl) \left(\frac{-1}{k^2} \right) \left(\frac{-1}{l^2} \right) \left(\frac{-1}{(k-l)^2} \right) \times \\
&\quad \times \sum_{n_1, n_2, n_3 = 0}^{\infty} c^{n_1 + n_2 + n_3} n_1! (-k v_-)^{n_1} n_2! (-l v_-)^{n_2} n_3! (l v_-)^{n_3} \times \\
&\quad \times \frac{1}{(\omega + k v_+)^{n_1 + 1}} \frac{1}{(\omega + l v_+)^{n_2 + 1}} \frac{1}{(\omega' + l v_+)^{n_3 + 1}}.
\end{aligned} \tag{5.166}$$

In expanding this expression up to the second order in r, the factor $c^{n_1+n_2+n_3}$ allows one to restrict the sum up to the fourth order. And there is still another restriction possible. If one extracts the vectors v_- from the integral, one ends up with a tensor integral of high rank. This can only be covariantly represented by the vector v_+ and the metric tensor. But by inserting this covariant representation back again, all scalar products $v_- \cdot v_+$ vanish and only coefficients with containing pure metric tensor components remain. This allows one to skip all terms in the sum from the very beginning for which $n_1 + n_2 + n_3$ is odd.

Partial fraction decomposition

The last step consists of a partial fraction decomposition of products of heavy propagators with the same residual momentum but different scalars ω and ω'. This will be done in order that the different parts shall fall into the two integral classes given in Ref. [190]. In the case considered here, the partial fraction decomposition is necessary for the product

$$\frac{1}{(\omega + l v_+)^{n_2+1}} \frac{1}{(\omega' + l v_+)^{n_3+1}}. \tag{5.167}$$

Again, the partial fraction decomposition will be shown for one specific example, namely the case $n_2 = 1$ and $n_3 = 0$. In this case one has

$$\frac{1}{(\omega + lv_+)^2(\omega' + lv_+)} = \frac{A_1}{\omega + lv_+} + \frac{A_2}{(\omega + lv_+)^2} + \frac{B_1}{\omega' + lv_+} \tag{5.168}$$

which results in

$$\begin{aligned} 1 &\overset{!}{=} A_1(\omega' + lv_+)(\omega + lv_+) + A_2(\omega' + lv_+) + B_1(\omega + lv_+)^2 \;= \\ &= A_1\omega'\omega + A_1(\omega' + \omega)(lv_+) + A_1(lv_+)^2 + A_2\omega' + A_2(lv_+) \;+ \\ &\qquad + B_1\omega^2 + 2B_1\omega(lv_+) + B_1(lv_+)^2. \end{aligned} \tag{5.169}$$

A comparison of coefficients in this example leads us to

$$A_1 = \frac{-1}{(\omega' - \omega)^2}, \quad A_2 = \frac{1}{\omega' - \omega}, \quad B_1 = \frac{1}{(\omega' - \omega)^2} \tag{5.170}$$

In general, one writes

$$\frac{1}{(\omega + lv_+)^{n_2+1}} \frac{1}{(\omega' + lv_+)^{n_3+1}} = \sum_{m_2=1}^{n_2+1} \frac{A_{m_2}}{(\omega + lv_+)^{m_2}} + \sum_{m_3=1}^{n_3+1} \frac{B_{m_3}}{(\omega' + lv_+)^{m_3}} \tag{5.171}$$

and finally obtains

$$\begin{aligned} I_g(\omega, \omega') &= c_+^{4-2D} \int \frac{d^Dk}{(2\pi)^D} \frac{d^Dl}{(2\pi)^D} (kl) \left(\frac{-1}{k^2}\right) \left(\frac{-1}{l^2}\right) \left(\frac{-1}{(k-l)^2}\right) \times \\ &\times \sum_{n_1,n_2,n_3=0}^{n_1+n_2+n_3 \le 4,\ \text{even}} c^{n_1+n_2+n_3} n_1!(-kv_-)^{n_1} n_2!(-lv_-)^{n_2} n_3!(lv_-)^{n_3} \times \\ &\times \frac{1}{(\omega + kv_+)^{n_1+1}} \left[\sum_{m_2=1}^{n_2+1} \frac{A_{m_2}}{(\omega + lv_+)^{m_2}} + \sum_{m_3=1}^{n_3+1} \frac{B_{m_3}}{(\omega' + lv_+)^{m_3}} \right]. \end{aligned} \tag{5.172}$$

5.5.2 The recurrence algebra for the integrals I'

While the recurrence of the integrals I is already implemented in the procedure `recursor`, the implementation of the recurrence of the integrals I' is still an outstanding task. Ref. [188] provides one with a calculation of these integrals under Borel transformation. But here at least the attempt will be made to find a recurrence algebra for these integrals in order to fill the gap mentioned in Ref. [169].

Generating a set of operator equations

The basis of each recurrence algebra is a set of operator equations. The operators are understood as acting on the entries of the integrals which on the other hand indicate the powers of the propagators contained in the integral. The most general two-loop integral considered here is given by

$$\begin{aligned} &\frac{-1}{(4\pi)^D} (-2\omega)^{2D-2a-2b} I'(a, b, c, p, q; \omega'/\omega) \;= \\ &= \int \frac{d^Dk}{(2\pi)^D} \frac{d^Dl}{(2\pi)^D} \left(\frac{-1}{k^2}\right)^a \left(\frac{-1}{l^2}\right)^b \left(\frac{-1}{(k-l)^2}\right)^c \left(\frac{\omega}{\omega + kv}\right)^p \left(\frac{\omega'}{\omega' + lv}\right)^q. \end{aligned} \tag{5.173}$$

In order to obtain the operator equations which are a consequence of a integration by parts, one first calculates the contributions of the different terms,

$$\begin{aligned}
\frac{\partial}{\partial k}\left(\frac{-1}{k^2}\right)^a &= (-1)^a\frac{\partial}{\partial k}(k^2)^{-a} = (-1)^a(-2ak)(k^2)^{-a-1} = \\
&= 2ak(-1)^{a+1}(k^2)^{-a-1} = 2ak\left(\frac{-1}{k^2}\right)^{a+1}, \\
\frac{\partial}{\partial l}\left(\frac{-1}{l^2}\right)^b &= 2bl\left(\frac{-1}{l^2}\right)^{b+1}, \\
\frac{\partial}{\partial k}\left(\frac{-1}{(k-l)^2}\right)^c &= 2c(k-l)\left(\frac{-1}{(k-l)^2}\right)^{c+1}, \\
\frac{\partial}{\partial l}\left(\frac{-1}{(k-l)^2}\right)^c &= 2c(l-k)\left(\frac{-1}{(k-l)^2}\right)^{c+1}, \\
\frac{\partial}{\partial k}\left(\frac{\omega}{\omega+kv}\right)^p &= \omega^p\frac{\partial}{\partial k}(\omega+kv)^{-p} = \omega^p(-pv)(\omega+kv)^{-p-1} = \\
&= -\frac{pv}{\omega}\left(\frac{\omega}{\omega+kv}\right)^{p+1}, \\
\frac{\partial}{\partial l}\left(\frac{\omega'}{\omega'+lv}\right)^q &= -\frac{qv}{\omega'}\left(\frac{\omega'}{\omega'+lv}\right)^{q+1}.
\end{aligned} \tag{5.174}$$

All these derivatives increase the corresponding entry by one. This will be used to define operators $\mathbf{A}^+$, $\mathbf{B}^+$, $\mathbf{C}^+$, $\mathbf{P}^+$, and $\mathbf{Q}^+$ acting on the integral and increasing the corresponding entry. The first equation to be constructed is the one corresponding to the operator,

$$k\frac{\partial}{\partial k}: \quad -\int k\frac{\partial}{\partial k}f(k,l) = \int f(k,l)\frac{\partial}{\partial k}k = D\int f(k,l) \tag{5.175}$$

where a fairly symbolic but obvious notation is used. The surface term always vanishes. To obtain the left hand side one has to multiply the calculated derivatives by the vector k. In the first derivative, the produced k^2 cancels one denominator factor and produces a minus sign. This derivative, therefore, is connected with an operator $-\mathbf{A}^-$. If one multiplies the third listed derivative with k, one has to deal with the product $k(k-l) = (k^2-l^2+(k-l)^2)/2$. Therefore, this derivative will result in an operator $-\mathbf{A}^-+\mathbf{B}^--\mathbf{C}^-$. The multiplication of the fifth equation with k, finally, results in

$$\frac{kv}{\omega} = \frac{\omega+kv}{\omega} - 1 \tag{5.176}$$

and therefore results in an operator $\mathbf{P}^- - 1$. Combining this with the right hand side of Eq. (5.175), one obtains

$$2a\mathbf{A}^+\mathbf{A}^- + c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^- + \mathbf{C}^-) + p\mathbf{P}^+(\mathbf{P}^- - 1) = D, \tag{5.177}$$

assuming that this operator acts on $I'(a,b,c,p,q;\omega'/\omega)$. One can of course also make use of $\mathbf{A}^+\mathbf{A}^- = \mathbf{C}^+\mathbf{C}^- = \mathbf{P}^+\mathbf{P}^- = 1$ and shift all these parts to the right hand side. Then one ends up with

$$c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-) - p\mathbf{P}^+ = D - 2a - c - p. \tag{5.178}$$

This equation is the first of the operator equations. For the second equation, considering

$$l\frac{\partial}{\partial k}: \qquad -\int l\frac{\partial}{\partial k}f(k,l) = \int f(k,l)\frac{\partial}{\partial k}l = 0, \tag{5.179}$$

one has to calculate

$$\begin{aligned} 2lk &= k^2+l^2-(k-l)^2 && \Rightarrow && -(\mathbf{A}^-+\mathbf{B}^--\mathbf{C}^-),\\ 2l(k-l) &= k^2-l^2-(k-l)^2 && \Rightarrow && -(\mathbf{A}^--\mathbf{B}^--\mathbf{C}^-),\\ \frac{lv}{\omega} &= \frac{\omega'}{\omega}\left(\frac{\omega'+lv}{\omega'}-1\right) && \Rightarrow && \frac{\omega'}{\omega}(\mathbf{Q}^--1). \end{aligned} \tag{5.180}$$

Therefore, one obtains

$$\begin{aligned} a\mathbf{A}^+(\mathbf{A}^-+\mathbf{B}^--\mathbf{C}^-)+c\mathbf{C}^+(\mathbf{A}^--\mathbf{B}^--\mathbf{C}^-)+\frac{\omega'}{\omega}p\mathbf{P}^+(\mathbf{Q}^--1) &= 0\\ \text{or}\quad a\mathbf{A}^+(\mathbf{B}^--\mathbf{C}^-)+c\mathbf{C}^+(\mathbf{A}^--\mathbf{B}^-)+\frac{\omega'}{\omega}p\mathbf{P}^+(\mathbf{Q}^--1) &= c-a. \end{aligned} \tag{5.181}$$

The next operator equation is the one related to

$$k\frac{\partial}{\partial l}: \qquad -\int k\frac{\partial}{\partial l}f(k,l) = \int f(k,l)\frac{\partial}{\partial l}k = 0. \tag{5.182}$$

Here one has to calculate

$$\begin{aligned} 2kl &= k^2+l^2-(k-l)^2 && \Rightarrow && -(\mathbf{A}^-+\mathbf{B}^--\mathbf{C}^-),\\ 2k(l-k) &= -k^2+l^2-(k-l)^2 && \Rightarrow && \mathbf{A}^--\mathbf{B}^-+\mathbf{C}^-,\\ \frac{kv}{\omega'} &= \frac{\omega}{\omega'}\left(\frac{\omega+kv}{\omega}-1\right) && \Rightarrow && \frac{\omega}{\omega'}(\mathbf{P}^--1) \end{aligned} \tag{5.183}$$

and obtains

$$\begin{aligned} b\mathbf{B}^+(\mathbf{A}^-+\mathbf{B}^--\mathbf{C}^-)-c\mathbf{C}^+(\mathbf{A}^--\mathbf{B}^-+\mathbf{C}^-)+\frac{\omega}{\omega'}q\mathbf{Q}^+(\mathbf{P}^--1) &= 0\\ \text{or}\quad b\mathbf{B}^+(\mathbf{A}^--\mathbf{C}^-)-c\mathbf{C}^+(\mathbf{A}^--\mathbf{B}^-)+\frac{\omega}{\omega'}q\mathbf{Q}^+(\mathbf{P}^--1) &= c-b. \end{aligned} \tag{5.184}$$

Finally, for

$$l\frac{\partial}{\partial l}: \qquad -\int l\frac{\partial}{\partial l}f(k,l) = \int f(k,l)\frac{\partial}{\partial l}l = D\int f(k,l) \tag{5.185}$$

one calculates

$$\begin{aligned} 2l^2 &= 2l^2 && \Rightarrow && -2\mathbf{B}^-,\\ 2l(l-k) &= -k^2+l^2+(k-l)^2 && \Rightarrow && \mathbf{A}^--\mathbf{B}^--\mathbf{C}^-,\\ \frac{lv}{\omega'} &= \frac{\omega'+lv}{\omega'}-1 && \Rightarrow && \mathbf{Q}^--1. \end{aligned} \tag{5.186}$$

Therefore, the last operator equation reads

$$\begin{aligned} 2b\mathbf{B}^+\mathbf{B}^- - c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^- - \mathbf{C}^-) + q\mathbf{Q}^+(\mathbf{Q}^- - 1) &= D \\ \text{or} \quad -c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-) - q\mathbf{Q}^+ &= D - 2b - c - q. \end{aligned} \tag{5.187}$$

To summarize, the operator equations are given by

$$c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-) - p\mathbf{P}^+ = D - 2a - c - p, \tag{5.188}$$

$$a\mathbf{A}^+(\mathbf{B}^- - \mathbf{C}^-) + c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-) + \frac{\omega'}{\omega}p\mathbf{P}^+(\mathbf{Q}^- - 1) = c - a, \tag{5.189}$$

$$b\mathbf{B}^+(\mathbf{A}^- - \mathbf{C}^-) - c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-) + \frac{\omega}{\omega'}q\mathbf{Q}^+(\mathbf{P}^- - 1) = c - b, \tag{5.190}$$

$$-c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-) - q\mathbf{Q}^+ = D - 2b - c - q. \tag{5.191}$$

The first two equations resp. the last two equations can be combined in the form described in the literature. This combination avoids the pure increase of one entry (p or q),

$$c - a - \frac{\omega'}{\omega}(D - 2a - c - p) = a\mathbf{A}^+(\mathbf{B}^- - \mathbf{C}^-) + \left(1 - \frac{\omega'}{\omega}\right)c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-) + \frac{\omega'}{\omega}p\mathbf{P}^+\mathbf{Q}^-, \tag{5.192}$$

$$c - b - \frac{\omega}{\omega'}(d - 2b - c - q) = b\mathbf{B}^+(\mathbf{A}^- - \mathbf{C}^-) - \left(1 - \frac{\omega}{\omega'}\right)c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-) + \frac{\omega}{\omega'}q\mathbf{Q}^+\mathbf{P}^-. \tag{5.193}$$

On the way to a recursion that works

In order to obtain an algorithm to reduce the integrals to those with one vanishing entry one should not use Eqs. (5.192) and (5.193). The reason is that they contain a seesaw mechanism. Looking e.g. at Eq. (5.192), the operators $\mathbf{A}^+\mathbf{C}^-$ and $\mathbf{C}^+\mathbf{A}^-$ appear both in the operator equation which means that this operation will keep on doing an infinite circle in changing these two entries forth and back which is not very helpful to get to an end. Instead of this one should take Eqs. (5.188) to (5.191) and try to eliminate as many operations as one can in order to "direct" the action of these operations. One can for instance combine the first and the second equation to eliminate the term proportional to $\mathbf{C}^+$, obtaining

$$a\mathbf{A}^+(\mathbf{B}^- - \mathbf{C}^-) + p\mathbf{P}^+\left(1 + \frac{\omega'}{\omega}(\mathbf{Q}^- - 1)\right) = a + 2c + p - D, \tag{5.194}$$

and in the same manner combine the third and fourth equation to obtain

$$b\mathbf{B}^+(\mathbf{A}^- - \mathbf{C}^-) + q\mathbf{Q}^+\left(1 + \frac{\omega}{\omega'}(\mathbf{P}^- - 1)\right) = b + 2c + q - D. \tag{5.195}$$

But one can alternatively combine the first and the third equation, resulting in

$$b\mathbf{B}^+(\mathbf{A}^- - \mathbf{C}^-) - p\mathbf{P}^+ + \frac{\omega}{\omega'}q\mathbf{Q}^+(\mathbf{P}^- - 1) = D - 2a - b - p, \tag{5.196}$$

and the second and fourth, obtaining

$$a\mathbf{A}^+(\mathbf{B}^- - \mathbf{C}^-) - q\mathbf{Q}^+ + \frac{\omega'}{\omega}p\mathbf{P}^+(\mathbf{Q}^- - 1) = D - a - 2b - q. \tag{5.197}$$

One can now rearrange these and the previous equations, keeping in mind that the action of $\mathbf{A}^+$ on such an operator equation will lead to the replacement $a \to a+1$ while the action of $\mathbf{A}^-$ does the opposite, $a \to a-1$. With this one obtains

$$\begin{aligned}
a-1 &= (a-1)\mathbf{C}^+\mathbf{B}^- + (D-(a-1)-2(c+1)-p)\mathbf{C}^+\mathbf{A}^- + \\
&\qquad + p\mathbf{P}^+\mathbf{C}^+\mathbf{A}^-\left(1+\frac{\omega'}{\omega}\left(\mathbf{Q}^- - 1\right)\right), && (5.198)\\
b-1 &= (b-1)\mathbf{C}^+\mathbf{A}^- + (D-(b-1)-2(c+1)-q)\mathbf{C}^+\mathbf{B}^- + \\
&\qquad + q\mathbf{Q}^+\mathbf{C}^+\mathbf{B}^-\left(1+\frac{\omega}{\omega'}\left(\mathbf{P}^- - 1\right)\right), && (5.199)\\
p-1 &= c\mathbf{C}^+\mathbf{P}^-(\mathbf{A}^- - \mathbf{B}^-) - (D-2a-c-(p-1))\mathbf{P}^-, && (5.200)\\
q-1 &= c\mathbf{C}^+\mathbf{Q}^-(\mathbf{B}^- - \mathbf{A}^-) - (D-2b-c-(q-1))\mathbf{Q}^-, && (5.201)\\
c-1 &= (c-1)\mathbf{A}^+\mathbf{B}^- - (D-2b-(c-1)-q)\mathbf{A}^+\mathbf{C}^- + q\mathbf{Q}^+\mathbf{A}^+\mathbf{C}^-, && (5.202)
\end{aligned}$$

These equations can be used to reduce the entries a, b, p, q, and c down to the value 1. But this does not end the recursion. Instead one observes that starting with an integral $I'(1,1,c,1,1;\omega'/\omega)$, one ends up with integrals with at least one vanishing entry and again the same integral $I'(1,1,c,1,1;\omega'/\omega)$. Therefore, the final step for the method is to solve the equation for $I'(1,1,c,1,1;\omega'/\omega)$ in order to obtain an explicit expression. In considering the already working recurrence procedures for the unprimed integrals, one can can learn much for the construction of a precisealgorithms for the primed integrals. The first two equations used for the unprimed integrals are given by

$$D-a-2c-p-q+1 = (2(D-a-b-c)-p-q+1)\mathbf{Q}^- - a\mathbf{A}^+(\mathbf{B}^- - \mathbf{C}^-) \quad (tI_5)$$

$$2(D-a-b-c)-p-q = (D-a-2c-p-q)\mathbf{Q}^+ + a\mathbf{A}^+(\mathbf{B}^- - \mathbf{C}^-)\mathbf{Q}^+ \quad (qI_5)$$

which are actually the same equation (the labels are taken from the recursion program). (tI_5) is applied for $b,c,q>0$ while (qI_5) for $q<0$. The next equation is

$$\begin{aligned}
a-1 &= (a-1)\mathbf{B}^-\mathbf{C}^+ - (2(D-a-b-c)-p-q+1)\mathbf{Q}^-\mathbf{A}^-\mathbf{C}^+ + \\
&\qquad + (D-a-2c-p-q)\mathbf{A}^-\mathbf{C}^+ \quad (cI_5)
\end{aligned}$$

which is used for $c<0$ and $a>1$. For one of the last entries one has

$$p-1 = c\mathbf{C}^+(\mathbf{A}^- - \mathbf{B}^-)\mathbf{P}^- - (D-2a-c-p+1)\mathbf{P}^- \quad (pI_5)$$

which is used for $c<0$ and $p>1$. Finally, for $c<0$ one uses

$$I(1,1,c,1,1) = \left(\frac{D}{2}-2-c\right)\frac{I(1,1,c+1,1,1)}{D-3-c}. \tag{5.203}$$

The final ends of the recursion

The recursion terminates if one of the entries vanishes. This paragraph contains these final expressions. The easiest case is the one where the entry c vanishes. In this case, the two momenta are no longer connected and one ends up with a product of one-loop integrals,

$$\begin{aligned} I'(a,b,0,p,q;\omega'/\omega) &= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^a\left(\frac{-1}{l^2}\right)^b\left(\frac{\omega}{\omega+kv}\right)^p\left(\frac{\omega'}{\omega'+lv}\right)^q = \\ &= \int \frac{d^Dk}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^a\left(\frac{\omega}{\omega+kv}\right)^p \times \int \frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{\omega'}{\omega'+lv}\right)^q = \\ &= \frac{-1}{(4\pi)^D}(-2\omega)^{D-2a}(-2\omega')^{D-2b}I(a,p)I(b,q). \end{aligned} \tag{5.204}$$

The next-to-simplest cases are those where one of the heavy entries p or q vanishes. For instance, for $q=0$ one obtains

$$\begin{aligned} I'(a,b,c,p,0;\omega'/\omega) &= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^a\left(\frac{-1}{l^2}\right)^b\left(\frac{-1}{(k-l)^2}\right)^c\left(\frac{\omega}{\omega+kv}\right)^p = \\ &= \int \frac{d^Dk}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^a\left(\frac{\omega}{\omega+kv}\right)^p \int \frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{-1}{(k-l)^2}\right)^c. \end{aligned} \tag{5.205}$$

The innermost integral is a massless one-loop integral with outer momentum k,

$$\int \frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{-1}{(k-l)^2}\right)^c = \frac{i}{(4\pi)^{D/2}}(-k^2)^{D/2-b-c}G(b,c). \tag{5.206}$$

Therefore, one continues with

$$\begin{aligned} I'(a,b,c,p,0;\omega'/\omega) &= \frac{i}{(4\pi)^{D/2}}G(b,c)\int \frac{d^Dk}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^{a+b+c-D/2}\left(\frac{\omega}{\omega+kv}\right)^p = \\ &= \frac{-1}{(4\pi)^D}(-2\omega)^{D-2(a+b+c-D/2)}G(b,c)I(a+b+c-D/2,p) = \\ &= \frac{-1}{(4\pi)^D}(-2\omega)^{2(D-a-b-c)}G(b,c)I(a+b+c-D/2,p). \end{aligned} \tag{5.207}$$

A corresponding result is obtained for $p=0$,

$$I'(a,b,c,0,q;\omega'/\omega) = \frac{-1}{(4\pi)^D}(-2\omega')^{2(D-a-b-c)}G(a,c)I(a+b+c-D/2,q). \tag{5.208}$$

The most complicated final expression results for $a=0$ and $b=0$, resp. In these cases one has to use hypergeometric functions. For $a=0$ one starts with

$$\begin{aligned} I'(0,b,c,p,q;\omega'/\omega) &= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{-1}{(k-l)^2}\right)^c\left(\frac{\omega}{\omega+kv}\right)^p\left(\frac{\omega'}{\omega'+lv}\right)^q = \\ &= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{-1}{k^2}\right)^c\left(\frac{\omega}{\omega+kv+lv}\right)^p\left(\frac{\omega'}{\omega'+lv}\right)^q = \qquad (k\to k+l) \\ &= \int \frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{-1}{k^2}\right)^c\left(\frac{\omega+lv}{\omega+kv+lv}\right)^p\left(\frac{\omega}{\omega+lv}\right)\left(\frac{\omega'}{\omega'+lv}\right)^q = \\ &= \int \frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{\omega}{\omega+lv}\right)^p\left(\frac{\omega'}{\omega'+lv}\right)^q \int \frac{d^Dk}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^c\left(\frac{\omega+lv}{\omega+lv+kv}\right)^p. \end{aligned} \tag{5.209}$$

The innermost integral is a heavy one-loop integral with shifted energy,

$$\begin{aligned}\int\frac{d^Dk}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^c\left(\frac{\omega+lv}{\omega+lv+kv}\right)^p &= \frac{i}{(4\pi)^{D/2}}\left(-2(\omega+lv)\right)^{D-2c}I(c,p) = \\ &= \frac{i}{(4\pi)^{D/2}}(-2\omega)^{D-2c}\left(\frac{\omega}{\omega+lv}\right)^{2c-D}I(c,p). \qquad (5.210)\end{aligned}$$

Therefore, one continues with

$$\begin{aligned}I'(0,b,c,p,q;\omega'/\omega) &= \frac{i(-2\omega)^{D-2c}}{(4\pi)^{D/2}}I(c,p)\int\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{\omega}{\omega+lv}\right)^{2c+p-D}\left(\frac{\omega'}{\omega'+lv}\right)^q = \\ &= -\frac{1}{(4\pi)^{D/2}}(-2\omega)^{D-2c}(-2\omega')^{D-2b}\left(\frac{\omega}{\omega'}\right)^{2c+p-D}\frac{\Gamma(2b+2c+p+q-2D)\Gamma(D/2-b)}{\Gamma(b)\Gamma(2c+p+q-D)}\times \\ &\quad\times {}_2F_1\left(2b+2c+p+q-2D,2c+p-D;2c+p+q-D;1-\frac{\omega}{\omega'}\right)I(c,p) \qquad (5.211)\end{aligned}$$

where for the last step the relation

$$\begin{aligned}\int\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{l^2}\right)^b\left(\frac{\omega}{\omega+lv}\right)^p\left(\frac{\omega'}{\omega'+lv}\right)^q &= \frac{i}{(4\pi)^{D/2}}\frac{\Gamma(2a+p+q-D)\Gamma(D/2-a)}{\Gamma(a)\Gamma(p+q)}\times \\ &\quad\times\left(\frac{\omega}{\omega'}\right)^p(-2\omega')^{D-2a}\,{}_2F_1\left(p+q+2a-D,p;p+q;1-\frac{\omega}{\omega'}\right) \qquad (5.212)\end{aligned}$$

has been used [190]. The corresponding calculation for $b=0$ reads

$$\begin{aligned}I'(a,0,c,p,q;\omega'/\omega) &= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dk}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^a\left(\frac{-1}{(k-l)^2}\right)^c\left(\frac{\omega}{\omega+kv}\right)^p\left(\frac{\omega'}{\omega'+lv}\right)^q = \\ &= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^a\left(\frac{-1}{l^2}\right)^c\left(\frac{\omega}{\omega+kv}\right)^p\left(\frac{\omega'}{\omega'+kv+lv}\right)^q = \qquad (l\to k+l) \\ &= \int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^a\left(\frac{-1}{l^2}\right)^c\left(\frac{\omega}{\omega+kv}\right)^p\left(\frac{\omega'}{\omega'+kv}\right)^q\left(\frac{\omega'+kv}{\omega'+kv+lv}\right) = \\ &= \frac{i}{(4\pi)^{D/2}}(-2\omega')^{D-2c}I(c,q)\int\frac{d^Dk}{(2\pi)^D}\left(\frac{-1}{k^2}\right)^a\left(\frac{\omega}{\omega+kv}\right)^p\left(\frac{\omega'}{\omega'+kv}\right)^{2c+q-D} = \\ &= -\frac{1}{(4\pi)^{D/2}}(-2\omega)^{D-2a}(-2\omega')^{D-2c}\left(\frac{\omega}{\omega'}\right)^p\frac{\Gamma(2a+2c+p+q-2D)\Gamma(D/2-a)}{\Gamma(a)\Gamma(2c+p+q-D)}\times \\ &\quad\times{}_2F_1\left(2a+2c+p+q-2D,2c+p-D;2c+p+q-D;1-\frac{\omega}{\omega'}\right)I(c,q). \\ &\qquad (5.213)\end{aligned}$$

A detailed calculation of these initial integrals is found in Appendix D.2.3. The initial integrals are important as input integrals for a future recurrence algorithm for the yet unresolved integrals $I'(a,b,c,d,e;\omega'/\omega)$.

Chapter 6

Anisotropic improved quark actions

The simulation of QCD on the lattice is a nonperturbative approach to this theory. Since its introduction between 1970 and 1980, this approach has lead to many interesting results, but always had to fight with the limited capacity of available computer facilities. But lattice QCD in the past, present and future is one of the most important users of more and more powerful computer systems. Also another idea for possible improvements for the precision of calculations on the QCD lattice came up rather soon. This idea is indeed related again to the main topic of this thesis, i.e. the effective theories. The idea was to improve the precision of calculations by improving the theory itself, in order to be able to obtain results also on computers which were available at that time.

Possible improvements are given in three areas. A classical improvement can be accomblished by combining extended Wilson loops, i.e. closed paths on the lattice (see for instance Refs. [191, 192, 193]). Using the tadpole improvement, quantum fluctuations can be reduced [194]. Finally, the coefficients of the effective theory can be improved by including radiative corrections into the coefficients. This is just the same concept as for other effective theories like HQET which was treated in the last chapter. For the lattice the effective theory is given by the nonrelativistic QCD (NRQCD), and the improvement just mentioned is called Symanzik improvement (for an overview and known possibilities to use this improvement see Refs. [194, 195]). In previous publications the currents for the decay of heavy into light hadrons given by the NRQCD were matched to QCD [196].

The use of anisotropic lattices plays an important role in the development of the QCD on the lattice [195]. It was noted that the difference between signal and noise decreases exponentially when the lattice is subdivided in the time direction while such an effect is not observed for the subdivision in spatial direction. This allows one to keep a coarse spatial lattice. Therefore, it is worthwile to apply the Symanzik improvement to the anisotropic lattice as well. Examples of a successful employment of anisotropic lattices in lattice QCD simulations have been increasing lately. They include extensive studies of the glueball spectrum [197], investigations of heavy hybrid states [198, 199] and calculations of the quarkonium fine structure [200].

In this chapter radiative corrections are calculated for the anisotropic lattice in order to establish a Symanzik improvement of the effective action. Such calculations are already done on the isotropic lattice [201, 202]. The calculation of the self energy of the quark (see Ref. [203] for the calculations on the isotropic lattice) is the tool to determine an appropriate energy scale for the running coupling constant (cf. the corresponding section in Ref. [194]), calculate the effective mass and the wave function renormalization and finally calculate also the relative factor between the spatial and temporal derivatives in the action, being a pure effect of the anisotropy [195].

Two different kinds of actions for the quarks are used in the following. The first one, applicable for heavy quarks, is an extension of the Wilson action such as the so-called D234 action. These actions will be described in detail in the following section. For light quarks the staggered quark action is used where the problematic CP violating contributions (mirror quarks) is solved by using fattened links. This quark action will be introduced in Sec. 6.5. For the gluon the naive action is extended by including higher Wilson loops.

6.1 Link operators, gauge and quark actions

The basic objects for lattice QCD to construct actions are operators which are defined on the *grids* or *links* of the lattice. They depend on the strong charge g, the QCD vector potential $A_\mu = T_a A_\mu^a$ given at the midst of the link and the lattice spacing. On the anisotropic lattice a distinction has to be made between lattice spacings in spatial and temporal direction. Therefore, the *lattice spacings* a_μ are characterized by the Lorentz index μ. The *link operator* at the spacetime point x in the μ direction is given by $U_\mu(x) := \exp(iga_\mu A_\mu(x + a_\mu/2))$ where a_μ is a short form for $a_\mu \hat{e}_\mu$ ($\hat{e}_\mu$ is the unit vector in direction of the μ-th coordinate axis). Because the potential A_μ is always accompanied by the lattice spacing in the same direction, one can *absorb this lattice spacing into the potential component* and write

$$U_\mu(x) = \exp(igA_\mu(x + a_\mu/2)). \tag{6.1}$$

This link operator, understood as a link from the spacetime point x to $x + a_\mu$, is used to construct all necessary quantities for the lattice QCD action. As one can show [195], the concatenation of link operators building a closed loop, the so-called *Wilson loop*, is a gauge invariant quantity. The reason why so much care is invested in gauge invariance is a very practical one. If gauge invariance is not guaranteed, the couplings at the quark-gluon, three-gluon and four-gluon vertices become independent of each other and have to be tuned independently. This is much more impractical than the absence of Lorentz invariance, rotation invariance etc. on the lattice.

6.1.1 The tadpole improvement

Before going into detail about how to construct Wilson loops and how to use them, one has to deal with a problem that was seen as a big disadvantange of lattice QCD until a solution was found for the problem in the early 1990's [204]. Looking for instance at the quark gluon vertex analogon on the lattice $\bar{\psi} U_\mu \gamma_\mu \psi / a$, this term contains of course the usual vertex $\bar{\psi} g A_\mu \gamma_\mu \psi$ but in addition vertices with any number of additional powers of agA_μ. For classical fields these additional powers are irrelevant because they are suppressed by powers of the lattice spacing a. For quantum fields, however, pairs of Lorentz-contracted fields A_μ generate UV divergent factors of $1/a^2$ that precisely cancel the powers of a in these terms. Therefore, the suppression of additional terms is done by g^2 instead of a^2 which leads to unconfortable large lattice artefacts, called *tadpole contributions*.

The simplest way to deal with the tadpole contributions is to cancel them out. Because tadpole contributions are generally process independent, they can be calculated once and used for the whole calculation. The tadpole improvement factor u_0 is given by the mean value of $\mathrm{Re}(\mathrm{Tr}(U_\mu))/N_c$ for a Monte Carlo simuation on the lattice. Practical methods to calculate u_0 are given later. This tadpole correction is applied in all cases where the link operator occurs,

$$U_\mu(x) \to U_\mu(x)/u_0 \tag{6.2}$$

and in the same manner for the adjoint operators and products. Note, however, that

$$U^\dagger_\mu(x)U_\nu(x) \to U^\dagger_\mu(x)U_\nu(x)/u_0^2 \quad \text{but} \quad U^\dagger_\mu(x)U_\mu(x) \to U^\dagger_\mu(x)U_\mu(x) = 1. \tag{6.3}$$

On an anisotropic lattice, different *tadpole improvement factors* u_μ occur which are related to the $U_\mu(x)$ and $U^\dagger_\mu(x)$. For an anisotropy in the time direction the factors read u_s and u_t.

6.1.2 Difference operators

The analogon to differential operators in continuum QCD are the *difference operators*. The simplest one, corresponding to the first derivative, acting on a field ψ is given by [204]

$$\nabla_\mu\psi(x) := \frac{\psi(x+a_\mu)-\psi(x-a_\mu)}{2a_\mu}. \tag{6.4}$$

However, this is only a first rough approximation for the derivative. The Taylor expansion shows that indeed

$$\nabla_\mu\psi(x) = \left(\partial_\mu + \frac{a_\mu^2}{6}\partial_\mu^3 + O(a_\mu^4)\right)\psi(x) \tag{6.5}$$

($\partial_\mu = \partial/\partial x^\mu$, there is *no* sum convention). In order to improve the convergence, the discretization can be improved by using higher difference operators. For the present example one has

$$\partial_\mu\psi(x) = \nabla_\mu\psi(x) - \frac{a_\mu^2}{6}\nabla_\mu^3\psi(x) + O(a_\mu^4). \tag{6.6}$$

A short estimate shows that it is fare more efficient to improve the discretization than to compactify the lattice in order to reduce the finite lattice spacing errors.

In order to formulate difference operators locally, chains of link operators are used to transfer the field ψ from x to the spacetime point where the difference is taken. Because of the gauge invariance of the closed Wilson loops, such transfers are gauge independent as well. The difference operators constructed by doing so are given by

$$\begin{aligned}
\nabla^{(1)}_\mu\psi(x) &= \nabla_\mu\psi(x) = \frac{1}{2u_\mu}\Big[U_\mu(x)\psi(x+a_\mu) - U^\dagger_\mu(x-a_\mu)\psi(x-a_\mu)\Big],\\
\nabla^{(2)}_\mu\psi(x) &= \frac{1}{u_\mu}\Big[U_\mu(x)\psi(x+a_\mu) + U^\dagger_\mu(x-a_\mu)\psi(x-a_\mu)\Big] - 2\psi(x),\\
\nabla^{(3)}_\mu\psi(x) &= \frac{1}{2u_\mu^2}\Big[U_\mu(x)U_\mu(x+a_\mu)\psi(x+2a_\mu) - U^\dagger_\mu(x-a_\mu)U^\dagger_\mu(x-2a_\mu)\psi(x-2a_\mu)\Big] +\\
&\quad -\frac{1}{u_\mu}\Big[U_\mu(x)\psi(x+a_\mu) - U^\dagger_\mu(x-a_\mu)\psi(x-a_\mu)\Big],
\end{aligned}$$

$$
\begin{aligned}
\nabla^{(4)}_\mu \psi(x) &= \frac{1}{u_\mu^2}\Big[U_\mu(x)U_\mu(x+a_\mu)\psi(x+2a_\mu) + U_\mu^\dagger(x-a_\mu)U_\mu^\dagger(x-2a_\mu)\psi(x-2a_\mu)\Big] + \\
&\quad -\frac{4}{u_\mu}\Big[U_\mu(x)\psi(x+a_\mu) + U_\mu^\dagger(x-a_\mu)\psi(x-a_\mu)\Big] + 6\psi(x). \qquad (6.7)
\end{aligned}
$$

The action of the difference operators on the link operator $U_\mu(x)$ itself is different,

$$
\begin{aligned}
\nabla^{(1)}_\rho U_\mu(x) &= \nabla_\rho U_\mu(x) = \frac{1}{2u_\rho^2}\Big[U_\rho(x)U_\mu(x+a_\rho)U_\rho^\dagger(x+a_\mu) + \\
&\qquad -U_\rho^\dagger(x-a_\rho)U_\mu(x-a_\rho)U_\rho(x+a_\mu-a_\rho)\Big], \\
\nabla^{(1)}_\rho U_\mu^\dagger(x) &= \nabla_\rho U_\mu^\dagger(x) = \frac{1}{2u_\rho^2}\Big[U_\rho(x+a_\mu)U_\mu^\dagger(x+a_\rho)U_\rho^\dagger(x) + \\
&\qquad -U_\rho^\dagger(x+a_\mu-a_\rho)U_\mu^\dagger(x-a_\rho)U_\rho(x-a_\rho)\Big], \\
\nabla^{(2)}_\rho U_\mu(x) &= \frac{1}{u_\rho^2}\Big[U_\rho(x)U_\mu(x+a_\rho)U_\rho^\dagger(x+a_\mu) + \\
&\qquad +U_\rho^\dagger(x-a_\rho)U_\mu(x-a_\rho)U_\rho(x+a_\mu-a_\rho)\Big] - 2U_\mu(x), \\
\nabla^{(2)}_\rho U_\mu^\dagger(x) &= \frac{1}{u_\rho^2}\Big[U_\rho(x+a_\mu)U_\mu^\dagger(x+a_\rho)U_\rho^\dagger(x) + \\
&\qquad +U_\rho^\dagger(x+a_\mu-a_\rho)U_\mu^\dagger(x-a_\rho)U_\rho(x-a_\rho)\Big] - 2U_\mu(x). \qquad (6.8)
\end{aligned}
$$

The corresponding Wilson loops can easily be constructed by taking a grid, indicating a starting point x and (in these case two) directions given by a_μ and a_ρ. $U_\nu(y)$ ($\nu = \mu, \rho$) is then represented by a directed line starting at the grid point y and going in a_ν direction to the next one while $U_\nu^\dagger(y)$ is represented by a directed line in the $-a_\nu$ direction *ending* in y. The factors of an operator product are read from the left to the right to obtain the Wilson loop. The Wilson loops which are building blocks in Eqs. (6.8) are shown in Fig. 6.1. Note that

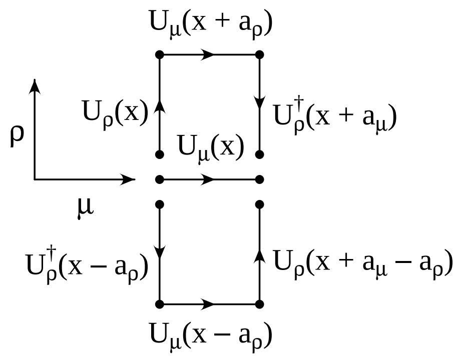


Figure 6.1: Wilson loops for the operators in Eqs. (6.8)

$$
\nabla^{(1)}_\mu U_\mu(x) = \nabla^{(1)}_\mu U_\mu^\dagger(x) = \nabla^{(2)}_\mu U_\mu(x) = \nabla^{(2)}_\mu U_\mu^\dagger(x) = 0. \qquad \text{(no summation over } \mu) \quad (6.9)
$$

Therefore the multiple application of the derivatives can be accomplished rather easily,

$$
\begin{aligned}
(\nabla_\rho)^2 U_\mu(x) &= \nabla_\rho\left(\nabla_\rho U_\mu(x)\right) = \\
&= \frac{1}{2u_\rho^2}\Big[U_\rho(x)\left(\nabla_\rho U_\mu(x+a_\rho)\right)U_\rho^\dagger(x+a_\mu) + \\
&\qquad -U_\rho^\dagger(x-a_\rho)\left(\nabla_\rho U_\mu(x-a_\rho)\right)U_\rho(x+a_\mu-a_\rho)\Big] = \\
&= \frac{1}{4u_\rho^4}\Big[U_\rho(x)U_\rho(x+a_\rho)U_\mu(x+2a_\rho)U_\rho^\dagger(x+a_\rho+a_\mu)U_\rho^\dagger(x+a_\mu) + \\
&\qquad -U_\rho(x)U_\rho^\dagger(x)U_\mu(x)U_\rho(x+a_\mu)U_\rho^\dagger(x+a_\mu) +
\end{aligned}
$$

$$
\begin{aligned}
& -U_\rho^\dagger(x-a_\rho)U_\rho(x-a_\rho)U_\mu(x)U_\rho^\dagger(x+a_\mu-a_\rho)U_\rho(x+a_\mu-a_\rho) + \\
& +U_\rho^\dagger(x-a_\rho)U_\rho^\dagger(x-2a_\rho)U_\mu(x-2a_\rho)U_\rho(x-2a_\rho+a_\mu)U_\rho(x+a_\mu-a_\rho)\big] = \\
= \; & \frac{1}{4u_\rho^4}\big[U_\rho(x)U_\rho(x+a_\rho)U_\mu(x+2a_\rho)U_\rho^\dagger(x+a_\rho+a_\mu)U_\rho^\dagger(x+a_\mu) + \qquad (6.10)\\
& +U_\rho^\dagger(x-a_\rho)U_\rho^\dagger(x-2a_\rho)U_\mu(x-2a_\rho)U_\rho(x-2a_\rho+a_\mu)U_\rho(x+a_\mu-a_\rho)\big] - \frac{1}{2}U_\mu(x).
\end{aligned}
$$

A final remark is in order here before proceeding to the different actions. The lattice is always considered as an *Euclidean lattice* which means that the temporal components are changed to Euclidean components by replacements like $A_0 \to iA_4$. The index "4" should indicate the Euclidean character of these components, the consequences of this change will be dealt on later. However, the temporal direction retains its special meaning on the anisotropic lattice, as the lattice spacings might be different.

6.1.3 Wilson actions for the lattice quark

The simplest action for a quark on the lattice, the *naive quark action*, is given by

$$
\begin{aligned}
S_{\text{naive0}} &= V_a \sum_{x,\mu} \bar\psi_c(x)\left\{\frac{1}{a_\mu}\gamma_\mu\nabla_\mu + m_0\right\}\psi_c(x) = \\
&= a_s^3 a_t \sum_x \bar\psi_c(x)\left\{\frac{1}{a_s}\sum_{j=1}^{3}\gamma_j\nabla_j + \frac{1}{a_t}\gamma_4\nabla_4 + m_0\right\}\psi_c(x) = \\
&= \sum_x \bar\psi_L(x)\left\{\frac{1}{\chi}\sum_{j=1}^{3}\gamma_j\nabla_j + \gamma_4\nabla_4 + a_t m_0\right\}\psi_L(x). \qquad (6.11)
\end{aligned}
$$

Here $V_a = a_1a_2a_3a_4 = a_s^3 a_t$ is the volume of the lattice units, $\psi(x)$ is the continuum quark field which is related to the quark field on the lattice by $\psi_L(x) = a_s^{3/2}\psi_c(x)$ making the quark field dimensionless, m_0 is the bare mass and

$$\chi := a_s/a_t \qquad (6.12)$$

is the *anisotropy parameter* which is an essential parameter for anisotropic lattices. In the following expressions the lattice fields are used in all cases and the index L is dropped again. Because of the (possible) anisotropy, in

$$S_{\text{naive}} = \sum_x \bar\psi(x)\left\{\frac{c_0}{\chi}\sum_{j=1}^{3}\gamma_j\nabla_j + \gamma_4\nabla_4 + a_t m_0\right\}\psi(x) \qquad (6.13)$$

a coefficient c_0 appears which is 1 at tree level but will obtain contributions higher loop corrections, as will be shown later. Because of its role as mediator between the temporal and spatial parts of the action this parameter is eventually called the *speed-of-light coefficient*.

Being derived from the NRQCD as an effective theory, the action also contains terms which do not occur in pure QCD. There is a second derivative and a Pauli coupling to the external field, and the corresponding action is known as the *clover action* [205],

$$S_{\text{clover}} = \sum_x \bar\psi(x)\Big\{\frac{c_0}{\chi}\sum_{j=1}^{3}\gamma_j\nabla_j + \gamma_4\nabla_4 + a_t m_0 +$$

$$-\frac{r}{2}\left[\frac{1}{\chi}\sum_{j=1}^{3}\nabla_j^{(2)}+\chi\nabla_4^{(2)}\right]-i\frac{rC_F a_s a_t}{4a_\mu a_\nu}\sigma_{\mu\nu}F_{\mu\nu}\Big\}\psi(x) \tag{6.14}$$

where $\sigma_{\mu\nu}=\frac{1}{2}[\gamma_\mu,\gamma_\nu]_-$. The field strength tensor $F_{\mu\nu}$ is commented on later. However, because ∇_μ/a_μ is not the derivative but a difference operator, the coincidence with the continuum limit is only valid up to the order a^2. There occur *lattice artefacts* which should be removed. This is done in the same way as described before by using difference operators of the third and fourth order, obtaining the so-called *"D234-action"* [192, 206]

$$\begin{aligned}S^I_{D234} &= \sum_x \bar\psi(x)\Big\{\frac{c_0}{\chi}\sum_{j=1}^{3}\gamma_j\Big(\nabla_j-\frac{1}{6}c_3\nabla_j^{(3)}\Big)+\gamma_4\Big(\nabla_4-\frac{1}{6}c_{3t}\nabla_4^{(3)}\Big)+a_t m_0+ \qquad (6.15)\\ &-\frac{r}{2}\left[\frac{1}{\chi}\sum_{j=1}^{3}\Big(\nabla_j^{(2)}-\frac{1}{12}c_4\nabla_j^{(4)}\Big)+\chi\Big(\nabla_4^{(2)}-\frac{1}{12}c_{4t}\nabla_4^{(4)}\Big)\right]-i\frac{rC_F a_s a_t}{a_\mu a_\nu}\sigma_{\mu\nu}\tilde F_{\mu\nu}\Big\}\psi(x).\end{aligned}$$

The field strength tensor $\tilde F^{\mu\nu}$ is tadpole improved and will be shown later. Again the coefficients c_3, c_{3t}, c_4, and c_{4t} are equal to 1 at tree level, the quark action itself is tree-level accurate through $O(a_s^3)$ and $O(a_t^3)$. In anticipation of working on anisotropic lattices with a_t much finer than a_s, one can drop the higher order improvement terms in the temporal derivatives by setting $c_{3t}=c_{4t}=0$ without loosing accuracy. The action then reads

$$\begin{aligned}S^{II}_{D234} &= \sum_x \bar\psi(x)\Big\{\frac{c_0}{\chi}\sum_{j=1}^{3}\gamma_j\Big(\nabla_j-\frac{1}{6}c_3\nabla_j^{(3)}\Big)+\gamma_4\nabla_4+a_t m_0+\\ &-\frac{r}{2}\left[\frac{1}{\chi}\sum_{j=1}^{3}\Big(\nabla_j^{(2)}-\frac{1}{12}c_4\nabla_j^{(4)}\Big)+\chi\nabla_4^{(2)}\right]-i\frac{rC_F a_s a_t}{a_\mu a_\nu}\sigma_{\mu\nu}\tilde F_{\mu\nu}\Big\}\psi(x). \qquad (6.16)\end{aligned}$$

6.1.4 The field strength tensor

The simplest expression one can figure out for the field strength tensor is the construction of a "clover leaf" glued together from four *plaquettes* (a fact that gave the name "clover action" to the action including this field strength tensor),

$$\Omega_{\mu\nu}(x)=\frac{1}{4u_\mu^2 u_\nu^2}\sum_{\{(\alpha,\beta)\}_{\mu\nu}}U_\alpha(x)U_\beta(x+a_\alpha)U_{-\alpha}(x+a_\alpha+a_\beta)U_{-\beta}(x+a_\beta) \tag{6.17}$$

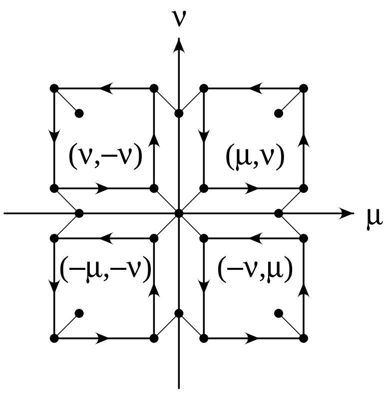


Figure 6.2: Clover leaf construction, the leafs are taken apart to see the details.

with $\{(\alpha,\beta)\}_{\mu\nu}=\{(\mu,\nu),(\nu,-\mu),(-\mu,-\nu),(-\nu,\mu)\}$ for $\mu\neq\nu$ and $U_{-\mu}(x+a_\mu)=U_\mu^\dagger(x)$. Using this "clover leaf" as shown in Fig. 6.2 (taken apart in order to make the different grids more visible), the field strength tensor is given by its imaginary part,

$$F_{\mu\nu}(x)=\frac{1}{2i}\Big(\Omega_{\mu\nu}(x)-\Omega_{\mu\nu}^\dagger(x)\Big). \tag{6.18}$$

It is not very easy to see that for the continuum limit $a_\mu\to 0$ this expression reduces to the field strength tensor as given by the commutator of the covariant derivatives, but this will become obvious when the Fourier transformation is performed.

An $O(a^2)$ improvement of the field strength tensor is obtained by using

$$
\begin{aligned}
\tilde{F}_{\mu\nu}(x) \;=\; & \left(\frac{4}{3}+\frac{1}{6u_\mu^2}+\frac{1}{6u_\nu^2}\right)F_{\mu\nu}(x)+ \\
& -\frac{1}{6u_\mu^2}\Big(U_\mu(x)F_{\mu\nu}(x+a_\mu)U_\mu^\dagger(x)+U_\mu^\dagger(x-a_\mu)F_{\mu\nu}(x-a_\mu)U_\mu(x-a_\mu)\Big)\;+ \\
& +\frac{1}{6u_\nu^2}\Big(U_\nu(x)F_{\nu\mu}(x+a_\nu)U_\nu^\dagger(x)+U_\nu^\dagger(x-a_\nu)F_{\nu\mu}(x-a_\nu)U_\nu(x-a_\nu)\Big).
\end{aligned}
\tag{6.19}
$$

6.1.5 Wilson actions for the lattice gluon

The same "clover leaf" construction is used for the action of the lattice gluon. The simplest lattice action, the *Wilson action*, is given by

$$
S_{\text{Wilson}} = -\frac{V_a N_c}{g^2}\sum_x\sum_{\mu\neq\nu}\left(\frac{1}{u_\mu^2u_\nu^2}\frac{P_{\mu\nu}^{11}(x)}{a_\mu^2a_\nu^2}-1\right) = -\frac{2V_aN_c}{g^2}\sum_x\sum_{\mu<\nu}\left(\frac{1}{u_\mu^2u_\nu^2}\frac{P_{\mu\nu}^{11}(x)}{a_\mu^2a_\nu^2}-1\right)
\tag{6.20}
$$

where $\beta = 2N_c/g^2$ is used in the following, N_c is the number of colours, and

$$
P_{\mu\nu}^{11}(x) = \frac{1}{N_c}\,\mathrm{Re}\left(\mathrm{Tr}\Big(U_\mu(x)U_\nu(x+a_\mu)U_\mu^\dagger(x+a_\nu)U_\nu^\dagger(x)\Big)\right).
\tag{6.21}
$$

Corrections of lattice artefacts of order a^2 are worked in by using an "eight-fold leaf" constructed from rectangles [192, 206, 194],

$$
\begin{aligned}
S_G^I \;=\; & -\beta\sum_x\sum_{i<j}^{3}\frac{1}{\chi}\left\{c_0^G\frac{P_{ij}^{11}(x)}{u_s^4}+c_1^G\frac{P_{ij}^{21}(x)}{u_s^6}+c_1^G\frac{P_{ji}^{21}(x)}{u_s^6}\right\}+ \\
& -\beta\sum_x\sum_{i=1}^{3}\chi\left\{c_0^G\frac{P_{i4}^{11}(x)}{u_s^2u_t^2}+c_1^G\frac{P_{i4}^{21}(x)}{u_s^4u_t^2}+c_1^G\frac{P_{4i}^{21}(x)}{u_t^4u_s^2}\right\}
\end{aligned}
\tag{6.22}
$$

where

$$
P_{\mu\nu}^{21}(x) = \frac{1}{N_c}\,\mathrm{Re}\left(\mathrm{Tr}\Big(U_\mu(x)U_\mu(x+a_\mu)U_\nu(x+2a_\mu)U_\mu^\dagger(x+a_\mu+a_\nu)U_\mu^\dagger(x+a_\nu)U_\nu^\dagger(x)\Big)\right).
\tag{6.23}
$$

The parameters c_0^G and c_1^G occuring in the action S_G^I are constrained to satisfy $c_0^G+8c_1^G=1$. The Symanzik improved gauge action, in which $O(a^2)$ errors are removed, corresponds to $c_0^G=5/3$ and $c_1^G=-1/12$ [207] whereas $c_0^G=3.648$ and $c_1^G=-0.331$ (for $\chi=1$) leads to one of the renormalization group improved Iwasaki actions [208]. Anticipating again that a_t is much finer than a_s and taking the Symanzik values, the simplified action

$$
\begin{aligned}
S_G^{II} \;=\; & -\beta\sum_x\sum_{i<j}^{3}\frac{1}{\chi}\left\{\frac{5}{3}\frac{P_{ij}^{11}(x)}{u_s^4}-\frac{1}{12}\frac{P_{ij}^{21}(x)}{u_s^6}-\frac{1}{12}\frac{P_{ji}^{21}(x)}{u_s^6}\right\}+ \\
& -\beta\sum_x\sum_{i=1}^{3}\chi\left\{\frac{4}{3}\frac{P_{i4}^{11}(x)}{u_s^2u_t^2}-\frac{1}{12}\frac{P_{i4}^{21}(x)}{u_s^4u_t^2}\right\}
\end{aligned}
\tag{6.24}
$$

can be used.

6.2 Fourier transform and Feynman rules

All considerations so far are formulated in configuration space. To obtain the usual set of Feynman rules in momentum space, a *Fourier transform* is applied to the operators. In this section *functions* ξ are constructed in order to phrase this transition [203]. The main ingredient is of course the Fourier transform of the link operator $U_\mu(x)$ and the potential $A_\mu(x)$ contained in it. To start with the last one,

$$A_\mu(x) = \int \frac{d^4q}{(2\pi)^4} e^{iqx} \tilde{A}_\mu(q), \tag{6.25}$$

the link operator $U_\mu(x)$ can be expanded in the charge g and transformed as well,

$$\begin{aligned}
U_\mu(x) &= 1 + igA_\mu(x + a_\mu/2) - \frac{g^2}{2} A_\mu(x + a_\mu/2)A_\mu(x + a_\mu/2) + O(g^3) = \\
&= 1 + ig \int \frac{d^4q}{(2\pi)^4} e^{ixq} e^{ia_\mu q_\mu/2} \tilde{A}_\mu(q) + \\
&\quad - \frac{g^2}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} e^{ix(q_1+q_2)} e^{ia_\mu q_{1\mu}/2} \tilde{A}_\mu(q_1) e^{ia_\mu q_{2\mu}/2} \tilde{A}_\mu(q_2) + O(g^3) = \\
&=: \xi^{(0)}(U_\mu) + ig \int \frac{d^4q}{(2\pi)^4} e^{ixq} e^{ia_\mu q_\mu/2} \tilde{A}_\nu(q) \xi^{(1)}(U_\mu; q, \nu) + \\
&\quad - g^2 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} e^{ix(q_1+q_2)} e^{ia_\mu q_{1\mu}/2} \tilde{A}_{\nu_1}(q_1) e^{ia_\mu q_{2\mu}/2} \tilde{A}_{\nu_2}(q_2) \xi^{(2)}(U_\mu; q_1, \nu_1; q_2, \nu_2) + O(g^3).
\end{aligned} \tag{6.26}$$

In the last line the ξ functions are defined. The first argument is always the operator which is transformed. The index (in parantheses) indicates the power in g as well as the power in $A_{\nu_i}(q_i)$. Therefore, this index also indicates the number of pairs of entries occuring in the argument. These pairs (separated by a semicolon) are the momenta and Lorentz indices related to the potential components. It is easy to see that $\xi^{(0)}$ is the *no-gluon contribution* (i.e. a fermion line insertion), $\xi^{(1)}$ is the *one-gluon contribution* and $\xi^{(2)}$ is the *two-gluon contribution* of the operator. Note, finally, that the potentials $A_{\nu_i}(q_i)$ are not the classical potentials but still contain the lattice spacing a_{ν_i}, and that they are non-commutative objects, because they contain the generators of $SU(3)$, $A_{\nu_i}(q_i) = T_{a_i} A^{a_i}_{\nu_i}(q_i)$. One can extend the pairs of arguments to triples containing the index a_i, or one can keep the order of the potentials. In this work the second possibility is favoured in most of the cases.

6.2.1 ξ functions for the link operator

In case of the link operator the result written in terms of ξ functions is very simple,

$$\xi^{(0)}(U_\mu) = 1, \quad \xi^{(1)}(U_\mu; q, \nu) = \delta_{\mu\nu}, \quad \xi^{(2)}(U_\mu; q_1, \nu_1; q_2, \nu_2) = \frac{1}{2} \delta_{\mu\nu_1} \delta_{\mu\nu_2}. \tag{6.27}$$

The different indices introduced in the ξ function and the momenta make sense only if one applies the transformation also to the different operators which occured so far.

6.2.2 ξ functions for $\nabla_\rho^{(1)}U_\mu(x)$, $\nabla_\rho^{(2)}U_\mu(x)$, and $(\nabla_\rho^{(1)})^2U_\mu(x)$

The calculation for the operator $\nabla_\rho^{(1)}U_\mu(x)$ is given by

$$\begin{aligned}
2\nabla_\rho^{(1)}U_\mu(x) \;&=\; U_\rho(x)U_\mu(x+a_\rho)U_\rho^\dagger(x+a_\mu) - U_\rho^\dagger(x-a_\rho)U_\mu(x-a_\rho)U_\rho(x+a_\mu-a_\rho) \;= \\
&= \exp\left(igA_\rho(x+a_\rho/2)\right)\exp\left(igA_\mu(x+a_\mu/2+a_\rho)\right)\exp\left(-igA_\rho(x+a_\mu+a_\rho/2)\right) \;+ \\
&\quad -\exp\left(-igA_\rho(x-a_\rho/2)\right)\exp\left(igA_\mu(x+a_\mu/2-a_\rho)\right)\exp\left(igA_\rho(x+a_\mu-a_\rho/2)\right) \;= \\
&= 1-1+ig\Big\{A_\rho(x+a_\rho/2)+A_\mu(x+a_\mu/2+a_\rho)-A_\rho(x+a_\mu+a_\rho/2) \;+ \\
&\quad +A_\rho(x-a_\rho/2)-A_\mu(x+a_\mu/2-a_\rho)-A_\rho(x+a_\mu-a_\rho/2)\Big\}+O(g^2) \;= \\
&= ig\int\frac{d^4q}{(2\pi)^4}\Big\{e^{iq_\rho/2}\tilde{A}_\rho(q)+e^{iq_\mu/2+iq_\rho}\tilde{A}_\mu(q)-e^{iq_\mu+iq_\rho/2}\tilde{A}_\rho(q) \;+ \\
&\quad +e^{-iq_\rho/2}\tilde{A}_\rho(q)-e^{iq_\mu/2-iq_\rho}\tilde{A}_\mu(q)-e^{iq_\mu-iq_\rho/2}\tilde{A}_\rho(q)\Big\}+O(g^2) \;= \qquad (6.28)\\
&= ig\int\frac{d^4q}{(2\pi)^4}\Big\{2\cos\left(\frac{q_\rho}{2}\right)\tilde{A}_\rho(q)+2ie^{iq_\mu/2}\sin(q_\rho)\tilde{A}_\mu(q)-2e^{iq_\mu}\cos\left(\frac{q_\rho}{2}\right)\tilde{A}_\rho(q)\Big\}+O(g^2) = \\
&= ig\int\frac{d^4q}{(2\pi)^4}e^{ixq}e^{iq_\mu/2}\Big\{2i\sin(q_\rho)\tilde{A}_\mu(q)-2\left(e^{iq_\mu/2}-e^{-iq_\mu/2}\right)\cos\left(\frac{q_\rho}{2}\right)\tilde{A}_\rho(q)\Big\}+O(g^2) \;= \\
&= ig\int\frac{d^4q}{(2\pi)^4}e^{ixq}e^{iq_\mu/2}4i\tilde{A}_\nu(q)\Big\{\sin\left(\frac{q_\rho}{2}\right)\cos\left(\frac{q_\rho}{2}\right)\delta_{\mu\nu}-\sin\left(\frac{q_\mu}{2}\right)\cos\left(\frac{q_\rho}{2}\right)\delta_{\rho\nu}\Big\}+O(g^2).
\end{aligned}$$

Note that for reasons of simplicity here and in the following *the lattice spacings a_ν are absorbed in q_ν* which now is a dimensionless quantity. Comparing this result with Eq. (6.26), one obtains

$$\begin{aligned}
\xi^{(0)}\left(\nabla_\rho^{(1)}U_\mu\right) \;&=\; 0, \\
\xi^{(1)}\left(\nabla_\rho^{(1)}U_\mu;q,\nu\right) \;&=\; 2i\cos\left(\frac{q_\rho}{2}\right)\left\{\sin\left(\frac{q_\rho}{2}\right)\delta_{\mu\nu}-\sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu}\right\}. \qquad (6.29)
\end{aligned}$$

The calculations for the operator $\nabla_\rho^{(2)}$ is quite similar to the previous one. One has

$$\begin{aligned}
&\nabla_\rho^{(2)}U_\mu(x) \;= \\
&= \; U_\rho(x)U_\mu(x+a_\rho)U_\rho^\dagger(x+a_\mu)+U_\rho^\dagger(x-a_\rho)U_\mu(x-a_\rho)U_\rho(x-a_\rho+a_\mu)-2U_\mu(x) \;= \\
&= \; \exp\left(igA_\rho(x+a_\rho/2)\right)\exp\left(igA_\mu(x+a_\rho+a_\mu/2)\right)\exp\left(-igA_\rho(x+a_\rho/2+a_\mu)\right) \;+ \\
&\quad +\exp\left(-igA_\rho(x-a_\rho/2)\right)\exp\left(igA_\mu(x-a_\rho+a_\mu/2)\right)\exp\left(igA_\rho(x-a_\rho/2+a_\mu)\right) \;+ \\
&\quad -2\exp\left(igA_\mu(x+a_\mu/2)\right) \;= \\
&= \; ig\Big\{A_\rho(x+a_\rho/2)+A_\mu(x+a_\mu/2+a_\rho)-A_\rho(x+a_\mu+a_\rho/2)-A_\rho(x-a_\rho/2) \;+ \\
&\quad +A_\mu(x+a_\mu/2-a_\rho)+A_\rho(x+a_\mu-a_\rho/2)-2A_\mu(x+a_\mu/2)\Big\}+O(g^2) \;= \\
&= \; ig\int\frac{d^4q}{(2\pi)^4}e^{ixq}\Big\{e^{iq_\rho/2}\tilde{A}_\rho(q)+e^{iq_\mu/2+iq_\rho}\tilde{A}_\mu(q)-e^{iq_\mu+iq_\rho/2}\tilde{A}_\rho(q) \;+ \\
&\quad -e^{-iq_\rho/2}\tilde{A}_\rho(q)+e^{iq_\mu/2-iq_\rho/2}\tilde{A}_\mu(q)+e^{iq_\mu-iq_\rho/2}\tilde{A}_\rho(q)-2e^{iq_\mu/2}\tilde{A}_\mu(q)\Big\}+O(g^2) \;=
\end{aligned}$$

$$
\begin{aligned}
&= \; ig\int \frac{d^4q}{(2\pi)^4} e^{ixq}\Big\{ \Big(e^{iq_\rho/2} - e^{iq_\mu+iq_\rho/2} - e^{-iq_\rho/2} + e^{iq_\mu-iq_\rho/2}\Big)\tilde{A}_\rho(q) \; + \\
&\quad + \Big(e^{iq_\mu/2+iq_\rho} - 2e^{iq_\mu} + e^{iq_\mu/2-iq_\rho}\Big)\tilde{A}_\mu(q)\Big\} + O(g^2) \; = \\
&= \; ig\int \frac{d^4q}{(2\pi)^4} e^{ixq}\Big\{(e^{iq_\rho/2} - e^{-iq_\rho/2})(1 - e^{iq_\mu})\tilde{A}_\rho(q) \; + \\
&\qquad\qquad + (e^{iq_\rho/2} - e^{-iq_\rho/2})^2 e^{iq_\mu/2}\tilde{A}_\mu(q)\Big\} + O(g^2) \; = \\
&= \; ig\int \frac{d^4q}{(2\pi)^4} e^{iq_\mu/2}\Big\{(e^{iq_\rho/2} - e^{-iq_\rho/2})(e^{-iq_\mu/2} - e^{iq_\mu/2})\tilde{A}_\rho(q) \; + \\
&\qquad\qquad + (e^{iq_\rho/2} - e^{-iq_\rho/2})^2\tilde{A}_\mu(q)\Big\} + O(g^2) \; = \\
&= \; ig\int \frac{d^4q}{(2\pi)^4} e^{ixq} e^{iq_\mu/2}\tilde{A}_\nu(q)\Big\{4\sin\left(\frac{q_\rho}{2}\right)\sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu} - 4\sin^2\left(\frac{q_\rho}{2}\right)\delta_{\mu\nu}\Big\} + O(g^2). \qquad (6.30)
\end{aligned}
$$

From this result one obtains $\xi^{(0)}(\nabla^{(2)}_\rho U_\mu) = 0$ and

$$
\xi^{(1)}\left(\nabla^{(2)}_\rho U_\mu; q, \nu\right) = -4\sin\left(\frac{q_\rho}{2}\right)\left\{\sin\left(\frac{q_\rho}{2}\right)\delta_{\mu\nu} - \sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu}\right\}. \qquad (6.31)
$$

The two-fold action of the operator $\nabla_\rho = \nabla^{(1)}_\rho$ is different from the action of the operator $\nabla^{(2)}_\rho$. Therefore, this action on $U_\mu(x)$ shall be calculated here as the last detailed calculation of ξ functions. Actually, a few steps have already been skipped in the calculation

$$
\begin{aligned}
\xi^{(1)}\left((\nabla_\rho)^2 U_\mu; q, \nu\right) \; &= \; \frac{1}{4}\Big[e^{iq_\mu/2-iq_\rho/2}\delta_{\rho\nu} + e^{-iq_\mu/23iq_\rho/2}\delta_{\rho\nu} + e^{2iq_\rho}\delta_{\mu\nu} \; + \\
&\quad - e^{iq_\mu/2+3iq_\rho/2}\delta_{\rho\nu} - e^{iq_\mu/2+iq_\rho/2}\delta_{\rho\nu} - e^{-iq_\mu/2-iq_\rho/2}\delta_{\rho\nu} \; + \\
&\quad - e^{-iq_\mu/2-3iq_\rho/2}\delta_{\rho\nu} + e^{-2iq_\rho}\delta_{\mu\nu} + e^{iq_\mu/2-3iq_\rho/2}\delta_{\rho\nu} + e^{iq_\mu/2-iq_\rho/2}\delta_{\rho\nu}\Big] \; = \\
&= \; \sin\left(\frac{q_\rho}{2}\right)\sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu} + \sin\left(\frac{3q_\rho}{2}\right)\sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu} + \frac{1}{2}\cos(2q_\rho)\delta_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu} \qquad (6.32)
\end{aligned}
$$

($\xi^{(0)}((\nabla_\rho)^2 U_\mu)$ vanishes again). One then can use

$$
\begin{aligned}
\sin\left(\frac{3q_\rho}{2}\right) &= 3\sin\left(\frac{q_\rho}{2}\right) - 4\sin^3\left(\frac{q_\rho}{2}\right), \\
\cos(2q_\rho) &= 1 - 8\sin^2\left(\frac{q_\rho}{2}\right) + 8\sin^4\left(\frac{q_\rho}{2}\right) \qquad (6.33)
\end{aligned}
$$

to obtain

$$
\begin{aligned}
\xi^{(1)}\left((\nabla_\rho)^2 U_\mu; q, \nu\right) \; &= \; 4\sin\left(\frac{q_\rho}{2}\right)\sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu} - 4\sin^3\left(\frac{q_\rho}{2}\right)\sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu} \; + \\
&\qquad - 4\sin^2\left(\frac{q_\rho}{2}\right)\delta_{\mu\nu} + 4\sin^4\left(\frac{q_\rho}{2}\right)\delta_{\mu\nu} \; = \\
&= \; 4\sin\left(\frac{q_\rho}{2}\right)\left(\sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu} - \sin\left(\frac{q_\rho}{2}\right)\delta_{\mu\nu}\right) \; + \\
&\qquad - 4\sin^3\left(\frac{q_\rho}{2}\right)\left(\sin\left(\frac{q_\mu}{2}\right)\delta_{\rho\nu} - \sin\left(\frac{q_\rho}{2}\right)\delta_{\mu\nu}\right). \qquad (6.34)
\end{aligned}
$$

This compact form will be important for the treatment of staggered quarks in Sec. 6.5.

6.2.3 New notation for the ξ functions

Because all of the functions $\xi^{(1)}$ calculated up to now have a common structure in terms of Kronecker delta symbols, a new notation is introduced by

$$\xi^{(1)}\left(O_\rho U_\mu; q, \nu\right) =: \xi_0^{(1)}\left(O_\rho U_\mu; q\right)\delta_{\mu\nu} + \xi_1^{(1)}\left(O_\rho U_\mu; q\right)\delta_{\rho\nu} \tag{6.35}$$

where $O_\rho = \nabla_\rho^{(1)}$, $\nabla_\rho^{(2)}$, and $(\nabla_\rho)^2$ are the cases calculated in this subsection. The notation for the functions $\xi^{(2)}$, though not calculated in this subsection, is given by

$$\begin{aligned}
\xi^{(2)}\left(O_\rho U_\mu; q_1, \nu_1; q_2, \nu_2\right) &= \\
=: \ &\xi_{00}^{(2)}\left(O_\rho U_\mu; q_1; q_2\right)\delta_{\mu\nu_1}\delta_{\mu\nu_2} + \xi_{01}^{(2)}\left(O_\rho U_\mu; q_1; q_2\right)\delta_{\mu\nu_1}\delta_{\rho\nu_2} + \\
&+\xi_{10}^{(2)}\left(O_\rho U_\mu; q_1; q_2\right)\delta_{\rho\nu_1}\delta_{\mu\nu_2} + \xi_{11}^{(2)}\left(O_\rho U_\mu; q_1; q_2\right)\delta_{\rho\nu_1}\delta_{\rho\nu_2}.
\end{aligned} \tag{6.36}$$

Finally, if two operators with different Lorentz indices are applied, the range of values for the lower indices of the functions $\xi^{(1)}$ and $\xi^{(2)}$ increases, so for instance for two such operators,

$$\begin{aligned}
\xi^{(1)}\left(O_{\rho_1} O_{\rho_2} U_\mu; q_1, \nu_1\right) &= \xi_0^{(1)}\left(O_{\rho_1} O_{\rho_2} U_\mu; q_1\right)\delta_{\mu\nu} + \sum_{i=1}^{2}\xi_i^{(1)}\left(O_{\rho_1} O_{\rho_2} U_\mu; q_1\right)\delta_{\rho_i\nu}, \\
\xi^{(2)}\left(O_{\rho_1} O_{\rho_2} U_\mu; q_1, \nu_1; q_2, \nu_2\right) &= \\
=: \ &\xi_{00}^{(2)}\left(O_{\rho_1} O_{\rho_2} U_\mu; q_1; q_2\right)\delta_{\mu\nu_1}\delta_{\mu\nu_2} + \sum_{j=1}^{2}\xi_{0j}^{(2)}\left(O_{\rho_1} O_{\rho_2} U_\mu; q_1; q_2\right)\delta_{\mu\nu_1}\delta_{\rho_j\nu_2} + \\
&+\sum_{i=1}^{2}\xi_{i0}^{(2)}\left(O_{\rho_1} O_{\rho_2} U_\mu; q_1; q_2\right)\delta_{\rho_i\nu_1}\delta_{\mu\nu_2} + \sum_{i,j=1}^{2}\xi_{ij}^{(2)}\left(O_{\rho_1} O_{\rho_2} U_\mu; q_1; q_2\right)\delta_{\rho_i\nu_1}\delta_{\rho_j\nu_2}.
\end{aligned} \tag{6.37}$$

This last notation proofs its usefulness first in Sec. 6.5 for staggered quarks.

6.2.4 The gluon action in ξ functions

To get back to more practical questions, the latter formalism can be used to calculate the Feynman rules that emerge for the gluon part of the action. The first step is to determine the unresolved ξ functions for the plaquettes, obtaining

$$\begin{aligned}
P_{\mu\nu}^{11}(x) &= -\frac{2}{\beta}\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}e^{ix(q_1+q_2)}e^{iq_{1\mu}/2}e^{iq_{2\mu}/2}\times \\
&\times\left(\sin\left(\frac{q_{1\mu}}{2}\right)\tilde{A}_\nu(q_1) - \sin\left(\frac{q_{1\nu}}{2}\right)\tilde{A}_\mu(q_1)\right)\times \\
&\times\left(\sin\left(\frac{q_{2\mu}}{2}\right)\tilde{A}_\nu(q_2) - \sin\left(\frac{q_{2\nu}}{2}\right)\tilde{A}_\mu(q_2)\right) + O(g^3)
\end{aligned} \tag{6.38}$$

($\beta = 2N_c/g^2$) and

$$P_{\mu\nu}^{21}(x) = -\frac{8}{\beta}\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}e^{ix(q_1+q_2)}e^{iq_{1\mu}/2}e^{iq_{2\mu}/2}\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\times$$

$$\times\left(\sin\left(\frac{q_{1\mu}}{2}\right)\tilde{A}_\nu(q_1) - \sin\left(\frac{q_{1\nu}}{2}\right)\tilde{A}_\mu(q_1)\right) \times$$
$$\times\left(\sin\left(\frac{q_{2\mu}}{2}\right)\tilde{A}_\nu(q_2) - \sin\left(\frac{q_{2\nu}}{2}\right)\tilde{A}_\mu(q_2)\right) + O(g^3) \qquad (6.39)$$

In this case the ξ functions are not used in order to keep the (multiplicative) structure. But one can use an appropiate short hand notation

$$\begin{aligned} P^{11}_{\mu\nu}(x) &= \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4} e^{ix(q_1+q_2)} e^{iq_{1\mu}/2} e^{iq_{2\mu}/2} \xi^{(2)}\left(P^{11}_{\mu\nu}; q_1; q_2\right) + O(g^3), \\ P^{21}_{\mu\nu}(x) &= \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4} e^{ix(q_1+q_2)} e^{iq_{1\mu}/2} e^{iq_{2\mu}/2} \xi^{(2)}\left(P^{21}_{\mu\nu}; q_1; q_2\right) + O(g^3). \end{aligned} \qquad (6.40)$$

For the gluon action on the isotropic lattice without tadpole improvement,

$$S_G = -\beta \sum_{\mu>\nu}\left[\frac{5}{3}P^{11}_{\mu\nu}(x) - \frac{1}{12}(P^{21}_{\mu\nu}(x) + P^{21}_{\nu\mu}(x))\right] \qquad (6.41)$$

and for $q_1 = -q_2 = q$ one obtains

$$\begin{aligned} \xi\left(\frac{5}{3}P^{11}_{\mu\nu} - \frac{1}{12}(P^{21}_{\mu\nu} + P^{21}_{\nu\mu}); q; q\right) &= \frac{2}{\beta}\left(\frac{5}{3} - \frac{1}{3}\cos^2\left(\frac{q_\mu}{2}\right) - \frac{1}{3}\cos^2\left(\frac{q_\nu}{2}\right)\right)\times \\ &\times\left(\sin\left(\frac{q_\mu}{2}\right)\tilde{A}_\nu(q) - \sin\left(\frac{q_\nu}{2}\right)\tilde{A}_\mu(q)\right) \times \\ &\times\left(\sin\left(\frac{q_\mu}{2}\right)\tilde{A}_\nu(q) - \sin\left(\frac{q_\nu}{2}\right)\tilde{A}_\mu(q)\right) + O(g^3) = \\ &= \frac{2}{\beta}\left(1 + \frac{1}{3}\sin^2\left(\frac{q_\mu}{2}\right) + \frac{1}{3}\sin^2\left(\frac{q_\nu}{2}\right)\right)\times \\ &\times\left(\sin\left(\frac{q_\mu}{2}\right)\tilde{A}_\nu(q) - \sin\left(\frac{q_\nu}{2}\right)\tilde{A}_\mu(q)\right) \times \\ &\times\left(\sin\left(\frac{q_\mu}{2}\right)\tilde{A}_\nu(q) - \sin\left(\frac{q_\nu}{2}\right)\tilde{A}_\mu(q)\right) + O(g^3) = \\ &= \frac{q_{\mu\nu}}{2\beta}\left(\tilde{q}_\mu\tilde{A}_\nu(q) - \tilde{q}_\nu\tilde{A}_\mu(q)\right)\left(\tilde{q}_\mu\tilde{A}_\nu(q) - \tilde{q}_\nu\tilde{A}_\mu(q)\right) + O(g^3) \end{aligned} \qquad (6.42)$$

where the abbreviations

$$q_{\mu\nu} := \left(1 + \frac{1}{12}\tilde{q}^2_\mu + \frac{1}{12}\tilde{q}^2_\nu\right), \qquad \tilde{q}_\mu := 2\sin\left(\frac{q_\mu}{2}\right) \qquad (a_\mu,\, a_\nu \text{ absorbed}) \qquad (6.43)$$

have been used. Including the tadpole improvement factors u_0, this changes to

$$\begin{aligned} &\xi\left(\frac{5}{3u_0^4}P^{11}_{\mu\nu} - \frac{1}{12u_0^6}(P^{21}_{\mu\nu} + P^{21}_{\nu\mu}); q; q\right) = \\ &= \frac{q^t_{\mu\nu}}{2\beta}\left(\tilde{q}_\mu\tilde{A}_\nu(q) - \tilde{q}_\nu\tilde{A}_\mu(q)\right)\left(\tilde{q}_\mu\tilde{A}_\nu(q) - \tilde{q}_\nu\tilde{A}_\mu(q)\right) + O(g^3) \end{aligned} \qquad (6.44)$$

where

$$q^t_{\mu\nu} := \frac{1}{u_0^6}\left(\frac{5u_0^2 - 2}{3} + \frac{1}{12}\tilde{q}^2_\mu + \frac{1}{12}\tilde{q}^2_\nu\right). \qquad (6.45)$$

Inserting this into the gluon action, one has to think about the asymmetric sum over $\mu > \nu$. If this sum is taken of an expression $M_{\mu\nu}$ which is symmetric in μ und ν as in the present case, one can simplify to obtain

$$\sum_{\mu>\nu} M_{\mu\nu} = \frac{1}{2}\sum_{\mu>\nu} M_{\mu\nu} + \frac{1}{2}\sum_{\mu>\nu} M_{\nu\mu} = \frac{1}{2}\sum_{\mu\neq\nu} M_{\mu\nu} \tag{6.46}$$

Also the restriction $\mu \neq \nu$ can be skipped in the present case because $P^{11}_{\mu\nu}$ and $P^{21}_{\mu\nu}$ vanish both for $\mu = \nu$. Summing over μ and ν, one thus obtains

$$\xi(S^I_G; q; q) = -\frac{1}{4}\sum_{\mu,\nu} q^t_{\mu\nu}\left(\tilde{q}_\mu\tilde{A}_\nu(q) - \tilde{q}_\nu\tilde{A}_\mu(q)\right)\left(\tilde{q}_\mu\tilde{A}_\nu(q) - \tilde{q}_\nu\tilde{A}_\mu(q)\right) + O(g). \tag{6.47}$$

For the action S^{II}_G the situation changes only slightly. The situation described above can be translated to the pure space part of

$$\begin{aligned} S^{II}_G \;&=\; -\beta\sum_x\sum_{i>j}\frac{1}{\chi}\left(\frac{5}{3}\frac{P^{11}_{ij}(x)}{u_s^4} - \frac{1}{12u_s^6}(P^{21}_{ij}(x) + P^{21}_{ji}(x))\right) + \\ &\quad -\beta\sum_x\sum_i \chi\left(\frac{4}{3}\frac{P^{11}_{i4}(x)}{u_s^2u_t^2} - \frac{1}{12u_s^4u_t^2}P^{21}_{i4}(x)\right). \end{aligned} \tag{6.48}$$

For the second, space-time mixed part one obtains

$$\xi\left(\frac{4}{3}\frac{P^{11}_{i4}(x)}{u_s^2u_t^2} - \frac{1}{12}\frac{P^{21}_{i4}(x)}{u_s^4u_t^2}; q; q\right) = \frac{2}{\beta}q^t_{i4}\left(\tilde{q}_i\tilde{A}_4(q) - \tilde{q}_4\tilde{A}_i(q)\right)\left(\tilde{q}_i\tilde{A}_4(q) - \tilde{q}_4\tilde{A}_i(q)\right) \tag{6.49}$$

with

$$q^t_{i4} := \frac{1}{u_s^4u_t^2}\left(\frac{4u_s^2-1}{3} + \frac{1}{12}\tilde{q}_i^2\right), \tag{6.50}$$

so that finally

$$\begin{aligned} \xi(S^{II}_G; q; q) \;&=\; -\frac{1}{4}\sum_{i,j}\frac{1}{\chi}q^t_{ij}\left(\tilde{q}_i\tilde{A}_j(q) - \tilde{q}_j\tilde{A}_i(q)\right)\left(\tilde{q}_i\tilde{A}_j(q) - \tilde{q}_j\tilde{A}_i(q)\right) + \\ &\quad -\frac{1}{2}\sum_i \chi q^t_{it}\left(\tilde{q}_i\tilde{A}_4(q) - \tilde{q}_4\tilde{A}_i(q)\right)\left(\tilde{q}_i\tilde{A}_4(q) - \tilde{q}_4\tilde{A}_i(q)\right) + O(g) \;= \\ &=\; -\frac{1}{4}\sum_{\mu,\nu}\tilde{q}_{\mu\nu}\left(\tilde{q}_\mu\tilde{A}_\nu(q) - \tilde{q}_\nu\tilde{A}_\mu(q)\right)\left(\tilde{q}_\mu\tilde{A}_\nu(q) - \tilde{q}_\nu\tilde{A}_\mu(q)\right) + O(g) \end{aligned} \tag{6.51}$$

where

$$\tilde{q}_{ij} \;=\; \frac{1}{\chi}q^t_{ij} \;=\; \frac{1}{u_s^6\chi}\left(\frac{5u_s^2-2}{3} + \frac{1}{12}\tilde{q}_i^2 + \frac{1}{12}\tilde{q}_j^2\right), \tag{6.52}$$

$$\tilde{q}_{i4} \;=\; \tilde{q}_{4i} \;=\; \chi q^t_{i4} \;=\; \frac{\chi}{u_s^4u_t^2}\left(\frac{4u_s^2-1}{3} + \frac{1}{12}\tilde{q}_i^2\right). \tag{6.53}$$

Restoring the (absorbed) dependence on the lattice spacings a_μ back into this expression, one obtains instead

$$\tilde{q}_{ij} \;=\; \frac{V_a}{u_s^6}\left(\frac{5u_s^2-2}{3} + \frac{1}{12}a_i^2\tilde{q}_i^2 + \frac{1}{12}a_j^2\tilde{q}_j^2\right), \tag{6.54}$$

$$\tilde{q}_{i4} \;=\; \tilde{q}_{4i} \;=\; \frac{V_a}{u_s^4u_t^2}\left(\frac{4u_s^2-1}{3} + \frac{1}{12}a_i^2\tilde{q}_i^2\right) \tag{6.55}$$

where $V_a = a_s^3 a_t$ and

$$\tilde{q}_\mu = \frac{2}{a_\mu}\sin\left(\frac{a_\mu q_\mu}{2}\right). \tag{6.56}$$

Before being able to determine the gluon propagator one has to use a *gauge fixing term*. Without this term the quadratic form given above is not invertible. For the gauge fixing term the analogon of $-(\partial_\mu A^\mu)^2/2\alpha_g$ is taken, namely

$$\xi\left(S_{gf};q;q\right) = -\frac{V_a}{2\alpha_g}\left(\sum_\mu \tilde{q}_\mu \tilde{A}_\mu(q)\right)^2 = -\frac{V_a}{2\alpha_g}\sum_{\mu,\nu}\tilde{q}_\mu\tilde{A}_\mu(q)\tilde{q}_\nu\tilde{A}_\nu(q). \tag{6.57}$$

Adding both, one ends up with

$$\xi(S_G + S_{gf};q;q) = \frac{1}{2}\sum_{\mu,\nu}\tilde{A}_\mu(q)M_{\mu\nu}(q)\tilde{A}_\nu(q). \tag{6.58}$$

The inverse of $M_{\mu\nu}(q)$ is the gluon propagator. Suprisingly, this gluon propagator has a form quite similar to the usual one,

$$G_{\mu\nu} = \frac{-1}{(\tilde{q}^2)^2}\left(\alpha_g\tilde{q}_\mu\tilde{q}_\nu + \frac{f_{\mu\nu}(\tilde{q}_\alpha,\tilde{q}_{\alpha\beta})}{f(\tilde{q}_\alpha,\tilde{q}_{\alpha\beta})}\right). \tag{6.59}$$

The structure of $f_{\mu\nu}(\tilde{q}_\alpha,\tilde{q}_{\alpha\beta})$ and $f(\tilde{q}_\alpha,\tilde{q}_{\alpha\beta})$, however, is more complicated.

6.2.5 Reflections on the Euclidean metric

Having the gluon propagator at hand, a discussion of the Euclidean metric used throughout these calculations is in order here. The reflections are done for the continuum theory, but the results can easily be translated to the lattice. For the momentum k one has [203] (journal version only)

$$\begin{aligned} k_i &= k^i = k_i^{(M)} = -k^i_{(M)}, \qquad k_4 = k^4 = -ik_0^{(M)} = -ik^0_{(M)}, \\ k^2 &= \sum_{i=1}^4 k_i^2 = -\sum_{i=1}^3 k_i^{(M)}k^i_{(M)} - k_0^{(M)}k^0_{(M)} = -k^2_{(M)} \end{aligned} \tag{6.60}$$

where the index "(M)" represents the Minkowskian metric. The question arises how to construct a gluon propagator in Euclidean space-time from this. In continuum theory one has

$$G_{\mu\nu}^{(M)} = \frac{-i}{k^2_{(M)}}\left(g_{\mu\nu}^{(M)} - (1-\alpha_g)\frac{k_\mu^{(M)}k_\nu^{(M)}}{k^2_{(M)}}\right), \qquad g_{\mu\nu}^{(M)} = \mathrm{diag}(-1;1,1,1). \tag{6.61}$$

Looking at the different parts

$$G_{ij}^{(M)} = \frac{-i}{k^2_{(M)}}\left(g_{ij}^{(M)} - (1-\alpha_g)\frac{k_i^{(M)}k_j^{(M)}}{k^2_{(M)}}\right) = \frac{-i}{k^2}\left(\delta_{ij} - (1-\alpha_g)\frac{k_ik_j}{k^2}\right), \tag{6.62}$$

$$G_{i0}^{(M)} = \frac{-i}{k^2_{(M)}}\left(g_{i0}^{(M)} - (1-\alpha_g)\frac{k_i^{(M)}k_0^{(M)}}{k^2_{(M)}}\right) = \frac{-1}{k^2}(1-\alpha_g)\left(\frac{k_ik_4}{k^2}\right) = \tag{6.63}$$

$$
\begin{aligned}
&= \frac{1}{k^2}\left(\delta_{i4} - (1-\alpha_g)\frac{k_i k_4}{k^2}\right) =: iG_{i4} \quad \Rightarrow \quad G_{i4} = \frac{-i}{k^2}\left(\delta_{i4} - (1-\alpha_g)\frac{k_i k_4}{k^2}\right) \\
G^{(M)}_{0i} &= iG_{4i} \quad \Rightarrow \quad G_{4i} = \frac{-i}{k^2}\left(\delta_{4i} - (1-\alpha_g)\frac{k_4 k_i}{k^2}\right) \qquad (6.64) \\
G^{(M)}_{00} &= \frac{-i}{k^2_{(M)}}\left(g^{(M)}_{00} - (1-\alpha_g)\frac{k^{(M)}_0 k^{(M)}_0}{k^2_{(M)}}\right) = \frac{i}{k^2}\left(\delta_{44} + (1-\alpha_g)\frac{k_4 k_4}{(-k^2)}\right) = \qquad (6.65) \\
&= \frac{i}{k^2}\left(\delta_{44} - (1-\alpha_g)\frac{k_4 k_4}{k^2}\right) = -G_{44} \quad \Rightarrow \quad G_{44} = \frac{-i}{k^2}\left(\delta_{44} - (1-\alpha_g)\frac{k_4 k_4}{k^2}\right),
\end{aligned}
$$

one ends up with the translations

$$
\begin{aligned}
G_{ij} &= D^{ij} = G^{(M)}_{ij} = D^{ij}_{(M)}, \qquad G_{i4} = D^{i4} = -iG^{(M)}_{i0} = iD^{i0}_{(M)}, \\
G_{4i} &= D^{4i} = -iG^{(M)}_{0i} = iD^{0i}_{(M)}, \qquad G_{44} = G_{44} = -G^{(M)}_{00} = -D^{00}_{(M)} \qquad (6.66)
\end{aligned}
$$

for the Euclidean gluon propagator

$$
G_{\mu\nu} = \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\alpha_g)\frac{k_\mu k_\nu}{k^2}\right), \qquad g_{\mu\nu} = \mathrm{diag}(1,1,1,1). \qquad (6.67)
$$

6.2.6 ξ functions for $\bar{\psi}(x)\nabla^{(n)}_\mu\psi(x)$

In order to deal with the part of the action related to the quark, the *spinor* ψ (considered as lattice spinor) has to be included into the Fourier transform as well. The Fourier transform of the spinor and the *adjoint spinor* is given by

$$
\psi(x) = \int \frac{d^4p_1}{(2\pi)^4} e^{ip_1x}\tilde{\psi}(p_1), \qquad \bar{\psi}(x) = \int \frac{d^4p_2}{(2\pi)^4} e^{-ip_2x}\tilde{\bar{\psi}}(p_2) \qquad (6.68)
$$

Using this and the Fourier transform of $U_\mu(x)$, one can calculate

$$
\begin{aligned}
2\bar{\psi}(x)\nabla_\mu\psi(x) &= \bar{\psi}(x)\left(U_\mu(x)\psi(x+a_\mu) - U^\dagger_\mu(x-a_\mu)\psi(x-a_\mu)\right) = \\
&= \int \frac{d^4p_2}{(2\pi)^4} e^{-ip_2x}\tilde{\bar{\psi}}(p_2)\Bigg\{\xi^{(0)}(U_\mu) + ig\int \frac{d^4q}{(2\pi)^4} e^{ixq} e^{iq_\mu/2}\tilde{A}_\nu(q)\xi^{(1)}(U_\mu;q,\nu) + \\
&\quad -g^2\int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4} e^{ix(q_1+q_2)} e^{iq_{1\mu}/2}\tilde{A}_{\nu_1}(q_1) e^{iq_{2\mu}/2}\tilde{A}_{\nu_2}(q_2)\xi^{(2)}(U_\mu;q_1,\nu_1;q_2,\nu_2)\Bigg\} \times \\
&\quad \times \int \frac{d^4p_1}{(2\pi)^4} e^{ip_1x} e^{ip_{1\mu}}\tilde{\psi}(p) + \\
&\quad - \int \frac{d^4p_2}{(2\pi)^4} e^{-ip_2x}\tilde{\bar{\psi}}(p_2)\Bigg\{\xi^{(0)}(U_\mu) - ig\int \frac{d^4q}{(2\pi)^4} e^{ixq} e^{-iq_\mu/2}\tilde{A}_\nu(q)\xi^{(1)}(U_\mu;q,\nu) + \\
&\quad -g^2\int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4} e^{ix(q_1+q_2)} e^{-iq_{2\mu}/2}\tilde{A}_{\nu_2}(q_2) e^{-iq_{1\mu}/2}\tilde{A}_{\nu_1}(q_1)\xi^{(2)}(U_\mu;q_1,\nu_1;q_2,\nu_2)\Bigg\} \times \\
&\quad \times \int \frac{d^4p_1}{(2\pi)^4} e^{ip_1x} e^{-ip_{1\mu}}\tilde{\psi}(p_1). \qquad (6.69)
\end{aligned}
$$

The summation over x then leads to Dirac delta functions which represent the momentum conservation. Using $(q_1,\nu_1)\leftrightarrow(q_2,\nu_2)$ for the second two-gluon part, one obtains

$$
\begin{aligned}
&\sum_x \bar\psi(x)\left(U_\mu(x)\psi(x+a_\mu)-U_\mu^\dagger(x-a_\mu)\psi(x-a_\mu)\right) \;=\\
&= \int\frac{d^4p_2}{(2\pi)^4}e^{-ip_2x}\tilde{\bar\psi}(p_2)\int\frac{d^4p_1}{(2\pi)^4}e^{ip_1x}\tilde\psi(p_1)\Big\{(2\pi)^4\delta(p_2-p_1)\left(e^{ip_{1\mu}}-e^{-ip_{1\mu}}\right)\xi^{(0)}(U_\mu)\;+\\
&\quad +ig(2\pi)^4\int\frac{d^4q}{(2\pi)^4}\delta(p_2-p_1-q)e^{ip_{1\mu}+iq_\mu/2}\tilde A_\nu(q)\xi^{(1)}(U_\mu;q,\nu)\;+\\
&\quad +ig(2\pi)^4\int\frac{d^4q}{(2\pi)^4}\delta(p_2-p_1-q)e^{-ip_{1\mu}-iq_\mu/2}\tilde A_\nu(q)\xi^{(1)}(U_\mu;q,\nu)\;+\\
&\quad -g^2(2\pi)^4\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\delta(p_2-p_1-q_1-q_2)e^{ip_{1\mu}+iq_{1\mu}/2+iq_{2\mu}/2}\;\times\\
&\qquad\qquad\times\;\tilde A_{\nu_1}(q_1)\tilde A_{\nu_2}(q_2)\xi^{(2)}(U_\mu;q_1,\nu_1;q_2,\nu_2)\;+\\
&\quad +g^2(2\pi)^4\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\delta(p_2-p_1-q_1-q_2)e^{-ip_{1\mu}-iq_{1\mu}/2-iq_{2\mu}/2}\;\times\\
&\qquad\qquad\times\;\tilde A_{\nu_1}(q_1)\tilde A_{\nu_2}(q_2)\xi^{(2)}(U_\mu;q_2,\nu_2;q_1,\nu_1)\Big\}\;=\\
&=:\;2\int\frac{d^4p_2}{(2\pi)^4}e^{ip_2x}\tilde{\bar\psi}(p_2)\int\frac{d^4p_1}{(2\pi)^4}e^{ip_1x}\tilde\psi(p_1)\Big\{(2\pi)^4\delta(p_2-p_1)\xi^{(0)}(\bar\psi\nabla_\mu\psi,p)\;+\\
&\quad +(2\pi)^4\int\frac{d^4q}{(2\pi)^4}\delta(p_2-p_1-q)\tilde A_\nu(q)\xi^{(1)}(\bar\psi U_\mu\psi,p;q,\nu)\;+\\
&\quad +(2\pi)^4\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\delta(p_2-p_1-q_1-q_2)\tilde A_{\nu_1}(q_1)\tilde A_{\nu_2}(q_2)\xi^{(2)}(\bar\psi U_\mu\psi,p;q_1,\nu_1;q_2;\nu_2)\Big\}.
\end{aligned}
\tag{6.70}
$$

In the last line new ξ functions are defined which are characterized by the argument $p=(p_1+p_2)/2$ for the averaged momentum as the second entry. For these ξ functions one obtains

$$
\begin{aligned}
\xi^{(0)}(\bar\psi\nabla_\mu\psi,p) &= \frac{1}{2}\left(e^{ip_\mu}-e^{-ip_\mu}\right)\xi^{(0)}(U_\mu) \;=\; i\sin(p_\mu)\xi^{(0)}(U_\mu),\\
\xi^{(1)}(\bar\psi\nabla_\mu\psi,p;q,\nu) &= ig\cos(p_\mu)\xi^{(1)}(U_\mu;q,\nu),\\
\xi^{(2)}(\bar\psi\nabla_\mu\psi,p;q_1,\nu_1;q_2,\nu_2) &= \\
= \;-ig^2\sin(p_\mu)\frac{1}{2}&\left(\xi^{(2)}(U_\mu;q_1,\nu_1;q_2,\nu_2)+\xi^{(2)}(U_\mu;q_2,\nu_2;q_1,\nu_1)\right)\;+\\
-ig^2\cos(p_\mu)\frac{i}{2}&\left(\xi^{(2)}(U_\mu;q_1,\nu_1;q_2,\nu_2)-\xi^{(2)}(U_\mu;q_2,\nu_2;q_1,\nu_1)\right).
\end{aligned}
\tag{6.71}
$$

At first sight this elaborate calculation looks unnecessary because the ξ functions of $U_\mu(x)$ are trivial. Because of this the results indeed reduce to

$$
\begin{aligned}
\xi^{(0)}\left(\bar\psi\nabla_\mu\psi,p\right) &= i\sin(p_\mu),\\
\xi^{(1)}\left(\bar\psi\nabla_\mu\psi,p;q,\nu\right) &= ig\cos(p_\mu)\delta_{\mu\nu},\\
\xi^{(2)}\left(\bar\psi\nabla_\mu\psi,p;q_1,\nu_1;q_2,\nu_2\right) &= -\frac{i}{2}g^2\sin(p_\mu)\delta_{\mu\nu_1}\delta_{\mu\nu_2}.
\end{aligned}
\tag{6.72}
$$

But it was useful to do this calculation because for the staggered quark the link operator U_μ is replaced by a more complicated one, and here the usefulness of the above result becomes obvious. For the higher derivatives, however, this calculation will not be performed. Here the "pure" results are sufficient. For the second derivative one obtains

$$\begin{aligned}
\xi^{(0)}\left(\bar\psi\nabla^{(2)}_\mu\psi,p\right) &= -4\sin^2\left(\frac{p_\mu}{2}\right), \\
\xi^{(1)}\left(\bar\psi\nabla^{(2)}_\mu\psi,p;q,\nu\right) &= -2g\sin(p_\mu)\delta_{\mu\nu}, \\
\xi^{(2)}\left(\bar\psi\nabla^{(2)}_\mu\psi,p;q_1,\nu_1;q_2,\nu_2\right) &= -g^2\cos(p_\mu)\delta_{\mu\nu_1}\delta_{\mu\nu_2},
\end{aligned} \tag{6.73}$$

for the third derivative one has

$$\begin{aligned}
\xi^{(0)}\left(\bar\psi\nabla^{(3)}_\mu\psi,p\right) &= -4i\sin(p_\mu)\sin^2\left(\frac{p_\mu}{2}\right), \\
\xi^{(1)}\left(\bar\psi\nabla^{(3)}_\mu\psi,p;q,\nu\right) &= -2ig\left(\cos(p_\mu)-\cos\left(\frac{q_\mu}{2}\right)\cos(2p_\mu)\right)\delta_{\mu\nu}, \\
\xi^{(2)}\left(\bar\psi\nabla^{(3)}_\mu\psi,p;q_1,\nu_1;q_2,\nu_2\right) &= ig^2\left(\sin(p_\mu)-2\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\sin(2p_\mu)\right)\delta_{\mu\nu_1}\delta_{\mu\nu_2}.
\end{aligned} \tag{6.74}$$

Finally, for the fourth derivative one obtains

$$\begin{aligned}
\xi^{(0)}\left(\bar\psi\nabla^{(4)}_\mu\psi,p\right) &= 16\sin^4\left(\frac{p_\mu}{2}\right), \\
\xi^{(1)}\left(\bar\psi\nabla^{(4)}_\mu\psi,p;q,\nu\right) &= 4g\left(2\sin(p_\mu)-\cos\left(\frac{q_\mu}{2}\right)\sin(2p_\mu)\right)\delta_{\mu\nu}, \\
\xi^{(2)}\left(\bar\psi\nabla^{(4)}_\mu\psi,p;q_1,\nu_1;q_2,\nu_2\right) &= 4g^2\left(\cos(p_\mu)-\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\cos(2p_\mu)\right)\delta_{\mu\nu_1}\delta_{\mu\nu_2}.
\end{aligned} \tag{6.75}$$

The non-vanishing ξ functions of the special combinations of operators used in the quark action are given by

$$\begin{aligned}
\xi^{(0)}\left(\bar\psi\Big(\nabla_\mu-\frac{c_3}{6}\nabla^{(3)}_\mu\Big)\psi,p\right) &= i\sin(p_\mu)\left(1+\frac{2c_3}{3}\sin^2\left(\frac{p_\mu}{2}\right)\right), \\
\xi^{(0)}\left(\bar\psi\Big(\nabla^{(2)}_\mu-\frac{c_4}{12}\nabla^{(4)}_\mu\Big)\psi,p\right) &= -4\sin^2\left(\frac{p_\mu}{2}\right)\left(1+\frac{c_4}{3}\sin^2\left(\frac{p_\mu}{2}\right)\right)
\end{aligned} \tag{6.76}$$

for the no-gluon contribution,

$$\begin{aligned}
\xi^{(1)}_0\left(\bar\psi\Big(\nabla_\mu-\frac{c_3}{6}\nabla^{(3)}_\mu\Big)\psi,p;q,\nu\right) &= ig\left(\left(1+\frac{c_3}{3}\right)\cos(p_\mu)-\frac{c_3}{3}\cos\left(\frac{q_\mu}{2}\right)\cos(2p_\mu)\right), \\
\xi^{(1)}_0\left(\bar\psi\Big(\nabla^{(2)}_\mu-\frac{c_4}{12}\nabla^{(4)}_\mu\Big)\psi,p;q,\nu\right) &= -2g\left(\left(1+\frac{c_4}{3}\right)\sin(p_\mu)-\frac{c_4}{6}\cos\left(\frac{q_\mu}{2}\right)\sin(2p_\mu)\right)
\end{aligned} \tag{6.77}$$

for the one-gluon contribution, and

$$\begin{aligned}
&\xi^{(2)}_{00}\left(\bar\psi\Big(\nabla_\mu-\frac{c_3}{6}\nabla^{(3)}_\mu\Big)\psi,p;q_1,\nu_1;q_2,\nu_2\right) = \\
&= -\frac{ig^2}{2}\left(\left(1+\frac{c_3}{3}\right)\sin(p_\mu)-\frac{2c_3}{3}\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\sin(2p_\mu)\right), \\
&\xi^{(2)}_{00}\left(\bar\psi\Big(\nabla^{(2)}_\mu-\frac{c_4}{12}\nabla^{(4)}_\mu\Big)\psi,p;q_1,\nu_1;q_2,\nu_2\right) = \\
&= -g^2\left(\left(1+\frac{c_4}{3}\right)\sin(p_\mu)-\frac{c_4}{3}\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\cos(2p_\mu)\right)
\end{aligned} \tag{6.78}$$

for the two-gluon contribution.

6.2.7 ξ functions for the mass and the field strength tensor

The last elements necessary for the determination of Feynman rules are the mass part $a_t m_0$ and the part $\sum_{\mu,\nu} \sigma_{\mu\nu} F_{\mu\nu}$ including the QCD field strength tensor. The results for the (constant) mass part is rather simple,

$$\begin{aligned} \xi^{(0)}\left(\bar{\psi}(a_t m_0)\psi, p\right) &= a_t m_0, \\ \xi^{(1)}\left(\bar{\psi}(a_t m_0)\psi, p; q, \nu\right) &= \xi^{(2)}\left(\bar{\psi}(a_t m_0)\psi, p; q_1, \nu_1; q_2, \nu_2\right) = 0. \end{aligned} \tag{6.79}$$

For the field strength tensor one obtains

$$\begin{aligned} \xi^{(0)}\left(\bar{\psi} F_{\mu\nu}\psi, p\right) &= 0, \\ \xi^{(1)}\left(\bar{\psi} F_{\mu\nu}\psi, p; q, \nu_1\right) &= \left(f^0_{\mu\nu}\delta_{\mu\nu_1} - f^0_{\mu\nu}\delta_{\nu\nu_1}\right), \\ \xi^{(2)}\left(\bar{\psi} F_{\mu\nu}\psi, p; q_1, \nu_1; q_2, \nu_2\right) &= -g\left(f^1_{\mu\nu}\delta_{\mu\nu_1}\delta_{\mu\nu_2} + f^2_{\mu\nu}\delta_{\mu\nu_1}\delta_{\nu\nu_2} - f^2_{\mu\nu}\delta_{\nu\nu_1}\delta_{\mu\nu_2} - f^1_{\mu\nu}\delta_{\nu\nu_1}\delta_{\nu\nu_2}\right) \end{aligned} \tag{6.80}$$

where

$$\begin{aligned} f^0_{\mu\nu} &= \cos\left(\frac{q_\mu}{2}\right)\sin(q_\nu), \\ f^1_{\mu\nu} &= \sin\left(\frac{q_{1\mu}}{2} + \frac{q_{2\mu}}{2}\right)\sin\left(\frac{q_{1\nu}}{2} - \frac{q_{2\nu}}{2}\right)\cos\left(\frac{q_{1\nu}}{2} + \frac{q_{2\nu}}{2}\right), \\ f^2_{\mu\nu} &= \frac{1}{2}\Bigg[\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\nu}}{2}\right) - \cos\left(\frac{q_{1\mu}}{2} + q_{2\mu}\right)\cos\left(q_{1\nu} + \frac{q_{2\nu}}{2}\right) + \\ &\quad - \cos\left(\frac{q_{1\mu}}{2} + q_{2\mu}\right)\cos\left(\frac{q_{2\nu}}{2}\right) - \cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(q_{1\nu} + \frac{q_{2\nu}}{2}\right)\Bigg]. \end{aligned} \tag{6.81}$$

In combination with the antisymmetric matrix σ this result simplifies,

$$\begin{aligned} &\xi^{(1)}\left(\bar{\psi}\sum_{\mu,\nu}\sigma_{\mu\nu}F_{\mu\nu}\psi, p; q, \nu_1\right) = \\ &= i\sum_{\mu,\nu}\left(\sigma_{\mu\nu}\cos\left(\frac{q_\mu}{2}\right)\sin(q_\nu)\delta_{\mu\nu_1} - \sigma_{\mu\nu}\sin(q_\mu)\cos\left(\frac{q_\nu}{2}\right)\delta_{\nu\nu_1}\right) = \\ &= -2i\sum_{\mu}\sigma_{\mu\nu_1}\sin(q_\mu)\cos\left(\frac{q_{\nu_1}}{2}\right), \\ &\xi^{(2)}\left(\bar{\psi}\sum_{\mu,\nu}\sigma_{\mu\nu}F_{\mu\nu}\psi, p; q_1, \nu_1; q_2, \nu_2\right) = \\ &= -ig\left(\sum_{\nu}\sigma_{\nu_1\nu}f^1_{\nu_1\nu}\delta_{\nu_1\nu_2} + \sigma_{\nu_1\nu_2}f^2_{\nu_1\nu_2} - \sigma_{\nu_2\nu_1}f^2_{\nu_2\nu_1} - \sum_{\mu}\sigma_{\mu\nu_2}f^1_{\nu_2\mu}\delta_{\nu_1\nu_2}\right) = \\ &= -2ig\left(\sum_{\nu}\sigma_{\nu_1\nu}f^1_{\nu_1\nu}\delta_{\nu_1\nu_2} + \frac{1}{2}\sigma_{\nu_1\nu_2}(f^2_{\nu_1\nu_2} + f^2_{\nu_2\nu_1})\right). \end{aligned} \tag{6.82}$$

6.2.8 Quark propagator and vertices

While the gluon propagator is already given in Eq. (6.59), the different Feynman diagram elements including the quark propagator and the one- and the two-gluon vertex are obtained by using the functions $\xi^{(0)}$, $\xi^{(1)}$, and $\xi^{(2)}$. For the inverse propagator one obtains

$$\begin{aligned}
\xi^{(0)}(S^I_{D234};p) &= i\frac{c_0}{\chi}\sum_{i=1}^{3}\gamma_i\sin(p_i)\left(1+\frac{2c_{3s}}{3}\sin^2\left(\frac{p_i}{2}\right)\right)+\\
&\quad +i\gamma_4\sin(p_4)\left(1+\frac{2c_{3t}}{3}\sin^2\left(\frac{p_4}{2}\right)\right)+\\
&\quad +m_0a_t+\frac{2r}{\chi}\sum_{i=1}^{3}\sin^2\left(\frac{p_i}{2}\right)\left(1+\frac{c_{4s}}{3}\sin^2\left(\frac{p_i}{3}\right)\right)+\\
&\quad +2r\chi\sin^2\left(\frac{p_4}{2}\right)\left(1+\frac{c_{4t}}{3}\sin^2\left(\frac{p_4}{2}\right)\right)=\\
&= i\sum_{\mu=1}^{4}\gamma_\mu P_\mu(p_\mu)+M(p)
\end{aligned}\tag{6.83}$$

where the most general action expression S^I_{D234} is taken, allowing for a simplification by setting appropiate coefficients to zero. The momentum part component P_μ (not the mass part M) depends only on the component p_μ. Therefore, the quark propagator reads

$$Q(p)=\frac{-i\sum_{\mu=1}^{4}\gamma_\mu P_\mu(p_\mu)+M(p)}{\sum_{\mu=1}^{4}P_\mu^2(p_\mu)+M^2(p)}.\tag{6.84}$$

The one-gluon vertex with incoming quark momentum p, incoming gluon momentum q_1 and Lorentz index μ_1 is given by

$$\xi^{(1)}(S;p;q_1,\mu_1)=\sum_{\mu,\nu=1}^{4}\sigma_{\mu\nu}V_{\mu\nu}(p;q_1,\mu_1)+\sum_{\mu=1}^{4}\gamma_\mu V_\mu(p;q_1,\mu_1)+V(p;q,\mu_1).\tag{6.85}$$

The two-gluon vertex with incoming quark momentum p, incoming gluon momenta q_1, q_2 and Lorentz indices μ_1, μ_2 is used in the one-loop case only for the tadpole diagram,

$$\begin{aligned}
\xi^{(2)}(S;p;q_1,\mu_1;q_2,\mu_2) &= \sum_{\mu,\nu=1}^{4}\sigma_{\mu\nu}V_{\mu\nu}(p;q_1,\mu_1;q_2,\mu_2)+\\
&\quad +\sum_{\mu=1}^{4}\gamma_\mu V_\mu(p;q_1,\mu_1;q_2,\mu_2)+V(p;q_1,\mu_1;q_2,\mu_2).
\end{aligned}\tag{6.86}$$

6.2.9 The tadpole improvement

As explained before, the tadpole improvement "renormalizes" the size of the link operator U_μ by dividing it by a correction factor u_μ which is called the *mean link*. This mean link is a number between 0 and 1. It can be calculated by a Monte Carlo simulation as the mean value of $\frac{1}{3}\mathrm{Tr}(U_\mu)$. Expanded in the strong coupling, it can (for the so-called *Landau mean link definition*) be expressed and calculated by VEGAS [211] as

$$u_\mu=1-\frac{\alpha_s}{3\pi}u_\mu^{(2)}+O(\alpha_s^2),\qquad u_\mu^{(2)}=\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}G_{\mu\mu}(\alpha_g=0).\tag{6.87}$$

This correction is implemented into the MATHEMATICA package. Using this, the contributions to the expansion coefficient $\alpha_s u_\mu^{(2)}/3\pi$ in the expansion with respect to the strong coupling are determined. A hat notation is used for the time being to distinguish these contributions from the original operators. The results are given by

$$\begin{aligned}
\xi^{(0)}\left(\bar\psi\Big(\tilde\nabla_\mu - \frac{c_3}{6}\tilde\nabla_\mu^3\Big)\psi, p\right) &= iu_\mu^{(2)}\left[\left(1+\frac{c_3}{3}\right)\sin(p_\mu) - \frac{c_3}{3}\sin(2p_\mu)\right],\\
\xi^{(0)}\left(\bar\psi\Big(\tilde\nabla_\mu^2 - \frac{1}{12}c_4\tilde\nabla_\mu^4\Big)\psi, p\right) &= 2u_\mu^{(2)}\left[\left(1+\frac{2c_3}{3}\right)\cos(p_\mu) - \frac{c_4}{3}\cos(2p_\mu)\right],\\
\xi^{(0)}\left(\tilde F_{\mu\nu}, k', k\right) &= 0
\end{aligned}\tag{6.88}$$

for the zeroth order coefficients,

$$\begin{aligned}
&\xi_0^{(1)}\left(\bar\psi\Big(\tilde\nabla_\mu - \frac{c_3}{6}\tilde\nabla_\mu^{(3)}\Big)\psi, p; q\right) =\\
&= igu_\mu^{(2)}\left[\left(1+\frac{c_3}{3}\right)\cos(p_\mu) - \frac{2c_3}{3}\cos(2p_\mu)\cos\left(\frac{q_\mu}{2}\right)\right],\\
&\xi_0^{(1)}\left(\bar\psi\Big(\tilde\nabla_\mu^{(2)} - \frac{c_4}{12}\tilde\nabla_\mu^{(4)}\Big)\psi, p; q\right) =\\
&= -2gu_\mu^{(2)}\left[\left(1+\frac{c_4}{3}\right)\sin(p_\mu) - \frac{c_4}{3}\sin(2p_\mu)\cos\left(\frac{q_\mu}{2}\right)\right],\\
&\xi^{(1)}\left(\bar\psi\tilde F_{\mu\nu}\psi, p; q, \rho\right) =\\
&= 2i(u_\mu^{(2)} + u_\nu^{(2)})\left[\sin(q_\nu)\cos\left(\frac{q_\mu}{2}\right)\delta_{\mu\rho} - \sin(q_\mu)\cos\left(\frac{q_\nu}{2}\right)\delta_{\nu\rho}\right]
\end{aligned}\tag{6.89}$$

for the first order coefficients, and finally

$$\begin{aligned}
&\xi_{00}^{(2)}\left(\bar\psi\Big(\tilde\nabla_\mu - \frac{1}{6}c_3\tilde\nabla_\mu^{(3)}\Big)\psi, p; q_1; q_2\right) = -\frac{ig^2u_\mu^{(2)}}{2}\times\\
&\quad\times\left[\left(1+\frac{c_3}{3}\right)\sin(p_\mu) - \frac{4c_3}{3}\sin(2p_\mu)\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\right],\\
&\xi_{00}^{(2)}\left(\bar\psi\Big(\tilde\nabla_\mu^{(2)} - \frac{1}{12}c_4\tilde\nabla_\mu^{(4)}\Big)\psi, p; q_1; q_2\right) = -g^2u_\mu^{(2)}\times\\
&\quad\times\left[\left(1+\frac{c_4}{3}\right)\cos(p_\mu) - \frac{2c_4}{3}\cos(2p_\mu)\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\right],\\
&\xi^{(2)}\left(\bar\psi\tilde F_{\mu\nu}\psi, p; q_1, \rho_1; q_2, \rho_2\right) =\\
&= -\frac{ig}{2}(u_\mu^{(2)} + u_\nu^{(2)})\left(\delta_{\rho_1\mu}\delta_{\rho_2\nu} - \delta_{\rho_1\nu}\delta_{\rho_2\mu}\right)\times\\
&\quad\times\left[\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\nu}}{2}\right) - \cos\left(\frac{q_{1\mu}}{2}+q_{2\mu}\right)\cos\left(\frac{q_{2\nu}}{2}\right) + \right.\\
&\quad - \cos\left(q_{1\mu}+\frac{q_{2\mu}}{2}\right)\cos\left(\frac{q_{1\nu}}{2}\right) + \cos\left(\frac{q_{2\mu}}{2}\right)\cos\left(\frac{q_{1\nu}}{2}\right) +\\
&\quad - \cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right) - \cos\left(\frac{q_{1\mu}}{2}+q_{2\mu}\right)\cos\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right) +\\
&\quad \left. - \cos\left(\frac{q_{2\mu}}{2}\right)\cos\left(\frac{q_{1\nu}}{2}+q_{2\nu}\right) - \cos\left(q_{1\mu}+\frac{q_{2\mu}}{2}\right)\cos\left(\frac{q_{1\nu}}{2}+q_{2\nu}\right)\right]
\end{aligned}\tag{6.90}$$

for the second order coefficients. Even though all the coefficients up to second order are written down, for the first order case only the zeroth order coefficients of this correction are necessary. The correction to S^I_{D234} reads

$$\begin{aligned}
\xi^{(0)}\left(\tilde{S}^I_{D234}, p\right) &= i\frac{c_0}{\chi}u_s^{(2)}\sum_i \gamma_i\left[\left(1+\frac{c_{3s}}{3}\right)\sin(p_i) - \frac{c_{3s}}{3}\sin(2p_i)\right] + \\
&\quad + i\gamma_4 u_t^{(2)}\left[\left(1+\frac{c_{3t}}{3}\right)\sin(p_4) - \frac{c_{3t}}{3}\sin(2p_4)\right] + \\
&\quad + \frac{r}{\chi}u_s^{(2)}\sum_i\left[\left(1+\frac{c_{4s}}{3}\right)\cos(p_i) - \frac{c_{4s}}{6}\cos(2p_i)\right] + \\
&\quad + r\chi u_t^{(2)}\left[\left(1+\frac{c_{4t}}{3}\right)\cos(p_4) - \frac{c_{4t}}{6}\cos(2p_4)\right]. \qquad (6.91)
\end{aligned}$$

This expression, multiplied with $\alpha_s/3\pi$, is the tadpole improvement which can then be added to the unimproved contribution.

6.3 The pole mass and wave function renormalization

This section is devoted to the calculation of the *renormalized pole mass* M_1 and the *wave function renormalization factor* Z_2. As one knows from continuum renormalization theory, both quantities are obtained by the consideration of the inverse quark propagator in the rest frame of the quark. The considerations will be done in four steps, for the isotropic and anisotropic case on the one hand and the free propagator and the first order correction on the other hand. These considerations are quite parallel to those found in Ref. [201].

6.3.1 The isotropic free propagator

For the isotropic case, the inverse free quark propagator

$$\tilde{S}_0(p)^{-1} = Q(p)^{-1} = i\sum_{\mu=1}^{4}\gamma_\mu P_\mu(p_\mu) + M(p) \qquad (6.92)$$

in the quark rest frame with $\vec{p} = \vec{0}$ has the components

$$P_i(p_i = 0) = 0, \quad P_4(p_4) = \sin(p_4), \quad M(\vec{p} = \vec{0}, p_4) = ma_t + 1 - \cos(p_4). \qquad (6.93)$$

Therefore, one has

$$\tilde{S}_0(\vec{0}, p_4) = \frac{-i\gamma_4\sin(p_4) + ma_t + 1 - \cos(p_4)}{\sin^2(p_4) + ((1+ma_t) - \cos(p_4))^2}. \qquad (6.94)$$

The denominator can be rewritten as

$$\begin{aligned}
D(e^{ip_4}) &= \sin^2(p_4) + (1 + ma_t - \cos(p_4))^2 = \\
&= (-i\sin(p_4) + ma_t + 1 - \cos(p_4))\,(i\sin(p_4) + ma_t + 1 - \cos(p_4)) = \\
&= (-e^{ip_4} + ma_t + 1)(-e^{-ip_4} + ma_t + 1). \qquad (6.95)
\end{aligned}$$

By substituting $ma_t + 1 = e^{M_1}$ one obtains

$$D(z) = \left(z - e^{M_1}\right)\left(\frac{1}{z} - e^{M_1}\right). \tag{6.96}$$

The numerator is written in a similar way as

$$N(z) = -\gamma_4 \frac{1}{2}\left(z - \frac{1}{z}\right) + e^{M_1} - \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{6.97}$$

The Fourier transform of the propagator leads to the free Greens function

$$G_0(\vec{0}, t) = \int_{-\pi/a_t}^{\pi/a_t} \frac{dk_4}{2\pi} e^{ik_4 t} \tilde{S}_0(\vec{0}, a_t k_4) = \oint_{|z|=1} \frac{dz}{2\pi i a_t} z^{t/a_t} \frac{N(z)}{zD(z)} \tag{6.98}$$

or

$$a_t G_0(\vec{0}, a_t\tau) = -\oint_{|z|=1} \frac{dz}{2\pi i} \frac{z^\tau N(z) e^{-M_1}}{(z - e^{M_1})(z - e^{-M_1})} \tag{6.99}$$

where the ordinary time is given by $t = a_t\tau$, $q_4 = a_t k_4$ is used. The substitution $z = e^{ip_4}$ converts the integration path along the real axis to a circle path in the complex plane. Therefore, one can apply Cauchy's theorem which states that the value of this integral is given by the residues of the integrand lying within this circle. These are very simple in this case. Because of $m > 0$ one has $M_1 > 0$. Therefore, the only residue in this case which has to be taken into account is the one at $z = e^{-M_1}$. One therefore obtains

$$a_t G_0(\vec{0}, a_t\tau) = -\mathrm{Res}\left[\frac{z^\tau N(z) e^{-M_1}}{(z - e^{M_1})(z - e^{-M_1})}\ ; z = e^{-M_1}\right] = e^{-M_1(\tau+1)} \frac{N(e^{-M_1})}{e^{M_1} - e^{-M_1}}. \tag{6.100}$$

The denominator is simply $2\sinh(M_1)$, for the numerator one obtains

$$N(e^{-M_1}) = -\gamma_4 \frac{1}{2}(e^{-M_1} - e^{M_1}) + e^{M_1} - \frac{1}{2}(e^{-M_1} + e^{M_1}) = (\gamma_4 + 1)\sinh(M_1). \tag{6.101}$$

Therefore, one ends up with the pole mass $M_1 = \ln(1 + ma_t)$ and

$$a_t G_0(\vec{0}, a_t\tau) = \frac{1 + \gamma_4}{2} e^{-M_1(\tau+1)} \quad \Rightarrow \quad Z_2(\vec{0}) = e^{-M_1}. \tag{6.102}$$

6.3.2 The anisotropic free propagator

For the case of the anisotropic lattice, the components are modified,

$$P_i(p_i = 0) = 0, \quad P_4(p_4) = \sin(p_4) \quad M(\vec{p} = \vec{0}, p_4) = ma_t + \chi - \chi\cos(p_4). \tag{6.103}$$

The free quark propagator is given by

$$\tilde{S}_0(\vec{0}, p_4) = \frac{-i\gamma_4 \sin(p_4) + ma_t + \chi - \chi\cos(p_4)}{\sin^2(p_4) + (ma_t + \chi - \chi\cos(p_4))^2}, \tag{6.104}$$

and the denominator can be rewritten as

$$\begin{aligned}
D(e^{ip_4}) &= (-i\sin(p_4) + ma_t + \chi - \chi\cos(p_4)) \ \times \\
&\qquad \times\ (i\sin(p_4) + ma_t + \chi - \chi\cos(p_4)) \ = \\
&= \left(\frac{1}{2}(1-\chi)e^{-ip_4} + ma_t + \chi - \frac{1}{2}(1+\chi)e^{ip_4}\right) \ \times \\
&\qquad \times\ \left(\frac{1}{2}(1-\chi)e^{ip_4} + ma_t + \chi - \frac{1}{2}(1+\chi)e^{-ip_4}\right) \quad \Leftrightarrow \qquad (6.105) \\
D(z) &= \left(\frac{1}{2}(1-\chi)\frac{1}{z} + ma_t + \chi - \frac{1}{2}(1+\chi)z\right)\left(\frac{1}{2}(1-\chi)z + ma_t + \chi - \frac{1}{2}(1+\chi)\frac{1}{z}\right).
\end{aligned}$$

There are four roots of the denominator. The first two are solutions of the equation

$$z^2 - 2\frac{ma_t + \chi}{1+\chi}z - \frac{1-\chi}{1+\chi} = 0 \tag{6.106}$$

and are given by

$$z_{1,2} = \frac{ma_t + \chi \pm \sqrt{m^2a_t^2 + 2\chi ma_t + 1}}{1+\chi}. \tag{6.107}$$

In the limit $\chi \longrightarrow 1$ this simplifies to

$$z_1 \longrightarrow ma_t + 1, \qquad z_2 \longrightarrow 0. \tag{6.108}$$

The second two roots are solutions of

$$z^2 + 2\frac{ma_t + \chi}{1-\chi}z - \frac{1+\chi}{1-\chi} = 0, \tag{6.109}$$

given by

$$z_{3,4} = \frac{-ma_t - \chi \pm \sqrt{m^2a_t^2 + 2\chi ma_t + 1}}{1-\chi} \tag{6.110}$$

In this case the limit $\chi \longrightarrow 1$ is more involved. One obtains

$$z_3 \longrightarrow \frac{1}{ma_t + 1}, \qquad z_4 \longrightarrow \infty. \tag{6.111}$$

It is easy to see that $z_1z_3 = z_2z_4 = 1$. The only root that approaches the root for the isotropic lattice in the limit $\chi \longrightarrow 1$ is the root z_3. But one has to take into account also z_2 which lies inside the unit circle as well. Therefore, one has to determine the value of the numerator and the cancelled denominator at these values. In chosing

$$z_1 = e^{M_1}, \qquad z_2 = e^{-M_2}, \qquad z_3 = e^{-M_1}, \quad \text{and} \quad z_4 = e^{M_2}, \tag{6.112}$$

the expressions for numerator and denominator can be simplified. For $z = z_2$ one obtains

$$\begin{aligned}(1+\chi)z_2 = ma_t + \chi - \sqrt{m^2a_t^2 + 2\chi ma_t + 1} \quad &\Leftrightarrow \\ (ma_t + \chi - (1+\chi)z_2)^2 = m^2a_t^2 + 2\chi ma_t + 1 \quad &\Leftrightarrow \\ (ma_t+\chi)^2 - 2(1+\chi)(ma_t+\chi)z_2 + (1+\chi)^2z_2^2 = (ma_t+\chi)^2 + 1 - \chi^2 &\end{aligned} \tag{6.113}$$

such that

$$\begin{aligned} ma_t + \chi &= \frac{1}{2}(1+\chi)z_2 - \frac{1}{2}(1-\chi)\frac{1}{z_2} = \frac{1}{2}\left(z_2 - \frac{1}{z_2}\right) + \frac{1}{2}\chi\left(z_2 + \frac{1}{z_2}\right) = \\ &= \frac{1}{2}\left(e^{-M_2} - e^{M_2}\right) + \frac{1}{2}\chi\left(e^{-M_2} + e^{M_2}\right) = -\sinh(M_2) + \chi\cosh(M_2). \end{aligned} \tag{6.114}$$

In this case the numerator and the cancelled denominator are given by

$$\begin{aligned} N(z_2) &= -\frac{1}{2}\gamma_4\left(e^{-M_2} - e^{M_2}\right) + ma_t + \chi - \frac{1}{2}\chi\left(e^{-M_2} + e^{M_2}\right) = \\ &= \gamma_4\sinh(M_2) - \sinh(M_2) + \chi\cosh(M_2) - \chi\cosh(M_2) = (\gamma_4 - 1)\sinh(M_2), \end{aligned} \tag{6.115}$$

$$\begin{aligned}
\frac{D(z)}{z-z_2}\Big|_{z=z_2} &= \frac{1+\chi}{2z_2}(z_1-z_2)\left(\frac{1}{2}(1-\chi)e^{-M_2}+ma_t+\chi-\frac{1}{2}(1+\chi)e^{M_2}\right) = \\
&= e^{M_2}\sqrt{m^2a_t^2+2\chi ma_t+1}\left(\frac{1}{2}\left(e^{-M_2}-e^{M_2}\right)+ma_t+\chi-\frac{1}{2}\chi\left(e^{-M_2}+e^{M_2}\right)\right) = \\
&= e^{M_2}\sqrt{m^2a_t^2+2\chi ma_t+1}\left(-\sinh(M_2)-\sinh(M_2)+\chi\cosh(M_2)-\chi\cosh(M_2)\right) = \\
&= -2e^{M_2}\sinh(M_2)\sqrt{m^2a_t^2+2\chi ma_t+1} \qquad (6.116)
\end{aligned}$$

The corresponding residue is therefore

$$R_2 := \mathrm{Res}\left[\frac{z^\tau N(z)}{zD(z)}\ ; z=e^{-M_2}\right] = \frac{1-\gamma_4}{2}\frac{e^{-M_2\tau}}{\sqrt{m^2a_t^2+2\chi ma_t+1}}. \qquad (6.117)$$

For the other root $z=z_3$ one obtains

$$ma_t+\chi = \sinh(M_1)+\chi\cosh(M_1). \qquad (6.118)$$

Thus

$$\begin{aligned}
N(z_3) &= -\frac{1}{2}\gamma_4\left(e^{-M_1}-e^{M_1}\right)+ma_t+\chi-\frac{1}{2}\chi\left(e^{-M_1}+e^{M_1}\right) = \qquad (6.119)\\
&= \gamma_4\sinh(M_1)+\sinh(M_1)+\chi\cosh(M_1)-\cosh(M_1) \ = \ (\gamma_4+1)\sinh(M_1)
\end{aligned}$$

and

$$\begin{aligned}
\frac{D(z)}{z-z_3}\Big|_{z=z_3} &= \frac{1-\chi}{2z_3}(z_3-z_4)\left(\frac{1}{2}(1-\chi)e^{M_1}+ma_t+\chi-\frac{1}{2}(1+\chi)e^{-M_1}\right) = \\
&= e^{M_1}\sqrt{m^2a_t^2+2\chi ma_t+1}\left(\frac{1}{2}\left(e^{M_1}-e^{-M_1}\right)+ma_t+\chi-\frac{1}{2}\chi\left(e^{M_1}+e^{-M_1}\right)\right) = \\
&= e^{M_1}\sqrt{m^2a_t^2+2\chi ma_t+1}\left(\sinh(M_1)+\sinh(M_1)+\chi\cosh(M_1)-\chi\cosh(M_1)\right) = \\
&= e^{M_1}\sinh(M_1)\sqrt{m^2a_t^2+2\chi ma_t+1}. \qquad (6.120)
\end{aligned}$$

The corresponding residue is given by

$$R_1 := \mathrm{Res}\left[\frac{z^\tau N(z)}{zD(z)}\ ; z=e^{-M_1}\right] = \frac{1+\gamma_4}{2}\frac{e^{-M_1\tau}}{\sqrt{m^2a_t^2+2\chi ma_t+1}}. \qquad (6.121)$$

One finally obtains

$$a_tG_0(\vec{0},a_t\tau) = \frac{1-\gamma_4}{2}\frac{e^{-M_2\tau}}{\sqrt{m^2a_t^2+2\chi ma_t+1}}+\frac{1+\gamma_4}{2}\frac{e^{-M_1\tau}}{\sqrt{m^2a_t^2+2\chi ma_t+1}} \qquad (6.122)$$

which leads to the pole masses M_1 and M_2 and a still common renormalization factor

$$Z_2(\vec{0}) = \frac{1}{\sqrt{m^2a_t^2+2\chi ma_t+1}}. \qquad (6.123)$$

6.3.3 The isotropic one-loop case

After having gained experience how to calculate the pole mass and renormalization factor for the free case, the one-loop case is considered, starting with the inverse propagator

$$\tilde{S}(\vec{0},p_4)^{-1} = i\gamma_4\sin(p_4) + ma_t + 1 - \cos(p_4) - \tilde{\Sigma}(\vec{0},p_4) \tag{6.124}$$

where $\Sigma(p)$ is the result we obtain from the first order radiative correction, written as

$$\tilde{\Sigma}(p) = i\sum_{i=1}^{3}\gamma_i\sin(p_i)\Sigma_i(p) + i\gamma_4\sin(p_4)\Sigma_4(p) + \Sigma_m(p). \tag{6.125}$$

For the special case $\vec{p} = \vec{0}$ a short hand notation is used,

$$\Sigma_i(\vec{0},p_4) = 0, \quad \Sigma_4(\vec{0},p_4) = \hat{\Sigma}_4(-ip_4), \quad \Sigma_m(\vec{0},p_4) = \hat{\Sigma}_m(-ip_4). \tag{6.126}$$

The arguments of the new functions are chosen for later convenience. For the inverse propagator one ends up with

$$\tilde{S}(\vec{0},p_4)^{-1} = i\gamma_4(1-\hat{\Sigma}_4(-ip_4))\sin(p_4) + ma_t + 1 - \hat{\Sigma}_m(-ip_4) - \cos(p_4). \tag{6.127}$$

Calculating the propagator itself, the denominator is given by

$$\begin{aligned} D(e^{ip_4}) &= \Big(1-\hat{\Sigma}_4(-ip_4)\Big)^2\sin^2(p_4) + \Big(ma_t + 1 - \hat{\Sigma}_m(-ip_4) - \cos(p_4)\Big)^2 \quad \Leftrightarrow \qquad (6.128)\\ D(z) &= -\frac{1}{4}\Big(1-\hat{\Sigma}_4(-\ln z)\Big)^2\left(z-\frac{1}{z}\right)^2 + \left(ma_t + 1 - \hat{\Sigma}_m(-\ln z) - \frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2. \end{aligned}$$

Now one makes an ansatz, assuming that the denominator has roots $z = e^{-M}$ with $M > 0$ as before. Solutions for the parameter M are to be determined in the following. Inserting $z = e^{-M}$, one obtains

$$D(e^{-M}) = -\Big(1-\hat{\Sigma}_4(M)\Big)^2\sinh^2(M) + \Big(ma_t + 1 - \hat{\Sigma}_m(M) - \cosh(M)\Big)^2 \stackrel{!}{=} 0 \tag{6.129}$$

as determining equation for M. Because of $\sinh(M) > 0$, Eq. (6.129) can be solved by

$$0 < \sinh(M) = \left|\frac{ma_t + 1 - \hat{\Sigma}_m(M) - \cosh(M)}{1-\hat{\Sigma}_4(M)}\right|. \tag{6.130}$$

There are two possible solutions. The first one is given by

$$\begin{aligned} (1-\hat{\Sigma}_4(M_1))\sinh(M_1) &= ma_t + 1 - \hat{\Sigma}_m(M_1) - \cosh(M_1) \quad \Leftrightarrow \\ e^{M_1} &= ma_t + 1 - \hat{\Sigma}_m(M_1) + \hat{\Sigma}_4(M_1)\sinh(M_1) \qquad (6.131) \end{aligned}$$

and the second by

$$\begin{aligned} -(1-\hat{\Sigma}_4(M_2))\sinh(M_2) &= ma_t + 1 - \hat{\Sigma}_m(M_2) - \cosh(M_2) \quad \Leftrightarrow \\ e^{-M_2} &= ma_t + 1 - \hat{\Sigma}_m(M_2) - \hat{\Sigma}_4(M_2)\sinh(M_2). \qquad (6.132) \end{aligned}$$

Not both of these equations have positive solutions M_i. In addition, the solutions are still given by implicit equations, but they can already be used to simplify the numerator. The numerator is given by

$$\begin{aligned} N(e^{ip_4}) &= -i\gamma_4\left(1-\hat{\Sigma}_4(-ip_4)\right)\sin(p_4)+ma_t+1-\hat{\Sigma}_m(-ip_4)-\cos(p_4) \quad \Leftrightarrow \\ N(z) &= -\frac{1}{2}\gamma_4\left(1-\hat{\Sigma}_4(-\ln z)\right)\left(z-\frac{1}{z}\right)+ma_t+1-\hat{\Sigma}_m(-\ln z)-\frac{1}{2}\left(z+\frac{1}{z}\right) \\ \Leftrightarrow \quad & N(e^{-M}) = \gamma_4\left(1-\hat{\Sigma}_4(M)\right)\sinh(M)+ma_t+1-\hat{\Sigma}_m(M)-\cosh(M). \end{aligned} \tag{6.133}$$

Using Eqs. (6.131) and (6.132) to replace for ma_t+1 in the cases $M=M_1$ and $M=M_2$, one obtains

$$\begin{aligned} N(e^{-M_1}) &= \gamma_4\left(1-\hat{\Sigma}_4(M_1)\right)\sinh(M_1)+e^{M_1}-\hat{\Sigma}_4(M_1)\sinh(M_1)-\cosh(M_1) = \\ &= \gamma_4\left(1-\hat{\Sigma}_4(M_1)\right)\sinh(M_1)+\sinh(M_1)-\hat{\Sigma}_4(M_1)\sinh(M_1) = \\ &= (\gamma_4+1)\left(1-\hat{\Sigma}_4(M_1)\right)\sinh(M_1) \end{aligned} \tag{6.134}$$

and

$$\begin{aligned} N(e^{-M_2}) &= \gamma_4\left(1-\hat{\Sigma}_4(M_2)\right)\sinh(M_2)+e^{-M_2}+\hat{\Sigma}_4(M_2)\sinh(M_2)-\cosh(M_2) = \\ &= \gamma_4\left(1-\hat{\Sigma}_4(M_2)\right)\sinh(M_2)-\sinh(M_2)+\hat{\Sigma}_4(M_2)\sinh(M_2) = \\ &= (\gamma_4-1)\left(1-\hat{\Sigma}_4(M_2)\right)\sinh(M_2). \end{aligned} \tag{6.135}$$

Of course, one has problems to factor out these roots from the denominator. But the denominator can be expanded in a series near these roots. In order to do this, one has to calculate the first derivative of $D(z)$ with respect to z,

$$\begin{aligned} D'(z) &= -\frac{1}{2}(1-\hat{\Sigma}_4(-\ln z))^2\left(z-\frac{1}{z}\right)\left(1+\frac{1}{z^2}\right)+ \\ &\quad -\frac{1}{2}\left(1-\hat{\Sigma}_4(-\ln z)\right)\left(\frac{\Sigma_4'(-\ln z)}{z}\right)\left(z-\frac{1}{z}\right)^2+ \\ &\quad +2\left(ma_t+1-\hat{\Sigma}_m(-\ln z)-\frac{1}{2}\left(z+\frac{1}{z}\right)\right)\left(\frac{\Sigma_m'(-\ln z)}{z}-\frac{1}{2}\left(1-\frac{1}{z^2}\right)\right) \end{aligned} \tag{6.136}$$

and thus $D(e^{-M})=0$ and

$$\begin{aligned} e^{-M}D'(e^{-M}) &= 2\left(1-\hat{\Sigma}_4(M)\right)^2\sinh(M)\cosh(M)-2\left(1-\hat{\Sigma}_4(M)\right)\Sigma_4'(M)\sinh^2(M)+ \\ &\quad +2\left(ma_t+1-\hat{\Sigma}_m(M)-\cosh(M)\right)\left(\Sigma_m'(M)+\sinh(M)\right). \end{aligned} \tag{6.137}$$

Inserting the two different roots, one obtains

$$\begin{aligned} & e^{-M_1}D'(e^{-M_1}) = \\ &= 2\left(1-\hat{\Sigma}_4(M_1)\right)^2\sinh(M_1)\cosh(M_1)-2\left(1-\hat{\Sigma}_4(M_1)\right)\Sigma_4'(M_1)\sinh^2(M_1)+ \end{aligned}$$

$$
\begin{aligned}
& +2\left(\sinh(M_1) - \hat{\Sigma}_4(M_1)\sinh(M_1)\right)\left(\Sigma'_m(M_1) + \sinh(M_1)\right) = \\
= & \ 2\left(1 - \hat{\Sigma}_4(M_1)\right)\sinh(M_1) \times \\
& \times \left(\left(1 - \hat{\Sigma}_4(M_1)\right)\cosh(M_1) - \Sigma'_4(M_1)\sinh(M_1) + \Sigma'_m(M_1) + \sinh(M_1)\right) = \\
= & \ 2\left(1 - \hat{\Sigma}_4(M_1)\right)\sinh(M_1) \times \\
& \times \left(e^{M_1} - \hat{\Sigma}_4(M_1)\cosh(M_1) - \Sigma'_4(M_1)\sinh(M_1) + \Sigma'_m(M_1)\right) = \\
= & \ 2\left(1 - \hat{\Sigma}_4(M_1)\right)\sinh(M_1)\frac{d}{dM_1}\left(e^{M_1} - \hat{\Sigma}_4(M_1)\sinh(M_1) + \hat{\Sigma}_m(M_1)\right), \\
e^{-M_2}D'(e^{-M_2}) = & \\
= & \ 2\left(1 - \hat{\Sigma}_4(M_2)\right)^2\sinh(M_2)\cosh(M_2) - 2\left(1 - \hat{\Sigma}_4(M_2)\right)\Sigma'_4(M_2)\sinh^2(M_2) + \\
& +2\left(-\sinh(M_2) + \hat{\Sigma}_4(M_2)\sinh(M_2)\right)\left(\Sigma'_m(M_2) + \sinh(M_2)\right) = \\
= & \ 2\left(1 - \hat{\Sigma}_4(M_2)\right)\sinh(M_2) \times \\
& \times \left(\left(1 - \hat{\Sigma}_4(M_2)\right)\cosh(M_2) - \Sigma'_4(M_2)\sinh(M_2) - \Sigma'_m(M_2) - \sinh(M_2)\right) = \\
= & \ -2\left(1 - \hat{\Sigma}_4(M_2)\right)\sinh(M_2) \times \\
& \times \left(e^{-M_2} + \hat{\Sigma}_4(M_2)\cosh(M_2) + \Sigma'_4(M_2)\sinh(M_2) + \Sigma'_m(M_2)\right) = \\
= & \ -2\left(1 - \hat{\Sigma}_4(M_2)\right)\sinh(M_2)\frac{d}{dM_2}\left(e^{-M_2} + \hat{\Sigma}_4(M_2)\sinh(M_2) + \hat{\Sigma}_m(M_2)\right). \quad (6.138)
\end{aligned}
$$

This is a rather suprisingly simple result. Exactly this first derivative of the denominator is necessary to calculate the residue, because

$$
\begin{aligned}
D(z) &= D(e^{-M}) + (z - e^{-M})D'(e^{-M}) + O\left((z - e^{-M})^2\right) = \\
&= (z - e^{-M})D'(e^{-M}) + O\left((z - e^{-M})^2\right) \quad (6.139)
\end{aligned}
$$

for the two roots $M = M_1$ and $M = M_2$. The residues and wave function renormalization factors are given by

$$
\begin{aligned}
R_1 &= \mathrm{Res}\left[\frac{z^\tau N(z)}{zD(z)}\,; z = e^{-M_1}\right] = \frac{1+\gamma_4}{2}e^{-M_1\tau}Z_2(\vec{0}, M_1) \quad \text{with} \\
Z_2(\vec{0}, M_1)^{-1} &= \frac{d}{dM_1}\left(e^{M_1} - \hat{\Sigma}_4(M_1)\sinh(M_1) + \hat{\Sigma}_m(M_1)\right) \quad (6.140)
\end{aligned}
$$

and

$$
\begin{aligned}
R_2 &= \mathrm{Res}\left[\frac{z^\tau N(z)}{zD(z)}\,; z = e^{-M_2}\right] = \frac{1-\gamma_4}{2}e^{-M_2\tau}Z_2(\vec{0}, M_2) \quad \text{with} \\
Z_2(\vec{0}, M_2)^{-1} &= \frac{d}{dM_2}\left(e^{-M_2} + \hat{\Sigma}_4(M_2)\sinh(M_2) + \hat{\Sigma}_m(M_2)\right). \quad (6.141)
\end{aligned}
$$

6.3.4 The anisotropic one-loop case

For the anisotropic case one starts with

$$\tilde{S}(\vec{0}, p_4)^{-1} = i\gamma_4(1 - \hat{\Sigma}_4(-ip_4))\sin(p_4) + ma_t + \chi - \hat{\Sigma}_m(-ip_4) - \chi\cos(p_4) \tag{6.142}$$

for the inverse propagator. For the denominator of the propagator one obtains

$$D(e^{ip_4}) = (1 - \hat{\Sigma}_4(-ip_4))^2 \sin^2(p_4) + (ma_t + \chi - \hat{\Sigma}_m(-ip_4) - \chi\cos(p_4))^2 \tag{6.143}$$

or

$$D(z) = -\frac{1}{4}\left(1 - \hat{\Sigma}_4(-\ln z)\right)^2 \left(z - \frac{1}{z}\right)^2 + \left(ma_t + \chi - \hat{\Sigma}_m(-\ln z) - \frac{\chi}{2}\left(z + \frac{1}{z}\right)\right)^2. \tag{6.144}$$

For $z = e^{-M}$ the equation to solve for the pole contributions is given by

$$D(e^{-M}) = -(1 - \hat{\Sigma}_4(M))^2 \sinh^2 M + (ma_t + \chi - \hat{\Sigma}_m(M) - \chi\cosh M)^2 \stackrel{!}{=} 0. \tag{6.145}$$

The solutions for M are implicitly given by M_1 and M_2 where

$$\begin{aligned} \chi\cosh M_1 + \sinh M_1 &= ma_t + \chi + \hat{\Sigma}_4(M_1)\sinh M_1 - \hat{\Sigma}_m(M_1), \\ \chi\cosh M_2 - \sinh M_2 &= ma_t + \chi - \hat{\Sigma}_4(M_2)\sinh M_2 - \hat{\Sigma}_m(M_2). \end{aligned} \tag{6.146}$$

Using Eqs. (6.146), one can also simplify the numerator

$$N(e^{-M}) = \gamma_4(1 - \hat{\Sigma}_4(M))\sinh M + ma_t + \chi - \hat{\Sigma}_m(M) - \chi\cosh M, \tag{6.147}$$

resulting in

$$\begin{aligned} N(e^{-M_1}) &= (1 + \gamma_4)(1 - \hat{\Sigma}_4(M_1))\sinh M_1, \\ N(e^{-M_2}) &= -(1 - \gamma_4)(1 - \hat{\Sigma}_4(M_2))\sinh M_2. \end{aligned} \tag{6.148}$$

The zeros of the denominator can finally obtained by calculating the derivative,

$$\begin{aligned} e^{-M} D'(e^{-M}) &= 2(1 - \hat{\Sigma}_4(M))^2 \sinh M \cosh M - 2(1 - \hat{\Sigma}_4(M))\Sigma_4'(M)\sinh^2 M \; + \\ &\quad + 2(ma_t + \chi - \hat{\Sigma}_m(M) - \chi\cosh M)(\Sigma_m'(M) + \chi\sinh M). \end{aligned} \tag{6.149}$$

For the two solutions the result is given by

$$\begin{aligned} e^{-M_1} D'(e^{-M_1}) &= 2(1 - \hat{\Sigma}_4(M_1))\sinh M_2 \;\times \\ &\quad \times \frac{d}{dM_1}\left(\chi\cosh M_1 + \sinh M_1 - \hat{\Sigma}_4(M_1)\sinh M_1 + \hat{\Sigma}_m(M_1)\right), \\ e^{-M_2} D'(e^{-M_2}) &= -2(1 - \hat{\Sigma}_4(M_2))\sinh M_2 \;\times \\ &\quad \times \frac{d}{dM_2}\left(\chi\cosh M_2 - \sinh M_2 + \hat{\Sigma}_4(M_2)\sinh M_2 + \hat{\Sigma}_m(M_2)\right). \end{aligned} \tag{6.150}$$

As a result for the *wave function renormalization factor* one obtains

$$\begin{aligned} Z_2(\vec{0}, M_1)^{-1} &= \frac{d}{dM_1}\left(\chi\cosh M_1 + \sinh M_1 - \hat{\Sigma}_4(M_1)\sinh M_1 + \hat{\Sigma}_m(M_1)\right), \\ Z_2(\vec{0}, M_2)^{-1} &= \frac{d}{dM_2}\left(\chi\cosh M_2 - \sinh M_2 + \hat{\Sigma}_4(M_2)\sinh M_2 + \hat{\Sigma}_m(M_2)\right). \end{aligned} \tag{6.151}$$

One can write both in a even shorter form as

$$Z_2(\vec{0}, M_1)^{-1} = \frac{d}{dM_1}(ma_t + \chi), \qquad Z_2(\vec{0}, M_2)^{-1} = \frac{d}{dM_2}(ma_t + \chi) \tag{6.152}$$

remembering the defining equations (6.146) for M_1 and M_2.

6.3.5 The pole mass regularization

One has to think about how to use these calculations to build up a program for the calculation of M_1. As a consequence of $z_1 = e^{M_1}$ and Eq. (6.107), at tree level

$$M_1^{(0)} = \ln\left(\frac{ma_t + \chi + \sqrt{m^2 a_t^2 + 2\chi m a_t + 1}}{1+\chi}\right) \tag{6.153}$$

is obtained for the *pole mass*. One now can use the first of Eqs. (6.146) to set up a relation between the expressions for M_1 in subsequent orders in α_s. In order to see this, note that $\hat\Sigma_4$ and $\hat\Sigma_m$ themselves contain a factor α_s. For this reason one inserts the expression to zeroth order on the right hand side and the term up to first order on the left hand side and obtains

$$\chi\cosh(M_1^{(1)}) + \sinh(M_1^{(1)}) = ma_t + \chi + \hat\Sigma_4(M_1^{(0)})\sinh(M_1^{(0)}) - \hat\Sigma_m(M_1^{(0)}) \tag{6.154}$$

Note that $M_1^{(1)}$ means the term *up to* first order. If one inserts $M_1^{(1)} = M_1^{(0)} + \Delta M_1^{(1)}$ and expands in the first order correction $\Delta M_1^{(1)}$, the result is

$$\begin{aligned}\chi\cosh(M_1^{(1)}) + \sinh(M_1^{(1)}) &= \\ &\approx \chi\cosh(M_1^{(0)}) + \sinh(M_1^{(0)}) + (\chi\sinh(M_1^{(0)}) + \cosh(M_1^{(0)}))\Delta M_1^{(1)}.\end{aligned} \tag{6.155}$$

Because of $(ma_t + \chi + \sqrt{m^2a_t^2 + 2\chi m a_t + 1})(ma_t + \chi - \sqrt{m^2a_t^2 + 2\chi m a_t + 1}) = -1 + \chi^2$ one has

$$\begin{aligned} e^{M_1^{(0)}} &= \frac{ma_t + \chi + \sqrt{m^2a_t^2 + 2\chi m a_t + 1}}{1+\chi} \qquad \text{and} \\ e^{-M_1^{(0)}} &= \frac{-ma_t - \chi + \sqrt{m^2a_t^2 + 2\chi m a_t + 1}}{1-\chi} \end{aligned} \tag{6.156}$$

and thus

$$\begin{aligned}\chi\cosh(M_1^{(0)}) + \sinh(M_1^{(0)}) &= \frac{1}{2}(1+\chi)e^{M_1^{(0)}} - \frac{1}{2}(1-\chi)e^{-M_1^{(0)}} = ma_t + \chi, \qquad (6.157)\\ \chi\sinh(M_1^{(0)}) + \cosh(M_1^{(0)}) &= \frac{1}{2}(1+\chi)e^{M_1^{(0)}} + \frac{1}{2}(1-\chi)e^{-M_1^{(0)}} = \sqrt{m^2a_t^2 + 2\chi m a_t + 1}.\end{aligned}$$

Inserting Eq. (6.157) and the expansion in Eq. (6.155) into Eq. (6.154), one obtains

$$\Delta M_1^{(1)}(M_1^{(0)}) = \frac{\hat\Sigma_4(M_1^{(0)})\sinh(M_1^{(0)}) - \hat\Sigma_m(M_1^{(0)})}{\chi\sinh(M_1^{(0)}) + \cosh(M_1^{(0)})}, \qquad \Delta M_1^{(1)}(0) = -\hat\Sigma_m(0). \tag{6.158}$$

Because of the last equality, it is not guaranteed that M_1 vanishes when m vanishes and vice versa. Instead, looking at the first of Eqs. (6.146), for $M_1 = 0$ one obtains

$$m(M_1 = 0)a_t = \hat\Sigma_m(0). \tag{6.159}$$

The first of Eqs. (6.146) can now be supplemented by this additional term,

$$\chi\cosh(M_1) + \sinh(M_1) = m'a_t + \chi + \hat\Sigma_4(M_1)\sinh M_1 - \hat\Sigma'_m(M_1) \tag{6.160}$$

where $m'a_t = ma_t - \tilde{\Sigma}_m(0)$ and $\hat{\Sigma}'_m(M_1) = \hat{\Sigma}_m(M_1) - \hat{\Sigma}_m(0)$. While the last term is a correction for the first-order contribution, m' is a first-order correction of a leading-order expression. The only additional appearence of m is given in the quark propagator and therefore in the radiative correction terms $\hat{\Sigma}_4$ and $\hat{\Sigma}_m$. In these first-order terms m can simply be replaced by m'. Therefore one can replace m everywhere by m' and drop the prime for m again, being left with the *subtracted mass correction*

$$\Delta M_{1,\mathrm{sub}}^{(1)}(M_1^{(0)}) = \Delta M_1^{(1)}(M_1^{(0)}) - \frac{\Delta M_1^{(1)}(0)}{\chi \sinh(M_1^{(0)}) + \cosh(M_1^{(0)})}. \tag{6.161}$$

Still, Eqs. (6.158) and (6.161) are not be the final expressions for the mass correction. The short hand notation $\hat{\Sigma}$ has to be translated back to the original self energy contributions. The part $\hat{\Sigma}_4$ can be obtained by calculating the trace of γ_4 with the expression

$$\Sigma(\vec{p}, p_4) = i \sum_i \gamma_i \sin(p_i)\Sigma_i(\vec{p}, p_4) + i\gamma_4 \sin(p_4)\Sigma_4(\vec{p}, p_4) + \Sigma_m(\vec{p}, p_4) \tag{6.162}$$

at $\vec{p} = \vec{0}$, because

$$\Sigma(\vec{0}, p_4) = i\gamma_4 \sin(p_4)\Sigma_4(\vec{0}, p_4) + \Sigma_m(\vec{0}, p_4) \tag{6.163}$$

and

$$\frac{1}{4}\mathrm{Tr}\left(\gamma_4\Sigma(\vec{0}, p_4)\right) = i \sin(p_4)\Sigma_4(\vec{0}, p_4). \tag{6.164}$$

At the point $p_4 = iM_1$ one has

$$\sinh M_1 \Sigma_4(\vec{0}, iM_1) = -\frac{1}{4}\mathrm{Tr}\left(\gamma_4\Sigma(\vec{0}, p_4)\right)\Big|_{p_4=iM_1}. \tag{6.165}$$

$\hat{\Sigma}_m(M_1)$ is obtained by taking the trace without any additional Dirac matrix,

$$\Sigma_m(\vec{0}, iM_1) = \frac{1}{4}\mathrm{Tr}\left(\Sigma(\vec{0}, p_4)\right)\Big|_{p_4=iM_1}. \tag{6.166}$$

With

$$\Sigma_4(\vec{0}, iM_1^{(0)}) = \hat{\Sigma}_4(M_1^{(0)}) \quad \text{and} \quad \Sigma_m(\vec{0}, iM_1^{(0)}) = \hat{\Sigma}_m(M_1^{(0)}) \tag{6.167}$$

one obtains

$$\begin{aligned} \hat{\Sigma}_4(M_1^{(0)})\sinh(M_1^{(0)}) - \hat{\Sigma}_m(M_1^{(0)}) \;&=\; \qquad (6.168) \\ = \;\; \Sigma_4(\vec{0}, iM_1^{(0)})\sinh(M_1^{(0)}) - \Sigma_m(\vec{0}, iM_1^{(0)}) \;&=\; -\frac{1}{4}\mathrm{Tr}\left((1+\gamma_4)\Sigma(\vec{0}, iM_1^{(0)})\right) \end{aligned}$$

and thus

$$\Delta M_1^{(1)} = -\frac{1}{4}\frac{\mathrm{Tr}\left((1+\gamma_4)\Sigma(\vec{0}, iM_1^{(0)})\right)}{\chi\sinh(M_1^{(0)}) + \cosh(M_1^{(0)})}. \tag{6.169}$$

As a last step one has to think about the physical content of the parameter M_1. Because m is made dimensionless by multiplying it with a_t, the same holds for M_1, i.e. one defines a *physical mass* M by $M_1 = Ma_t$. But then the parameter M_1 makes no sense in the limit $\chi \to \infty$ because in this limit M_1 vanishes. Therefore, one defines a dimensionless *physical* *mass parameter* $\hat{M}_1 := Ma_s = M_1\chi$ which is related to the dimensional *unphysical mass* m by

$$ma_t = \sinh\left(\frac{\hat{M}_1}{\chi}\right) + \chi\left(\cosh\left(\frac{\hat{M}_1}{\chi}\right) - 1\right). \tag{6.170}$$

Finally, the correction $\Delta M_1^{(1)}$ is changed to $\Delta\hat{M}_1^{(1)}$ by multiplying the self energy contributions by a factor χ. The multiplication with χ, i.e. the "shift" to a stable lattice spacing, makes both the physical mass parameter and the singularities independent of the lattice spacing ratio. In the isotropic limit the result shown in Eq. (6.169) agrees with formulas given in the literature (see e.g Refs. [201, 212]).

6.3.6 The wave function renormalization

For the calculation of the *wave function renormalization factor* Z_2 one has to calculate derivatives of the self energy contributions with respect to the time component. This becomes obvious by looking at Eqs. (6.151). For the first of Eqs. (6.151), for instance, one obtains

$$\begin{aligned}
Z_2(\vec{0}, M_1)^{-1} &= \frac{d}{dM_1}\Big(\chi\cosh M_1 + \sinh M_1 - \hat{\Sigma}_4(M_1) + \hat{\Sigma}_m(M_1)\Big) = \\
&= \chi\sinh(M_1) + \cosh(M_1) + \frac{d}{dM_1}\Big(-\sinh(M_1)\Sigma_4(\vec{0}, iM_1) + \Sigma_m(\vec{0}, iM_1)\Big) = \\
&= \chi\sinh(M_1) + \cosh(M_1) + \frac{1}{4}\frac{d}{dM_1}\mathrm{Tr}\Big((1+\gamma_4)\Sigma(\vec{0}, iM_1)\Big) = \\
&= \chi\sinh(M_1) + \cosh(M_1) + \frac{i}{4}\frac{d}{dp_4}\mathrm{Tr}\Big((1+\gamma_4)\Sigma(\vec{0}, p_4)\Big)\Big|_{p_4=iM_1}.
\end{aligned} \tag{6.171}$$

The one-loop approximation for Z_2^{-1} is obtained by expanding once more in $\Delta M_1^{(1)}$, an expansion that only effects the first two terms,

$$\begin{aligned}
Z_2(\vec{0}, M_1^{(1)})^{-1} &= \chi\sinh(M_1^{(0)}) + \cosh(M_1^{(0)}) + \Delta M_1^{(1)}\Big(\chi\cosh(M_1^{(0)}) + \sinh(M_1^{(0)})\Big) + \\
&\quad + \frac{i}{4}\frac{d}{dp_4}\mathrm{Tr}\Big((1+\gamma_4)\Sigma(\vec{0}, p_4)\Big)\Big|_{p_4=iM_1}.
\end{aligned} \tag{6.172}$$

The change from the leading order wave function renormalization $Z_2^{(0)} = Z_2(\vec{0}, M_1^{(0)}) = (\chi\sinh(M_1^{(0)}) + \cosh(M_1^{(0)}))$ to the one-loop approximation $Z_2^{(1)} = Z_2(\vec{0}, M_1^{(1)})$ is given by

$$\begin{aligned}
Z_2^{(1)} &= Z_2^{(0)}\Bigg\{1 - \Delta M_1^{(1)}\frac{\chi\cosh(M_1^{(0)}) + \sinh(M_1^{(0)})}{\chi\sinh(M_1^{(0)}) + \cosh(M_1^{(0)})} + \\
&\quad + \frac{1}{4i}\frac{d}{dp_4}\mathrm{Tr}\Big((1+\gamma_4)\Sigma(\vec{0}, p_4)\Big)\Big|_{p_4=iM_1}\Bigg\}
\end{aligned} \tag{6.173}$$

For the isotropic limit, Eq. (6.173) reduces to the expression given in Ref. [201]. Note that for the case of wave function renormalization there will be IR divergences even in the massive case. These divergences cannot be cancelled by another term.

Treatment of IR divergences

One can treat the IR-divergences by using a subtraction method introduced by Kuramashi [212]. This method consists of the construction of a counter term by using the continuum theory with an appropiate choice for the mass which guarantees that the IR singularity is exactly cancelled. This appropiate mass parameter can be calculated by taking one step further in the consideration of the denominator of the quark propagator. In the previous subsections the pole contributions has been calculated in terms of the pole mass M_1 where

$$m_0a_t + \chi = \sinh M_1 + \chi \cosh M_1. \tag{6.174}$$

If one expands the denominator close to this pole position, i.e. at a four-momentum $p = (\vec{0}, iM_1) + q$ for a small additional momentum q, one can compare with the continuum result for the (scalar) denominator factor which is proportional to $2i\tilde{m}q_4 + q^2$ where $\tilde{m}$ is the effective mass one needs in order to adjust the continuum result to the lattice result. The expansion of the different parts of the expression given in Eq. (6.83) leads to

$$\begin{aligned}
P_i(q_i) &\approx \frac{1}{\chi}q_i, \\
P_4(iM_1 + q_4) &\approx \sin(iM_1 + q_4) \;=\; i\sinh M_1 \cos q_4 + \cosh M_1 \sin q_4 \;= \\
&\approx i\left(1 - \frac{1}{2}q_4^2\right)\sinh M_1 + q_4 \cosh M_1, \\
M(iM_1 + q) &\approx \frac{1}{\chi}\sum_{i=1}^{3}(1 - \cos q_i) + m_0a_t + \chi\left(1 - \cos(iM_1 + q_4)\right) \;= \\
&\approx \frac{1}{2\chi}\sum_{i=1}^{3} q_i^2 + m_0a_t + \chi\left(1 - \cosh M_1\left(1 - \frac{1}{2}q_4^2\right) + iq_4 \sinh M_1\right) \;= \\
&= \frac{1}{2}\chi q^2 - \frac{1}{2}\chi q_4^2 + m_0a_t + \chi - \chi\cosh M_1 + \frac{1}{2}\chi q_4^2 \cosh M_1 + i\chi q_4 \sinh M_1 \;= \\
&= \sinh M_1 + i\chi q_4 \sinh M_1 + \frac{1}{2}\chi q^2 - \frac{1}{2}\chi q_4^2\left(1 - \cosh M_1\right)
\end{aligned} \tag{6.175}$$

where in the two last steps the mass relation (6.174) and

$$q^2 = \frac{1}{\chi^2}\sum_{i=1}^{3} q_i^2 + q_4^2 \tag{6.176}$$

have been used. The denominator is given by

$$\begin{aligned}
\sum_{i=1}^{3} P_i^2 + P_4^2 + M^2 &\approx \frac{1}{\chi^2}\sum_{i=1}^{3} q_i^2 + \left(i\left(1 - \frac{1}{2}q_4^2\right)\sinh M_1 + \cosh M_1 q_4\right)^2 + \\
&\quad + \left(\sinh M_1 + i\chi q_4 \sinh M + \frac{1}{2}\chi q^2 - \frac{1}{2}\chi q_4^2\left(1 - \cosh M_1\right)\right)^2 \;= \\
&\approx q^2 - q_4^2 - (1 - q_4^2)\sinh^2 M_1 + 2i\sinh M_1 \cosh M_1 q_4 \;+ \\
&\quad + \cosh^2 M_1 q_4^2 + \sinh^2 M_1 + 2i\chi q_4 \sinh^2 M_1 + \chi q^2 \sinh M_1 \;+ \\
&\quad - \chi q_4^2 \sinh M_1\left(1 - \cosh M_1\right) - \chi^2 q_4^2 \sinh^2 M_1 \;= \\
&= 2iq_4 \sinh M_1\left(\cosh M_1 + \chi\sinh M_1\right) + q^2\left(1 + \chi\sinh M_1\right) \;+
\end{aligned}$$

$$+q_4^2 \sinh M_1 \left((2-\chi^2)\sinh M_1 - \chi(1-\cosh M_1)\right). \tag{6.177}$$

The contribution proportional to q_4^2 does not give leading order contributions neither for the case $q_4 = 0$ nor for $q_4 \neq 0$ if q is considered to be small. Therefore, the comparison leads to the effective mass (also called *continuum mass*)

$$\tilde{m} = \sinh M_1 \frac{\cosh M_1 + \chi \sinh M_1}{1 + \chi \sinh M_1}. \tag{6.178}$$

The counter term has to be subtracted from the integrand and added as an integral, both within a circle of finite radius Λ which keeps the UV divergences under control.

IR correction for Feynman gauge

The sum given by Kuramashi [212] in Eq. (55) can be calculated to be

$$\begin{aligned}\sum_\rho \tilde{v}_\rho \tilde{S}(p_4+q,\tilde{m})\tilde{v}_\rho \tilde{D}(q,\lambda) &= \sum_\rho i\gamma_\rho \frac{1}{i\gamma_4 p_4 + i\not{q} + \tilde{m}} i\gamma_\rho \frac{1}{q^2+\lambda^2} = \\ = -\sum_\rho \frac{\gamma_\rho(-i\gamma_4 p_4 - i\not{q} + \tilde{m})\gamma_\rho}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)(q^2+\lambda^2)} &= -\frac{2i\gamma_4 p_4 + 2i\not{q} + 4\tilde{m}}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)(q^2+\lambda^2)}.\end{aligned} \tag{6.179}$$

The contraction with $1+\gamma_4$ and the multiplication with i leads to

$$\frac{2(p_4+q_4-2i\tilde{m})}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)(q^2+\lambda^2)} \tag{6.180}$$

The derivative with respect to p_4 is seperately written for numerator (first) and denominator (second),

$$\frac{2}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)(q^2+\lambda^2)} - \frac{4(p_4+q_4-2i\tilde{m})(p_4+q_4)}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)^2(q^2+\lambda^2)} \tag{6.181}$$

Inserting $p_4 = i\tilde{m}$, one obtains

$$\begin{aligned}&\frac{2}{(-\tilde{m}^2+2i\tilde{m}q_4+q^2+\tilde{m}^2)(q^2+\lambda^2)} - \frac{4(q_4-i\tilde{m})(q_4+i\tilde{m})}{(-\tilde{m}^2+2i\tilde{m}q_4+q^2+\tilde{m}^2)^2(q^2+\lambda^2)} = \\ &= \frac{2}{(2i\tilde{m}q_4+q^2)(q^2+\lambda^2)} - \frac{4(q_4^2+\tilde{m}^2)}{(2i\tilde{m}q_4+q^2)^2(q^2+\lambda^2)} = \\ &= \frac{2q^2}{(q^4+4\tilde{m}^2q_4^2)(q^2+\lambda^2)} - \frac{4(q_4^2+\tilde{m}^2)(q^4-4\tilde{m}^2q_4^2)}{(q^4+4\tilde{m}^2q_4^2)^2(q^2+\lambda^2)} + i\tilde{m}q_4(\cdots).\end{aligned} \tag{6.182}$$

This expression is the one which is subtracted from the integrand. In order to calculate the corresponding counter term for the integral, the integration of this expression over a finite sphere of radius Λ is done for polar coordinates. The measure in four-dimensional polar coordinates is given by $q^3 dq\, d\varphi \sin\theta_1 d\theta_1 \sin^2\theta_2 d\theta_2$ where $\varphi_1 \in [0,2\pi]$, $\theta_i \in [0,\pi]$ for $i = 1,2$. The integrand will not depend on the two first angles, the integration over these can therefore be performed in advance, it results in a factor 4π. The calculation is done as an example for the second (denominator) part and the representation $q_4 = q\cos\theta_2$ is used (q is the absolute value of the four-vector). Then the last angular integration can be performed to obtain

$$f(q) := 4\pi^2 \frac{(3q^4 + 2q^2\tilde{m}^2 - 4\tilde{m}^4)\sqrt{q^2 + 4\tilde{m}^2} - (3q^4 + 8q^2\tilde{m}^2 - 4\tilde{m}^4)q}{4q^2(q^2 + \lambda^2)\tilde{m}^4\sqrt{q^2 + 4\tilde{m}^2}}. \tag{6.183}$$

One has to to keep the gluon mass λ non-vanishing only in cases where it regularizes the integration. This is the case for the part

$$f_0(q) := 4\pi^2 \frac{q - \sqrt{q^2 + 4\tilde{m}^2}}{q^2(q^2 + \lambda^2)\sqrt{q^2 + 4\tilde{m}^2}} \tag{6.184}$$

of the above cited function $f(q)$. For this part one obtains

$$\begin{aligned}
\int_0^\Lambda f_0(q)q^3 dq \;&=\; \pi \int_0^\Lambda \frac{q^2 - q\sqrt{q^2 + 4\tilde{m}^2}}{(q^2 + \lambda^2)\sqrt{q^2 + 4\tilde{m}^2}} dq \;=\\
&= 4\pi^2 \int_0^{\ln a} \frac{4\tilde{m}^2 \sinh^2\zeta - 4\tilde{m}^2 \sinh\zeta\cosh\zeta}{4\tilde{m}^2(\sinh^2\zeta + \lambda^2/4\tilde{m}^2)2\tilde{m}\cosh\zeta} 2\tilde{m}\cosh\zeta\, d\zeta \;=\\
&= 4\pi^2 \int_0^{\ln a} \frac{\sinh^2\zeta - \sinh\zeta\cosh\zeta}{\sinh^2\zeta + \lambda^2/4\tilde{m}^2} d\zeta = 4\pi^2 \int_0^{\ln a} \frac{e^{2\zeta} - 2 + e^{-2\zeta} - e^{2\zeta} + e^{-2\zeta}}{e^{2\zeta} - 2 + e^{-2\zeta} + \lambda^2/\tilde{m}^2} d\zeta \;=\\
&= \pi \int_0^{\ln a} \frac{2(e^{-2\zeta} - 1)d\zeta}{e^{2\zeta} - 2 + e^{-2\zeta} + \lambda^2/\tilde{m}^2} = 4\pi^2 \int_{1/a^2}^1 \frac{(z-1)dz}{z^2 + (\lambda^2/\tilde{m}^2 - 2)z + 1} \;=\\
&\approx 2\pi^2 \int_{1/a^2}^1 \frac{(2z + (\lambda^2/\tilde{m}^2 - 2))dz}{z^2 + (\lambda^2/\tilde{m}^2 - 2)z + 1} = 2\pi^2 \ln\left(z^2 + (\lambda^2/\tilde{m}^2 - 2)z + 1\right)\Big|_{z=1/a^2}^1 \;=\\
&\approx 2\pi^2 \left(\ln(1 + \lambda^2/\tilde{m}^2 - 2 + 1) - \ln\left(a^{-4} - 2a^{-2} + 1\right)\right) \;=\\
&= 2\pi^2 \left(\ln\left(\frac{\lambda^2}{\tilde{m}^2}\right) - 2\ln\left(1 - \frac{1}{a^2}\right)\right) = 2\pi^2 \left(\ln\left(\frac{\lambda^2}{\tilde{m}^2}\right) - 2\ln\left(\frac{2\Lambda(\Lambda + \sqrt{\Lambda^2 + 4\tilde{m}^2})}{(\Lambda + \sqrt{\Lambda^2 + 4\tilde{m}^2})^2}\right)\right) =\\
&= 2\pi^2 \left(\ln\left(\frac{\lambda^2}{\tilde{m}^2}\right) - 2\ln\left(\frac{2\Lambda}{\Lambda + \sqrt{\Lambda^2 + 4\tilde{m}^2}}\right)\right)
\end{aligned} \tag{6.185}$$

where the substitutions

$$\begin{aligned}
q \;&=\; 2\tilde{m}\sinh\zeta, \qquad \zeta(q) \;=\; \ln\left(\frac{q + \sqrt{q^2 + 4\tilde{m}^2}}{2\tilde{m}}\right), \qquad dk \;=\; 2\tilde{m}\cos\zeta\, d\zeta,\\
&\zeta(0) = 0, \qquad \zeta(\Lambda) = \ln\left(\frac{\Lambda + \sqrt{\Lambda^2 + 4\tilde{m}^2}}{2\tilde{m}}\right) =: \ln a
\end{aligned} \tag{6.186}$$

with

$$a^2 - 1 = \frac{1}{4\tilde{m}^2}\left(\Lambda^2 + \Lambda^2 + 4\tilde{m}^2 + 2\Lambda\sqrt{\Lambda^2 + 4\tilde{m}^2} - 4\tilde{m}^2\right) = \frac{2\Lambda(\Lambda + \sqrt{\Lambda^2 + 4\tilde{m}^2})}{4\tilde{m}^2} \tag{6.187}$$

and

$$z = e^{-2\zeta}, \qquad dz = -2e^{-2\zeta}d\zeta \tag{6.188}$$

have been used. The rest of $f(q)$, called $f_1(q)$, can be calculated by setting $\lambda = 0$,

$$f_1(q) = 4\pi^2 \frac{(3q^2 + 2\tilde{m}^2)\sqrt{q^2 + 4\tilde{m}^2} - (3q^2 + 8\tilde{m}^2)q}{4q^2\tilde{m}^4\sqrt{q^2 + 4\tilde{m}^2}}, \tag{6.189}$$

it results in

$$\begin{aligned}\int f_1(q)q^3dq &= 2\pi^2\ln(2\tilde{m}) - 8\tilde{m}^4\ln(\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}) + \\ &\quad +\frac{\pi^2}{4\tilde{m}^4}\Big(\Lambda(3\Lambda^3+4\Lambda\tilde{m}^2-3\Lambda^2\sqrt{\Lambda^2+4\tilde{m}^2}+2\tilde{m}^2\sqrt{\Lambda^2+4\tilde{m}^2})\Big) = \qquad (6.190)\\ &= 2\pi^2\left(-\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right)+\frac{\Lambda^2}{8\tilde{m}^4}(3\Lambda^2+4\tilde{m}^2)-\frac{\Lambda}{8\tilde{m}^4}(3\Lambda^2-2\tilde{m}^2)\sqrt{\Lambda^2+4\tilde{m}^2}\right).\end{aligned}$$

Together with the integral over f_0, and up to a general factor $2\pi^2$, one thus ends up with

$$2\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right)-\frac{\Lambda^2}{2\tilde{m}^2}+\frac{\Lambda}{2\tilde{m}^2}\sqrt{\Lambda^2+4\tilde{m}^2} \qquad (6.191)$$

for the numerator derivative and

$$\ln\left(\frac{\lambda^2}{\Lambda^2}\right)+\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right)+\frac{\Lambda^2(3\Lambda^2+4\tilde{m}^2)}{8\tilde{m}^4}-\frac{\Lambda(3\Lambda^2-2\tilde{m}^2)}{8\tilde{m}^4}\sqrt{\Lambda^2+4\tilde{m}^2} \qquad (6.192)$$

for the denominator derivative.

IR correction for general gauge

Because the calculations are already performed for the Feynman gauge ($\alpha_g = 1$), for the general covariant gauge one has to concentrate only on the additional term (up to the factor $-(1-\alpha_g)$). This is given by

$$\begin{aligned}i\not{q}\frac{1}{i\gamma_4p_4+i\not{q}+\tilde{m}}i\not{q}\frac{1}{q^2(q^2+\lambda^2)} &= \frac{i\not{q}(-i\gamma_4p_4-i\not{q}+\tilde{m})i\not{q}}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)q^2(q^2+\lambda^2)} = \qquad (6.193)\\ &= \frac{ip_4(2q_4\not{q}-\gamma_4q^2)+iq^2\not{q}-\tilde{m}q^2}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)q^2(q^2+\lambda^2)} = \frac{2ip_4q_4\not{q}-i\gamma_4p_4q^2+iq^2\not{q}-\tilde{m}q^2}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)q^2(q^2+\lambda^2)}.\end{aligned}$$

Contraction with $1+\gamma_4$ and multiplication with i gives

$$\frac{-2p_4q_4^2+p_4q^2-q^2q_4-i\tilde{m}q^2}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)q^2(q^2+\lambda^2)} \qquad (6.194)$$

The derivative with respect to p_4 for the numerator and the denominator leads to

$$\frac{-(2q_4^2-q^2)}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)q^2(q^2+\lambda^2)}+\frac{2(2p_4q_4^2-p_4q^2+q^2q_4+i\tilde{m}q^2)(p_4+q_4)}{(p_4^2+2p_4q_4+q^2+\tilde{m}^2)^2q^2(q^2+\lambda^2)} \qquad (6.195)$$

By inserting $p_4 = i\tilde{m}$ one obtains

$$\begin{aligned}&\frac{-(2q_4^2-q^2)}{(2i\tilde{m}q_4+q^2)q^2(q^2+\lambda^2)}+\frac{2(2i\tilde{m}q_4^2-i\tilde{m}q^2+q^2q_4+i\tilde{m}q^2)(q_4+i\tilde{m})}{(2i\tilde{m}q_4+q^2)^2q^2(q^2+\lambda^2)} =\\ &= \frac{-(2q_4^2-q^2)}{(2i\tilde{m}q_4+q^2)q^2(q^2+\lambda^2)}+\frac{2q_4(2i\tilde{m}q_4+q^2)(q_4+i\tilde{m})}{(2i\tilde{m}q_4+q^2)^2q^2(q^2+\lambda^2)} =\\ &= \frac{-(2q_4^2-q^2)}{(2i\tilde{m}q_4+q^2)q^2(q^2+\lambda^2)}+\frac{2q_4(q_4+i\tilde{m})}{(2i\tilde{m}q_4+q^2)q^2(q^2+\lambda^2)} =\\ &= \frac{-(2q_4^2-q^2)(q^2-2i\tilde{m}q_4)}{(q^4+4\tilde{m}^2q_4^2)q^2(q^2+\lambda^2)}+\frac{2q_4(q_4+i\tilde{m})(q^2-2i\tilde{m}q_4)}{(q^4+4\tilde{m}^2q_4^2)q^2(q^2+\lambda^2)} =\\ &= \frac{-(2q_4^2-q^2)}{(q^4+4\tilde{m}^2q_4^2)(q^2+\lambda^2)}+\frac{2q_4^2(q^2+2\tilde{m}^2)}{(q^4+4\tilde{m}^2q_4^2)q^2(q^2+\lambda^2)}+i\tilde{m}q_4(\cdots). \qquad (6.196)\end{aligned}$$

The integrated result is

$$\frac{1}{2}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right)-\frac{\Lambda^2(\Lambda^2+8\tilde{m}^2)}{16\tilde{m}^4}+\frac{\Lambda(\Lambda^2+6\tilde{m}^2)}{16\tilde{m}^4}\sqrt{\Lambda^2+4\tilde{m}^2} \tag{6.197}$$

for the numerator derivative and

$$-\frac{1}{2}\ln\left(\frac{\lambda^2}{\Lambda^2}\right)-\frac{1}{2}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right)+\frac{\Lambda^2(\Lambda^2+8\tilde{m}^2)}{16\tilde{m}^4}-\frac{\Lambda(\Lambda^2+6\tilde{m}^2)}{16\tilde{m}^4}\sqrt{\Lambda^2+4\tilde{m}^2} \tag{6.198}$$

for the denominator derivative. Looking at the IR divergences, one observes that in the massive case these divergences are only present in the derivative parts of the denominator. They combine to

$$\left(1+\frac{1}{2}(1-\alpha_g)\right)\ln\left(\frac{\lambda^2}{\Lambda^2}\right) \tag{6.199}$$

which vanishes for $\alpha_g = 3$. There is also an interesting feature for the total sum of the gauge dependent parts. In looking at Eq. (6.196) one recognizes that there is a substancial cancellation of terms. The only remaining term is the singular one,

$$\frac{q^4+4\tilde{m}^2q_4^2}{(q^4+4\tilde{m}^2q_4^2)q^2(q^2+\lambda^2)}=\frac{1}{q^2(q^2+\lambda^2)}. \tag{6.200}$$

This is also reflected in the integrated results.

Counter term for the massless case

Because needed later on, the counter terms are considered also in the case $m_0 = 0$. In this case the non-integrated denominator derivative is given by

$$\frac{-4q_4^2}{q^4(q^2+\lambda^2)}-(1-\alpha_g)\frac{2q_4^2}{q^4(q^2+\lambda^2)}. \tag{6.201}$$

For the angular integration (normalized such that $\int\sqrt{1-t^2}dt = 1$) one obtains

$$\begin{aligned}\int_{-1}^{+1}\frac{2}{\pi}q_4^2\sqrt{1-t^2}dt &= \int_{-1}^{+1}\frac{2q^2t^2}{\pi}\sqrt{1-t^2}dt = \frac{2q^2}{\pi}\int_0^{\pi}\cos^2\theta\sin^2\theta\,d\theta = \\ &= \frac{q^2}{2\pi}\int_0^{\pi}\sin^2(2\theta)d\theta = \frac{q^2}{4\pi}\int_0^{2\pi}\sin^2\theta' d\theta' = \frac{q^2}{4}.\end{aligned} \tag{6.202}$$

Therefore, one arrives at

$$-\frac{1}{q^2(q^2+\lambda^2)}-(1-\alpha_g)\frac{1}{2q^2(q^2+\lambda^2)}=\left(-1-\frac{1}{2}(1-\alpha_g)\right)\frac{1}{q^2(q^2+\lambda^2)}. \tag{6.203}$$

Together with

$$\int_0^{\Lambda}\frac{q^3dq}{q^2(q^2+\lambda^2)}=-\frac{1}{2}\ln\left(\frac{\lambda^2}{\Lambda^2}\right) \tag{6.204}$$

one ends up with

$$\frac{1}{2}\left(1+\frac{1}{2}(1-\alpha_g)\right)\ln\left(\frac{\lambda^2}{\Lambda^2}\right). \tag{6.205}$$

For the numerator derivative one obtains

$$\frac{2}{q^2(q^2+\lambda^2)}+(1-\alpha_g)\frac{2q_4^2-q^2}{q^4(q^2+\lambda^2)}. \tag{6.206}$$

After angular integration this expression results in

$$\frac{2}{q^2(q^2+\lambda^2)}-(1-\alpha_g)\frac{1}{2q^2(q^2+\lambda^2)}=\left(2-\frac{1}{2}(1-\alpha_g)\right)\frac{1}{q^2(q^2+\lambda^2)} \tag{6.207}$$

and after radial integration in

$$\frac{1}{2}\left(-2+\frac{1}{2}(1-\alpha_g)\right)\ln\left(\frac{\lambda^2}{\Lambda^2}\right). \tag{6.208}$$

The sum of both is given by

$$\frac{1}{2}\left(-1+(1-\alpha_g)\right)\ln\left(\frac{\lambda^2}{\Lambda^2}\right). \tag{6.209}$$

6.3.7 Corrections to the speed of light

As a further issue in this section the first order radiative correction to the coefficient c_0, simply denominated as the *speed-of-light coefficient*, is calculated. The name "speed-of-light" was coinde because this coefficient is determined by the condition that the energy relation $p_4^2=\vec{p}^{\,2}+m^2$ should hold also for a quark propagator changed by the first order correction to the self energy. The corrected inverse quark propagator is given by

$$\begin{aligned}
Q(\vec{p},p_4)^{-1} \;=\;& ic_0\frac{a_t}{a_s}\sum_i\gamma_i\sin(p_i)\left(1+\frac{c_{3s}}{3}-\frac{c_{3s}}{3}\cos(p_i)\right)-i\sum_i\gamma_i\sin(p_i)\Sigma_i(\vec{p},p_4)\;+\\
&+i\gamma_4\sin(p_4)\left(1+\frac{c_{3t}}{3}-\frac{c_{3t}}{3}\cos(p_4)\right)-i\gamma_4\sin(p_4)\Sigma_4(\vec{p},p_4)\;+\\
&+2r\frac{a_t}{a_s}\sum_i\sin^2\left(\frac{p_i}{2}\right)\left(1+\frac{c_{4s}}{3}-\frac{c_{4s}}{3}\cos^2\left(\frac{p_i}{2}\right)\right)+ma_t\;+\\
&+2r\frac{a_s}{a_t}\sin^2\left(\frac{p_4}{2}\right)\left(1+\frac{c_{4t}}{3}-\frac{c_{4t}}{3}\cos^2\left(\frac{p_4}{2}\right)\right)-\Sigma_m(\vec{p},p_4).
\end{aligned} \tag{6.210}$$

The requirement results in the condition

$$c_0\frac{a_t}{a_s}\left(1+\frac{c_{3s}}{3}-\frac{c_{3s}}{3}\cos(p_i)\right)-\Sigma_i(\vec{p},p_4)=\frac{a_t}{a_s}\left(1+\frac{c_{3t}}{3}-\frac{c_{3t}}{3}\cos(p_4)-\Sigma_4(\vec{p},p_4)\right) \tag{6.211}$$

or for $\vec{p}=\vec{0}$ and $p_4=iM_1$ in

$$c_0=1+\frac{c_{3t}}{3}\left(1-\cosh M_1\right)+\chi\hat{\Sigma}_i(M_1)-\hat{\Sigma}_4(M_1). \tag{6.212}$$

Again the question arises how these quantities $\hat{\Sigma}_i(M_1)$ are connected to the calculated self energy contributions. To determine $\hat{\Sigma}_i$ one first has to construct the derivative of Eq. (6.162) with respect to p_i,

$$\begin{aligned}
\frac{\partial}{\partial p_i}\Sigma(\vec{p},p_4) \;=\;& i\gamma_i\cos(p_i)\Sigma_i(\vec{p},p_4)+i\sum_j\gamma_j\sin(p_j)\frac{\partial}{\partial p_i}\Sigma_j(\vec{p},p_4)\;+\\
&+i\gamma_4\sin(p_4)\frac{\partial}{\partial p_i}\Sigma_4(\vec{p},p_4)+\frac{\partial}{\partial p_i}\Sigma_m(\vec{p},p_4).
\end{aligned} \tag{6.213}$$

The trace together with γ_i at $\vec{p}=\vec{0}$ then results in

$$\frac{1}{4}\mathrm{Tr}\left(\gamma_i\frac{\partial}{\partial p_i}\Sigma(\vec{0},p_4)\right) = i\Sigma_i(\vec{0},p_4) \tag{6.214}$$

so that

$$\hat{\Sigma}_i(M_1) = \frac{1}{4i}\mathrm{Tr}\left(\gamma_i\frac{\partial}{\partial p_i}\Sigma(\vec{0},p_4)\right)\Big|_{p_4=iM_1}. \tag{6.215}$$

To determine c_0, one thus has to calculate

$$c_0 = 1+\frac{c_{3t}}{3}(1-\cosh M_1)+\frac{\chi}{4i}\mathrm{Tr}\left(\gamma_i\frac{\partial}{\partial p_i}\Sigma(\vec{0},p_4)\right)\Big|_{p_4=iM_1}+\frac{\mathrm{Tr}\left(\gamma_4\Sigma(\vec{0},p_4)\right)}{4\sinh M_1}\Big|_{p_4=iM_1}. \tag{6.216}$$

where the index i is arbitrary. p_i could be for instance the first space component. In the case of a massless quark there would be a division by zero in the last term. Therefore, the procedure changes a bit. In this case one calculates the derivative of Eq. (6.162) with respect to p_4,

$$\begin{aligned}\frac{\partial}{\partial p_4}\Sigma(\vec{p},p_4) &= i\sum_i \gamma_i\sin(p_i)\frac{\partial}{\partial p_4}\Sigma_i(\vec{p},p_4)+ \\ &+i\gamma_4\cos(p_4)\Sigma_4(\vec{p},p_4)+i\gamma_4\sin(p_4)\frac{\partial}{\partial p_4}\Sigma_4(\vec{p},p_4)+\frac{\partial}{\partial p_4}\Sigma_m(\vec{p},p_4)\end{aligned} \tag{6.217}$$

at the point $(\vec{p},p_4)=(\vec{0},iM_1)=(\vec{0},0)$ to obtain

$$\frac{\partial}{\partial p_4}\Sigma(\vec{0},p_4)\Big|_{p_4=0} = i\gamma_4\Sigma_4(\vec{0},0)+\frac{\partial}{\partial p_4}\Sigma_m(\vec{0},p_4)\Big|_{p_4=0}. \tag{6.218}$$

Thus one has

$$\hat{\Sigma}(0)=\Sigma_4(\vec{0},0)=\frac{1}{4i}\mathrm{Tr}\left(\gamma_4\frac{\partial}{\partial p_4}\Sigma(\vec{0},p_4)\right)\Big|_{p_4=0} \tag{6.219}$$

and finally

$$c_0 = 1+\frac{\chi}{4i}\mathrm{Tr}\left(\gamma_i\frac{\partial}{\partial p_i}\Sigma(\vec{0},p_4)\right)\Big|_{p_4=0}-\frac{1}{4i}\mathrm{Tr}\left(\gamma_4\frac{\partial}{\partial p_4}\Sigma(\vec{0},p_4)\right)\Big|_{p_4=0}. \tag{6.220}$$

The IR singularities appearing in the massless case does exactly cancel within the two derivative terms.

6.3.8 The effective energy scale

Up to now the general factor $\alpha_s/3\pi$ has not been considered. The reason is that this factor is not a constant but a quantity running with the energy scale. A priori one does not know what energy scale to chose. But one can find an energy scale which is closely related to the problem under consideration by following the subsequent steps. Suppose the first order correction of a quantity A (written in integral form) is given by

$$A^{(1)} = A^{(0)}+\alpha_s\Delta A^{(1)}, \qquad \alpha_s\Delta A^{(1)} = \int \alpha_V(p^2)\Delta I(p)d^4p, \tag{6.221}$$

where $\alpha_V(p^2)$ is taken from the naive potential model to be $\alpha_V(p^2) := V_{Q\bar{Q}}p^2$. Assume further that the asymptotic expansion of $\alpha_V(p^2)$ is terminated after the second term,

$$\alpha_V(p^2) \approx \alpha_V(\mu^2) + \beta^* \ln\left(\frac{p^2}{\mu^2}\right) a_V^2(\mu^2) \tag{6.222}$$

where β^* is related to the leading order coefficient of the beta function. Then one obtains

$$\begin{aligned} \alpha_s A^{(1)} &\approx \int \left(\alpha_V(\mu^2) + \beta^* \ln\left(\frac{p^2}{\mu^2}\right)\alpha_V^2(\mu^2)\right)\Delta I(p)d^4p = \qquad (6.223)\\ &= \left(\alpha_V(\mu^2) + \beta^* \ln\left(\frac{p^{*2}}{\mu^2}\right)\alpha_V(\mu^2)\right)\int \Delta I(p)d^4p \approx \alpha_V(p^{*2})\int \Delta I(p)d^4p. \end{aligned}$$

Therefore, it makes sense to chose $\alpha_s = \alpha_V(p^{*2})$ where p^{*2} is determined by

$$\ln(p^{*2}) = \frac{\int \ln(p^2)\Delta I(p)d^4p}{\int \Delta I(p)d^4p}. \tag{6.224}$$

It is now an easy task to calculate this effective energy scale by weighting the integrands by this logarithmic factor.

6.4 The one-loop diagrams

The one-loop Feynman diagrams which are to be calculated to correct the quark propagator are only of two kinds: The rainbow diagram and the tadpole diagram, as shown in Fig. 6.3. In doing so, the MATHEMATICA packages written for the purpose of these calculations are introduced in this section.

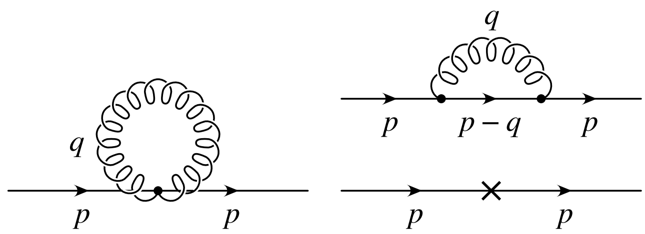


Figure 6.3: Rainbow diagram (top right), tadpole diagram (bottom left) and tadpole improvement counter term (bottom right)

6.4.1 The trace of the self energy diagram

The self energy diagram will be split up into two parts, the gluon propagator and the trace along the quark line. While the first is common also for the tadpole diagram, the latter will be considered first. The trace has to be taken in the opposite direction of the fermion flow. Looking at the calculations of the previous section, the different contributing parts are given by

- the quark-gluon vertices $\sum_{\mu,\nu}\sigma_{\mu\nu}V_{\mu\nu}(p;q_1,\mu_1) + \sum_\mu \gamma_\mu V_\mu(p;q_1,\mu_1) + V(p;q_1,\mu_1)$

- the quark propagator Q with numerator $-i\sum_\mu \gamma_\mu P_\mu(p_\mu) + M(p)$ and denominator $\sum_\mu P_\mu^2(p_\mu) + M^2(p)$. The packages will handle them in this separate form for simplicity of the results, especially the derivative of the quark propagator correction.

The package `gamma.add` does not use a formal Clifford algebra treatment of the Dirac structure of the Fermion line because it is complicated in this case (multiply occuring indices without summation) and also not necessary for the rather simple applications. Instead of this, it makes use of the explicit form of the gamma matrices, given by

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{6.225}$$

which is a Hermitean basis. The package combines all the expressions and calculates the trace of the total expression. This trace will be taken with or without the inclusion of some γ_λ in order to extract either the four momentum parts Σ_λ or the mass part Σ_m of the self energy contribution. Some work has to be done in order to evaluate this trace. Modules of trigonometric functions are defined by

$$\mathtt{sin}[\mathtt{p},\mathtt{mu}] = \sin\left(\frac{p_\mu}{2}\right), \qquad \mathtt{cos}[\mathtt{p},\mathtt{mu}] = \cos\left(\frac{p_\mu}{2}\right) \tag{6.226}$$

The trigonometric relations are implemented into a MATHEMATICA package and are performed by the command `collect[]`. This command also converts the trigonometric functions with momentum q of the gluon to expressions which can be handled by the VEGAS package [211]. There is an additional package `string.add` which converts the result of this calculation to the (bare) input form of VEGAS (Fortran77) using the newly defined MATHEMATICA command `FortranWrite[]`.

6.4.2 Options for quark propagator contributions

Options are left to select either the clover action (i.e. $c_{3t} = c_{3s} = c_{4t} = c_{4s} = 0$, all other constants equal to 1 for the moment) or the D234 action (in this simple case also $c_{3t} = c_{3s} = c_{4t} = c_{4s} = 1$). One can also change between the simple and the improved cloverleaf field strength tensor (see before, Eq. (6.19)). The change to

$$\begin{aligned}\tilde{F}_{\mu\nu}(x) = \frac{5}{3}F_{\mu\nu}(x) - \frac{1}{6}\Big(&U_\mu(x)F_{\mu\nu}(x+a_\mu)U_\mu^\dagger(x) + U_\mu^\dagger(x-a_\mu)F_{\mu\nu}(x-a_\mu)U_\mu(x-a_\mu) + \\ &-U_\nu(x)F_{\nu\mu}(x+a_\nu)U_\nu^\dagger(x) - U_\nu^\dagger(x-a_\nu)F_{\nu\mu}(x-a_\nu)U_\nu(x-a_\nu)\Big)\end{aligned} \tag{6.227}$$

within the Feynman term can be acomplished by replacing $\sigma_{\mu\nu}$ by

$$\frac{1}{3}\left(5 - \cos(q_\mu) - \cos(q_\nu)\right)\sigma_{\mu\nu} = \left(1 + \frac{2}{3}\sin\left(\frac{q_\mu}{2}\right) + \frac{2}{3}\sin\left(\frac{q_\nu}{2}\right)\right)\sigma_{\mu\nu}. \tag{6.228}$$

6.4.3 The gluon part

The gluon part can be calculated by adding the gauge fixing term to the pure gluon action and performing the inversion decribed above in Sec. 6.2.4. This gluon propagator then depends on the gluon gauge parameter α_g, and one can use this parameter in the calculations to test for gauge independence of the calculated quantities. A nice fact is that the coefficients $f_{\mu\nu}$ in $G_{\mu\nu}$ and the gauge dependent part turn out to be proportional to $\tilde{q}_\mu\tilde{q}_\nu$ in the case $\mu \neq \nu$. The same holds for the quark trace $Q_{\mu\nu}$ in what follows. Because the remaining parts of these expressions are shown to be only functions of squared sines as well, the contraction of these two terms is a totally symmetric function in all four integration variables, so that one can restrict the integration range to the range $[0, \pi]$ for each of these inner momentum components. The integration will be shown later. One should mention in advance that the factors $\tilde{q}_\mu \tilde{q}_\nu$ are removed from the single input expressions to the VEGAS package and instead are included in the contracted integrand.

6.4.4 The rainbow diagram

The *rainbow diagram* can be calculated as

$$\begin{aligned}\Sigma &= \frac{4}{3}g^2 \int_{-\pi}^{\pi} \frac{d^4q^{\rm dl}}{(2\pi)^4} \sum_{\mu,\nu=1}^{4} Q_{\mu\nu}^{\rm dl} G_{\mu\nu}^{\rm dl} = \frac{16\pi\alpha_s}{3} \int_{-\pi}^{\pi} \frac{d^4q^{\rm dl}}{16\pi^4} \sum_{\mu,\nu=1}^{4} Q_{\mu\nu}^{\rm dl} G_{\mu\nu}^{\rm dl} = \\ &= \frac{\alpha_s}{3\pi}\left(\frac{4}{\pi}\right)^2 \int_0^{\pi} d^4q^{\rm dl} \sum_{\mu,\nu=1}^{4} Q_{\mu\nu}^{\rm dl} G_{\mu\nu}^{\rm dl}. \end{aligned} \tag{6.229}$$

The VEGAS output is given up to a factor $\alpha_s/3\pi$. This formula is valid also for the tadpole diagram. For the self energy diagram, $Q_{\mu\nu}$ represents the trace part, i.e. the contribution of the quark propagator together with the quark-gluon vertices, while for the tadpole diagram $Q_{\mu\nu}$ only contains the quark-two-gluon vertex. $G_{\mu\nu}$ represents the gluon propagator. In order to keep the gluon propagator out of consideration in possible derivatives with respect to the outer momentum, the outer momentum flow is considered to run through the quark propagator. Note, though, that in this expression all quantities are chosen to be dimensionless (indicated by the index "dl"). Therefore, Σ itself is a dimensionless quantity, defined similar to the inverse quark propagator by a_t times the corresponding dimensional quantity. The trace part

$$Q_{\mu\nu}^{\rm dl} = {\rm Tr}(V_\mu^{\rm dl} Q^{\rm dl} V_\nu^{\rm dl}), \qquad V_\mu^{\rm dim} = a_\mu V_\mu^{\rm dl}, \quad Q^{\rm dim} = a_t Q^{\rm dl} \tag{6.230}$$

is already dimensionless according to the construction with dimensionless Q and V_μ, while this is not the case for the gluon propagator. But one can transform the initial expression $M_{\mu\nu}^{-1}$ (cf. Eq. (6.58)) to a dimensionless quantity by calculating

$$G_{\mu\nu}^{\rm dl} = G_{\mu\nu}^{\rm dim} a_\mu a_\nu = \frac{1}{V_a} M_{\mu\nu}^{-1} a_\mu a_\nu. \tag{6.231}$$

The quantity $G_{\mu\nu}^{\rm dim}$ is actually the one obtained as output from the package. Inserting all these quantities, one obtains

$$\begin{aligned}\Sigma &= \frac{\alpha_s}{3\pi}\left(\frac{4}{\pi}\right)^2 \int_0^{\pi} d^4q^{\rm dl} \sum_{\mu,\nu=1}^{4} a_\mu a_\nu Q_{\mu\nu}^{\rm dl} G_{\mu\nu}^{\rm dim} = \\ &= \frac{\alpha_s}{3\pi}\left(\frac{4}{\pi}\right)^2 \int_0^{\pi} d^4q \sum_{\mu,\nu=1}^{4} a_\mu a_\nu Q_{\mu\nu} G_{\mu\nu} = \frac{\alpha_s}{3\pi} \int_0^{\pi} d^4q\, \sigma(q) \end{aligned} \tag{6.232}$$

where the indices "dl" and "dim" are omitted again, assuming that from now on one knows the meaning of the different contributions. The integration itself is then done by VEGAS. This package written by Peter Lepage [211] is able to perform up to ten nested integrations by using a Monte Carlo method. It is written in Fortran77 and for the present case compiled together with a main program called `vegmain.f` and different subroutines, which will be included in one file `vegasfcn.f`. Depending on the term which should be calculated, different ingredients can be combined automatically: the main function f in `vqvg.fcn` for the whole diagram as explained in this subsection, together with the gluon propagator (`gprop.fcn`) and one of the quark line parts (e.g. the mass part `vqv0.fcn`). VEGAS needs several parameters to determine the precision of the calculation (e.g. the number of iterations) which are given in the input file `inveg.dat`.

The two ingredients, namely the quark and the gluon part of the diagrams, have to be combined in a main integrand function $\sigma(q)$ which can be integrated by VEGAS. Not included in this function will be the factor $\alpha_s/3\pi$ because additional considerations have to be done for the running coupling α_s. The following calculation also includes the different "preparations" made for the contributions. These are

Restoration of the lattice spacing

$$\tilde{q}_\mu = \frac{\hat{k}_\mu}{a_\mu} \quad \Rightarrow \quad \tilde{q}_i = \frac{1}{a_t\chi}\hat{k}_i, \quad \tilde{q}_4 = \frac{1}{a_t}\hat{k}_4 \quad \Rightarrow \quad \tilde{q}^2 = \frac{1}{a_t^2\chi^2}\hat{k}^2, \quad \hat{k}^2 := \hat{k}_1^2+\hat{k}_2^2+\hat{k}_3^2+\hat{k}_4^2$$
$$\text{where} \qquad \hat{k}_\mu = 2\sin\left(\frac{q_\mu}{2}\right) = 2\sin\left(\frac{a_\mu k_\mu}{2}\right) \tag{6.233}$$

Change of the quark propagator

$$Q_{\mu\nu} = \frac{\tilde{Q}_{\mu\nu}}{\tilde{Q}} \qquad \begin{array}{cc} \tilde{Q} = \hat{Q} & \tilde{\hat{Q}}_{ij} = \hat{Q}_{ij} \\ \tilde{Q}_{\mu\mu} = \tilde{\hat{Q}}_{\mu\mu} & \tilde{\hat{Q}}_{i4} = \chi\hat{Q}_{i4} \\ \tilde{Q}_{\mu\nu} = \hat{k}_\mu\hat{k}_\nu\tilde{\hat{Q}}_{\mu\nu} & \tilde{\hat{Q}}_{44} = \chi^2\hat{Q}_{44} \end{array} \tag{6.234}$$

Change of the gluon propagator

$$G_{\mu\nu} = \frac{\alpha_g\tilde{q}_\mu\tilde{q}_\nu}{V_a(\tilde{q}^2)^2} + \frac{\tilde{G}_{\mu\nu}}{\tilde{G}}, \qquad \begin{array}{cc} \tilde{G} = (\hat{k}^2)^2\hat{G} & \tilde{\hat{G}}_{ij} = \hat{G}_{ij} \\ \tilde{G}_{\mu\mu} = a_t^2\chi^2\tilde{\hat{G}}_{\mu\mu} & \tilde{\hat{G}}_{i4} = \chi\hat{G}_{i4} \\ \tilde{G}_{\mu\nu} = a_t^2\chi^2\hat{k}_\mu\hat{k}_\nu\tilde{\hat{G}}_{\mu\nu} & \tilde{\hat{G}}_{44} = \chi^2\hat{G}_{44} \end{array} \tag{6.235}$$

The dimensionless momenta $\hat{k}$ and the other hatted quantities are those which are output of the program. The integrand function is thus calculated to be

$$\left(\frac{\pi}{4}\right)^2\sigma(q) = \sum_{\mu,\nu=1}^{4} a_\mu a_\nu \frac{\tilde{Q}_{\mu\nu}}{\tilde{Q}}\left(\frac{\alpha_g\tilde{q}_\mu\tilde{q}_\nu}{V_a(q^2)^2} + \frac{\tilde{G}_{\mu\nu}}{\tilde{G}}\right) =$$
$$= \sum_{i=1}^{3} a_i^2\frac{\tilde{Q}_{ii}}{\tilde{Q}}\left(\frac{\alpha_g\hat{k}_i^2}{a_t^2\chi(\hat{k}^2)^2} + \frac{\tilde{G}_{ii}}{\tilde{G}}\right) + 2\sum_{i<j}^{3} a_i a_j\frac{\tilde{Q}_{ij}}{\tilde{Q}}\left(\frac{\alpha_g\hat{k}_i\hat{k}_j}{a_t^2\chi(\hat{k}^2)^2} + \frac{\tilde{G}_{ij}}{\tilde{G}}\right) +$$
$$+2\sum_{i=1}^{3} a_i a_4\frac{\tilde{Q}_{i4}}{\tilde{Q}}\left(\frac{\alpha_g\hat{k}_i\hat{k}_4}{a_t^2(\hat{k}^2)^2} + \frac{\tilde{G}_{i4}}{\tilde{G}}\right) + a_4^2\frac{\tilde{Q}_{44}}{\tilde{Q}}\left(\frac{\alpha_g\chi\hat{k}_4^2}{a_t^2(\hat{k}^2)^2} + \frac{\tilde{G}_{44}}{\tilde{G}}\right) =$$
$$= \sum_{i=1}^{3} a_t^2\chi^2\frac{\hat{Q}_{ii}}{\hat{Q}}\left(\frac{\alpha_g\hat{k}_i^2}{a_t^2\chi(\hat{k}^2)^2} + \frac{a_t^2\chi^2\hat{G}_{ii}}{a_t^4\chi^3(\hat{k}^2)^2\hat{G}}\right) + 2\sum_{i<j}^{3} a_t^2\chi^2\frac{\hat{Q}_{ij}}{\hat{Q}}\left(\frac{\alpha_g\hat{k}_i\hat{k}_j}{a_t^2\chi(\hat{k}^2)^2} + \frac{a_t^2\chi^2\hat{k}_i\hat{k}_j\hat{G}_{ij}}{a_t^4\chi^3(\hat{k}^2)^2\hat{G}}\right) +$$

$$+2\sum_{i=1}^{3} a_t^2 \chi \frac{\chi \hat{Q}_{i4}}{\hat{Q}} \left(\frac{\alpha_g \hat{k}_i \hat{k}_4}{a_t^2 (\hat{k}^2)^2} + \frac{a_t^2 \chi^3 \hat{k}_i \hat{k}_4 \hat{G}_{i4}}{a_t^4 \chi^3 (\hat{k}^2)^2 \hat{G}} \right) + a_t^2 \frac{\chi^2 \hat{Q}_{44}}{\hat{Q}} \left(\frac{\alpha_g \chi \hat{k}_4^2}{a_t^2 (\hat{k}^2)^2} + \frac{a_t^2 \chi^4 \hat{G}_{44}}{a_t^4 \chi^3 (\hat{k}^2)^2 \hat{G}} \right) =$$

$$= \chi \left[\sum_{i=1}^{3} \frac{\hat{Q}_{ii}}{\hat{Q}} \left(\alpha_g \hat{k}_i^2 + \frac{\hat{G}_{ii}}{\hat{G}} \right) + 2 \sum_{i<j}^{3} \frac{\hat{Q}_{ij}}{\hat{Q}} \left(\alpha_g + \frac{\hat{G}_{ij}}{\hat{G}} \right) \hat{k}_i \hat{k}_j + \right.$$

$$\left. +2\chi \sum_{i=1}^{3} \frac{\hat{Q}_{i4}}{\hat{Q}} \left(\alpha_g + \frac{\hat{G}_{i4}}{\hat{G}} \right) \hat{k}_i \hat{k}_4 + \chi^2 \frac{\hat{Q}_{44}}{\hat{Q}} \left(\alpha_g \hat{k}_4^2 + \frac{\hat{G}_{44}}{\hat{G}} \right) \right] \frac{1}{(\hat{k}^2)^2}. \qquad (6.236)$$

6.4.5 The tadpole diagram

The *tadpole diagram* calculation contains the two-gluon vertex. The clover part $V_{\mu\nu}$ does not contribute for a vanishing momentum change, one has

- the two-gluon vertices $\sum_\mu V_\mu(p; q_1, \mu_1; q_2, \mu_2) + V(p; q_1, \mu_1, q_2, \mu_2)$.

The non-clover part has a very simple form. The non-derivative expression is diagonal for vanishing external momentum (the case always considered to determine the renormalization) because the γ_μ term contains at least one sine function of the sum of the external momenta. This is of course not the case if one calculates the derivatives with respect to the outer momentum components. Both contributions have to be integrated by VEGAS in the way decribed above. The gauge independence of the summed expression of rainbow and tadpole contributions is checked in all cases. The rather large contribution due to the tadpole will be cancelled on the lattice by the tadpole improvement terms.

6.4.6 The tadpole improvement

The tadpole improvement is given by the corresponding first order correction of the self energy diagram due to the insertion of the tadpole improvement factor u_μ up to this order. Therefore, is simply the expression in Eq. (6.91) at $p = 0$ multiplied by $\alpha_s/3\pi$. Because $\alpha_s/3\pi$ is the general factor left out of all results, one obtains for the mass part

$$\frac{\chi}{4}\mathrm{Tr}(\Sigma) = r \left[\frac{3}{\chi} \left(1 + \frac{c_{4s}}{6} \right) u_s^{(2)} + \chi \left(1 + \frac{c_{4t}}{6} \right) u_t^{(2)} \right]. \qquad (6.237)$$

For the derivative of the momentum parts one obtains

$$\frac{\chi}{4ia_s} \frac{\partial}{\partial k_i} \mathrm{Tr}(\gamma_i \Sigma) = \frac{c_0}{\chi} \left(1 + \frac{c_{3s}}{3} \right) u_s^{(2)}, \qquad \frac{\chi}{4ia_t} \frac{\partial}{\partial k_4} \mathrm{Tr}(\gamma_4 \Sigma) = \left(1 + \frac{c_{3t}}{3} \right) u_t^{(2)}. \qquad (6.238)$$

6.4.7 Steps of the calculation

In this subsection the steps of the calculation will be detailed. First of all, starting with a definite set of numbers for the anisotropy χ and the physical mass parameter M_1, the unphysical mass parameter m necessary for the calculations is calculated by using Eq. (6.170). With these input parameters, the following quantities (specified by their file name, listed alphabetically and named by a few key words) are calculated:

- the full rainbow `vqv0r`, tadpole `vqv0t`, and tadpole improvement `vqv0u`

$$\frac{-\chi}{4(\chi \sinh(M_1) + \cosh(M_1))} \mathrm{Tr} \left((1 + \gamma_4) \Sigma^{(p)} \right), \qquad p = r, t, u \qquad (6.239)$$

- the 1-derivative of the 1-rainbow `vqv1d` + `vqv1n`, 1-tadpole `vqv1t`, and 1-tadpole improvement `vqv1u`

$$\frac{\chi}{4ia_s}\mathrm{Tr}\left(\gamma_1\frac{\partial}{\partial k_1}\Sigma^{(p)}\right), \qquad p=r,t,u \tag{6.240}$$

where the first one is separated in two parts according to the quotient rule,

$$\frac{\partial}{\partial k_1}\left(\frac{\hat{Q}_{\mu\nu}}{\hat{Q}}\right)=\frac{1}{\hat{Q}}\frac{\partial\hat{Q}_{\mu\nu}}{\partial k_1}-\frac{\hat{Q}_{\mu\nu}}{\hat{Q}^2}\frac{\partial\hat{Q}}{\partial k_1}=:\frac{ia_s}{\chi}\texttt{vqv1n}+\frac{ia_s}{\chi}\texttt{vqv1d} \tag{6.241}$$

- the 4-derivative of the full rainbow `vqv4d`+`vqv4n`, tadpole `vqv4p`, and tadpole improvement `vqv4q`

$$\frac{1}{4ia_t(\chi\sinh(M_1)+\cosh(M_1))}\mathrm{Tr}\left((1+\gamma_4)\frac{\partial}{\partial k_4}\Sigma^{(p)}\right), \qquad p=r,t,u \tag{6.242}$$

- the 4-rainbow `vqv4r`, 4-tadpole `vqv4t`, and 4-tadpole improvement `vqv4u`

$$\frac{1}{4\sinh(M_1)}\mathrm{Tr}\left(\gamma_4\Sigma^{(p)}\right), \qquad p=r,t,u \tag{6.243}$$

- the 4-derivative of the 4-rainbow `vqv4s`

$$\frac{-1}{4ia_t}\mathrm{Tr}\left(\gamma_4\frac{\partial}{\partial k_4}\Sigma^{(r)}\right) \tag{6.244}$$

Given the results for all these parts (for different values for the anisotropy parameter χ and/or the mass parameter), the remaining work is a matter of combination of these contributions. Because this parameter is needed for the wave function renormalization, the mass renormalization is the first quantity that has to be calculated. Considering Eq. (6.158), the ingredients for the unsubtracted $\Delta M^{(1)}$ are just `vqv0r`, `vqv0t`, and `vqv0u`. Adding up the rainbow contribution `vqv0r` and the tadpole contribution `vqv0t`, the result turns out to be gauge independent, i.e. independent of the parameter α_g. The full tadpole improvement `vqv0u` taken in Landau gauge ($\alpha_g=0$) cancels the whole or almost the whole tadpole contribution. In order to calculate the subtracted mass correction, the subtraction shown in Eq. (6.161) has to be done, using the already obtained value for the unsubtracted mass correction for vanishing input mass parameter. An example for values obtained in this procedure is shown in Table 6.1.

The next step in the procedure is to calculate the speed-of-light renormalization. This calculation does not need a subtraction of IR-divergences and the mass renormalization results, therefore it is performed at this point. For the massive case, Eq. (6.216) shows that the elements `vqv1d` and `vqv1n` for the rainbow, `vqv1t` for the tadpole, and `vqv1u` for the tadpole improvement constitute the second term of this expression, while the third term is given by `vqv4r` for the rainbow, `vqv4t` for the tadpole, and `vqv4u` for the tadpole improvement. The first term vanishes as long as one uses $c_{3t}=0$ as it is done in most of the cases, namely for the actions S_{clover} in Eq. (6.14) and S^{II}_{D234} in Eq. (6.16). This term has to be taken into account as well for $c_{4t}\neq 0$, i.e. for the action S^{I}_{D234} in Eq. (6.15).

$\alpha_g = 1$	vqv0r	vqv0t	sum	vqv0u	res	sub
0.00	−3.062	6.348	3.285	−4.529	−1.243	0.000
0.01	−3.001	6.286	3.285	−4.484	−1.200	0.031
0.05	−2.752	6.051	3.299	−4.316	−1.017	0.167
0.10	−2.493	5.781	3.287	−4.122	−0.835	0.295
0.50	−1.147	4.253	3.106	−3.023	0.083	0.906
1.00	−0.278	3.188	2.910	−2.254	0.656	1.262
2.00	0.512	2.108	2.620	−1.469	1.151	1.532
5.00	1.138	1.020	2.158	−0.662	1.496	1.636
10.00	1.323	0.588	1.911	−0.330	1.581	1.614
$\alpha_g = 0$	vqv0r	vqv0t	sum	vqv0u	res	sub
0.00	−1.244	4.529	3.285	−4.529	−1.244	0.000
0.01	−1.191	4.484	3.294	−4.484	−1.191	0.041
0.05	−1.013	4.316	3.303	−4.316	−1.013	0.171
0.10	−0.832	4.122	3.290	−4.122	−0.832	0.299
0.50	0.085	3.023	3.108	−3.023	0.085	0.908
1.00	0.657	2.254	2.911	−2.254	0.657	1.263
2.00	1.150	1.469	2.619	−1.469	1.150	1.532
5.00	1.496	0.662	2.158	−0.662	1.496	1.636
10.00	1.581	0.330	1.911	−0.330	1.581	1.615

Table 6.1: Mass renormalization for the action $S_G^{II} + S_{\rm clover}$ for different values of the mass parameter $M_1^{(0)}$ and $\chi = 3.6$. The first column indicates the value for $M_1^{(0)}$, the fourth column is the sum of the second and third column, the sixth column is the sum of the fourth and the fifth column. Finally, the seventh column is the subtracted result.

Rainbow and tadpole contributions are summed separately in Table 6.2. However, note that in case of $M_1^{(0)} = 0$ the contribution `vqv4s` is used instead of `vqv4r`. In Table 6.3 the results of the rainbow and the tadpole diagram calculation are summed and the tadpole improvement is calculated as well, again only for Landau gauge ($\alpha_g = 0$). Note that all these tables are only examples for the whole calculation which is given by many tables of this kind in Ref. [209], as it is useful for practical applications in lattice calculations.

The calculation of the wave function renormalization, finally, needs the mass correction as well as the subtraction of the IR-divergence, before the integration can be performed. Looking at Eq. (6.173), the ingredients for this calculation are `vqv4d` and `vqv4n` for the rainbow, `vqv4p` for the tadpole, and `vqv4q` for the tadpole improvement. The values for `vqv4d` and `vqv4n` shown in Table 6.4 are the IR-subtracted values. Therefore, Table 6.4 only shows the finite part $Z_2^{(1)}$ of the correction to the wave function renormalization in

$$Z_2 = Z_2^{(0)}\Big\{1 + \alpha_s(Z_2^{(1)} + Z_2^{(1)\rm IR}) + O(\alpha_s^2)\Big\}. \tag{6.245}$$

The singular part is given by

$$Z_2^{(1)\rm IR} = \begin{cases} (1 + (\alpha_g - 1))\ln(\lambda^2/\Lambda^2)/3\pi & \text{for } m = 0 \\ (-2 + (\alpha_g - 1))\ln(\lambda^2/\Lambda^2)/3\pi & \text{for } m > 0 \end{cases} \tag{6.246}$$

$\alpha_g = 1$	vqv1d	vqv1n	vqv4r	reg	vqv1t	vqv4t	tad
0.00	0.313	−0.785	−0.124	−0.596	1.565	−0.128	1.437
0.01	1.237	−2.889	1.058	−0.594	1.565	−0.128	1.437
0.05	0.890	−2.086	0.601	−0.595	1.565	−0.128	1.437
0.10	0.735	−1.683	0.352	−0.596	1.565	−0.128	1.437
0.50	0.351	−0.614	−0.325	−0.588	1.565	−0.128	1.437
1.00	0.190	−0.211	−0.549	−0.571	1.565	−0.128	1.437
2.00	0.073	0.039	−0.649	−0.538	1.565	−0.128	1.437
5.00	0.012	0.130	−0.672	−0.530	1.565	−0.128	1.437
10.00	0.002	0.121	−0.706	−0.583	1.565	−0.128	1.437
$\alpha_g = 0$	vqv1d	vqv1n	vqv4r	reg	vqv1t	vqv4t	tad
0.00	0.463	−0.113	−0.682	−0.332	1.235	−0.064	1.171
0.01	1.945	−1.719	−0.567	−0.340	1.235	−0.064	1.171
0.05	1.431	−1.119	−0.649	−0.338	1.235	−0.064	1.171
0.10	1.202	−0.819	−0.720	−0.337	1.235	−0.064	1.171
0.50	0.643	−0.023	−0.942	−0.322	1.235	−0.064	1.171
1.00	0.404	0.281	−0.989	−0.304	1.235	−0.064	1.171
2.00	0.212	0.470	−0.955	−0.272	1.235	−0.064	1.171
5.00	0.067	0.545	−0.877	−0.264	1.235	−0.064	1.171
10.00	0.016	0.543	−0.877	−0.318	1.235	−0.064	1.171

Table 6.2: The speed-of-light correction for the action $S_G^{II} + S_{\text{clover}}$ for different values of the mass parameter $M_1^{(0)}$ (first column). Columns two to four contain the contributions from the rainbow diagram, added in column five. Columns six and seven contain the contributions from the tadpole diagram, added in column eight.

(cf. Eqs. (6.199) and (6.209)). Finally, the term

$$Z_{2,M_1}^{(1)} = -\Delta M_1^{(1)} \frac{\chi \cosh(M_1^{(0)}) + \sinh(M_1^{(0)})}{\chi \sinh(M_1^{(0)}) + \cosh(M_1^{(0)})} \tag{6.247}$$

is added, using the results from Table 6.1. In the literature, the contribution $Z_{2,M_1}^{(1)}$ is not always included as part of the definition of $Z_2^{(1)}$. Therefore it makes sense to give these values separately, as it is done in Table 6.4. If one takes $Z_2^{(1)} - Z_{2,M_1}^{(1)}$ for the unimproved Wilson action S_{Wilson} in Eq. (6.20), the results of Ref. [163] are reproduced. Including $Z_{2,M_1}^{(1)}$ leads to the static result of Ref. [213] which has been used in many subsequent static calculations, for instance in Ref. [214]. This latter static value also corresponds to the large mass limit of the one-loop Z_2 calculated in many versions of the NRQCD actions [203].

For the action $S_G^I + S_{\text{clover}}$ the results for the massless case in Ref. [209] agree with Ref. [202]. Note, however, that the massive data do not tend towards the massless result as $a_s M_1^{(0)}$ decreases. This is a result of the fact that the massive case expressions contain a contribution $\ln(a_t m)$ which eventually diverges, while for the massless case the mass is set identically to zero from the beginning which is a usual practice in most of the literatur on massless lattice perturbation theory. It leads to different IR structures, seen for instance

$\alpha_g = 0$	reg	tad	sum	vqv1u	vqv4u	t.i.	res
0.00	−0.596	1.437	0.842	−1.235	0.064	−1.171	−0.330
0.01	−0.594	1.437	0.843	−1.235	0.064	−1.171	−0.328
0.05	−0.595	1.437	0.842	−1.235	0.064	−1.171	−0.329
0.10	−0.596	1.437	0.842	−1.235	0.064	−1.171	−0.330
0.50	−0.588	1.437	0.849	−1.235	0.064	−1.171	−0.322
1.00	−0.571	1.437	0.867	−1.235	0.064	−1.171	−0.304
2.00	−0.538	1.437	0.899	−1.235	0.064	−1.171	−0.272
5.00	−0.530	1.437	0.907	−1.235	0.064	−1.171	−0.264
10.00	−0.583	1.437	0.854	−1.235	0.064	−1.171	−0.317
$\alpha_g = 0$	reg	tad	sum	vqv1u	vqv4u	t.i.	res
0.00	−0.332	1.171	0.839	−1.235	0.064	−1.171	−0.332
0.01	−0.340	1.171	0.831	−1.235	0.064	−1.171	−0.341
0.05	−0.338	1.171	0.834	−1.235	0.064	−1.171	−0.338
0.10	−0.337	1.171	0.835	−1.235	0.064	−1.171	−0.337
0.50	−0.322	1.171	0.850	−1.235	0.064	−1.171	−0.322
1.00	−0.304	1.171	0.867	−1.235	0.064	−1.171	−0.305
2.00	−0.272	1.171	0.899	−1.235	0.064	−1.171	−0.273
5.00	−0.264	1.171	0.907	−1.235	0.064	−1.171	−0.264
10.00	−0.318	1.171	0.854	−1.235	0.064	−1.171	−0.318

Table 6.3: Continuation of Table 6.2, the results of this table are listed in the second and third column. They are summed in the fourth column. Columns five and six contain the tadpole improvement contributions for Landau gauge. These are summed in column seven. The final result is found in column eight.

in Eq. (6.246). The alternative way, namely to take the limit $a_t m \to 0$ and $\lambda \to 0$ keeping $a_t m > \lambda$, is not used in this report but can be taken in order to analyse the anomalous contributions (see Ref. [209]).

In a matching calculation one will be looking at differences between the lattice and continuum Z_2. As long as IR divergences are handled in the same manner in the lattice and continuum evaluations, one should not run into any problems and the $m \to 0$ limit should be smooth. For instance, using dimensional regularization in the $\overline{\rm MS}$ scheme, in Feynman gauge one finds the UV finite continuum results

$$Z_2^{(1)\rm cont} = \begin{cases} \left(\ln(\lambda^2/\mu^2) + 1/2\right)/3\pi & \text{for } m = 0 \\ \left(\ln(m^2/\mu^2) + 2\ln(m^2/\lambda^2) - 4\right)/3\pi & \text{for } m > 0 \end{cases} \tag{6.248}$$

The expression for the massive case can be rewritten as $\left(3\ln(m^2/\mu^2) - 2\ln(\lambda^2/\mu^2) - 4\right)/3\pi$. Comparing with the singular part calculated for the wave function renormalization on the lattice in Eq. (6.246), is makes sense to consider the subtracted Z_2 factor

$$Z_{2,\rm sub}^{(1)} = Z_2^{(1)} - \begin{cases} (1/2)/3\pi & \text{for } m = 0 \\ \left(3\ln\left((a_s M_1^{(0)})^2\right) - 4\right)/3\pi & \text{for } m > 0. \end{cases} \tag{6.249}$$

It is shown in Ref. [209] that for this subtracted Z_2 factor the transition to $m = 0$ is smooth.

$\alpha_g = 1$	vqv4d	vqv4n	vqv4p	sum	vqv4q	res	z2m	sum
0.00	0.580	−0.454	0.128	0.253	−0.064	0.190	0.000	0.190
0.01	−0.698	−2.547	0.128	−3.118	−0.064	−3.182	−0.031	−3.213
0.05	−0.464	−1.662	0.128	−1.999	−0.064	−2.062	−0.160	−2.222
0.10	−0.420	−1.184	0.128	−1.476	−0.064	−1.540	−0.271	−1.810
0.50	−0.459	0.116	0.128	−0.216	−0.064	−0.279	−0.628	−0.907
1.00	−0.416	0.517	0.128	0.228	−0.064	0.165	−0.687	−0.522
2.00	−0.226	0.635	0.128	0.536	−0.064	0.473	−0.620	−0.147
5.00	0.140	0.540	0.128	0.807	−0.064	0.744	−0.487	0.256
10.00	0.307	0.520	0.128	0.954	−0.064	0.891	−0.450	0.440
$\alpha_g = 0$	vqv4d	vqv4n	vqv4p	sum	vqv4q	res	z2m	sum
0.00	1.005	−0.321	0.064	0.747	−0.064	0.683	0.000	0.683
0.01	−0.885	−1.803	0.064	−2.624	−0.064	−2.688	−0.040	−2.728
0.05	−0.455	−1.113	0.064	−1.505	−0.064	−1.568	−0.164	−1.732
0.10	−0.312	−0.734	0.064	−0.982	−0.064	−1.045	−0.273	−1.319
0.50	−0.089	0.304	0.064	0.279	−0.064	0.215	−0.630	−0.415
1.00	0.052	0.606	0.064	0.722	−0.064	0.658	−0.688	−0.029
2.00	0.301	0.665	0.064	1.029	−0.064	0.966	−0.620	0.346
5.00	0.684	0.553	0.064	1.301	−0.064	1.237	−0.487	0.750
10.00	0.847	0.538	0.064	1.449	−0.064	1.385	−0.450	0.935

Table 6.4: Wave function renormalization for the action $S_G^{II} + S_{\text{clover}}$ for different values of the mass parameter $M_1^{(0)}$ (first column) and $\chi = 3.6$. The rainbow and tadpole contributions are shown in columns two to four, in the fifth column the sum is shown. Column six contains the tadpole improvement which is added to give the results shown in column seven. In column eight the correction $Z_{2,M_1}^{(1)}$ is shown (cf. Eq. (6.247)), the sum of this result with the result in column seven is shown in column nine.

6.4.8 Insight into some technical details

The calculation has to be performed not for a vanishing outer momentum but for a remaining temporal component. This will give rise to a dependence on this momentum in terms of powers of trigonometric functions in $a_t k_4/2$. After using $\cos^2(a_t k_4/2) = 1 - \sin^2(a_t k_4/2)$ and other trigonometric relations one can distinguish between different powers of $\sin(a_t k_4/2)$ where the odd powers are accompained by a factor $\cos(a_t k_4/2)$ whereas the even are not. One then inserts the value $-i a_t k_4 = M_1 = \hat{M}_1/\chi$ which means that one converts

$$\sin\left(\frac{a_t k_4}{2}\right) \to i \sinh\left(\frac{\hat{M}_1^{(0)}}{2\chi}\right) = i\,\texttt{sh1}, \qquad \cos\left(\frac{a_t k_4}{2}\right) \to \cosh\left(\frac{\hat{M}_1^{(0)}}{2\chi}\right) = \texttt{ch1}. \quad (6.250)$$

This conversion affects only the trace part of the diagram. These conversions are done separately for the numerator and the denominator and are combined afterwards. In doing so one sees that the imaginary parts vanish again. The reason is that the imaginary unit in each of these parts (up to some general imaginary factor for the derivatives) is always accompained by a factor $c_s := \texttt{k4} * \texttt{c4} = 2\sin(a_t k_4/2)\, 2\cos(a_t k_4/2)$ where k_4 is the fourth component of the loop momentum. An odd power of such a term vanishes because of the symmetry of the integration. Therefore, the imaginary parts vanish altogether (which of course should not lead to the erroneous assumption that one can skip the single imaginary parts from the very beginning). Different cases are considered in the following.

Treatment of imaginary parts for the tadpole contributions

Because there is no denominator, the imaginary part can be skipped, since they are odd in c_i.

Treatment of imaginary parts for the rainbow contributions

The result of this calculation can be written as

$$\frac{n_r+ic_in_i}{d_r+ic_id_i}=\frac{d_rn_r+c_i^2d_in_i+ic_i(d_rn_i-d_in_r)}{d_r^2+c_i^2d_i^2}. \tag{6.251}$$

The second part of the numerator vanishes because of the symmetry property, and one has to look only on the first part and the denominator using

$$\begin{aligned} c_i^2 &= (\mathtt{k4}*\mathtt{c4})^2 = 2\sin^2\left(\frac{a_tk_4}{2}\right)\,2\cos^2\left(\frac{a_tk_4}{2}\right) = \\ &= 4\sin^2\left(\frac{a_tk_4}{2}\right)\left(1-\sin^2\left(\frac{a_tk_4}{2}\right)\right) = \mathtt{sk4}(4-\mathtt{sk4})/4 =: \mathtt{relf}. \end{aligned} \tag{6.252}$$

Treatment of imaginary parts for the derivative of the rainbow numerator

Here one has the same situation as in the previous case except for the general factor ia_s which one has to keep track of.

Treatment of imaginary parts for the derivative of the rainbow denominator

The derivative of the denominator gives rise to three different contributions, namely the numerator $n_r+ic_in_i$, the derivative of the denominator $ia_s(g_r+ic_ig_i)$, and the denominator itself, $d_r+ic_id_i$. Up to a general factor ia_s one obtains

$$\begin{aligned} \frac{(g_r+ic_ig_i)(n_r+ic_in_i)}{(d_r+ic_id_i)^2} &= \frac{(d_r-ic_id_i)^2(g_r+ic_ig_i)(n_r+ic_in_i)}{(d_r^2+c_i^2d_i^2)^2} = \\ &= \frac{(d_r-2ic_id_rd_i-c_i^2d_i^2)(g_rn_r+ic_ig_in_r+ic_ig_rn_i-c_i^2g_in_i)}{(d_i^2+c_i^2d_i^2)^2} = \\ &= \frac{(d_r^2-c_i^2d_i^2)(g_rn_r-c_i^2g_in_i)+2c_i^2d_rd_i(g_rn_i+g_in_r)}{(d_r^2+c_i^2d_i^2)^2} + \\ &\quad +ic_i\frac{(d_r^2-c_i^2d_i^2)(g_rn_i+g_in_r)-2d_rd_i(g_rn_r-c_i^2g_in_i)}{(d_r^2+c_i^2d_i^2)^2}. \end{aligned} \tag{6.253}$$

All these calculations were automatized up to the very end. The FORTRAN input code is generated by MATHEMATICA, the quark and gluon parts are joined and general settings are added, as well as IR divergence subtraction features as being necessary for the calculation of the wave function renormalization. After compilation, this program is run for different physical mass parameters and lattice spacing ratios. The output is then used to automatically generate the corresponding tables for the pole mass and wave function renormalization as well as the speed-of-light correction (see Ref. [209]).

6.5 Staggered quarks

While the Wilson method to create quark actions (as shown and used in the previous sections) is quite involved, the method to discretize QCD by *staggering quark actions* is the simplest way to obtain an improvement of the lattice spacing behaviour of lattice QCD starting from the naive quark action in Eq. (6.11). The reason why this action was disfavoured for a long time was the fact that the quarks always appear in groups of four identical quark flavours. The multiplicity of flavours is not a real problem, it can be easily adjusted in simulations. The real problem consists in the occurence of flavour changing interactions which greatly complicate the interpretation of the simulations and has to be understood more deeply.

The question why it is worthwile to spend effort on this kind is answered by a couple of advantages of the staggered quark quantization. Besides the easy construction, the application of staggered quarks for light flavours can use the fact that in close analogy to the continuum, chiral symmetry prohibits additive mass renormalizations and implies that errors caused by a nonzero grid spacing a are automatically quadratic in a, rather than linear as in Wilson's formulation. Fortunately, the above mentioned problem has got solved. It could be shown that the flavour-changing interaction is mainly given by the one-gluon exchange between quarks [194, 215]. This interaction is a lattice artefact of order $O(a^2)$ which can be removed by a tree-level modification of the lattice action, namely the *fattening of the link operators*, as will be detailed in the following. The tree-level modification is proved to work for the isotropic case in Ref. [210].

6.5.1 The staggered quark action

The naive quark action in Eq. (6.11) has an exact symmetry under the transformation

$$\psi(x) \to \psi'(x) = i\gamma_5\gamma_\rho(-1)^{x_\rho/a_\rho}\psi(x) = i\gamma_5\gamma_\rho \exp(ix_\rho\pi/a_\rho)\psi(x) \tag{6.254}$$

This symmetry is called *"doubling symmetry"*. Therefore, any low energy momentum mode $\psi(x)$ of the theory is equivalent to another mode $\psi'(x)$ with momentum $p_\rho \approx \pi/a_\rho$, the maximally allowed momentum on the lattice in the direction ρ. This new mode is one of the "doublers" of the naive quark. The doubling transformation can be appied successively in all directions, the general transformation is given by

$$\psi(x) \to \psi'(x) := \prod_\rho (i\gamma_5\gamma_\rho)^{\zeta_\rho} \exp\left(i\zeta_\rho x_\rho \pi/a_\rho\right)\psi(x) \tag{6.255}$$

where ζ is a vector with one or more components equal to 1 and all the others equal to 0. Consequently, there are 15 doublers in four dimensions, the whole amount has to be interpreted as sixteen equivalent quark flavours. As it is shown in Ref. [210], the fifteen redundant flavours can be removed again by replacing the link operator $U_\mu(x)$ in the naive quark action (6.11) by the *fat link*

$$V_\mu(x) := \left(1 + \sum_\zeta (1-\zeta_\mu)c(\zeta^2)P(\zeta)\right) U_\mu(x) \tag{6.256}$$

where $c(\zeta^2) = 1 + O(\alpha_s(\pi/a))$ and where

$$P(\zeta) := \prod_\rho \left(\frac{\nabla_\rho^{(2)}}{4}\right)^{\zeta_\rho}\Bigg|_{\text{symm}} \tag{6.257}$$

is a correction of lattice artefacts of even order in a_ρ, symmetrized over all possible orderings of the operators. At tree level, $c(\zeta^2)$ can be omitted. The corrections to $V_\mu(x)$ with $\zeta_\mu = 1$ are dropped explicitly since the other parts of the corresponding quark-gluon vertex vanish when the gluon has momentum $q_\mu = \pi/a_\mu$, as is the case in the naive action. At tree level one therefore obtains

$$V_\mu(x) = \left(1 + \sum_\zeta \prod_{\rho\neq\mu} \left(\frac{\nabla_\rho^{(2)}}{4}\right)^{\zeta_\rho}\right)\Bigg|_{\mathrm{symm}} U_\mu(x) = \prod_{\rho\neq\mu}\left(1 + \frac{1}{4}\nabla_\rho^{(2)}\right)\Bigg|_{\mathrm{symm}} U_\mu(x). \qquad (6.258)$$

Using $V_\mu(x)$ instead of $U_\mu(x)$ in the derivative contained in the naive quark action (6.11), all flavour-changing interactions of the order $O(a^2)$ are removed. It is straightforward to remove the remaining lattice artefacts in that order to obtain an $O(a^2)$ Symanzik improved quark action. First, the flavour-conserving lattice artefacts of order $O(a^2)$ due to the $\zeta^2 = 1$ parts of V_μ are removed by

$$V_\mu(x) \quad \to \quad V'_\mu(x) := V_\mu(x) - \sum_{\rho\neq\mu} \frac{(\nabla_\rho)^2}{4} U_\mu(x). \qquad (6.259)$$

As will be seen later on, these corrections cancel the low-energy effects of the single operators $\Delta_\rho^{(2)}$ without affecting their high-momentum behaviour and therefore are called *low energy corrections*, the derivative ∇_μ in the naive quark action is changed to ∇'_μ. Second, the discretization of the derivative through $O(a^2)$ is corrected by replacing

$$\nabla'_\mu \quad \to \quad \left(\nabla'_\mu - \frac{1}{6}(\nabla_\mu)^3\right) \qquad (6.260)$$

in the action [216, 217] (see Eq. (6.313)). This correction is called *Naik term* (note that the replacement of $U_\mu(x)$ by $V'_\mu(x)$ is not necessary for the $(\nabla_\mu)^3$ term).

6.5.2 ξ functions for the fat link

The construction of the fat link $V_\mu(x)$ calls for the calculation of multiple derivatives up to the third order which are symmetrized with respect to all occuring indices. In the following the symmetrization will be indicated by round brackets enclosing the indices. For these operators, the ξ functions have to be calculated up to the two-gluon exchange in order to obtain Feynman rules for the one-loop correction.

The no-gluon vertex component

Using the formalism developed in Sec. 6.2, the no-gluon contributions are given by

$$\begin{aligned}
\xi^{(0)}(U_\mu) &= 1, \\
\xi^{(0)}\left(\frac{1}{4}\nabla_{(\rho)}^{(2)} U_\mu\right) &= \xi^{(0)}\left(\frac{1}{4}\nabla_\rho^{(2)} U_\mu\right) = 0, \\
\xi^{(0)}\left(\frac{1}{16}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2)}^{(2)} U_\mu\right) &= \xi^{(0)}\left(\frac{1}{16}\nabla_{\rho_1}^{(2)}\nabla_{\rho_2}^{(2)} U_\mu\right) = 0, \\
\xi^{(0)}\left(\frac{1}{64}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2}^{(2)}\nabla_{\rho_3)}^{(2)} U_\mu\right) &= \xi^{(0)}\left(\frac{1}{64}\nabla_{\rho_1}^{(2)}\nabla_{\rho_2}^{(2)}\nabla_{\rho_3}^{(2)} U_\mu\right) = 0.
\end{aligned} \qquad (6.261)$$

The no-gluon component for the fat link is therefore given by $\xi^{(0)}(V_\mu) = 1$. Going from here to the ξ function of the action part, one obtains

$$\xi^{(0)}\left(\bar{\psi}\nabla'_\mu\psi, p\right) = i\sin(p_\mu)\xi^{(0)}(V_\mu) = i\sin(p_\mu) \tag{6.262}$$

(cf. Eq. (6.71)) where the prime throughout this subsection indicates *only* the fattening without the low energy correction which is treated separately.

The one-gluon vertex component

The contributions with one gluon line attached contain the momentum q and the Lorentz index ν of the gluon. The contributions are given by

$$\begin{aligned}
\xi^{(1)}\left(U_\mu; q, \nu\right) &= \xi_0^{(1)}\delta_{\mu\nu}, \\
\xi^{(1)}\left(\frac{1}{4}\nabla_\rho^{(2)}U_\mu; q, \nu\right) &= \xi_0^{(1)}\delta_{\mu\nu} + \xi_1^{(1)}\delta_{\rho\nu}, \\
\xi^{(1)}\left(\frac{1}{16}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2)}^{(2)}U_\mu; q, \nu\right) &= \xi_0^{(1)}\delta_{\mu\nu} + \sum_{i=1}^{2}\xi_i^{(1)}\delta_{\rho_i\nu}, \\
\xi^{(1)}\left(\frac{1}{64}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2}^{(2)}\nabla_{\rho_3)}^{(2)}U_\mu; q, \nu\right) &= \xi_0^{(1)}\delta_{\mu\nu} + \sum_{i=1}^{3}\xi_i^{(1)}\delta_{\rho_i\nu}
\end{aligned} \tag{6.263}$$

where

$$\begin{aligned}
\xi_0^{(1)}\left(U_\mu; q\right) &= 1, \\
\xi_0^{(1)}\left(\frac{1}{4}\nabla_\rho^{(2)}U_\mu; q\right) &= -\sin^2\left(\frac{q_\rho}{2}\right), \\
\xi_1^{(1)}\left(\frac{1}{4}\nabla_\rho^{(2)}U_\mu; q\right) &= \sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\rho}{2}\right), \\
\xi_0^{(1)}\left(\frac{1}{16}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2)}^{(2)}U_\mu; q\right) &= \sin^2\left(\frac{q_{\rho_1}}{2}\right)\sin^2\left(\frac{q_{\rho_2}}{2}\right), \\
\xi_i^{(1)}\left(\frac{1}{16}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2)}^{(2)}U_\mu; q\right) &= -\frac{1}{2}\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_{\rho_1}}{2}\right)\sin^2\left(\frac{q_{\rho_2}}{2}\right), \\
\xi_0^{(1)}\left(\frac{1}{64}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2}^{(2)}\nabla_{\rho_3)}^{(2)}U_\mu; q\right) &= -\sin^2\left(\frac{q_{\rho_1}}{2}\right)\sin^2\left(\frac{q_{\rho_2}}{2}\right)\sin^2\left(\frac{q_{\rho_3}}{2}\right), \\
\xi_i^{(1)}\left(\frac{1}{64}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2}^{(2)}\nabla_{\rho_3)}^{(2)}U_\mu; q\right) &= \frac{1}{3}\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_{\rho_1}}{2}\right)\sin^2\left(\frac{q_{\rho_2}}{2}\right)\sin^2\left(\frac{q_{\rho_3}}{2}\right).
\end{aligned} \tag{6.264}$$

In calculating

$$\begin{aligned}
\xi^{(1)}\left(V_\mu; q, \nu\right) &= \xi^{(1)}\left(U_\mu; q, \nu\right) + \sum_{\rho\neq\mu}{}' \xi^{(1)}\left(\frac{1}{4}\nabla_\rho^{(2)}U_\mu; q, \nu\right) + \\
&+\frac{1}{2}\sum_{\rho_i\neq\mu}{}' \xi^{(1)}\left(\frac{1}{16}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2)}^{(2)}U_\mu; q, \nu\right) + \frac{1}{6}\sum_{\rho_i\neq\mu}{}' \xi^{(1)}\left(\frac{1}{64}\nabla_{(\rho_1}^{(2)}\nabla_{\rho_2}^{(2)}\nabla_{\rho_3)}^{(2)}U_\mu; q, \nu\right),
\end{aligned} \tag{6.265}$$

two cases have to be distinguished, namely the case $\nu = \mu$ and the case $\nu \neq \mu$. The primed sums $\sum'_{\rho_i\neq\sigma}$ run over all occuring indices ρ_i distinct from each other and from σ.

The one-gluon vertex component (μ)

Using the $\xi_0^{(1)}$ parts, for the first case one obtains

$$\begin{aligned}
\xi^{(1)}(V_\mu;q,\nu) \;&=\; 1-{\sum_{\rho}}'\sin^2\left(\frac{q_\rho}{2}\right)+\frac{1}{2}{\sum_{\rho_i}}'\sin^2\left(\frac{q_{\rho_1}}{2}\right)\sin^2\left(\frac{q_{\rho_2}}{2}\right)+\\
&\qquad -\frac{1}{6}{\sum_{\rho_i}}'\sin^2\left(\frac{q_{\rho_1}}{2}\right)\sin^2\left(\frac{q_{\rho_2}}{2}\right)\sin^2\left(\frac{q_{\rho_3}}{2}\right) \;=\\
&= 1-\sin^2\left(\frac{q_{\nu_1}}{2}\right)-\sin^2\left(\frac{q_{\nu_2}}{2}\right)-\sin^2\left(\frac{q_{\nu_3}}{2}\right)+\sin^2\left(\frac{q_{\nu_1}}{2}\right)\sin^2\left(\frac{q_{\nu_2}}{2}\right)+\\
&\quad+\sin^2\left(\frac{q_{\nu_1}}{2}\right)\sin^2\left(\frac{q_{\nu_3}}{2}\right)+\sin^2\left(\frac{q_{\nu_2}}{2}\right)\sin^2\left(\frac{q_{\nu_3}}{2}\right)-\sin^2\left(\frac{q_{\nu_1}}{2}\right)\sin^2\left(\frac{q_{\nu_2}}{2}\right)\sin^2\left(\frac{q_{\nu_3}}{2}\right) \;=\\
&= \left(1-\sin^2\left(\frac{q_{\nu_1}}{2}\right)\right)\left(1-\sin^2\left(\frac{q_{\nu_2}}{2}\right)\right)\left(1-\sin^2\left(\frac{q_{\nu_3}}{2}\right)\right) \;=\\
&= \cos^2\left(\frac{q_{\nu_1}}{2}\right)\cos^2\left(\frac{q_{\nu_2}}{2}\right)\cos^2\left(\frac{q_{\nu_3}}{2}\right). \qquad (6.266)
\end{aligned}$$

The ξ function for the action is given by

$$\begin{aligned}
\xi^{(1)}\left(\bar\psi\nabla'_\mu\psi,p;q,\mu\right) \;&=\; ig\cos(p_\mu)\xi^{(1)}(V_\mu;q,\mu) \;=\\
&=\; ig\cos(p_\mu)\cos^2\left(\frac{q_{\nu_1}}{2}\right)\cos^2\left(\frac{q_{\nu_2}}{2}\right)\cos^2\left(\frac{q_{\nu_3}}{2}\right). \qquad (6.267)
\end{aligned}$$

The one-gluon vertex component (ν)

For the second case $(\nu\neq\mu)$ the summation reads

$$\begin{aligned}
\xi^{(1)}(V_\mu;q,\nu) \;&=\; \sum_{\rho\neq\mu}\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\rho}{2}\right)\delta_{\rho\nu}+\\
&\quad-\frac{1}{4}{\sum_{\rho_i\neq\mu}}'\sum_{i=1}^{2}\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_{\rho_i}}{2}\right)\sin^2\left(\frac{q_{\rho_j}}{2}\right)\delta_{\rho_i\nu}\;+\\
&\quad+\frac{1}{18}{\sum_{\rho_i\neq\mu}}'\sum_{i=1}^{3}\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_{\rho_i}}{2}\right)\sin^2\left(\frac{q_{\rho_j}}{2}\right)\sin^2\left(\frac{q_{\rho_k}}{2}\right)\delta_{\rho_i\nu} \;=\\
&= \sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)-\frac{1}{2}\sum_{\rho_i\neq\mu,\nu}\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\sin^2\left(\frac{q_\rho}{2}\right)+\\
&\quad+\frac{1}{6}{\sum_{\rho_i\neq\mu,\nu}}'\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\sin^2\left(\frac{q_{\rho_2}}{2}\right)\sin^2\left(\frac{q_{\rho_3}}{2}\right) \;=\\
&= \sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\left(1-\frac{1}{2}{\sum_{\rho_i\neq\mu,\nu}}'\sin^2\left(\frac{q_{\rho_1}}{2}\right)+\frac{1}{6}{\sum_{\rho_i\neq\mu,\nu}}'\sin^2\left(\frac{q_{\rho_1}}{2}\right)\sin^2\left(\frac{q_{\rho_2}}{2}\right)\right) \;=\\
&= \sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\left(1-\frac{1}{2}\sin^2\left(\frac{q_{\nu_1}}{2}\right)-\frac{1}{2}\sin^2\left(\frac{q_{\nu_1}}{2}\right)+\frac{1}{3}\sin^2\left(\frac{q_{\nu_1}}{2}\right)\sin^2\left(\frac{q_{\nu_1}}{2}\right)\right) \;=\\
&= \frac{1}{6}\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\left(2+\cos^2\left(\frac{q_{\nu_1}}{2}\right)+\cos^2\left(\frac{q_{\nu_2}}{2}\right)+2\cos^2\left(\frac{q_{\nu_1}}{2}\right)\cos^2\left(\frac{q_{\nu_2}}{2}\right)\right).
\end{aligned}$$
$$(6.268)$$

Here the ξ function for the action reads

$$\begin{aligned}\xi^{(1)}\left(\bar{\psi}\nabla'_\mu\psi, p; q, \nu\right) &= ig\cos(p_\mu)\xi^{(1)}(V_\mu; q, \nu) = \frac{ig}{6}\cos(p_\mu)\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\times \\ &\times\left(2+\cos^2\left(\frac{q_{\nu_1}}{2}\right)+\cos^2\left(\frac{q_{\nu_2}}{2}\right)+2\cos^2\left(\frac{q_{\nu_1}}{2}\right)\cos^2\left(\frac{q_{\nu_2}}{2}\right)\right).\end{aligned} \tag{6.269}$$

The two-gluon vertex components

The results for contributions including gluons with quantum numbers (q_1, ν_1) and (q_2, ν_2) can be organized as

$$\begin{aligned}
&\xi^{(2)}\left(U_\mu; q_1, \nu_1; q_2, \nu_2\right) = \xi^{(2)}_{00}\delta_{\mu\nu_1}\delta_{\mu\nu_2}, \\
&\xi^{(2)}\left(\frac{1}{4}\nabla^{(2)}_\rho U_\mu; q_1, \nu_1; q_2, \nu_2\right) = \xi^{(2)}_{00}\delta_{\mu\nu_1}\delta_{\mu\nu_2}+\xi^{(2)}_{01}\delta_{\mu\nu_1}\delta_{\rho\nu_2}+\xi^{(2)}_{10}\delta_{\rho\nu_1}\delta_{\mu\nu_2}+\xi^{(2)}_{11}\delta_{\rho\nu_1}\delta_{\rho\nu_2}, \\
&\xi^{(2)}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu; q_1, \nu_1; q_2, \nu_2\right) = \\
&= \xi^{(2)}_{00}\delta_{\mu\nu_1}\delta_{\mu\nu_2}+\sum_{j=1}^{2}\xi^{(2)}_{0j}\delta_{\mu\nu_1}\delta_{\rho_j\nu_2}+\sum_{i=1}^{2}\xi^{(2)}_{i0}\delta_{\rho_i\nu_1}\delta_{\mu\nu_2}+\sum_{i,j=1}^{2}\xi^{(2)}_{ij}\delta_{\rho_i\nu_1}\delta_{\rho_j\nu_2}, \\
&\xi^{(2)}\left(\frac{1}{64}\nabla^{(2)}_{\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3}U_\mu; q_1, \nu_1; q_2, \nu_2\right) = \\
&= \xi^{(2)}_{00}\delta_{\mu\nu_1}\delta_{\mu\nu_2}+\sum_{j=1}^{3}\xi^{(2)}_{0j}\delta_{\mu\nu_1}\delta_{\rho_j\nu_2}+\sum_{i=1}^{3}\xi^{(2)}_{i0}\delta_{\rho_i\nu_1}\delta_{\mu\nu_2}+\sum_{i,j=1}^{3}\xi^{(2)}_{ij}\delta_{\rho_i\nu_1}\delta_{\rho_j\nu_2}.
\end{aligned} \tag{6.270}$$

For no derivative one obtains

$$\xi^{(2)}_{00}\left(U_\mu; q_1; q_2\right) = \frac{1}{2}, \tag{6.271}$$

for one derivative one gets

$$\begin{aligned}
\xi^{(2)}_{00}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu; q_1; q_2\right) &= -\frac{1}{2}\sin^2\left(\frac{q_{1\rho_1}+q_{2\rho_1}}{2}\right), \\
\xi^{(2)}_{01}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu; q_1; q_2\right) &= \frac{i}{2}e^{-iq_{2\mu}/2}\sin\left(q_{1\rho_1}+\frac{q_{2\rho_1}}{2}\right), \\
\xi^{(2)}_{10}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu; q_1; q_2\right) &= -\frac{i}{2}e^{iq_{1\mu}/2}\sin\left(\frac{q_{1\rho_1}}{2}+q_{2\rho_1}\right), \\
\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu; q_1; q_2\right) &= \frac{1}{2}\left(\cos\left(\frac{q_{1\mu}+q_{2\mu}}{2}\right)-e^{iq_{1\mu}/2-iq_{2\mu}/2}\right)\cos\left(\frac{q_{1\rho_1}+q_{2\rho_1}}{2}\right).
\end{aligned} \tag{6.272}$$

For two derivatives the result reads

$$\begin{aligned}
&\xi^{(2)}_{00}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu; q_1; q_2\right) = -\xi^{(2)}_{00}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu; q_1; q_2\right)\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right), \\
&\xi^{(2)}_{0i}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu; q_1; q_2\right) = \\
&= -\frac{1}{2}\xi^{(2)}_{01}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu; q_1; q_2\right)\left(\sin^2\left(\frac{q_{1\rho_j}}{2}\right)+\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right)\right), \\
&\xi^{(2)}_{i0}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu; q_1; q_2\right) =
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\xi^{(2)}_{10}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\left(\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right)+\sin^2\left(\frac{q_{2\rho_j}}{2}\right)\right), \\
\xi^{(2)}_{ii}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu;q_1;q_2\right) &= -\frac{1}{2}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right), \\
\xi^{(2)}_{ij}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu;q_1;q_2\right) &= \frac{1}{2}\xi^{(2)}_{10}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\sin\left(\frac{q_{2\mu}}{2}\right)\sin\left(\frac{q_{2\rho_j}}{2}\right)+ \\
&\quad +\frac{1}{2}\xi^{(2)}_{01}\left(\frac{1}{4}\nabla^{(2)}_{\rho_j}U_\mu;q_1;q_2\right)\sin\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{1\rho_i}}{2}\right)
\end{aligned}
\tag{6.273}
$$

where (i,j) is a cyclic permutation of $(1,2)$, and for three derivatives one obtains

$$
\begin{aligned}
&\xi^{(2)}_{00}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1;q_2\right) = \\
&= \xi^{(2)}_{00}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right)\sin^2\left(\frac{q_{1\rho_k}+q_{2\rho_k}}{2}\right), \\
&\xi^{(2)}_{0i}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1;q_2\right) = \frac{1}{6}\xi^{(2)}_{01}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\times \\
&\quad\times\left\{\sin^2\left(\frac{q_{1\rho_j}}{2}\right)\sin^2\left(\frac{q_{1\rho_k}}{2}\right)+\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right)\sin^2\left(\frac{q_{1\rho_k}+q_{2\rho_k}}{2}\right)+\right. \\
&\quad\left.+\left(\sin^2\left(\frac{q_{1\rho_j}}{2}\right)+\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right)\right)\left(\sin^2\left(\frac{q_{1\rho_k}}{2}\right)+\sin^2\left(\frac{q_{1\rho_k}+q_{2\rho_k}}{2}\right)\right)\right\}, \\
&\xi^{(2)}_{i0}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1;q_2\right) = \frac{1}{6}\xi^{(2)}_{10}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\times \\
&\quad\times\left\{\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right)\sin^2\left(\frac{q_{1\rho_k}+q_{2\rho_k}}{2}\right)+\sin^2\left(\frac{q_{2\rho_j}}{2}\right)\sin^2\left(\frac{q_{2\rho_k}}{2}\right)+\right. \\
&\quad\left.+\left(\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right)+\sin^2\left(\frac{q_{2\rho_j}}{2}\right)\right)\left(\sin^2\left(\frac{q_{1\rho_k}+q_{2\rho_k}}{2}\right)+\sin^2\left(\frac{q_{2\rho_k}}{2}\right)\right)\right\}, \\
&\xi^{(2)}_{ii}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1;q_2\right) = \\
&= \frac{1}{3}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\sin^2\left(\frac{q_{1\rho_j}+q_{2\rho_j}}{2}\right)\sin^2\left(\frac{q_{1\rho_k}+q_{2\rho_k}}{2}\right), \\
&\xi^{(2)}_{ij}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1;q_2\right) = \\
&= -\frac{1}{6}\xi^{(2)}_{10}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\sin\left(\frac{q_{2\mu}}{2}\right)\sin\left(\frac{q_{2\rho_j}}{2}\right)\left(\sin^2\left(\frac{q_{1\rho_k}+q_{2\rho_k}}{2}\right)+\sin^2\left(\frac{q_{2\rho_k}}{2}\right)\right)+ \\
&\quad -\frac{1}{6}\xi^{(2)}_{01}\left(\frac{1}{4}\nabla^{(2)}_{\rho_j}U_\mu;q_1;q_2\right)\sin\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{1\rho_i}}{2}\right)\left(\sin^2\left(\frac{q_{1\rho_k}}{2}\right)+\sin^2\left(\frac{q_{1\rho_k}+q_{2\rho_k}}{2}\right)\right)
\end{aligned}
\tag{6.274}
$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$. For the calculation of the fat link ξ function $\xi^{(2)}(V_\mu;q_1,\nu_1;q_2,\nu_2)$, several cases have to be considered.

The two-gluon vertex component $(\mu\mu)$

The ξ function for V_μ is given by

$$
\xi^{(2)}(V_\mu;q_1,\mu;q_2,\mu) = \xi^{(2)}(U_\mu;q_1,\mu;q_2,\mu)+\sum_{\rho_i\neq\mu}{}' \xi^{(2)}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu;q_1,\mu;q_2,\mu\right)+
$$

$$+\frac{1}{2}\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu;q_1,\mu;q_2,\mu\right)+\frac{1}{6}\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1,\mu;q_2,\mu\right) =$$
$$= \xi^{(2)}_{00}(U_\mu;q_1;q_2)+\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}_{00}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu;q_1;q_2\right)+\frac{1}{2}\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}_{00}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu;q_1;q_2\right)+$$
$$+\frac{1}{6}\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}_{00}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1;q_2\right)+\ldots =$$
$$= \frac{1}{2}-\frac{1}{2}\sum_{\rho_i\neq\mu}^{\prime}\sin^2(q_{\rho_1})+\frac{1}{4}\sum_{\rho_i\neq\mu}^{\prime}\sin^2(q_{\rho_i})\sin^2(q_{\rho_j})-\frac{1}{12}\sum_{\rho_i\neq\mu}^{\prime}\sin^2(q_{\rho_i})\sin^2(q_{\rho_j})\sin^2(q_{\rho_k}) =$$
$$= \frac{1}{2}-\frac{1}{2}\sum_{\rho_i\neq\mu}^{\prime}\sin^2(q_{\rho_1})+\frac{1}{2}\sum_{\rho_i\neq\mu}^{\prime}\sin^2(q_{\rho_1})\sin^2(q_{\rho_2})-\frac{1}{2}\sum_{\rho_i\neq\mu}^{\prime}\sin^2(q_{\rho_1})\sin^2(q_{\rho_2})\sin^2(q_{\rho_3}) =$$
$$= \frac{1}{2}\prod_{i=1}^{3}\left(1-\sin^2(q_{\rho_i})\right) = \frac{1}{2}\prod_{i=1}^{3}\cos^2(q_{\rho_i})$$

where the ellipses indicate components for ν_1 or ν_2 unequal to μ. In this calculation the short-hand notation $q=(q_1+q_2)/2$ is used. The result on the level of the action reads

$$\xi^{(2)}(\bar\psi\nabla'_\mu\psi,p;q_1,\mu;q_2,\mu)=\frac{ig^2}{2}\sin(p_\mu)\cos^2(q_{\nu_1})\cos^2(q_{\nu_2})\cos^2(q_{\nu_3}). \qquad (6.275)$$

The two-gluon vertex diagonal components ($\nu\nu$)

Next, one considers the case where the (common) Lorentz index ν of the gluon legs is different from the Lorentz index μ of the Dirac gamma matrix. In this case one obtains

$$\xi^{(2)}(V_\mu;q_1,\nu;q_2,\nu) = \xi^{(2)}(U_\mu;q_1,\nu;q_2,\nu)+\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu;q_1,\nu;q_2,\nu\right)+$$
$$+\frac{1}{2}\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu;q_1,\nu;q_2,\nu\right)+\frac{1}{6}\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1,\nu;q_2,\nu\right) =$$
$$= \sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu;q_1;q_2\right)\delta_{\rho_1\nu}-\frac{1}{4}\sum_{\rho_i\neq\mu}^{\prime}\sum_{i=1}^{2}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\sin^2(q_{\rho_j})\delta_{\rho_i\nu}+$$
$$+\frac{1}{18}\sum_{\rho_i\neq\mu}^{\prime}\sum_{i=1}^{3}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_i}U_\mu;q_1;q_2\right)\sin^2(q_{\rho_j})\sin^2(q_{\rho_k})\delta_{\rho_i\nu} =$$
$$= \sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu;q_1;q_2\right)\delta_{\rho_1\nu}-\frac{1}{2}\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu;q_1;q_2\right)\sin^2(q_{\rho_2})\delta_{\rho_1\nu}+$$
$$+\frac{1}{6}\sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu;q_1;q_2\right)\sin^2(q_{\rho_2})\sin^2(q_{\rho_3})\delta_{\rho_1\nu} = \qquad (6.276)$$
$$= \sum_{\rho_i\neq\mu}^{\prime}\xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1}U_\mu;q_1;q_2\right)\delta_{\rho_1\nu}\left\{1-\frac{1}{2}\sum_{\rho_i\neq\rho_1}^{\prime}\sin^2(q_{\rho_2})+\frac{1}{6}\sum_{\rho_i\neq\rho_1}^{\prime}\sin^2(q_{\rho_2})\sin^2(q_{\rho_3})\right\}.$$

In the second step of the previous calculation the sums over i were done explicitly, but then the indices ρ_1 and ρ_2 resp. ρ_1, ρ_2 and ρ_3 were interchanged, resulting in two or three identical expressions, resp. Therefore the factors were cancelled. There are finally two possibilities left for each of the two sums in the parantheses (while $\rho_1 = \nu$, ρ_2 and ρ_3 can take two possible values different from μ and ν). If one takes ν_1 and ν_2 to be these values, one ends up with

$$\begin{aligned}
&\xi^{(2)}(V_\mu; q_1, \nu; q_2, \nu) = \\
&= \xi^{(2)}_{11}\left(\frac{1}{4}\nabla^{(2)}_\nu U_\mu; q_1; q_2\right)\left\{1 - \frac{1}{2}\sin^2(q_{\nu_1}) - \frac{1}{2}\sin^2(q_{\nu_2}) + \frac{1}{3}\sin^2(q_{\nu_1})\sin^2(q_{\nu_2})\right\} = \\
&= \frac{1}{2}\left(\cos(q_\mu) - e^{iq_{1\mu}/2 - iq_{2\mu}/2}\right)\cos(q_\nu) \;\times \\
&\qquad \times \left\{1 - \frac{1}{2}\sin^2(q_{\nu_1}) - \frac{1}{2}\sin^2(q_{\nu_2}) + \frac{1}{3}\sin^2(q_{\nu_1})\sin^2(q_{\nu_2})\right\}.
\end{aligned} \tag{6.277}$$

After using trigonometric relations, one obtains

$$\begin{aligned}
1 - \frac{1}{2}\sin^2(q_{\nu_1}) - \frac{1}{2}\sin^2(q_{\nu_2}) + \frac{1}{3}\sin^2(q_{\nu_1})\sin^2(q_{\nu_2}) &= \\
= 1 - \frac{1}{2} + \frac{1}{2}\cos^2(q_{\nu_1}) - \frac{1}{2} + \frac{1}{2}\cos^2(q_{\nu_2}) &+ \\
+\frac{1}{3} - \frac{1}{3}\cos^2(q_{\nu_1}) - \frac{1}{3}\cos^2(q_{\nu_2}) + \frac{1}{3}\cos^2(q_{\nu_1})\cos^2(q_{\nu_2}) &= \\
= \frac{1}{3} + \frac{1}{6}\cos^2(q_{\nu_1}) + \frac{1}{6}\cos^2(q_{\nu_2}) + \frac{1}{3}\cos^2(q_{\nu_1})\cos^2(q_{\nu_2}) &= \\
= \frac{1}{6}\left(2 + \cos^2(q_{\nu_1}) + \cos^2(q_{\nu_2}) + 2\cos^2(q_{\nu_1})\cos^2(q_{\nu_2})\right)&,
\end{aligned} \tag{6.278}$$

such that

$$\xi^{(2)}(V_\mu; q_1, \nu; q_2, \nu) = \frac{1}{12}\left(\cos(q_\mu) - e^{iq_{1\mu}/2 - iq_{2\mu}/2}\right)\cos(q_\nu) \;\times \tag{6.279}$$

$$\begin{aligned}
&\times \left\{2 + \cos^2(q_{\nu_1}) + \cos^2(q_{\nu_2}) + 2\cos^2(q_{\nu_1})\cos^2(q_{\nu_2})\right\} = \\
&= \frac{1}{12}\left(\cos(q_\mu) - e^{iq_{1\mu}/2 - iq_{2\mu}/2}\right)\cos(q_\nu) C^{(\nu\nu)}_{\nu_1\nu_2}
\end{aligned}$$

where

$$C^{(\nu\nu)}_{\nu_1\nu_2} := 2 + \cos^2(q_{\nu_1}) + \cos^2(q_{\nu_2}) + 2\cos^2(q_{\nu_1})\cos^2(q_{\nu_2}) \tag{6.280}$$

is used. All factors except the first (non-trivial) one are symmetric under the interchange $(q_1, \mu) \leftrightarrow (q_2, \mu)$. Therefore, according to Eq. (6.71),

$$\begin{aligned}
\xi^{(2)}(\bar\psi \nabla'_\mu \psi, p; q_1, \nu; q_2, \nu) = &-\frac{ig^2}{6}\sin(p_\mu)\sin\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{2\mu}}{2}\right)\cos(q_\nu)C^{(\nu\nu)}_{\nu_1\nu_2} + \\
&+\frac{ig^2}{12}\cos(p_\mu)\sin\left(\frac{q_{1\mu} - q_{2\mu}}{2}\right)\cos(q_\nu)C^{(\nu\nu)}_{\nu_1\nu_2}.
\end{aligned} \tag{6.281}$$

The two-gluon vertex components $(\mu\nu)$

Next the case is considered where one of the indices is μ, the other $\nu \neq \mu$. One obtains

$$\xi^{(2)}(V_\mu; q_1, \mu; q_2, \nu) = \xi^{(2)}(U_\mu; q_1, \mu; q_2, \nu) + \sum_{\rho_i \neq \mu}{}' \xi^{(2)}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1} U_\mu; q_1, \mu; q_2, \nu\right) +$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{\rho_i\neq\mu}{}^{'}\xi^{(2)}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)}U_\mu;q_1,\mu;q_2,\nu\right)+\frac{1}{6}\sum_{\rho_i\neq\mu}{}^{'}\xi^{(2)}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)}U_\mu;q_1,\mu;q_2,\nu\right) = \\
&= \frac{1}{6}\xi^{(2)}_{01}\left(\frac{1}{4}\nabla^{(2)}_\nu U_\mu;q_1;q_2\right)\left\{2\cos^2\left(\frac{q_{1\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}}{2}\right)+\right.\\
&\qquad +\cos^2\left(\frac{q_{1\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)+\cos^2\left(\frac{q_{1\nu_2}+q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}}{2}\right)+\\
&\qquad \left.+2\cos^2\left(\frac{q_{1\nu_2}+q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)\right\} = \\
&= \frac{i}{12}e^{-iq_{2\mu}/2}\sin\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right)\left\{2\cos^2\left(\frac{q_{1\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}}{2}\right)+\right.\\
&\qquad +\cos^2\left(\frac{q_{1\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)+\cos^2\left(\frac{q_{1\nu_2}+q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}}{2}\right)+\\
&\qquad \left.+2\cos^2\left(\frac{q_{1\nu_2}+q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)\right\} = \\
&= \frac{i}{12}e^{-iq_{2\mu}/2}\sin\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right)C^{(\mu\nu)}_{\nu_1\nu_2}.
\end{aligned}
\tag{6.282}
$$

The corresponding result for the component $(\nu\mu)$ reads

$$
\xi^{(2)}(V_\mu;q_1,\nu;q_2,\mu) = -\frac{i}{12}e^{iq_{1\mu}/2}\sin\left(\frac{q_{1\nu}}{2}+q_{2\nu}\right)C^{(\nu\mu)}_{\nu_1\nu_2}
\tag{6.283}
$$

where the relations

$$
\begin{aligned}
C^{(\mu\nu)}_{\nu_1\nu_2} &= \left\{2\cos^2\left(\frac{q_{1\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}}{2}\right)+\cos^2\left(\frac{q_{1\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)+\right.\\
&\quad \left.+\cos^2\left(\frac{q_{1\nu_2}+q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}}{2}\right)+2\cos^2\left(\frac{q_{1\nu_2}+q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)\right\},\\
C^{(\nu\mu)}_{\nu_1\nu_2} &= \left\{2\cos^2\left(\frac{q_{1\nu_2}+q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)+\cos^2\left(\frac{q_{1\nu_2}+q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{2\nu_3}}{2}\right)+\right.\\
&\quad \left.+\cos^2\left(\frac{q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)+2\cos^2\left(\frac{q_{2\nu_2}}{2}\right)\cos^2\left(\frac{q_{2\nu_3}}{2}\right)\right\}
\end{aligned}
\tag{6.284}
$$

are used. For this case one therefore has to use two different ξ functions which, however, on interchange of arguments turn out to be equal except for the first factor,

$$
\begin{aligned}
\xi^{(2)}(V_\mu;q_1,\mu;q_2,\nu) &= \frac{i}{12}e^{-iq_{2\mu}/2}\sin\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right)C^{(\mu\nu)}_{\nu_1\nu_2},\\
\xi^{(2)}(V_\mu;q_2,\mu;q_1,\nu) &= -\frac{i}{12}e^{iq_{2\mu}/2}\sin\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right)C^{(\mu\nu)}_{\nu_1\nu_2}.
\end{aligned}
\tag{6.285}
$$

Therefore (cf. again Eq. (6.71))

$$
\begin{aligned}
\xi^{(2)}(\bar\psi\nabla'_\mu\psi,p;q_1,\mu;q_2,\nu) &= \frac{ig^2}{12}\sin(p_\mu)\sin\left(\frac{q_{2\mu}}{2}\right)\sin\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right)C^{(\mu\nu)}_{\nu_1\nu_2}+\\
&\quad -\frac{ig^2}{12}\cos(p_\mu)\cos\left(\frac{q_{2\mu}}{2}\right)\sin\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right)C^{(\mu\nu)}_{\nu_1\nu_2}.
\end{aligned}
\tag{6.286}
$$

The two-gluon vertex components $(\nu_1\nu_2)$

Here the result reads

$$\begin{aligned}
\xi^{(2)}\left(V_\mu; q_1,\nu_1; q_2,\nu_2\right) &= \xi^{(2)}(U_\mu; q_1,\nu_1; q_2,\nu_2) + \\
&+ \sum_{\rho_1\neq\mu} \xi^{(2)}\left(\frac{1}{4}\nabla^{(2)}_{\rho_1} U_\mu; q_1,\nu_1; q_2,\nu_2\right) + \frac{1}{2}\sum_{\rho_2\neq\rho_1\neq\mu} \xi^{(2)}\left(\frac{1}{16}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2)} U_\mu; q_1,\nu_1; q_2,\nu_2\right) + \\
&+ \frac{1}{6}\sum_{\rho_3\neq\rho_2\neq\rho_1\neq\mu} \xi^{(2)}\left(\frac{1}{64}\nabla^{(2)}_{(\rho_1}\nabla^{(2)}_{\rho_2}\nabla^{(2)}_{\rho_3)} U_\mu; q_1,\nu_1; q_2,\nu_2\right) = \\
&= -\frac{i}{12} e^{iq_{1\mu}/2} \sin\left(\frac{q_{1\nu_1}}{2} + q_{2\nu_1}\right) \sin\left(\frac{q_{2\mu}}{2}\right) \sin\left(\frac{q_{2\nu_2}}{2}\right) \times \\
&\qquad \times \left(1 + \cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right) + \cos^2\left(\frac{q_{2\nu_3}}{2}\right)\right) + \\
&+ \frac{i}{12} e^{-iq_{2\mu}/2} \sin\left(q_{1\nu_2} + \frac{q_{2\nu_2}}{2}\right) \sin\left(\frac{q_{1\mu}}{2}\right) \sin\left(\frac{q_{1\nu_1}}{2}\right) \times \\
&\qquad \times \left(1 + \cos^2\left(\frac{q_{1\nu_3}}{2}\right) + \cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right)\right)
\end{aligned} \tag{6.287}$$

which can be written as

$$\begin{aligned}
\xi^{(2)}(V_\mu; q_1,\nu_1; q_2,\nu_2) &= -\frac{i}{12} e^{iq_{1\mu}/2} \sin\left(\frac{q_{1\nu_1}}{2} + q_{2\nu_1}\right) \sin\left(\frac{q_{2\mu}}{2}\right) \sin\left(\frac{q_{2\nu_2}}{2}\right) C^{(\nu_1\nu_2)}_{1\nu_3} + \\
&+ \frac{i}{12} e^{-iq_{2\mu}/2} \sin\left(q_{1\nu_2} + \frac{q_{2\nu_2}}{2}\right) \sin\left(\frac{q_{1\mu}}{2}\right) \sin\left(\frac{q_{1\nu_1}}{2}\right) C^{(\nu_1\nu_2)}_{2\nu_3}, \qquad (6.288) \\
\xi^{(2)}(V_\mu; q_2,\nu_2; q_1,\nu_1) &= -\frac{i}{12} e^{iq_{2\mu}/2} \sin\left(q_{1\nu_2} + \frac{q_{2\nu_2}}{2}\right) \sin\left(\frac{q_{1\mu}}{2}\right) \sin\left(\frac{q_{1\nu_1}}{2}\right) C^{(\nu_1\nu_2)}_{2\nu_3} + \\
&+ \frac{i}{12} e^{-iq_{1\mu}/2} \sin\left(\frac{q_{1\nu_1}}{2} + q_{2\nu_1}\right) \sin\left(\frac{q_{2\mu}}{2}\right) \sin\left(\frac{q_{2\nu_2}}{2}\right) C^{(\nu_1\nu_2)}_{1\nu_3} \qquad (6.289)
\end{aligned}$$

where

$$\begin{aligned}
C^{(\nu_1\nu_2)}_{1\nu_3} &= 1 + \cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right) + \cos^2\left(\frac{q_{2\nu_3}}{2}\right), \\
C^{(\nu_1\nu_2)}_{2\nu_3} &= 1 + \cos^2\left(\frac{q_{1\nu_3}}{2}\right) + \cos^2\left(\frac{q_{1\nu_3}+q_{2\nu_3}}{2}\right).
\end{aligned} \tag{6.290}$$

Therefore, one obtains

$$\begin{aligned}
&\xi^{(2)}(\bar\psi\nabla'_\mu\psi, p; q_1,\nu_1; q_2,\nu_2) = \\
&= \frac{ig^2}{12}\sin(p_\mu)\sin\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{2\mu}}{2}\right)\sin\left(\frac{q_{1\nu_1}}{2}+q_{2\nu_1}\right)\sin\left(\frac{q_{2\nu_2}}{2}\right) C^{(\nu_1\nu_2)}_{1\nu_3} + \\
&\quad + \frac{ig^2}{12}\sin(p_\mu)\sin\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{2\mu}}{2}\right)\sin\left(\frac{q_{1\nu_1}}{2}\right)\sin\left(q_{1\nu_2}+\frac{q_{2\nu_2}}{2}\right) C^{(\nu_1\nu_2)}_{2\nu_3} + \\
&\quad + \frac{ig^2}{12}\cos(p_\mu)\cos\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{2\mu}}{2}\right)\sin\left(\frac{q_{1\nu_1}}{2}+q_{2\nu_1}\right)\sin\left(\frac{q_{2\nu_2}}{2}\right) C^{(\nu_1\nu_2)}_{1\nu_3} + \\
&\quad - \frac{ig^2}{12}\cos(p_\mu)\sin\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\sin\left(\frac{q_{1\nu_1}}{2}\right)\sin\left(q_{1\nu_2}+\frac{q_{2\nu_2}}{2}\right) C^{(\nu_1\nu_2)}_{2\nu_3}.
\end{aligned} \tag{6.291}$$

6.5.3 The low energy contributions

Low energy contributions come from $\Delta V = V' - V$, i.e. the matrix

$$\Delta V_\mu(x) := -\sum_{\rho\neq\mu} \frac{1}{4}(\nabla_\rho)^2 U_\mu(x). \tag{6.292}$$

The contributions with no and one gluon

There is no contribution from the low energy term in case of no gluon,

$$\xi^{(0)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu\right) = 0 \quad \Rightarrow \quad \xi^{(0)}\left(\bar\psi \Delta\nabla'_\mu \psi, p\right) = 0. \tag{6.293}$$

The contribution for one gluon is given by

$$\begin{aligned}
\xi^{(1)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q, \nu\right) &= \xi_0^{(1)}\delta_{\mu\nu} + \xi_1^{(1)}\delta_{\rho\nu} \quad \text{with} \\
\xi_0^{(1)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q\right) &= \sin^2\left(\frac{q_\rho}{2}\right) - \sin^4\left(\frac{q_\rho}{2}\right), \\
\xi_1^{(1)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q\right) &= -\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\rho}{2}\right) + \sin\left(\frac{q_\mu}{2}\right)\sin^3\left(\frac{q_\rho}{2}\right).
\end{aligned} \tag{6.294}$$

For the two cases $\nu = \mu$ and $\nu \neq \mu$ one obtains

$$\begin{aligned}
\xi^{(1)}\left(\Delta V_\mu; q, \mu\right) &= \sum_{\rho\neq\mu} \xi^{(1)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q, \mu\right) = \\
&= \sum_{\rho\neq\mu} \sin^2\left(\frac{q_\rho}{2}\right)\left(1 - \sin^2\left(\frac{q_\rho}{2}\right)\right) = \sum_{\rho\neq\mu} \sin^2\left(\frac{q_\rho}{2}\right)\cos^2\left(\frac{q_\rho}{2}\right) = \\
&= \frac{1}{4}\sum_{\rho\neq\mu} \sin^2(q_\rho) = \frac{1}{4}\left(\sin^2(q_{\nu_1}) + \sin^2(q_{\nu_2}) + \sin^2(q_{\nu_3})\right), \\
\xi^{(1)}\left(\Delta V_\mu; q, \nu\right) &= -\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right) \sin\left(\frac{q_\mu}{2}\right)\sin^3\left(\frac{q_\nu}{2}\right) = \\
&= -\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\left(1 - \sin^2\left(\frac{q_\nu}{2}\right)\right) = \\
&= -\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\cos^2\left(\frac{q_\nu}{2}\right).
\end{aligned} \tag{6.295}$$

Therefore, the ξ functions for the action are given by

$$\begin{aligned}
\xi^{(1)}\left(\bar\psi\Delta\nabla'_\mu\psi, p; q, \mu\right) &= \frac{ig}{4}\cos(p_\mu)\left(\sin^2(q_{\nu_1}) + \sin^2(q_{\nu_2}) + \sin^2(q_{\nu_3})\right), \\
\xi^{(1)}\left(\bar\psi\Delta\nabla'_\mu\psi, p; q, \nu\right) &= -ig\cos(p_\mu)\sin\left(\frac{q_\mu}{2}\right)\sin\left(\frac{q_\nu}{2}\right)\cos^2\left(\frac{q_\nu}{2}\right).
\end{aligned} \tag{6.296}$$

The two-gluon vertex components

In general one has

$$\begin{aligned}
&\xi^{(2)}(\Delta V_\mu; q_1, \nu_1; q_2, \nu_2) = \\
&= \sum_{\rho\neq\mu}\left(\xi_{00}^{(2)}\delta_{\mu\nu_1}\delta_{\mu\nu_2} + \xi_{01}^{(2)}\delta_{\mu\nu_1}\delta_{\rho\nu_2} + \xi_{10}^{(2)}\delta_{\rho\nu_1}\delta_{\mu\nu_2} + \xi_{11}^{(2)}\delta_{\rho\nu_1}\delta_{\rho\nu_2}\right).
\end{aligned} \tag{6.297}$$

where

$$
\begin{aligned}
\xi^{(2)}_{00}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q_1; q_2\right) &= \\
&= \frac{1}{2}\sin^2\left(\frac{q_{1\rho}+q_{2\rho}}{2}\right)\left(1-\sin^2\left(\frac{q_{1\rho}+q_{2\rho}}{2}\right)\right) = -\frac{1}{8}\sin^2(q_{1\rho}+q_{2\rho}), \\
\xi^{(2)}_{01}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q_1; q_2\right) &= \\
&= -\frac{i}{2}e^{-iq_{2\mu}/2}\sin\left(q_{1\rho}+\frac{q_{2\rho}}{2}\right)\left(1-\sin^2\left(\frac{q_{1\rho}}{2}\right)-\sin^2\left(\frac{q_{1\rho}+q_{2\rho}}{2}\right)\right), \\
\xi^{(2)}_{10}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q_1; q_2\right) &= \\
&= \frac{i}{2}e^{iq_{1\mu}/2}\sin\left(\frac{q_{1\rho}}{2}+q_{2\rho}\right)\left(1-\sin^2\left(\frac{q_{1\rho}+q_{2\rho}}{2}\right)-\sin^2\left(\frac{q_{2\rho}}{2}\right)\right), \\
\xi^{(2)}_{11}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q_1; q_2\right) &= \\
&= -\frac{1}{2}\left(\cos\left(\frac{q_{1\mu}+q_{2\mu}}{2}\right)-e^{iq_{1\mu}/2-iq_{2\mu}/2}\right)\left[\cos\left(\frac{q_{1\rho}+q_{2\rho}}{2}\right)-\sin\left(\frac{q_{1\rho}}{2}\right)\sin\left(\frac{q_{2\rho}}{2}\right)+\right. \\
&\quad \left. -\cos\left(\frac{q_{1\rho}+q_{2\rho}}{2}\right)\left(\sin^2\left(\frac{q_{1\rho}+q_{2\rho}}{2}\right)+\sin^2\left(\frac{q_{1\rho}}{2}\right)+\sin^2\left(\frac{q_{2\rho}}{2}\right)\right)\right]+ \\
&\quad -\frac{i}{2}\sin\left(\frac{q_{1\mu}+q_{2\mu}}{2}\right)\cos\left(\frac{q_{1\rho}+q_{2\rho}}{2}\right)\left(\sin^2\left(\frac{q_{1\rho}}{2}\right)-\sin^2\left(\frac{q_{2\rho}}{2}\right)\right).
\end{aligned}
\tag{6.298}
$$

With this one obtains

$$
\begin{aligned}
\xi^{(2)}(\Delta V_\mu; q_1,\mu; q_2,\mu) &= \sum_{\rho\neq\mu}\xi^{(2)}_{00}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu; q_1; q_2\right), \\
\xi^{(2)}(\Delta V_\mu; q_1,\nu; q_2,\nu) &= \xi^{(2)}_{11}\left(-\frac{1}{4}(\nabla_\nu)^2 U_\mu; q_1; q_2\right), \\
\xi^{(2)}(\Delta V_\mu; q_1,\mu; q_2,\nu) &= \xi^{(2)}_{01}\left(-\frac{1}{4}(\nabla_\nu)^2 U_\mu; q_1; q_2\right), \\
\xi^{(2)}(\Delta V_\mu; q_1,\nu; q_2,\mu) &= \xi^{(2)}_{10}\left(-\frac{1}{4}(\nabla_\nu)^2 U_\mu; q_1; q_2\right).
\end{aligned}
\tag{6.299}
$$

Proceeding to the ξ functions to $\bar{\psi}\Delta\nabla_\mu\psi$, the component $(\mu\mu)$ is given by

$$
\begin{aligned}
\xi^{(2)}(\bar{\psi}\Delta\nabla_\mu\psi, p; q_1,\mu; q_2,\mu) &= \\
&= \frac{ig^2}{8}\sin(p_\mu)\left(\sin^2(q_{1\nu_1}+q_{2\nu_1})+\sin^2(q_{1\nu_2}+q_{2\nu_2})+\sin^2(q_{1\nu_3}+q_{2\nu_3})\right)
\end{aligned}
\tag{6.300}
$$

while for the coefficient $(\nu\nu)$ one obtains

$$
\begin{aligned}
\xi^{(2)}(\bar{\psi}\Delta\nabla_\mu\psi, p; q_1,\nu; q_2,\nu) &= \\
&= ig^2\sin(p_\mu)\sin\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{2\mu}}{2}\right)\left[\cos\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)-\sin\left(\frac{q_{1\nu}}{2}\right)\sin\left(\frac{q_{2\nu}}{2}\right)+\right. \\
&\quad \left. -\cos\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)\left(\sin^2\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)+\sin^2\left(\frac{q_{1\nu}}{2}\right)+\sin^2\left(\frac{q_{2\nu}}{2}\right)\right)\right]+
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{ig^2}{2}\cos(p_\mu)\sin\left(\frac{q_{1\mu}-q_{2\mu}}{2}\right)\left[\cos\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)-\sin\left(\frac{q_{1\nu}}{2}\right)\sin\left(\frac{q_{2\nu}}{2}\right)+\right.\\
&\qquad\left.-\cos\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)\left(\sin^2\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)+\sin^2\left(\frac{q_{1\nu}}{2}\right)+\sin^2\left(\frac{q_{2\nu}}{2}\right)\right)\right]+\\
&+\frac{ig^2}{2}\cos(p_\mu)\sin\left(\frac{q_{1\mu}+q_{2\mu}}{2}\right)\cos\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)\left(\sin^2\left(\frac{q_{1\nu}}{2}\right)-\sin^2\left(\frac{q_{2\nu}}{2}\right)\right).
\end{aligned}
\tag{6.301}
$$

The off-diagonal components $(\mu\nu)$ and $(\nu\mu)$ are given by

$$
\begin{aligned}
&\xi^{(2)}(\bar{\psi}\Delta\nabla_\mu\psi,p;q_1,\mu;q_2,\nu) \;=\\
&= -\frac{ig^2}{2}\sin(p_\mu)\sin\left(\frac{q_{2\mu}}{2}\right)\sin\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right)\left(1-\sin^2\left(\frac{q_{1\nu}}{2}\right)-\sin^2\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)\right)+\\
&\quad+\frac{ig^2}{2}\cos(p_\mu)\cos\left(\frac{q_{2\mu}}{2}\right)\sin\left(q_{1\nu}+\frac{q_{2\nu}}{2}\right)\left(1-\sin^2\left(\frac{q_{1\nu}}{2}\right)-\sin^2\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)\right),\\
&\xi^{(2)}(\bar{\psi}\Delta\nabla_\mu\psi,p;q_1,\nu;q_2,\mu) \;=\\
&= -\frac{ig^2}{2}\sin(p_\mu)\sin\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{1\nu}}{2}+q_{2\nu}\right)\left(1-\sin^2\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)-\sin^2\left(\frac{q_{2\nu}}{2}\right)\right)+\\
&\quad-\frac{ig^2}{2}\cos(p_\mu)\cos\left(\frac{q_{1\mu}}{2}\right)\sin\left(\frac{q_{1\nu}}{2}+q_{2\nu}\right)\left(1-\sin^2\left(\frac{q_{1\nu}+q_{2\nu}}{2}\right)-\sin^2\left(\frac{q_{2\nu}}{2}\right)\right).
\end{aligned}
\tag{6.302}
$$

6.5.4 The Naik term contributions

The Naik term contributions read

$$
\begin{aligned}
&\xi^{(0)}\left(-\frac{1}{6}\bar{\psi}(\nabla_\mu)^3\psi,p\right) \;=\; -\frac{i}{6}\sin^3(p_\mu),\\
&\xi^{(1)}_0\left(-\frac{1}{6}\bar{\psi}(\nabla_\mu)^3\psi,p;q\right) \;=\\
&= -\frac{ig}{6}\cos(p_\mu)\left(\left(1-4\sin^2(p_\mu)\right)\cos^2\left(\frac{q_\mu}{2}\right)-\cos^2(p_\mu)\right),\\
&\xi^{(2)}_{00}\left(-\frac{1}{6}\bar{\psi}(\nabla_\mu)^3\psi,p;q_1;q_2\right) \;=\\
&= \frac{ig^2}{48}\left\{\sin(p_\mu)\left(3-\left(4\cos^2(p_\mu)-1\right)\left(4\cos^2\left(\frac{q_{1\mu}}{2}\right)-1\right)\left(4\cos^2\left(\frac{q_{2\mu}}{2}\right)-1\right)\right)+\right.\\
&\quad\left.+8\cos(p_\mu)\left(1-4\sin^2(p_\mu)\right)\cos\left(\frac{q_{1\mu}}{2}\right)\cos\left(\frac{q_{2\mu}}{2}\right)\sin\left(\frac{q_{1\mu}-q_{2\mu}}{2}\right)\right\}
\end{aligned}
\tag{6.303}
$$

where

$$
\sin(3p_\mu)=3\sin(p_\mu)-4\sin^3(p_\mu), \qquad \cos(3p_\mu)=4\cos^3(p_\mu)-3\cos(p_\mu) \tag{6.304}
$$

has been used to obtain the final form.

6.5.5 The tadpole improvement

Finally, the tadpole improvement has to be worked out which, to the considered order g^2, is given by the expansion of the no-gluon contribution in the parameters

$$\Delta u_\rho^{(2)} = \frac{g^2}{4\pi^2} u_0^{(2)}. \tag{6.305}$$

One obtains (note the comma notation)

$$\begin{aligned}
\xi_{,}^{(0)}\left(U_\mu\right) &= \xi_{,0}^{(0)} \Delta u_\mu^{(2)} \quad \text{with} \quad \xi_{,0}^{(0)}\left(U_\mu\right) \;=\; 1, \\
\xi_{,}^{(0)}\left(\frac{1}{4}\nabla_\rho^{(2)} U_\mu\right) &= \xi_{,0}^{(0)} \Delta u_\mu^{(2)} + \xi_{,1}^{(0)} \Delta u_\rho^{(2)} \quad \text{with} \\
\xi_{,0}^{(0)}\left(\frac{1}{4}\nabla_\rho^{(2)} U_\mu\right) &= 0, \qquad \xi_{,1}^{(0)}\left(\frac{1}{4}\nabla_\rho^{(2)} U_\mu\right) \;=\; 1,
\end{aligned} \tag{6.306}$$

while all functions $\xi_{,}^{(0)}$ of higher derivatives vanish. Therefore, one ends up with

$$\xi_{,}^{(0)}(V_\mu) = \xi_{,}^{(0)}\left(U_\mu\right) + \sum_{\rho\neq\mu} \xi_{,}^{(0)}\left(\frac{1}{4}\nabla_\rho^{(2)} U_\mu\right) = \Delta u_\mu^{(2)} + \sum_{\rho\neq\mu} \Delta u_\rho^{(2)}. \tag{6.307}$$

The ξ function of the corresponding action is given by

$$\xi_{,}^{(0)}\left(\bar\psi \nabla'_\mu \psi, p\right) = -i\sin(p_\mu)\left(\Delta u_\mu^{(2)} + \sum_{\rho\neq\mu}\Delta u_\rho^{(2)}\right) = \frac{-ig^2}{4\pi}\sin(p_\mu)\left(4u_0^{(2)}\right). \tag{6.308}$$

The tadpole improvement contribution of the low energy part is calculated via

$$\begin{aligned}
\xi_{,}^{(0)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu\right) &= \xi_{,0}^{(0)} \Delta u_\mu^{(2)} + \xi_{,1}^{(0)} \Delta u_\rho^{(2)} \quad \text{with} \\
\xi_{,0}^{(0)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu\right) &= 0, \qquad \xi_{,1}^{(0)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu\right) \;=\; -\frac{1}{2}.
\end{aligned} \tag{6.309}$$

Therefore

$$\xi_{,}^{(0)}(\Delta V_\mu) = \sum_{\rho\neq\mu} \xi_{,}^{(0)}\left(-\frac{1}{4}(\nabla_\rho)^2 U_\mu\right) = -\frac{1}{2}\sum_{\rho\neq\mu}\Delta u_\rho^{(2)} = \frac{g^2}{4\pi}\left(-\frac{3}{2}u_0^{(2)}\right) \tag{6.310}$$

and thus

$$\xi_{,}^{(0)}\left(\bar\psi \Delta\nabla'_\mu \psi, p\right) = -i\sin(p_\mu)\xi_{,}^{(0)}(\Delta V_\mu) = \frac{-ig^2}{4\pi}\sin(p_\mu)\left(-\frac{3}{2}u_0^{(2)}\right). \tag{6.311}$$

The Naik term, finally, gives a contribution

$$\begin{aligned}
\xi_{,}^{(0)}\left(-\frac{1}{6}\bar\psi(\nabla_\mu)^3\psi, p\right) &= \frac{1}{16}\left(e^{3ip_\mu} - e^{ip_\mu} + e^{-ip_\mu} - e^{-3ip_\mu}\right)\Delta u_\mu^{(2)} \;= \\
&= \frac{1}{16}\left(e^{ip_\mu} - e^{-ip_\mu}\right)\left(e^{2ip_\mu} + e^{-2ip_\mu}\right)\Delta u_\mu^{(2)} \;= \\
&= \frac{i}{4}\sin(p_\mu)\cos(2p_\mu)\Delta u_\mu^{(2)} \;=\; \frac{-ig^2}{4\pi}\sin(p_\mu)\left(-\frac{1}{4}\cos(2p_\mu)u_0^{(2)}\right).
\end{aligned} \tag{6.312}$$

6.6 Staggered quark action and vertices

After applying all changes of the naive action mentioned in the previous section, the staggered quark action reads

$$\begin{aligned} S &= a_s^3 a_t \sum_x \bar{\psi}_c(x) \left\{ \frac{c_0}{a_s} \sum_{i+1}^3 \gamma_i \left(\nabla_i' - \frac{1}{6}(\nabla_i)^3 \right) + \frac{1}{a_t}\gamma_4 \left(\nabla_4' - \frac{1}{6}(\nabla_4)^3 \right) + m_0 \right\} \psi_c(x) = \\ &= \sum_x \bar{\psi}(x) \left\{ \frac{c_0}{\chi} \sum_{i=1}^3 \gamma_i \left(\nabla_i' - \frac{1}{6}(\nabla_i)^3 \right) + \gamma_4 \left(\nabla_4' - \frac{1}{6}(\nabla_4)^3 \right) + m_0 a_t \right\} \psi(x) \qquad (6.313) \end{aligned}$$

where again $\psi_c(x)$ is the continuum spinor and $\psi(x) = a_s^{3/2}\psi_c(x)$ is the (dimensionless) lattice spinor. The anisotropy is given by $\chi = a_s/a_t$. ∇_μ', finally, is the lattice derivative using $V_\mu'(x)$ instead of $U_\mu(x)$. According to this action together with the results of the previous section, the no-gluon component reads

$$\begin{aligned} \xi^{(0)}(S,p) &= -\frac{ic_0}{\chi} \sum_{i=1}^3 \gamma_i \left(\sin(p_i)\xi^{(0)}(V_i') + \frac{1}{6}\sin^3(p_i) \right) + \\ &\quad -i\gamma_4 \left(\sin(p_4)\xi^{(0)}(V_4') + \frac{1}{6}\sin^3(p_4) \right) + m_0 a_t = \\ &= -\frac{ic_0}{\chi} \sum_{i=1}^3 \gamma_i \sin(p_i) \left(1 + \frac{1}{6}\sin^2(p_i) \right) + \\ &\quad -i\gamma_4 \sin(p_4) \left(1 + \frac{1}{6}\sin^2(p_4) \right) + m_0 a_t \qquad (6.314) \end{aligned}$$

There is no low-energy contribution, and the contribution from the Naik term is obvious in these two steps. This result is just the inverse free quark propagator and can be written as

$$\tilde{S}_0(p)^{-1} = \xi^{(0)}(S,p) = i \sum_{\mu=1}^4 P_{0\mu}(p_\mu) + M_0(p). \qquad (6.315)$$

The results for the one- and two-gluon vertices are not written out explicitly. They can be combined from the different parts of the previous section and are implemented in a MATHEMATICA code which generates the FORTRAN input codes for the numerical calculation. Details of this are given later on. The consideration is continued with the mass- and wavefunction renormalization at this point.

6.6.1 Green function and residues

After having calculated the self energy correction

$$\Sigma(p) = \frac{i}{\chi} \sum_{i=1}^3 \gamma_i \sin(p_i)\Sigma_i(p) + i\gamma_4 \sin(p_4)\Sigma_4(p) + \Sigma_m(p), \qquad (6.316)$$

this correction can be subtracted from the inverse free quark propagator $S_0(p)^{-1}$ in order to obtain the inverse resummed quark propagator $\tilde{S}(p)^{-1}$ which can be written as

$$\tilde{S}(p)^{-1} = i \sum_{\mu=1}^4 \gamma^\mu P_\mu(p) + M(p). \qquad (6.317)$$

Taking $\vec{p} = 0$, this inverse propagator simplifies to

$$\begin{aligned}
\tilde{S}(\vec{0}, p_4)^{-1} &= i\gamma_4 P_4(\vec{0}, p_4) + M(\vec{0}, p_4) = \\
&= i\gamma_4 \sin(p_4)\left(1 + \frac{1}{6}\sin^2(p_4) - \Sigma_4(\vec{0}, p_4)\right) + m_0 a_t - \Sigma_m(\vec{0}, p_4). \qquad (6.318)
\end{aligned}$$

The part $\sin^2(p_4)/6$ is due to the Naik term and vanishes in the absence of the Naik term. This has to be taken into account in the following. The general procedure in getting to the pole mass and the wave function renormalization is to calculate the Green function

$$G(\vec{0}, t) = \int_{-\pi/a_t}^{\pi/a_t} \frac{dk_4}{2\pi} e^{ik_4 t} \tilde{S}(\vec{0}, a_t k_4) = \oint_{|z|=1} \frac{dz}{2\pi i a_t} z^{t/a_t - 1} \tilde{S}(0, -i\ln z) \qquad (6.319)$$

(k_4 is the momentum vector while $p_4 = a_t k_4$ is made dimensionless). In the second step the substitution $z = e^{ip_4} = e^{ia_t k_4}$ leads to a closed circle integral with radius $|z| = 1$ which according to Cauchy's law reduces to the sum of residues of poles of order one at $z = z_n$ lying within this circle. One therefore has

$$a_t G(\vec{0}, a_t\tau) = \oint_{|z|=1} \frac{dz}{2\pi i} z^{\tau-1} \tilde{S}(\vec{0}, -i\ln z) = \sum_n \mathrm{Res}\left[z^{\tau-1}\tilde{S}(\vec{0}, -i\ln z);\ z = z_n\right] \qquad (6.320)$$

In the case $\tilde{S}(\vec{0}, p_4)$, the denominator and numerator are considered separately,

$$\tilde{S}(\vec{0}, -i\ln z) = \frac{N(z)}{D(z)} \qquad (6.321)$$

where for the free case

$$\begin{aligned}
N(e^{ip_4}) &= -i\gamma_4 \sin(p_4)\left(1 + \frac{1}{6}\sin^2(p_4)\right) + m_0 a_t, \\
D(e^{ip_4}) &= \sin^2(p_4)\left(1 + \frac{1}{6}\sin^2(p_4)\right)^2 + m_0^2 a_t^2 = \qquad (6.322) \\
&= \left(-i\sin(p_4)\left(1 + \frac{1}{6}\sin^2(p_4)\right) + m_0 a_t\right)\left(i\sin(p_4)\left(1 + \frac{1}{6}\sin^2(p_4)\right) + m_0 a_t\right).
\end{aligned}$$

In order to obtain residues of the integrand and therefore results for the Green function, z is replaced by e^{-M} and (implicit) zeros of $D(e^{-M})$ are searched for in terms of M. For the denominator one obtains

$$\hat{D}(M) := D(e^{-M}) = \sinh^2(M)\left(1 - \frac{1}{6}\sinh^2(M) - \hat{\Sigma}_4(M)\right)^2 + \left(m_0 a_t - \hat{\Sigma}_m(M)\right)^2. \qquad (6.323)$$

where $\sin(iM) = i\sinh(M)$ is used and $\hat{\Sigma}_i(M) := \Sigma_i(\vec{0}, iM)$ (for $i = 4, m$) as well as $\hat{D}(M) = D(e^{-M})$ are defined the same way as in the Wilson action case. The two possible solutions M_1 and M_2 for $D(e^{-M}) = 0$ are given by

$$m_0 a_t - \Sigma_m(\vec{0}, iM_1) = \sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \Sigma_4(\vec{0}, iM_1)\right), \qquad (6.324)$$

$$m_0 a_t - \Sigma_m(\vec{0}, iM_2) = -\sinh(M_2)\left(1 - \frac{1}{6}\sinh^2(M_2) - \Sigma_4(\vec{0}, iM_2)\right). \qquad (6.325)$$

These are implicit equations for M_1 and M_2. However, looking at the right hand sides (without self energy corrections), it is obvious that the appropiate solution with $M > 0$ for $m > 0$ is given by $M = M_1$. Because one cannot explicitly factor out the corresponding zero as factor $(z - e^{-M})$ in the denominator, the alternative way is chosen, namely to calculate the Taylor series expansion of $D(z)$ at this specific zero up to first order,

$$D(z) = D(e^{-M}) + (z - e^{-M})D'(e^{-M}) + O\left((z - e^{-M})^2\right). \tag{6.326}$$

The derivative of $D(z)$ at e^{-M} is not equal to the derivative of the previously defined $\hat{D}(M) = D(e^{-M})$. Instead, one has $(M = -\ln z)$

$$\begin{aligned} e^{-M}D'(e^{-M}) &= z\frac{dD(z)}{dz}\Big|_{z=e^{-M}} = z\frac{dM}{dz}\frac{d\hat{D}(M)}{dM} = -\frac{d\hat{D}(M)}{DM} = \\ &= 2\sinh(M)\cosh(M)\left(1 - \frac{1}{6}\sinh^2(M) - \hat{\Sigma}_4(M)\right)^2 + \\ &\quad +2\sinh^2(M)\left(1 - \frac{1}{6}\sinh^2(M) - \hat{\Sigma}_4(M)\right)\left(-\frac{1}{3}\sinh(M)\cosh(M) - \hat{\Sigma}_4'(M)\right) + \\ &\quad +2\left(m_0a_t - \hat{\Sigma}_m(M)\right)\Sigma_m'(M). \end{aligned} \tag{6.327}$$

One now can insert the implicit equation for $M = M_1$,

$$\begin{aligned} &e^{-M_1}D'(e^{-M_1}) = \\ &= 2\sinh(M_1)\cosh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right)^2 + \\ &\quad +2\sinh^2(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right)\left(-\frac{1}{3}\sinh(M_1)\cosh(M_1) - \hat{\Sigma}_4'(M_1)\right) + \\ &\quad +2\sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right)\Sigma_m'(M_1) = \\ &= 2\sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right)\Big\{\cosh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right) + \\ &\quad + \sinh(M_1)\left(-\frac{1}{3}\sinh(M_1)\cosh(M_1) - \hat{\Sigma}_4'(M_1)\right) + \Sigma_m'(M_1)\Big\} = \\ &= 2\sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right) \times \\ &\quad \times \frac{d}{dM_1}\Big\{\sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right) + \hat{\Sigma}_m(M_1)\Big\}. \end{aligned} \tag{6.328}$$

The numerator is given by

$$\hat{N}(M) := N(e^{-M}) = \gamma_4\sinh(M)\left(1 - \frac{1}{6}\sinh^2(M) - \hat{\Sigma}_4(M)\right) + m_0a_t - \hat{\Sigma}_m(M), \tag{6.329}$$

therefore

$$\begin{aligned} \hat{N}(M_1) &= \gamma_4\sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right) + \\ &\quad + \sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right) = \\ &= (1 + \gamma_4)\sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right). \end{aligned} \tag{6.330}$$

Using this, the relevant residue is given by

$$R_1 = \mathrm{Res}\left[\frac{z^\tau N(z)}{zD(z)};\ z = e^{-M_1}\right] = \frac{1+\gamma_4}{2} e^{-M_1\tau} Z_2(\vec{0}, M_1) \tag{6.331}$$

with the inverse wave function renormalization factor

$$Z_2(\vec{0}, iM_1)^{-1} = \frac{d}{dM_1}\left\{\hat{\Sigma}_m(M_1) + \sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \hat{\Sigma}_4(M_1)\right)\right\}. \tag{6.332}$$

6.6.2 The mass renormalization

In order to calculate the mass renormalization, one iterates Eq. (6.324) in using the ansatz $M_1 = M_1^{(0)} + \Delta M_1^{(1)} + O(\alpha^2)$ and performs an order by order comparison. One obtains

$$m_0 a_t = \sinh(M_1^{(0)})\left(1 - \frac{1}{6}\sinh^2(M_1^{(0)})\right) \tag{6.333}$$

and

$$\begin{aligned} -\hat{\Sigma}_m(M_1^{(0)}) &= \Delta M_1^{(1)} \frac{d}{dM_1}\left\{\sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1)\right)\right\}_{M_1 = M_1^{(0)}} - \sinh(M_1^{(0)})\hat{\Sigma}_4(M_1^{(0)}) = \\ &= \Delta M_1^{(1)}\left\{\cosh(M_1^{(0)}) - \frac{1}{2}\sinh^2(M_1^{(0)})\cosh(M_1^{(0)})\right\} - \sinh(M_1^{(0)})\hat{\Sigma}_4(M_1^{(0)}). \end{aligned} \tag{6.334}$$

Therefore, one has

$$\Delta M_1^{(1)} = \frac{-\hat{\Sigma}_m(M_1^{(0)}) + \sinh(M_1^{(0)})\hat{\Sigma}_4(M_1^{(0)})}{\cosh(M_1^{(0)})\left(1 - \sinh^2(M_1^{(0)})/2\right)} = \frac{-\Sigma_m(\vec{0}, iM_1^{(0)}) + \sinh(M_1^{(0)})\Sigma_4(\vec{0}, iM_1^{(0)})}{\cosh(M_1^{(0)})\left(1 - \sinh^2(M_1^{(0)})/2\right)}. \tag{6.335}$$

The part $\Sigma_m(\vec{0}, iM_1^{(0)}) - \sinh(M_1^{(0)})\Sigma_4(\vec{0}, iM_1^{(0)})$ is given by one quarter of the trace of the self energy contribution in Eq. (6.316) multiplied by $(1 + \gamma_4)$,

$$\mathrm{Tr}\left((1+\gamma_4)\Sigma(\vec{0}, iM_1^{(0)})\right) = 4\Sigma_m(\vec{0}, iM_1^{(0)}) - 4\sinh(M_1^{(0)})\Sigma_4(\vec{0}, iM_1^{(0)}). \tag{6.336}$$

The final result for the *mass renormalization correction* reads

$$\Delta M_1^{(1)}(M_1^{(0)}) = \frac{-\mathrm{Tr}\left((1+\gamma_4)\Sigma_m(\vec{0}, iM_1^{(0)})\right)}{4\cosh(M_1^{(0)})\left(1 - \sinh^2(M_1^{(0)})/2\right)}. \tag{6.337}$$

Because $m = 0$ implies $\Delta M_1^{(1)} = 0$ for staggered quarks, one does not need to subtract the correction according to

$$\Delta M_{1,\mathrm{sub}}^{(1)}(M_1^{(0)}) = \Delta M_1^{(1)}(M_1^{(0)}) - \frac{\Delta M_1^{(1)}(0)}{\cosh(M_1^{(0)})\left(1 - \sinh^2(M_1^{(0)})/2\right)} \tag{6.338}$$

in this case.

6.6.3 The wave function renormalization

The *wave function renormalization* the expression is given by Eq. (6.332), i.e.

$$\begin{aligned} Z_2(\vec{0}, iM_1)^{-1} &= \frac{d}{dM_1}\left\{\Sigma_m(\vec{0}, iM_1) + \sinh(M_1)\left(1 - \frac{1}{6}\sinh^2(M_1) - \Sigma_4(\vec{0}, iM_1)\right)\right\} = \\ &= \cosh(M_1)\left(1 - \frac{1}{2}\sinh^2(M_1)\right) + \frac{d}{dM_1}\left\{\Sigma_m(\vec{0}, iM_1) - \sinh(M_1)\Sigma_4(\vec{0}, iM_1)\right\} = \\ &= \cosh(M_1^{(0)})\left(1 - \frac{1}{2}\sinh^2(M_1^{(0)})\right) - \frac{3}{2}\Delta M_1^{(1)}\sinh^3(M_1^{(0)}) + \\ &\quad + \frac{d}{dM_1}\left\{\Sigma_m(\vec{0}, iM_1^{(0)}) - \sinh(M_1^{(0)})\Sigma_4(\vec{0}, iM_1^{(0)})\right\}. \end{aligned} \tag{6.339}$$

There is again a zeroth order contribution,

$$\begin{aligned} Z_2^{(0)}(\vec{0}, iM_1)^{-1} &= \frac{d}{dM_1}\left\{\sinh(M_1^{(0)})\left(1 - \frac{1}{6}\sinh^2(M_1^{(0)})\right)\right\} = \\ &= \cosh(M_1^{(0)})\left(1 - \frac{1}{2}\sinh^2(M_1^{(0)})\right). \end{aligned} \tag{6.340}$$

In terms of this contribution one can invert Eq. (6.340) to obtain

$$\begin{aligned} Z_2^{(1)}(\vec{0}, iM_1) &= Z_2^{(0)}\left\{1 + \frac{3}{2}Z_2^{(0)}\Delta M_1^{(1)}\sinh^3(M_1^{(0)}) + \right. \\ &\quad \left. -Z_2^{(0)}\frac{d}{dM_1}\left(\Sigma_m(\vec{0}, iM_1^{(0)}) - \sinh(M_1^{(0)})\Sigma_4(\vec{0}, iM_1^{(0)})\right)\right\}. \end{aligned} \tag{6.341}$$

The part $\Sigma_m(\vec{0}, iM_1^{(0)})$ can be obtained as before by taking one quarter of the trace of $\Sigma(\vec{0}, iM_1^{(0)})$. The part $-\sinh(M_1^{(0)})\Sigma_4(\vec{0}, iM_1^{(0)})$ is obtained by taking one quarter of the trace of $\Sigma(\vec{0}, iM_1)$ multiplied with γ_4. One has to take the derivative of both terms. Combining these steps one obtains

$$Z_2^{(1)}(\vec{0}, iM_1) = Z_2^{(0)}\left\{1 + \frac{3}{2}Z_2^{(0)}\Delta M_1^{(1)}\sinh^3(M_1^{(0)}) - \frac{Z_2^{(0)}}{4}\frac{d}{dM_1}\mathrm{Tr}\left((1+\gamma_4)\Sigma(\vec{0}, iM_1)\right)\right\}. \tag{6.342}$$

Written in terms of a derivative with respect to the (dimensionless) quantity $p_4 = iM_1^{(0)}$, one obtains

$$\begin{aligned} Z_2^{(1)}(\vec{0}, iM_1) &= Z_2^{(0)}\left\{1 + \frac{3}{2}Z_2^{(0)}\Delta M_1^{(1)}\sinh^3(M_1^{(0)}) + \right. \\ &\quad \left. + \frac{Z_2^{(0)}}{4i}\frac{d}{dp_4}\mathrm{Tr}\left((1+\gamma_4)\Sigma(\vec{0}, p_4)\right)\Big|_{p_4 = iM_1^{(0)}}\right\}. \end{aligned} \tag{6.343}$$

The derivative will be IR-divergent, and one has to think about whether the same method as before works, especially how the effective continuum mass reads in this case. This is done in the following subsection.

6.6.4 The IR counter term

The subtraction of the IR contribution is done by calculating an appropiate counter term within the continuum theory. In order to cancel the IR divergences, the mass parameter of the continuum contribution has to be adjusted to the lattice result. This is done by comparing the (scalar) denominator of the quark propagator in the continuum to the expansion of the denominator of the quark propagator on the lattice close to the calculated pole. Starting point for the latter one is again the ξ function of the action S in Eq. (6.314). If one uses the representation in terms of momentum part and mass parts given in Eq. (6.315) and $p = (\vec{0}, iM_1) + q$ for the momentum where q is small, the different parts read

$$\begin{aligned} P_{0i}(q_i) &= \frac{1}{\chi}\sin q_i\left(1+\frac{1}{6}\sin^2 q_i\right) \approx \frac{1}{\chi}q_i, \\ P_{04}(iM_1+q_4) &= \sin(iM_1+q_4)\left(1+\frac{1}{6}\sin^2(iM_1+q_4)\right) = \\ &\approx (i\sinh M_1+q_4\cosh M_1)\Bigg(1-\frac{1}{6}\sinh^2 M_1 + \\ &\qquad +\frac{i}{3}q_4\sinh M_1\cosh M_1+\frac{1}{6}q_4^2\left(\sinh^2 M_1+\cosh^2 M_1\right)\Bigg), \\ M_0(iM_1+q) &= m_0a_t = \sinh M_1\left(1-\frac{1}{6}\sinh^2 M_1\right). \end{aligned} \tag{6.344}$$

Therefore, for the scalar denominator factor one obtains

$$\begin{aligned} &\sum_{i=1}^{3} P_{0i}^2(q_i)+P_{04}^2(iM_1+q_4)+M_0^2(iM_1+q) = \\ &= 2iq_4\left(1-\frac{1}{6}\sinh^2 M_1\right)\left(1-\frac{1}{2}\sinh^2 M_1\right)\sinh M_1\cosh M_1+q^2+O(q_4^2) \end{aligned} \tag{6.345}$$

where terms proportional to q_4^2 can be neglected because they do not contribute to the leading order expression (cf. Ref. [212]). This expression has to be compared with the denominator factor $(2i\tilde{m}q_4+q^2)$ from the continuum result for the quark propagator where $\tilde{m}$ is called the (effective) *continuum mass*. The comparison results in

$$\tilde{m} = \sinh M_1\cosh M_1\left(1-\frac{1}{6}\sinh^2 M_1\right)\left(1-\frac{1}{2}\sinh^2 M_1\right). \tag{6.346}$$

Without the Naik term the calculations simplify significantly. One then has

$$\begin{aligned} P_{0i}(q_i) &\approx \frac{1}{\chi}q_i, \\ P_{04}(iM_1+q_4) &\approx i\sinh M_1+q_4\cosh M_1, \\ M_0(iM_1+q) &= m_0a_t = \sinh M_1 \end{aligned} \tag{6.347}$$

and therefore

$$\begin{aligned} \sum_{i=1}^{3} P_{0i}^2+P_{04}^2+M_0^2 &= \frac{1}{\chi^2}\sum_{i=1}^{3} q_i^2+(i\sinh M_1+q_4\cosh M_1)^2+\sinh^2 M_1 = \\ &= 2iq_4\sinh M_1\cosh M_1+q^2+q_4^2\sinh^2 M_1. \end{aligned} \tag{6.348}$$

In this case the continuum mass is given by

$$\tilde{m} = \sinh M_1 \cosh M_1. \tag{6.349}$$

Finally, note that χ does not explicitly occur in these expressions, even though one started with the anisotropic action. Using this continuum mass, the expressions for the subtraction term of the integrand for

$$\frac{1}{4i}\frac{d}{dp_4}\mathrm{Tr}\left((1+\gamma_4)\Sigma(\vec{0},p_4)\right)\bigg|_{p_4=iM_1^{(0)}} \tag{6.350}$$

are just the same as for the non-staggered quarks. This holds true also for the re-added singular contribution. Therefore, the result can be taken from there. The subtraction term for the numerator derivative (first term according to the quotient rule where the derivative of the numerator is taken) reads

$$\tilde{\Delta}_n(q) = \frac{2q^2}{(q^4+4\tilde{m}^2q_4^2)(q^2+\lambda^2)} + (1-\alpha_g)\frac{2q_4^2-q^2}{(q^4+4\tilde{m}^2q_4^2)(q^2+\lambda^2)}, \tag{6.351}$$

the singular contribution is given by

$$\begin{aligned}\Delta_n(\Lambda) \;&=\; 2\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right) - \frac{\Lambda^2}{2\tilde{m}^2} + \frac{\Lambda}{2\tilde{m}^2}\sqrt{\Lambda^2+4\tilde{m}^2} + \\ &-\frac{1}{2}(1-\alpha_g)\left\{\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right) - \frac{\Lambda^2(\Lambda^2+8\tilde{m}^2)}{8\tilde{m}^4} + \frac{\Lambda(\Lambda^2+6\tilde{m}^2)}{8\tilde{m}^4}\sqrt{\Lambda^2+4\tilde{m}^2}\right\}.\end{aligned} \tag{6.352}$$

Here λ is the gluon mass, and Λ is the cutoff parameter which have to be chosen to be less than π. For the denominator derivative the subtraction term reads

$$\tilde{\Delta}_d(q) = -4\frac{(q_4^2+\tilde{m}^2)(q^4-4\tilde{m}^2q_4^2)}{(q^4+4\tilde{m}^2q_4^2)^2(q^2+\lambda^2)} - 2(1-\alpha_g)\frac{q_4^2(q^2+2\tilde{m}^2)}{(q^4+4\tilde{m}^2q_4^2)q^2(q^2+\lambda^2)}, \tag{6.353}$$

while the singular contribution is given by

$$\begin{aligned}\Delta_d(\Lambda) \;&=\; \left(1+\frac{1}{2}(1-\alpha_g)\right)\ln\left(\frac{\lambda^2}{\Lambda^2}\right) + \\ &+\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right) + \frac{\Lambda^2(3\Lambda^2+4\tilde{m}^2)}{8\tilde{m}^4} - \frac{\Lambda(3\Lambda^2-2\tilde{m}^2)}{8\tilde{m}^4}\sqrt{\Lambda^2+4\tilde{m}^2} + \\ &+\frac{1}{2}(1-\alpha_g)\left\{\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+4\tilde{m}^2}}{2\tilde{m}}\right) - \frac{\Lambda^2(\Lambda^2+8\tilde{m}^2)}{8\tilde{m}^4} + \frac{\Lambda(\Lambda^2+6\tilde{m}^2)}{8\tilde{m}^4}\sqrt{\Lambda^2+4\tilde{m}^2}\right\}.\end{aligned} \tag{6.354}$$

It turns out that in the massive case only the denominator derivative is IR singular. This changes for the massless case. The corresponding expressions are then given by

$$\tilde{\Delta}_n(q) \;=\; \frac{2}{q^2(q^2+\lambda^2)} + (1-\alpha_g)\frac{2q_4^2-q^2}{q^4(q^2+\lambda^2)}, \tag{6.355}$$

$$\tilde{\Delta}_d(q) \;=\; \frac{-4q_4^2}{q^4(q^2+\lambda^2)} - (1-\alpha_g)\frac{2q_4^2}{q^4(q^2+\lambda^2)}, \tag{6.356}$$

$$\Delta_n(\Lambda) \;=\; \frac{1}{2}\left(-2+\frac{1}{2}(1-\alpha_g)\right)\ln\left(\frac{\lambda^2}{\Lambda^2}\right), \tag{6.357}$$

$$\Delta_d(\Lambda) \;=\; \frac{1}{2}\left(1+\frac{1}{2}(1-\alpha_g)\right)\ln\left(\frac{\lambda^2}{\Lambda^2}\right). \tag{6.358}$$

6.6.5 The speed of light correction

Following Ref. [209], the *correction of the speed-of-light coefficient* is given by the requirement that the spatial and the temporal parts of the inverse quark propagator are related by this factor, $c_0 a_s P_i(p) = a_t P_4(p)$. This condition leads to

$$c_0 \left(1 + \frac{1}{6}\sin^2(p_i)\right) - \Sigma_i(\vec{p}, p_4) = 1 + \frac{1}{6}\sin^2(p_4) - \Sigma_4(\vec{p}, p_4) \tag{6.359}$$

where i is either 1, 2, or 3. For $\vec{p} = \vec{0}$ and $p_4 = iM_1$ one obtains

$$c_0 = 1 - \frac{1}{6}\sinh^2(M_1) + \hat{\Sigma}_i(M_1) - \hat{\Sigma}_4(M_1). \tag{6.360}$$

The part $\hat{\Sigma}_i(M_1)$ can only be extracted from $\Sigma(\vec{0}, iM_1)$ by taking the derivative with respect to the spatial component,

$$\begin{aligned} \frac{d}{dp_i}\Sigma(\vec{p}, p_4) &= \frac{i}{\chi}\gamma_i \cos(p_i)\Sigma_i(\vec{p}, p_4) + \\ &+ \frac{i}{\chi}\sum_{j=1}^{3}\gamma_j \sin(p_j)\frac{d}{dp_i}\Sigma_j(\vec{p}, p_4) + i\gamma_4 \sin(p_4)\frac{d}{dp_j}\Sigma_4(\vec{p}, p_4) + \frac{d}{dp_4}\Sigma_m(\vec{p}, p_4). \end{aligned} \tag{6.361}$$

Taking $\vec{p} = \vec{0}$ and the trace with γ_i, all the parts in the second line vanish such that one obtains

$$\frac{d}{dp_i}\mathrm{Tr}\left(\gamma_i \Sigma(\vec{0}, p_4)\right) = \frac{4i}{\chi}\Sigma_i(\vec{0}, p_4). \tag{6.362}$$

For the part $\hat{\Sigma}_4(M_1)$ one takes the trace with γ_4,

$$\mathrm{Tr}\left(\gamma_4 \Sigma(\vec{0}, p_4)\right) = 4i\sin(p_4)\Sigma_4(\vec{0}, p_4). \tag{6.363}$$

Therefore, one obtains

$$c_0 = 1 - \frac{1}{6}\sinh^2(M_1) + \frac{\chi}{4i}\frac{d}{dp_i}\mathrm{Tr}\left(\gamma_i \Sigma(\vec{0}, p_4)\right)\Bigg|_{p=(\vec{0}, iM_1)} + \frac{1}{4}\frac{\mathrm{Tr}\left(\gamma_4 \Sigma(\vec{0}, iM_1)\right)}{\sinh(M_1)}. \tag{6.364}$$

In the massless case the division by $\sinh(M_1)$ is ill-defined. In this case one has to take the derivative here as well, the result then reads

$$c_0 = 1 + \frac{\chi}{4i}\frac{d}{dp_i}\mathrm{Tr}\left(\gamma_i \Sigma(\vec{0}, p_4)\right)\Bigg|_{p=(\vec{0},0)} - \frac{1}{4i}\frac{d}{dp_4}\mathrm{Tr}\left(\gamma_4 \Sigma(\vec{0}, p_4)\right)\Bigg|_{p=(\vec{0},0)}. \tag{6.365}$$

Chapter 7

QCD sum rules for heavy quarks

Sum rules have been investigated years before Quantum Chromodynamics was introduced to describe elementary particle physics. In order to understand, why the method considered in this chapter is called *"sum rule method"*, i.e. to understand, what is summed here and to what purpose, it is useful to take a short glimpse at a historical example just as an illustration, without going too much into details. In 1966 Drell and Hearn [218] and Gerasimov [219] independently constructed an exact sum rule for the magnetic moment of the nucleon. The sum rule presented in Ref. [218] was written as

$$\int_0^\infty \frac{d\nu}{\nu}\left(\sigma_P(\nu)-\sigma_A(\nu)\right)=\frac{2\pi^2\alpha}{M_p^2}\kappa_p^2$$

where the quantities on the left hand side were the total cross section for the absorption of a circularly polarized photon on a proton of laboratory energy $h\nu$ with spin parallel or antiparallel to the photon spin, while the right hand side contained the fine structure constant α, the proton mass m_p, and the anomalous magnetic moment κ_p of the proton in units of the nucleon magneton. The authors stressed that this sum rule containing only experimental quantities could be analyzed in the laboratory. This sum rule already contained elements which appear also in more modern treatments.

The general principle shown in this example is that perturbatively calculable quantities like the cross sections on the left hand side are compared with pure experimental quantities which cannot be determined by means of perturbation theory. Sum rules, therefore, can be used for the nonperturbative determination of physical parameters. The kind of sum rules shown in this example were later on called *asymptotic sum rules* because they use the assumption that the integral over the infinite range converges asymptotically.

The highly cited publication of Bjørken about asymptotic sum rules [220] was the starting point for activities of constructing and analyzing sum rules for all kinds of phenomena. The branch of sum rules which are considered in this chapter are related to QCD. In 1979 Shifman, Vainshtain and Zakharov (SVZ) developed the method of *QCD sum rules* [155]. This method became a widely used working tool in hadron phenomenology.

After saying a few words about the SVZ approach to QCD sum rules in the first section, the following sections will deal with the preparation for and the construction and analysis of QCD sum rules for different applications.

7.1 SVZ approach and operator product expansion

The objects QCD sum rules are applied to are the two-point correlators of hadronic currents, an object which was already introduced in the previous chapters. Originally, the SVZ approach [155] was only intended for light quark systems but it turned out to be applicaple to heavy quark systems as well. Because it relates perturbative and nonperturbative quantities, the QCD sum rule method is deeply related to an expansion in inverse powers of the squared momentum q^2 (actually, the Euclidean analogue $Q^2 = -q^2$), the *Operator Product Expansion (OPE)*. For the vector-vector correlation function

$$i\int T\left\{j_\mu(x)j_\nu(0)\right\}e^{iqx}d^4x = \mathcal{O}_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu})\mathcal{O}, \tag{7.1}$$

the operator product expansion is given by

$$\begin{aligned}\mathcal{O} &= \sum_n C_n(q^2)\mathcal{O}_n = \\ &= C_I(q^2)I + C_M(q^2)\mathcal{O}_M + C_G(q^2)\mathcal{O}_G + C_\sigma(q^2)\mathcal{O}_\sigma + C_\Gamma(q^2)\mathcal{O}_\Gamma + C_f(q^2)\mathcal{O}_f + \ldots\end{aligned} \tag{7.2}$$

where the order of the expansion is given by the *mass dimension d*. Operators up to the sixth order in the mass dimension are given by

$$\begin{aligned}&I \text{ (the unit operator)} \quad (d=0), \qquad \mathcal{O}_M = \bar\psi M\psi, \qquad \mathcal{O}_G = G^a_{\mu\nu}G^a_{\mu\nu} \quad (d=4),\\ &\mathcal{O}_\sigma = \bar\psi\sigma_{\mu\nu}t^a\tilde M\psi G^a_{\mu\nu}, \qquad \mathcal{O}_\Gamma = \bar\psi\Gamma_1\psi\bar\psi\Gamma_2\psi, \qquad \mathcal{O}_f = f^{abc}G^a_{\mu\nu}G^b_{\nu\rho}G^c_{\rho\mu} \quad (d=6)\end{aligned} \tag{7.3}$$

where t^a are the generators of $SU(3)$. The coefficients C_n are known as *Wilson coefficients* and can be calculated perturbatively. The operators take various forms, depending on whether the currents are vector or axial vector currents. For the vector current, especially for the ρ meson current

$$j^{(\rho)}_\mu = \frac{1}{2}(\bar u\gamma_\mu u - \bar d\gamma_\mu d) \tag{7.4}$$

one obtains the contributions

$$\begin{aligned}C_I &= \frac{1}{8\pi^2}\ln\left(\frac{\mu^2}{Q^2}\right) + O(\alpha_s)\\ C_M\mathcal{O}_M &= \frac{1}{2Q^4}(m_u\bar uu + m_d\bar dd) + O(\alpha_s),\\ C_G\mathcal{O}_G &= \frac{\alpha_s}{24\pi Q^4}G^a_{\mu\nu}G^a_{\mu\nu},\\ C_\sigma\mathcal{O}_\sigma &= \frac{ig_s}{12Q^8}(m_u^3\bar u\sigma_{\mu\nu}t^a u + m_d^3\bar d\sigma_{\mu\nu}t^a d)G^a_{\mu\nu} + O(g_s\alpha),\\ C_\Gamma\mathcal{O}_\Gamma &= -\frac{\pi\alpha_s}{2Q^6}(\bar u\gamma_\mu\gamma_5 t^a u - \bar d\gamma_\mu\gamma_5 t^a d)^2 - \frac{\pi\alpha_s}{9Q^6}(\bar u\gamma_\mu t^a u + d\gamma_\mu t^a d)\sum_{q=u,d,s}\bar q\gamma_\mu t^a q.\end{aligned} \tag{7.5}$$

The operator $C_\sigma\mathcal{O}_\sigma$ as well as the (not even listed) operator $C_f\mathcal{O}_f$ do not play any significant role here. For the axial vector current of the a_1 meson,

$$j^{(a_1)}_\mu = \frac{1}{2}(\bar u\gamma_\mu\gamma_5 u - \bar d\gamma_\mu\gamma_5 d) \tag{7.6}$$

one obtains the following difference to the result for the vector current,

$$\begin{aligned} i\int (T\{j_\mu^{(a_1)}(x)j_\nu^{(a_1)}(0)\} - T\{j_\mu^{(\rho)}(x)j_\nu^{(\rho)}(0)\})e^{iqx}d^4x &= -\frac{g_{\mu\nu}}{Q^2}(m_u\bar{u}u + m_d\bar{d}d) + \\ -(q_\mu q_\nu - g_{\mu\nu}q^2)\frac{2\pi\alpha_s}{Q^6}(\bar{u}_L\gamma_\mu t^a u_L - \bar{d}_L\gamma_\mu t^a d_L)(\bar{u}_R\gamma_\mu t^a u_R - \bar{d}_R\gamma_\mu t^a d_R) \end{aligned} \tag{7.7}$$

where $q_{L,R} = \frac{1}{2}(1 \pm \gamma_5)q$. It is obvious that for this case one obtains a deviation from the transversality, given by the $(d = 4)$-term. The result for the pseudoscalar current for the π meson,

$$j^{(\pi)} = \frac{i}{2}(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d) \tag{7.8}$$

reads

$$\begin{aligned} i\int T\{j^{(\pi)}(x)j^{(\pi)}(0)\}e^{iqx}d^4x &= -\frac{3Q^2}{16\pi^2}\ln\left(\frac{\mu^2}{Q^2}\right) - \frac{1}{4Q^2}(m_u\bar{u}u + m_d\bar{d}d) + \frac{\alpha_s}{16\pi Q^2}G^a_{\mu\nu}G^a_{\mu\nu} + \\ +\frac{\pi\alpha_s}{4Q^4}(\bar{u}\sigma_{\mu\nu}\gamma_5 t^a u - \bar{d}\sigma_{\mu\nu}\gamma_5 t^a d)^2 &+ \frac{\pi\alpha_s}{6Q^4}(\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d)\sum_{q=u,d,s}\bar{q}\gamma_\mu t^a q. \end{aligned} \tag{7.9}$$

Actually, one is interested in the vacuum expectation values (or *condensates*) of all these quantities. The basic quantities are $\langle I\rangle = 1$, $\langle\bar{q}q\rangle$, $\langle(\alpha_s/\pi)G^a_{\mu\nu}G^a_{\mu\nu}\rangle$, and $\langle\bar{q}q\rangle^2$. In order to reduce the more complicated vacuum expectation values of the operators to these values, one can use assumptions such as the equality of the light quark masses and the *PCAC (partially conserved axial-vector current) hypothesis* [221] to obtain

$$\langle m_u\bar{u}u + m_d\bar{d}d\rangle = \frac{1}{2}(m_u + m_d)\langle\bar{u}u + \bar{d}d\rangle = -\frac{1}{2}m_\pi^2 f_\pi^2. \tag{7.10}$$

The *isotopic invariance* $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = \langle\bar{q}q\rangle$ is used as well. The *gluon condensate* is abbreviated by $\langle(\alpha_s/\pi)G^2\rangle$. A approximation has to be done for the *four-quark operators.* One represents this vacuum expectation value by the square of vacuum expectation values $\langle\bar{\psi}\psi\rangle$, with

$$\langle\bar{\psi}\Gamma_1\psi\bar{\psi}\Gamma_2\psi\rangle = \frac{1}{N^2}\left((\mathrm{Tr}(\Gamma_1)\mathrm{Tr}(\Gamma_2) - \mathrm{Tr}(\Gamma_1\Gamma_2)\right)\langle\bar{\psi}\psi\rangle^2 \tag{7.11}$$

where the normalization factor N is defined by

$$\langle\bar{\psi}_A\psi_B\rangle = \frac{\delta_{AB}}{N}\langle\bar{\psi}\psi\rangle. \tag{7.12}$$

The subsripts A and B denote spin, colour, and flavour. For the $SU(3)$ symmetric case one obtains $N = 3\times 3\times 4 = 36$ and $\langle\bar{\psi}\psi\rangle = \langle\bar{u}u + \bar{d}d + \bar{s}s\rangle$. If the $SU(3)$ breaking is taken into account explicitly, A and B do no longer contain the flavour indices. Therefore, one obtains $N = 12$. Examples are given in Ref. [155] for $\Gamma_1 = \Gamma_2 = \gamma_\mu$, $\Gamma_1 = \Gamma_2 = \gamma_\mu\gamma_5$, and $\Gamma_1 = \Gamma_2 = \sigma_{\mu\nu}\gamma_5$. If one compares the simplified results

$$\begin{aligned} \mathrm{Tr}(\gamma_\mu) = 0, \quad \mathrm{Tr}(\gamma_\mu\gamma_\mu) &= 4\mathrm{Tr}(\mathbb{1}_4) = 16, \\ \mathrm{Tr}(\gamma_\mu\gamma_5) = 0, \quad \mathrm{Tr}(\gamma_\mu\gamma_5\gamma_\mu\gamma_5) &= -\mathrm{Tr}(\gamma_\mu\gamma_\mu) = -16, \\ \mathrm{Tr}(\sigma_{\mu\nu}\gamma_5) = 0, \quad \mathrm{Tr}(\sigma_{\mu\nu}\gamma_5\sigma_{\mu\nu}\gamma_5) &= \mathrm{Tr}(\sigma_{\mu\nu}\sigma_{\mu\nu}) = \\ (\text{where } \sigma_{\mu\nu} = i[\gamma_\mu,\gamma_\nu]/2) \quad &= -\frac{1}{4}\mathrm{Tr}([\gamma_\mu,\gamma_\nu][\gamma_\mu,\gamma_\nu]) = 48 \end{aligned} \tag{7.13}$$

with the vacuum condensates

$$\begin{aligned}\langle\bar q\gamma_\mu\gamma_5t^aq\bar q\gamma_\mu\gamma_5t^aq\rangle &= -\langle\bar q\gamma_\mu t^aq\bar q\gamma_\mu t^aq\rangle = \frac{16}{9}\langle\bar qq\rangle^2,\\ \langle\bar q\sigma_{\mu\nu}\gamma_5t^aq\bar q\sigma_{\mu\nu}\gamma_5t^aq\rangle &= \frac{16}{3}\langle\bar qq\rangle^2,\end{aligned}\tag{7.14}$$

one obtains an effective normalization[1] $N=9$ and

$$\begin{aligned}\langle C_\Gamma\mathcal{O}_\Gamma\rangle^V &= -\frac{\pi\alpha_s}{2Q^6}\Big(\langle(\bar u\gamma_\mu\gamma_5t^au)^2\rangle+\langle(\bar d\gamma_\mu\gamma_5t^ad)^2\rangle\Big)-\frac{\pi\alpha_s}{9Q^6}\Big(\langle(\bar u\gamma_\mu t^au)^2\rangle+\langle(\bar d\gamma_\mu t^ad)^2\rangle\Big)=\\ &= -\frac{\pi\alpha_s}{2Q^6}\left(\frac{32}{9}\langle\bar qq\rangle^2\right)-\frac{\pi\alpha_s}{9Q^6}\left(-\frac{32}{9}\langle\bar qq\rangle^2\right) = -\frac{112\pi\alpha_s}{81Q^6}\langle\bar qq\rangle^2,\end{aligned}\tag{7.15}$$

$$\begin{aligned}\langle C_\Gamma\mathcal{O}_\Gamma\rangle^A &= \langle C_\Gamma\mathcal{O}_\Gamma\rangle^V-\frac{2\pi\alpha_s}{Q^6}\Big(\langle\bar u_L\gamma_\mu t^au_L\bar u_R\gamma_\mu t^au_R\rangle+\langle\bar d_L\gamma_\mu t^ad_L\bar d_R\gamma_\mu t^ad_R\rangle\Big) =\\ &= -\frac{112\pi\alpha_s}{81Q^6}\langle\bar qq\rangle^2-\frac{2\pi\alpha_s}{Q^6}\left(-\frac{16}{9}\langle\bar qq\rangle^2\right) = \frac{176\pi\alpha_s}{81Q^6}\langle\bar qq\rangle^2,\end{aligned}\tag{7.16}$$

$$\begin{aligned}\langle C_\Gamma\mathcal{O}_\Gamma\rangle^P &= \frac{\pi\alpha_s}{4Q^4}\Big(\langle(\bar u\sigma_{\mu\nu}\gamma_5t^au)^2\rangle-\langle(\bar d\sigma_{\mu\nu}\gamma_5t^ad)^2\rangle\Big)+\frac{\pi\alpha_s}{6Q^4}\Big(\langle(\bar u\gamma_\mu t^au)^2\rangle+\langle(\bar d\gamma_\mu t^ad)^2\rangle\Big)=\\ &= \frac{\pi\alpha_s}{4Q^4}\left(\frac{32}{3}\langle\bar qq\rangle^2\right)+\frac{\pi\alpha_s}{6Q^4}\left(-\frac{32}{9}\langle\bar qq\rangle^2\right) = \frac{56\pi\alpha_s}{27Q^4}\langle\bar qq\rangle^2.\end{aligned}\tag{7.17}$$

For the axial vector current one has used

$$\bar q_L\gamma_\mu t^aq_L=\frac14\bar q(1+\gamma_5)\gamma_\mu(1-\gamma_5)t^aq=\frac12\bar q\gamma_\mu(1-\gamma_5)t^aq,\quad \bar q_R\gamma_\mu t^aq_R=\frac12\bar q\gamma_\mu(1+\gamma_5)t^aq\tag{7.18}$$

and

$$\begin{aligned}\mathrm{Tr}(\Gamma_1) &= \mathrm{Tr}(\Gamma_2)=0\qquad\text{for}\quad \Gamma_1=\frac12\gamma_\mu(1-\gamma_5),\quad \Gamma_2=\frac12\gamma_\mu(1+\gamma_5),\\ \mathrm{Tr}(\Gamma_1\Gamma_2) &= \frac14\mathrm{Tr}\left(\gamma_\mu(1-\gamma_5)\gamma_\mu(1+\gamma_5)\right) = \frac14\mathrm{Tr}\left(\gamma_\mu\gamma_\mu(1+\gamma_5)(1+\gamma_5)\right) =\\ &= \frac12\mathrm{Tr}\left(\gamma_\mu\gamma_\mu(1+\gamma_5)\right) = \frac12\mathrm{Tr}(\gamma_\mu\gamma_\mu)=2\mathrm{Tr}(\mathbb{1}_4)=8\end{aligned}\tag{7.19}$$

to obtain

$$\langle\bar q_L\gamma_\mu t^aq_L\bar q_R\gamma_\mu t^aq_R\rangle=-\frac89\langle\bar qq\rangle^2.\tag{7.20}$$

In normalizing the whole operator product expansion to the leading order term, one obtains

$$\langle\mathcal{O}_{\mu\nu}\rangle=\frac{q_\mu q_\nu-g_{\mu\nu}q^2}{8\pi^2}\left(\ln\left(\frac{\mu^2}{Q^2}\right)+\frac{\langle\mathcal{O}_2\rangle}{Q^2}+\frac{\langle\mathcal{O}_4\rangle}{Q^4}+\frac{\langle\mathcal{O}_6\rangle}{Q^6}+\ldots\right)+\frac{g_{\mu\nu}q^2}{8\pi^2}\left(\frac{\langle\mathcal{O}_4'\rangle}{Q^4}+\ldots\right)\tag{7.21}$$

(actually, $\langle\mathcal{O}_2\rangle$ is absent in all cases). Taking the value $\alpha_s\langle\bar qq\rangle^2\approx4\times10^{-4}\,GeV^6$ one has

$$\begin{aligned}\langle\mathcal{O}_6\rangle^V &= -\frac{896\pi^3\alpha_s}{81}\langle\bar qq\rangle^2 \approx -0.137\,GeV^6,\\ \langle\mathcal{O}_6\rangle^A &= \frac{1408\pi^3\alpha_s}{81}\langle\bar qq\rangle^2 \approx 0.216\,GeV^6,\\ \langle\mathcal{O}_6'\rangle^P &= \frac{448\pi^3\alpha_s}{27}\langle\bar qq\rangle^2 \approx 0.206\,GeV^6.\end{aligned}\tag{7.22}$$

[1]This result is different from the one in Ref. [155] because the trace $\mathrm{Tr}(t^at^a)=16$ is not taken into account. Note in addition that in Ref. [155] the tensor $\sigma_{\mu\nu}$ is defined without the factor i.

7.2 The polynomial adjustment method

After the presentation of the main ideas of the SVZ approach to the QCD sum rule method, the first concrete application will be given. The quantities which will be determined in this application are the *fine structure constant* α at the pole of the Z^0 boson [222] and the *anomalous magnetic moment of the muon* [223]. The values of both quantities are of paramount importance for precision tests of the Standard Model. In this section the general method for the QCD sum rule determination of these two quantities is presented before details about the two different determinations are given.

7.2.1 The integral representation

The corrections $\Delta\alpha(-q^2)$ to the QED coupling constant are given by the corrections to the photon propagator due to the calculation of a chain of inserted vacuum-polarization terms [224, 225, 226],

$$\begin{aligned}\frac{-\alpha(-q^2)}{q^2} &= \alpha_0\left(\frac{-1}{q^2}+\frac{-1}{q^2}\Pi_\gamma(-q^2)\frac{-1}{q^2}+\dots\right) = \\ &= \frac{-\alpha_0}{q^2(1+\Pi_\gamma(-q^2)/q^2)} = \frac{-\alpha_0}{q^2(1-\Delta\alpha(q^2))}\end{aligned} \tag{7.23}$$

where $\Pi_\gamma(-q^2)=-e^2q^2\Pi(-q^2)=-4\pi\alpha_0q^2\Pi(-q^2)$ and $\Pi(-q^2)$ is the two-point correlator. The function $\Pi(-q^2)/q^2$ is a meromorphic function except for a cut along the real axis for $q^2>4m_\pi^2$, and it vanishes for $|q^2|\to\infty$.

In order to understand what this means, one can resort to a simplified example, namely the correlator function

$$\Pi_M(-q^2)=\sqrt{4m_\pi^2-q^2}. \tag{7.24}$$

This function takes an imaginary value for $q^2>4m_\pi^2$. The sign of the imaginary value depends on whether one approaches the real axis from the upper or lower half plane, as it is shown in Fig. 7.1. This is indicated by setting $q^2=se^{\pm i0}$, $s>4m_\pi^2$. Therefore, one obtains

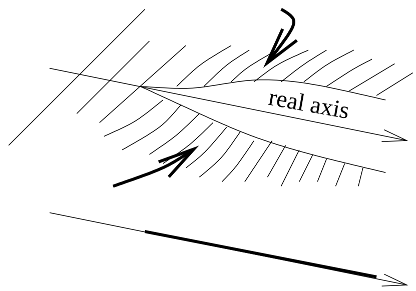


Figure 7.1: cut along the real axis

$$\begin{aligned}\sqrt{4m_\pi^2-se^{i0}} &= \sqrt{4m_\pi^2+se^{i\pi}} = \sqrt{(s-4m_\pi^2)e^{i\pi}} = \\ &= e^{i\pi/2}\sqrt{s-4m_\pi^2} = i\sqrt{s-4m_\pi^2}, \\ \sqrt{4m_\pi^2-se^{-i0}} &= e^{-i\pi/2}\sqrt{s-4m_\pi^2} = -i\sqrt{s-4m_\pi^2}.\end{aligned} \tag{7.25}$$

As explained in Chapter 2, the discontinuity is defined by

$$\operatorname{Disc}\Pi_M(s)=\Pi_M(-se^{i0})-\Pi_M(-se^{-i0})=2i\sqrt{s-4m_\pi^2} \tag{7.26}$$

and the spectral density is given by

$$\rho_M(s)=\frac{1}{2\pi i}\operatorname{Disc}\Pi_M(s)=\frac{1}{\pi}\sqrt{s-4m_\pi^2}. \tag{7.27}$$

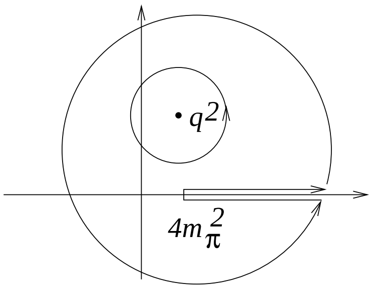


Figure 7.2: circle paths

After having parametrized the cut by introducing the discontinuity, one can use Cauchy's theorem in order to express the correlator function by the corresponding spectral density. For this one takes the circle about the specific point $z = q^2$ and expands this circle to a circle with infinite radius (see Fig. 7.2). If the circle reaches the cut, an alternative path along the real axis can be chosen to circumvent this cut, resulting in two line integrals. In assuming that in the limit of an infinite radius the circle part of the integral vanishes, one obtains

$$\begin{aligned}\Pi_M(-q^2) &= \frac{1}{2\pi i}\oint\frac{\Pi_M(-z)dz}{z-q^2} = \\ &= \frac{1}{2\pi i}\int_{\infty e^{-i0}}^{4m_\pi^2e^{-i0}}\frac{\Pi_M(-z)dz}{z-q^2}+\frac{1}{2\pi i}\int_{4m_\pi^2e^{i0}}^{\infty e^{i0}}\frac{\Pi_M(-z)dz}{z-q^2} = \\ &= \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty}\frac{\Pi_M(-s^{i0})-\Pi_M(-s^{-i0})}{s-q^2}ds = \int_{4m_\pi^2}^{\infty}\frac{\rho_M(s)}{s-q^2}. \end{aligned} \tag{7.28}$$

This *dispersion relation* is valid not only for the specific example but is valid in general if $\Pi_M(-q^2)$ falls off sufficiently fast for $|q^2|\to 0$. However, this is not the case for the correlator function one is dealing with in this application. Therefore, $\Pi_M(-q^2)$ is singular. But one can redefine $\Pi_M(-q^2)$ by a subtracted quantity,

$$\begin{aligned}\Pi_M(-q^2) \;\to\; \Pi_M(-q^2)-\Pi_M(0) &= \int_{4m_\pi^2}^{\infty}\left(\frac{1}{s-q^2}-\frac{1}{-q^2}\right)\rho(s)ds = \\ &= \int_{4m_\pi^2}^{\infty}\frac{q^2\rho(s)ds}{s(s-q^2)}\end{aligned} \tag{7.29}$$

which can also be obtained if one writes a dispersion relation for $\Pi_M(-q^2)/q^2$ instead of $\Pi_M(-q^2)$. Finally note that the index M represents the Minkowskian space. Usually, the correlator function is calculated in Euclidean space. For these correlator functions the cut runs along the negative real axis, so the discontinuity is defined along this cut with arguments $se^{\mp i\pi}$. For the two-point correlator in Euclidean space one then has the dispersion relation [59]

$$\frac{\Pi(-q^2)}{-q^2} = \int_{4m_\pi^2}^{\infty}\frac{\rho(s)ds}{s(s+q^2)} \quad \text{or} \quad \Pi(-q^2) = -\int_{4m_\pi^2}^{\infty}\frac{q^2\rho(s)ds}{s(s+q^2)}. \tag{7.30}$$

Taking into account only hadronic contributions, the spectral density is related to the relative hadronic cross section in e^+e^- annihilations

$$R = \frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} \tag{7.31}$$

by $R(s) = \rho(s)$. For low energies the relative cross section is determined by the pion form factor F_π,

$$R(s) = \frac{v_\pi^3}{4}|F_\pi(s)|^2, \qquad v_\pi = \sqrt{1-\frac{4m_\pi^2}{s}}. \tag{7.32}$$

Therefore, the expression one deals with for the hadronic contribution to the fine structure correction is given by

$$\Delta\alpha(M_Z^2) = -\frac{\alpha}{3\pi}\int_{4m_\pi^2}^\infty H(s)\rho(s)ds, \qquad H(s) = \frac{M_Z^2}{s(M_Z^2-s)}. \tag{7.33}$$

The integrand consists of the spectral density $\rho(s)$ and a factor $H(s)$ which is called *weight function*. This weight function has a single pole at the origin, a fact that will become important in the argumentation that follows. In case of the anomalous magnetic moment of the muon, the weight function can be determined from the first order QED correction to the muon-photon vertex with a self energy insertion in the photon loop. In Fig. 7.3, the contribution of the shaded bubble reads

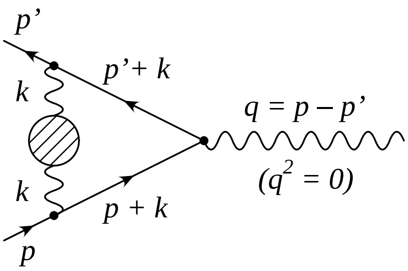


Figure 7.3: hadronic corrections to the muon anomalous moment

$$\Pi_{\mu\nu}(-q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(-q^2). \tag{7.34}$$

For the scalar correlator function one has the subtracted dispersion relation

$$\Pi(-q^2) = \int \frac{q^2\rho(s)ds}{s(s-q^2)}. \tag{7.35}$$

Ignoring for the moment the Dirac structure of the fermion lines, the integral representing the loop in the Feynman diagram is given by

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\,\frac{\Pi(-q^2)}{((p+k)^2-m^2)((p'+k)^2-m^2)(k^2)^2} = \\
&= \int \frac{d^4k}{(2\pi)^4}\int ds \frac{\rho(s)}{s(s-k^2)((p+k)^2-m^2)((p'+k)^2-m^2)k^2} = \\
&= \int ds\frac{\rho(s)}{s}\int \frac{d^4k}{(2\pi)^4}\frac{-1}{((p+k)^2-m^2)((p'+k)^2-m^2)k^2(k^2-s)}
\end{aligned} \tag{7.36}$$

where the dispersion relation has been inserted and the integrations have been interchanged. In the case $p', p \to 0$, the innermost integral is given by

$$\begin{aligned}
K(s) \;=\; & \frac{(1+x^2)(1+x)^2}{x^2}\ln(1+x) + \frac{x^2(1+x)}{1-x}\ln(x) \; + \\
& + \frac{x^2}{2}(2-x^2) + \frac{(1+x^2)(1+x)^2}{x^2}\left(-x + \frac{x^2}{2}\right)
\end{aligned} \tag{7.37}$$

with $x = (v-1)/(v+1$, $v = \sqrt{1-4m_\mu^2/s}$. This situation is represented by the same diagram as in Fig. 7.3 where the bubble and one of the attached gluon lines is replaced by a gluon line with "mass" s. Therefore, one obtains

$$a_\mu = \frac{\alpha_0^2}{3\pi^2}\int_{4m_\pi^2}^\infty H(s)\rho(s)ds \tag{7.38}$$

where the weight function is now given by $H(s) = K(s)/s$. $K(s)$ has a cut along the real axis so that the decay of the weight function $H(s) = K(s)/s$ with increasing s is faster than in the case of the QED coupling. If one again only takes the hadronic contribution, $\rho(s)$ is meant to be the spectral density corresponding to the hadronic correlator function.

7.2.2 Local and global duality

The question may arise why one cannot exclude the experimental values from the considerations at all and take relations like the one in Eq. (7.27) in order to calculate the spectral density from a correlator function, a quantity that could be calculated perturbatively. The reason is that there is actually an obstacle in using this relation. As depending on methods of functional analysis, the dispersion relation (7.35) can be solved for $\rho(s)$ only if there are no poles encircled by the path in the complex plane. These poles can have their origin from weight functions in combination with the spectral density. This means that if there is such a weight function included in the integration of the spectral density, the inverse relation shown above is only valid *locally* and not *globally*. This is called *local* resp. *global duality* (see e.g. Ref. [55]).

The occurence of a singularity in the encircled part of the complex plane is actually the fact for the weight functions for the two parameters under consideration. Therefore, one cannot completely keep out the experimental measurements from the considerations. There is, however, a way to include them in an optimal way. This is done by the *polynomial adjustment method* that will be introduced in the following. In the meantime, this method has also been used by other authors (see e.g. Ref. [227]).

7.2.3 Introduction of the method

The polynomial adjustment method is based on the fact that one can use global duality when the weight function is non-singular. This is the case for a polynomial function. Therefore, one can mimic the weight function by a polynomial function obeying several conditions which will be explained later. By adding and subtracting a polynomial function $P_N(s)$ of given order N to the weight function $H(s)$, one obtains exactly

$$\int_{s_a}^{s_b}\rho(s)H(s)ds=\int_{s_a}^{s_b}\rho(s)\left(H(s)-P_N(s)\right)ds+\int_{s_a}^{s_b}\rho(s)P_N(s)ds \tag{7.39}$$

where $[s_a,s_b]$ is any interval from the total integration range. But because the second term has a polynomial weight, one can use global duality to write

$$\begin{aligned}\int_{s_a}^{s_b}\rho(s)P_N(s)ds &= \frac{1}{2\pi i}\int_{s_a}^{s_b}\operatorname{Disc}\Pi(s)P_N(s)ds = \\ &= -\frac{1}{2\pi i}\oint_{|s|=s_a}\Pi(-s)P_N(s)ds+\frac{1}{2\pi i}\oint_{|s|=s_b}\Pi(-s)P_N(s)ds.\end{aligned} \tag{7.40}$$

Therefore, this part can be represented by the difference of two circle integrals in the complex plane. On the other hand, the difference $H(s)-P_N(s)$ suppresses the contribution of the first part. Thus, the method consists of the following steps:

- replace $\rho(s)$ in the first part of Eq. (7.39) by the value of the experimentally measured total cross section $R(s)$ (see e.g. Ref. [224]),
- replace the circle integral contribution by zero, if the radius s_a of this circle integral corresponds to the threshold energy of the corresponding flavour,
- insert the QCD perturbative and non-perturbative parts of $\Pi(-s)$ on the circle for the second part of Eq. (7.39) in all other cases.

These replacements can be seen as a concept within QCD sum rules. To obtain the best efficiency for the method, one should consider to restrict the polynomial function due to the following contraints:

- The method of least squares fit should be used to mimic the weight.
- However, the degree N should not be higher than the order of the highest perturbative resp. non-perturbative contribution increased by one (this is a consequence of Cauchy's theorem involved in the analytical integration of the circle integrals).
- Especially for the low energy region, the polynomial function should vanish on the real axis to avoid instanton effects [79].
- In regions where resonances occur, the polynomial function should fit the weight function to suppress those contributions which constitute the highest uncertainty of the experimental data.

As just mentioned, the integration on the circle can be done analytically by using Cauchy's theorem. But one has to keep in mind that the result for $\Pi(-s)$ used here depends logarithmically on the renormalization scale μ and on the parameters of the theory that are renormalized at the scale μ. These are the strong coupling constant, the quark masses and the condensates. As advocated in Ref. [228], the renormalization group improvement for the moments of the electromagnetic correlator is implemented by performing the integrations over the circle with radius $|s| = s_b$ with constant parameters, i.e. they are renormalized at a fixed scale μ. Subsequently these parameters are evolved from this scale to $\mu^2 = s_a$ using the four-loop β function. In other words, one imposes the renormalization group equation on the moments rather than on the correlator itself. This procedure is not only technically simpler but avoids also possible inconsistencies inherent to the usual approach where one applies the renormalization group to the correlator, expands in powers of $\ln(s/\mu^2)$ and carries out the integration in the complex plane only at the end. In the present case the reference scale is given by $\Lambda_{\overline{\rm MS}}$.

7.2.4 The experiment side

The polynomial adjustment method needs theoretical as well as experimental input data. Data from experiments are crucial in the low energy range and in the threshold regions where perturbative QCD cannot be applied. Combined data sets from various electron-positron annihilation experiments [224] are used which are supplemented by recent BES measurements [229]. In addition, the use of precise τ-decay data from Ref. [230] by isospin rotation promises to be a rewarding step in the low energy region. The vector spectral functions of τ decay are related to the isovector e^+e^- cross sections for the corresponding hadronic states X by [230]

$$\sigma^{I=1}(e^+e^- \longrightarrow X^0) = \frac{4\pi\alpha_0^2}{s} v_{J=1}(\tau^- \to X^-\nu_\tau). \tag{7.41}$$

$v_{J=1}(\tau^- \to X^-\nu_\tau)$ is obtained by dividing the normalized invariant mass-squared distribution $dN_{X^-}/N_{X^-}ds$ for a given hadronic mass $\sqrt{s}$ by the appropriate kinematic factor,

$$v_{J=1}(\tau^- \to X^-\nu_\tau) \;=\; \frac{m_\tau^2}{6|V_{ud}|^2 S_{\rm EW}} \frac{B(\tau^- \to X^-\nu_\tau)}{B(\tau^- \to e^-\bar\nu_e\nu_\tau)} \times$$

$$\times \frac{dN_{X^-}}{N_{X^-}ds}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1} \tag{7.42}$$

where $|V_{ud}| = 0.9752 \pm 0.0007$ denotes the CKM weak mixing matrix element [127] and $S_{EW} = 1 + \delta_{EW} = 1.0194$ accounts for electroweak second order corrections [231]. The spectral functions are normalized by the ratio of the respective vector branching fraction $B(\tau^- \to X^-\nu_\tau)$ to the branching fraction of the electron channel $B(\tau^- \to e^-\bar{\nu}_e\nu_\tau) = 17.79 \pm 0.04$ [232].

7.2.5 The theory side

The two-point correlator [108] is given by

$$12\pi^2 i\int\langle 0|j_\mu^{\rm em}(x)j_\nu^{\rm em}(0)|0\rangle e^{iqx}d^4x = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(-q^2) \tag{7.43}$$

where one only includes the isospin contribution $I = 1$ in order to make it comparable to considerations for the τ decay. The scalar correlator function $\Pi(-q^2)$ consists of perturbative and non-perturbative contributions which are included to the extent needed for the required accuracy. For massless quarks the imaginary part of the two-point function is known up to four loops in QCD perturbation theory [233]. For the strange quark the $O(m_q^2/s)$ power correction to three-loop order is included [233]. The *perturbative contributions* to the current-current correlator read [233, 234, 235]

$$\begin{aligned}
\Pi^{\rm P}(s) \;=\; & \frac{9}{4}\sum_{i=1}^{N_f}Q_i^2\Bigg[\frac{20}{9}+\frac{4}{3}L+C_F\left(\frac{55}{12}-4\zeta_3+L\right)\frac{\alpha_s}{\pi}+\\
& -C_F^2\left(\frac{143}{72}+\frac{37}{6}\zeta_3-10\zeta_5+\frac{1}{8}L\right)\left(\frac{\alpha_s}{\pi}\right)^2+\\
& +C_AC_F\left(\frac{44215}{2592}-\frac{227}{18}\zeta_3-\frac{5}{3}\zeta_5+\frac{41}{8}L-\frac{11}{3}\zeta_3L+\frac{11}{24}L^2\right)\left(\frac{\alpha_s}{\pi}\right)^2+\\
& -C_FT_FN_f\left(\frac{3701}{648}-\frac{38}{9}\zeta_3+\frac{11}{6}L-\frac{4}{3}\zeta_3L+\frac{1}{6}L^2\right)\left(\frac{\alpha_s}{\pi}\right)^2+\\
& +\Bigg\{8+C_F(16+12L)\frac{\alpha_s}{\pi}+\\
& +C_F^2\left(\frac{1667}{24}-\frac{5}{3}\zeta_3-\frac{70}{3}\zeta_5+\frac{51}{2}L+9L^2\right)\left(\frac{\alpha_s}{\pi}\right)^2+\\
& +C_AC_F\left(\frac{1447}{24}+\frac{16}{3}\zeta_3-\frac{85}{3}\zeta_5+\frac{185}{6}L+\frac{11}{2}L^2\right)\left(\frac{\alpha_s}{\pi}\right)^2+\\
& -C_FT_F\left(\frac{64}{3}-16\zeta_3+N_f\left(\frac{95}{6}+\frac{26}{3}L+2L^2\right)\right)\left(\frac{\alpha_s}{\pi}\right)^2\Bigg\}\frac{m_q^2}{s}+ \qquad (7.44)\\
& +\left(c_3+k_2L+\frac{1}{2}(k_0\beta_1+2k_1\beta_0)L^2+\frac{1}{3}k_0\beta_0^2L^3\right)\left(\frac{\alpha_s}{\pi}\right)^3+O({\alpha_s}^4)+O(m_q^4/s^2)\Bigg]
\end{aligned}$$

with $L = \ln(-\mu^2/s)$, $k_0 = 1$, $k_1 = 1.63982$ and $k_2 = 6.37101$ (for the β_i see Eq. (2.20)). The yet unknown constant term in the four-loop contribution has been denoted by c_3. However, the constant non-logarithmic terms do not contribute to the circle integrals. The condensate contributions which will be referred to as the *non-perturbative contributions* are given by [234]

$$
\begin{aligned}
\Pi^{\mathrm{NP}}(s) &= \frac{2\pi^2}{3s^2}\left(1+\frac{7\alpha_s}{6\pi}\right)\langle\frac{\alpha_s}{\pi}G^2\rangle + \\
&+\frac{32\pi^2}{3s^2}\left(1+\frac{\alpha_s}{4\pi}C_F+\dots\right)\langle m_u\bar{u}u\rangle+\frac{8\pi^2}{3s^2}\left(1+\frac{\alpha_s}{4\pi}C_F+\dots\right)\langle m_d\bar{d}d\rangle + \\
&+\frac{8\pi^2}{3s^2}\left(1+\frac{\alpha_s}{4\pi}C_F+(5.8+0.92L)\frac{\alpha_s^2}{\pi^2}\right)\langle m_s\bar{s}s\rangle + \\
&+\frac{4\alpha_s^2}{3s^2}(0.6+0.333L)\langle m_u\bar{u}u+m_d\bar{d}d\rangle + \\
&-\frac{C_A m_s^4}{3s^2}\left(1+2L+(0.7+7.333L+4L^2)\frac{\alpha_s}{\pi}\right)+\frac{1792\pi^3}{81s^3}\alpha_s|\langle\bar{q}q\rangle|^2+O(s^{-4})
\end{aligned} \tag{7.45}
$$

where the m_s^4/s^2-contribution arising from the unit operator has been included. The $SU(3)$ colour factors $C_F=4/3$, $C_A=3$, $T_F=1/2$ have been used. The number of active flavours is denoted by N_f. For the coupling constant α_s of the strong interaction the four-loop formula in Ref. [48] is used, although a three-loop accuracy would be sufficient for the present application,

$$
\begin{aligned}
\frac{\alpha_s(\mu^2)}{\pi} &= \frac{1}{\beta_0 L_\Lambda}-\frac{\beta_1\ln L_\Lambda}{\beta_0(\beta_0 L_\Lambda)^2}+\frac{1}{(\beta_0 L_\Lambda)^3}\left[\frac{\beta_1^2}{\beta_0^2}(\ln^2 L_\Lambda-\ln L_\Lambda-1)+\frac{\beta_2}{\beta_0}\right] + \\
&-\frac{1}{(\beta_0 L_\Lambda)^4}\left[\frac{\beta_1^3}{\beta_0^3}\left(\ln^3 L_\Lambda-\frac{5}{2}\ln^2 L_\Lambda-2\ln L_\Lambda+\frac{1}{2}\right)+3\frac{\beta_1\beta_2}{\beta_0^2}\ln L_\Lambda-\frac{\beta_3}{2\beta_0}\right]
\end{aligned} \tag{7.46}
$$

where $L_\Lambda=\ln(\mu^2/\Lambda^2_{\overline{\mathrm{MS}}})$ and the coefficients of the QCD beta function are given in Eq. (2.20). For the running quark mass the four-loop expression [236]

$$
\frac{\bar{m}(\mu^2)}{\bar{m}(m^2)}=\frac{c(\alpha_s(\mu^2)/\pi)}{c(\alpha_s(m^2)/\pi)} \tag{7.47}
$$

is used where [237]

$$
\begin{aligned}
c(x) = x^{\gamma_0/\beta_0}\Bigg\{&1+\left[\frac{\gamma_1}{\beta_0}-\frac{\gamma_0\beta_1}{\beta_0^2}\right]x+\frac{1}{2}\left[\frac{\gamma_2}{\beta_0}-\frac{\gamma_1\beta_1+\gamma_0\beta_2}{\beta_0^2}+\frac{\gamma_0\beta_1^2}{\beta_0^3}+\left(\frac{\gamma_1}{\beta_0}-\frac{\gamma_0\beta_1}{\beta_0^2}\right)^2\right]x^2 + \\
&+\Bigg[\frac{1}{3}\left(\frac{\gamma_3}{\beta_0}-\frac{\gamma_2\beta_1+\gamma_1\beta_2+\gamma_0\beta_3}{\beta_0^2}+\frac{\gamma_1\beta_1^2+2\gamma_0\beta_1\beta_2}{\beta_0^3}-\frac{\gamma_0\beta_1^3}{\beta_0^4}\right) + \\
&+\frac{1}{2}\left(\frac{\gamma_1}{\beta_0}-\frac{\gamma_0\beta_1}{\beta_0^2}\right)\left(\frac{\gamma_2}{\beta_0}-\frac{\gamma_1\beta_1+\gamma_0\beta_2}{\beta_0^2}+\frac{\gamma_0\beta_1^2}{\beta_0^3}\right)+\frac{1}{6}\left(\frac{\gamma_1}{\beta_0}-\frac{\gamma_0\beta_1}{\beta_0^2}\right)^3\Bigg]x^3+\dots\Bigg\}
\end{aligned} \tag{7.48}
$$

and where $\gamma_0=1$,

$$
\begin{aligned}
\gamma_1 &= \frac{1}{6}\left[\frac{202}{3}-\frac{20}{9}N_f\right], \\
\gamma_2 &= \frac{1}{64}\left[1249-\left(\frac{2216}{27}+\frac{160}{3}\zeta(3)\right)-\frac{140}{81}N_f^2\right],
\end{aligned}
$$

$$\begin{aligned}\gamma_3 &= \frac{1}{256}\Bigg[\frac{4603055}{162}+\frac{135680}{27}\zeta(3)-8800\zeta(5)+ \\ &\quad -\left(\frac{91723}{27}+\frac{34192}{9}\zeta(3)-880\zeta(4)-\frac{18400}{9}\zeta(5)\right)N_f+ \\ &\quad +\left(\frac{5242}{243}+\frac{800}{9}\zeta(3)-\frac{160}{3}\zeta(4)\right)N_f^2-\left(\frac{332}{243}-\frac{64}{27}\zeta(3)\right)N_f^3\Bigg]. \end{aligned} \tag{7.49}$$

$\zeta(z)$ is Riemann's zeta function. Again one could have stayed with three-loop accuracy in the running of the quark mass.

7.2.6 The evaluation of the QED coupling

Besides being of paramount importance for precision tests of the Standard Model, an accurate knowledge of $\alpha(M_Z)$ is instrumental in narrowing down the mass mindows for the last missing particle of the Standard Model, the Higgs boson. The calculation of hadronic contributions to the running fine structure constant or QED coupling using the polynomial adjustment method is published in Ref. [222]. An important is the selection of points s_a and s_b for the limits of the integrals resp. the radii of the circles. Except for the threshold regions there is a wide range for placing these points. One can show that the results are fairly independent of this choice. However, in some cases the method is limited by computational constraints as for instance the fact that matrices cannot be inverted in some special cases. For this reason it is not advisable to make the intervals too narrow.

As a first interval, a range is selected which starts from the light flavour production threshold $s_0 = 4m_\pi^2$ and ranges to the appearence of a new flavour, denoted as threshold and marked by the mass of the ψ, $s_1 = m_\psi^2 \approx (3.1\,GeV)^2$. Because there are no quark flavours active below the threshold s_0, the inner circle integral can be set to zero and one obtains

$$\begin{aligned}\int_{s_0}^{s_1} R(s)H(s)ds &= \\ &= \int_{s_0}^{s_1} R^{\rm exp}(s)\left(H(s)-P_N(s)\right)ds + 6\pi i\oint_{|s|=s_1}\Pi^{\rm QCD}(-s)P_N(s)ds.\end{aligned} \tag{7.50}$$

As mentioned before, the constraints imposed on the polynomial function are such that it vanishes on the real axis at $s = s_1$ and coincides with the weight function $H(S)$ at the ρ resonance, i.e. for $s = m_\rho^2 \approx (1\,GeV)^2$. Fig. 7.4 shows polynomials of different order in comparison to the weight function. The results shown in Fig. 7.5 are compared with the result obtained by using only experimental data. For the up and down quarks one only keeps the mass zero part of the QCD contribution while for the strange quark also the terms to order $O(m_s^2/q^2)$ are included.

The second interval runs from s_1 to the flavour threshold marked by the mass of the Υ, $s_2 = m_\Upsilon^2 \approx (9.46\,GeV)^2$. For the charm quark, one again can set the inner circle integral to zero, but for the lighter quarks one has to keep both circle integrals. The perturbative series for the charm quark is used up to its known extent.

The third interval given between s_2 and $(40\,GeV)^2$ is again subdivided into two intervals, in this case because of the length of the interval. For the first of these two

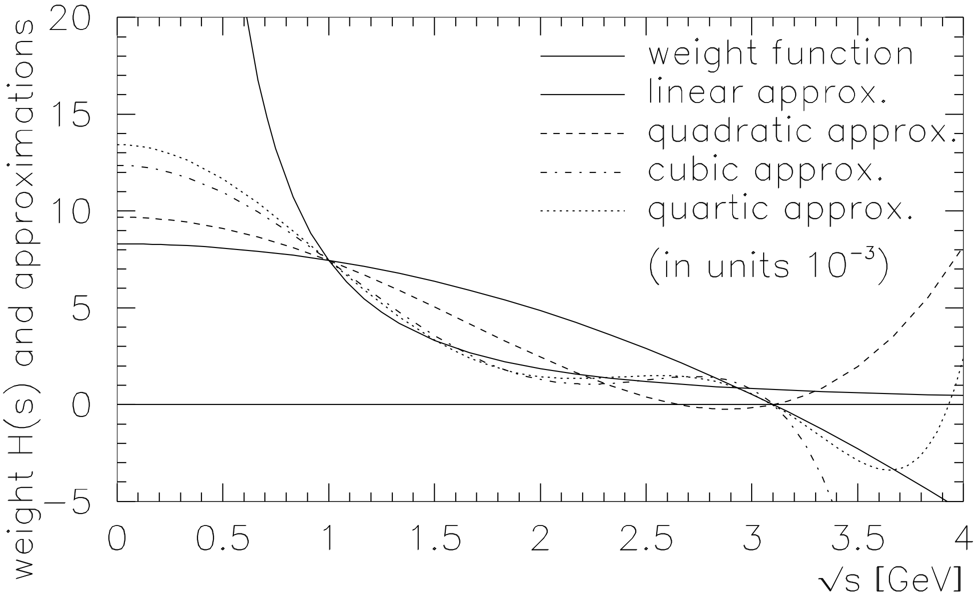


Figure 7.4: Weight function $H(s)$ and polynomial approximations $P_N(s)$ in the lowest energy interval $2m_\pi \le \sqrt{s} \le 3.1\,GeV$. The least square fit was done in the interval $m_\rho \le \sqrt{s} \le 3.1\,GeV$ with further constraints $H(s) = P_N(s)$ at $\sqrt{s} = 1\,GeV$ and $P_N(s) = 0$ at $\sqrt{s} = 3.1\,GeV$. The quality of the polynomial approximation is shown up to $N = 4$. The scaled variable s/s_1 is used for the polynomial approximation where s_1 is the upper end point of the interval such that $P_N(s/s_1)$ is dimensionless.

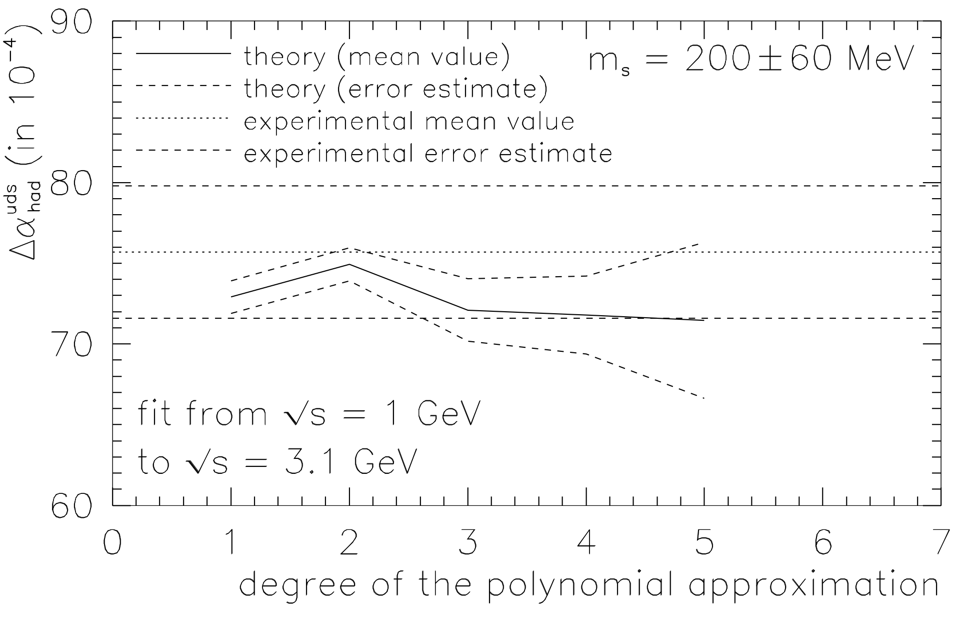


Figure 7.5: Comparison of the l.h.s. and r.h.s. of the sum rule given by Eq. (7.50) in the interval $0.28\,GeV \le \sqrt{s} \le 3.1\,GeV$. Dotted horizontal line: value of integrating the l.h.s. using experimental data including error bars [224]. The points give the values of the r.h.s. integration for various orders N of the polynomial approximation. Straight line interpolations between the points are for illustration only. The dashed lines indicate the error estimate of the calculation.

interval for $\sqrt{s}$	N	data contr.	contribution to $\Delta\alpha_{\rm had}^{(5)}(M_Z)$	error due to $\Lambda_{\overline{\rm MS}}$
$[0.28\,{\rm GeV}, 3.1\,{\rm GeV}]$	$1,2$	24%	$(73.9 \pm 1.1) \times 10^{-4}$	0.9×10^{-4}
$[3.1\,{\rm GeV}, 9.46\,{\rm GeV}]$	$3,4$	0.3%	$(69.5 \pm 3.0) \times 10^{-4}$	1.4×10^{-4}
$[9.46\,{\rm GeV}, 30\,{\rm GeV}]$	$3,4$	1.1%	$(71.6 \pm 0.5) \times 10^{-4}$	0.06×10^{-4}
$[30\,{\rm GeV}, 40\,{\rm GeV}]$	$3,4$	0.15%	$(19.93 \pm 0.01) \times 10^{-4}$	0.02×10^{-4}
$\sqrt{s} > 40\,{\rm GeV}$			$(42.67 \pm 0.09) \times 10^{-4}$	
total range			$(277.6 \pm 3.2) \times 10^{-4}$	1.67×10^{-4}

Table 7.1: Contributions of different energy intervals to $\alpha_{\rm had}^{(5)}(M_Z)$. Second column: choice of neighbouring pairs of the polynomial degree N. Third column: fraction of the contribution of experimental data [224]. Fourth column: contribution to $\Delta\alpha_{\rm had}^{(5)}(M_Z)$ with all errors included except for the systematic error due to the dependence on $\Lambda_{\overline{\rm MS}}$ which is separately listed in the fifth column.

intervals one chooses $[(9.46\,GeV)^2, (30\,GeV)^2]$, for the second $[(30\,GeV)^2, (40\,GeV)^2]$. For the bottom quark the "threshold rule" (i.e. leaving out the inner circle) applies again. The remaining part of the integral starting from $s_4 = (40\,GeV)^2$ up to infinity is done by inserting the function $R(s)$ into the second part of Eq. (7.39), proportional to

$$\rho^{\rm had}(s) = \pi\alpha_0 N_c \sum_f Q_f^2 \sqrt{1 - \frac{4m_f^2}{s}} \left(1 + \frac{2m_f^2}{s}\right) \tag{7.51}$$

(N_c is the number of colours).

The results are collected in Table 7.1. To obtain these results, the values

$$\langle\frac{\alpha_s}{\pi}GG\rangle = 0.04 \pm 0.04\,GeV^4, \qquad \alpha_s\langle\bar{q}q\rangle^2 = (4 \pm 4) \times 10^{-4}\,GeV^6 \tag{7.52}$$

for the condensates are used. For the errors coming from the uncertainty of the QCD scale one takes

$$\Lambda_{\overline{\rm MS}} = 380 \pm 60\,MeV \tag{7.53}$$

The errors resulting from the uncertainty in the QCD scale in different energy intervals are clearly correlated and will have to be added linearly in the end. Also included is the error of the strange quark mass at the scale of $1\,GeV$ which is taken to be

$$\bar{m}_s(1\,GeV) = 200 \pm 60\,MeV. \tag{7.54}$$

For the charm and bottom quark masses the values

$$\bar{m}_c(m_c) = 1.4 \pm 0.2\,GeV, \quad \bar{m}_b(m_b) = 4.8 \pm 0.3\,GeV \tag{7.55}$$

are used. Summing up the contributions from the five flavours u, d, s, c and b, the result for the hadronic contribution to the dispersion integral including the systematic error due to the dependence on $\Lambda_{\overline{\rm MS}}$ (column 5 in Table 7.1) reads

$$\Delta\alpha_{\rm had}^{(5)}(M_Z) = (277.6 \pm 4.1) \times 10^{-4}. \tag{7.56}$$

In order to obtain the total result for $\alpha(M_Z)$, one has to add the lepton and top contributions. Since there is nothing new about the calculation of these contributions, one simply can take the values cited in Ref. [238],

$$\Delta\alpha_{\rm had}^t(M_Z) = (-0.70 \pm 0.05) \times 10^{-4}, \qquad \Delta\alpha_{\rm lep}(M_Z) \approx 314.97 \times 10^{-4}. \tag{7.57}$$

Writing $\Delta\alpha(M_Z) = \Delta\alpha_{\rm lep}(M_Z) + \Delta\alpha_{\rm had}(M_Z)$, the final result reads

$$\alpha(M_Z)^{-1} = \alpha(0)^{-1}(1 - \Delta\alpha(M_Z)) = 128.925 \pm 0.056 \tag{7.58}$$

where $\alpha(0)^{-1} = \alpha_0 = 137.036$.

7.2.7 The evaluation of the magnetic moment

The Brookhaven National Laboratory (BNL) just recently reported on a precision measurement of the anomalous magnetic moment [239],

$$a_\mu^{\rm exp} = (116\,592\,023 \pm 151) \times 10^{-11} \tag{7.59}$$

which has to be contrasted with the theory prediction. From the theoretical point of view the uncertainty in the determination of a_μ (as well as $\alpha(M_Z)$) is dominated by the uncertainty of the hadronic contribution. The new measurement would correspond to a hadronic contribution of

$$a_\mu^{\rm exp,had} = (7350 \pm 153) \times 10^{-11} \tag{7.60}$$

The well-known QED contribution to the anomalous magnetic moment is the largest contribution with $a_\mu^{\rm QED} = (11658470 \pm 0.2) \times 10^{-10}$. The weak contribution is $a_\mu^{\rm weak} = (15.1 \pm 0.4) \times 10^{-10}$. While these contributions are rather well-known (see e.g. [241]), the uncertainty for the hadronic contribution is still high. As an example for the actual calculations of the hadronic contribution the value $a_\mu^{\rm had} = (6967 \pm 119) \times 10^{-11}$ of Ref. [242] is cited. This value as well as the value $a_\mu^{\rm had} = (6924 \pm 62) \times 10^{-11}$ cited by Davier and Höcker [227] no longer overlaps with the experimental measurement. Instead, as pointed out in the paper of Czarnecki and Marciano [240], the value given in Ref. [227] leads to

$$a_\mu^{\rm SM} = (116\,591\,597 \pm 67) \times 10^{-11} \tag{7.61}$$

which leads to a difference of

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = (426 \pm 165) \times 10^{-11}, \tag{7.62}$$

which is roughly 2.6σ. This observation has caused a flood of papers in the *hep-ph* preprint archive in 2001, containing models which could explain this difference with non-SM models. However, as found by several groups (see e.g. Ref. [243, 244]), the sign for the light-by-light contribution calculated in Ref. [245] figured out to be wrong. This has been corrected in the meantime by the authors (see Ref. [246]) and changed the difference between the experimental value and the Standard Model (SM) prediction to $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (247 \pm 165) \times 10^{-11}$.

The procedure for the polynomial adjustment method chosen in case of the anomalous magnetic moment of the muon differs from the one used for the QED coupling because of two reasons. As mentioned earlier, on the one hand the singularity of the weight function

interval for $\sqrt{s}$	contributions to $a_\mu^{\rm had}$	comments
$[0.28\,GeV, 1.4\,GeV]$	$(5303 \pm 61) \times 10^{-11}$	τ decay data
ω resonance	$(388.9 \pm 13.6) \times 10^{-11}$	Breit-Wigner
ϕ resonances	$(403.7 \pm 12.6) \times 10^{-11}$	narrow resonances
$[1.4\,GeV, 3.1\,GeV]$	$(519.6 \pm 20.4) \times 10^{-11}$	polynomial method
J/ψ resonances	$(88.1 \pm 6.1) \times 10^{-11}$	narrow resonances
$[3.1\,GeV, 40\,GeV]$	$(220.9 \pm 2.0) \times 10^{-11}$	e^+e^- annihilation data
Υ resonances	$(1.07 \pm 0.06) \times 10^{-11}$	narrow resonances
$[40\,GeV, \infty]$	1.5×10^{-11}	theory
top quark contr.	$< 10^{-13}$	theory
whole range	$\pm 18.3 \times 10^{-11}$	uncertainty from $\Lambda_{\overline{\rm MS}}$
hadronic contr.	$(6927 \pm 70) \times 10^{-11}$	

Table 7.2: The different contributions to the hadronic part of the anomalous magnetic moment $a_\mu^{\rm had}$ of the muon.

decays much faster with increasing s, it therefore would need more effort to approximate the weight function by a polynomial in the lower energy region. On the other hand one can (and, therefore, should) make use of the fact that the τ data between $s = (0.28\,GeV)^2$ and $s_1 = (1.4\,GeV)^2$ have been measured with high precision. The less precise e^+e^- data from the above energy region, however, are worthwile to be replaced by QCD expressions as much as possible. The methods and results shown here are published in Ref. [223].

Because of the reasons just mentioned, a first integration interval is fixed from the pion production threshold $s = 4m_\pi^2$ up to $s = (1.4\,GeV)^2$ where the τ data sets of Ref. [230] are taken to calculate the contribution. In this range these data set is in excellent shape. However, because this contribution will give the dominant part of the calculation, one has to be very careful with the treatment of the error estimates. In order to do this in a reliable fashion, the data set has been treated by taking into account the corresponding covariance matrix. The application of the polynomial adjustment method actually starts at $s = (1.4\,GeV)^2$. As in the case of the QED coupling, one takes the next threshold $s_\psi = (3.1\,GeV)^2$ as an upper limit. It turns out that in the case of the anomalous magnetic moment there is no need for further subdivision up to the end of the data sets in Ref. [224] at $s = (40\,GeV)^2$. However, the contribution of this range is already so small that one can take pure experimental e^+e^- data for this interval. The remaining contribution from above $s = (40\,GeV)^2$ is then a pure theoretical prediction obtained by integrating up the spectral density.

The results are presented in Table 7.2. The consideration of the error estimates differs from the precious case of the QED coupling in some points because different data sets are used. Only error estimates from the uncertainties for the quark masses and a systematic error over the whole interval from the uncertainty of the parameter $\Lambda_{\overline{\rm MS}}$ are kept. In comparison to this, all other error estimates for the theory contribution of the “mixed” regions (like errors of vacuum expectation values) can be neglected. The obtained value and the error estimate is comparable with the predictions in Ref. [227]. The result can therefore be seen as one of the most accurate estimates on this field.

7.3 Moments and power corrections

In the previous section, weight functions have been used extensively. These weight functions are special combinations of moments of the spectral density for which the considerations of Section 2.6 apply. Therefore, it is useful to continue with the consideration of power corrections analyzed by moments of the spectral density. The question is what power corrections (as ingredients of QCD sum rules) can tell us about the different condensate contributions. It will be shown that simple models for the vector and the axial vector reproduce the leading order power corrections of the operator product expansion to reasonable accuracy. But because the models for the vector and the axial vector channel are different, this shows that an extrapolation of perturbation theory to low energies, as it is suggested by the IR fix point method, does not work in the sense that this method leads to physically unreliable results in low energy regions.

The two models presented here are simple enough to allow for explicit evaluation. With these models it is possible to calculate power corrections to all orders and to determinate moments for these power corrections. On the other hand, the models are detailed enough to reproduce the main features of the spectra. Once again, they can be considered as test cases for the "quality" of different kinds of moments for use in QCD sum rules. Note that these rough models can only capture the gross features of the spectra while the fine details (visible at high resolution in the energy) can be different. It is expected that for the first few terms of the power expansion the accuracy is rather good while for high-order terms it can only be an order of magnitude approximation. Nevertheless, such simple models based on the gross features of the spectrum are definitely useful for a general analysis.

7.3.1 A vector channel model

As a model for the vector channel the spectral density can be taken to be

$$\rho_V(s) = 2m_V^2\delta(s-m_V^2) + \theta(s-2m_V^2) \tag{7.63}$$

(see Fig. 7.6 top). By using a dispersion relation one obtains the correlation function

$$\Pi_V(Q^2) = \frac{2m_V^2}{m_V^2+Q^2} + \ln\left(\frac{\mu^2}{2m_V^2+Q^2}\right) + \text{subtractions} \tag{7.64}$$

with necessary subtractions. The expression in Eq. (7.64) is used to generate all power corrections. Indeed, for $Q^2 \gg m_V^2$ the expansion in m_V^2/Q^2 leads to

$$\Pi_V(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) + \sum_{n=1}^{\infty}\left(-\frac{2m_V^2}{Q^2}\right)^n\left(\frac{1}{n}-\frac{1}{2^{n-1}}\right) \tag{7.65}$$

where the first term is the leading order perturbative contribution in α_s and the remaining terms are power corrections. Note that the analytic properties of the expansion in Eq. (7.65) up to any finite order are different from the exact (though model-dependent) result in Eq. (7.64). This is rather a general feature: analytic properties of approximations for the correlators can be different from those of the exact result. In some instances this restricts the precision and may lead to a misuse of approximations in areas where they do not work.

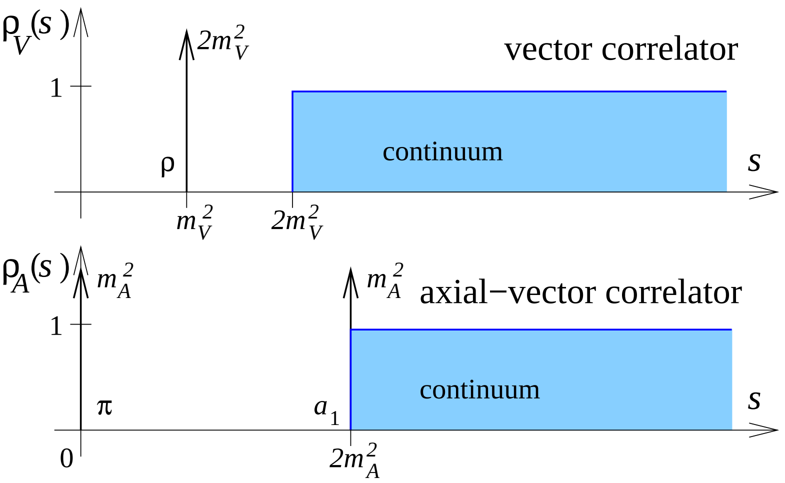


Figure 7.6: Spectral densities for the models for the vector channel (corresponding to Eq. (7.63) and the axial-vector channel (corresponding to Eq. (7.71). The arrows indicate the narrow width resonances, given by the δ distributions.

Writing the operator product expansion for the correlator in the general form

$$\Pi_V(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) + \sum_{n=1}^{\infty} \frac{c_n}{(Q^2)^n}, \tag{7.66}$$

one finds

$$c_n = \left(\frac{1}{n} - \frac{1}{2^{n-1}}\right)(-2m_V^2)^n \tag{7.67}$$

for the model in Eq. (7.64). The first two coefficients vanish, $c_1 = c_2 = 0$. The vanishing of c_1 is in full agreement with the fact that there exist no dimension-two operators in realistic cases while the vanishing of c_2 means that the gluon condensate is neglected in this model (which is justified numerically for the case of τ decays). Note that these two constraints are built-in requirements for the simple models – they were just constructed in this way. The third coefficient reads

$$c_3 = -\frac{2}{3}m_V^6. \tag{7.68}$$

Phenomenologically this coefficient is related to the value of the vacuum expectation of local four-quark operators which in factorized approximation is given by

$$\langle \mathcal{O}_6^V \rangle = -\frac{896\pi^3}{81}\alpha_s\langle\bar{q}q\rangle^2 \tag{7.69}$$

(see Eqs. (7.22)). Its numerical value is approximated reasonably well by the expression in Eq. (7.68) with $m_V = m_\rho$. The first few terms of the model operator product expansion in the vector channel read explicitly

$$\Pi_V(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) - \frac{2m_V^6}{3Q^6} + \frac{2m_V^8}{Q^8} - \frac{22m_V^{10}}{5Q^{10}} + \frac{26m_V^{12}}{3Q^{12}} + \ldots \tag{7.70}$$

The term of dimension 8 in the operator product expansion is sometimes taken into account as expressed through the vacuum expectation value of local operators, even though it is very poorly known numerically [247]. Higher order terms were never used in phenomenological applications. The expression in Eq. (7.65) shows also the actual scale of the expansion in the vector channel, $s_0 = 2m_V^2$. For the first few terms there is a numerical cancellation between the resonance and continuum contributions to the coefficients c_n while for higher order terms (large values of n) the scale $s_0 = 2m^2$ dominates. (This cancellation is one of the reasons for the success of the Borel sum rules for the ρ meson in the vector channel). The scale Λ with $\Lambda \sim \Lambda_{\rm QCD} \sim \Lambda_{\overline{\rm MS}} \sim 350 \div 400\, MeV$ or $\Lambda^2 \sim 0.25 m_V^2$ does not fit the scale of the power corrections in this model.

7.3.2 An axial channel model

Because of the presence of the pion, in the case of the axial part of the correlator (axial channel) the spectrum at low energies is drastically different from the one for the vector channel. All axial-vector resonances (with spin 1) have a finite mass. In the massless limit there is theoretically a Goldstone mode – corresponding to the observed pion – with spin zero contributing to the correlator of the axial-vector current (this is the reason why the nomenclature “axial correlator” and “axial channel” are chosen). The main mass scale is the mass m_{a_1} of the axial-vector meson a_1 which will be expressed by $m_{a_1}^2 = 2m_A^2$ for further convenience. The model for the spectrum in the axial channel reads (Fig. 7.6 bottom)

$$\rho_A(s) = m_A^2 \delta(s) + m_A^2 \delta(s - 2m_A^2) + \theta(s - 2m_A^2) \tag{7.71}$$

where the first term is the pion contribution, the second one is contribution of the a_1 meson, and the third represents the continuum. There is no gap between the second resonance and the continuum. The correlator in the axial channel is given by

$$\Pi_A(Q^2) = \frac{m_A^2}{Q^2} + \frac{m_A^2}{2m_A^2 + Q^2} + \ln\left(\frac{\mu^2}{2m_A^2 + Q^2}\right). \tag{7.72}$$

The expansion at large Q^2 reads

$$\begin{aligned} \Pi_A(Q^2) &= \ln\left(\frac{\mu^2}{Q^2}\right) + \sum_{n=1}^{\infty}\left(-\frac{2m_A^2}{Q^2}\right)^n \left(\frac{1}{n} - \frac{1}{2}(1+\delta_{n1})\right) = \\ &= \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{4m_A^6}{3Q^6} - \frac{4m_A^8}{Q^8} + \frac{48m_A^{10}}{5Q^{10}} - \frac{64m_A^{12}}{3Q^{12}} + \dots \end{aligned} \tag{7.73}$$

where δ_{n1} is the Kronecker symbol. Here the contribution of the dimension-four operator is again zero while the dimension-six contribution is positive and larger than that in the vector channel, which is the case also in the (model independent) operator product expansion,

$$\langle \mathcal{O}_6^A \rangle = \frac{1408\pi^3}{81} \alpha_s \langle \bar{q} q \rangle^2. \tag{7.74}$$

While the continuum contribution (logarithm and $1/n$ part in Eq. (7.65)) remains the same, the factor $-1/2^{n-1}$ in case of the vector channel is replaced by $-1/2$ (for $n > 1$) in Eq. (7.73). Therefore, higher order power corrections for the vector channel are dominated by the continuum while for the axial channel they are dominated by the resonance contributions and are generally larger – the mass of the a_1 meson gives the scale both for the resonance contributions and the continuum threshold in this particular model. However, this cannot be quantitatively checked at present because the numerical values of the higher order condensates are not known phenomenologically with sufficient accuracy.

7.3.3 Comparing the moments

One can compare the results for the model operator product expansion given by Eqs. (7.65) and (7.73) because of Weinberg's relation $m_{a_1}^2 = 2m_\rho^2$ [248] simply by identifying the scales $m_V = m_A = m$. The direct moments (cf. Eq. (2.145)) corresponding to the model spectral density for the vector channel for $k = 0$ then read (assuming $2m^2 < M_\tau^2$)

$$M_{0l} = 1 - \left(\frac{2m^2}{M_\tau^2}\right)^{l+1} \left(1 - \frac{l+1}{2^l}\right). \tag{7.75}$$

One observes that the perturbative contribution is represented by the first term on the right hand side of Eq. (7.75). The power corrections are given in what follows. The combined perturbative and power correction structure is a natural order for the direct s^l-moments. For large l the contribution of the power corrections decreases and the moments are saturated by perturbation theory, i.e. if $m^2 \ll M_\tau^2$, the power corrections for the moments M_{0l} die out fast.

For modified moments M_{kl} with $l = 0$ and arbitrary k (see Eq. (2.149)) one obtains

$$M_{k0} = \left(1 - \frac{2m^2}{M_\tau^2}\right)^{k+1} + (k+1)\frac{2m^2}{M_\tau^2}\left(1 - \frac{m^2}{M_\tau^2}\right)^k. \tag{7.76}$$

The magnitude of these moments tends to zero for large values of k and definitely cannot be represented perturbatively. It is obvious that a decomposition $M_{k0} = 1 + \Delta_{k0}$ with small values of Δ_{k0} fails in this case. If $m^2 \ll M_\tau^2$, the power corrections for the moments M_{k0} are still basically given by the lowest order term. But if $2m^2$ is close to M_τ^2 as it is actually the case for τ decays (the conclusion is based on the model spectrum), power corrections for the moments M_{k0} are given by a linear combination of *all* operators up to the specified order while for the moments M_{0l} power corrections are given by a *single* operator of dimension l and are relatively small. Keeping only the first few contributions can therefore give a completely wrong answer for Δ_{0k} at large values of k.

The dominance of the resonances for the axial channel becomes obvious if one considers the moments for the axial case. For $2m_A^2 < M_\tau^2$ the direct moments are given by

$$M_{00} = 1, \qquad M_{0l} = 1 + \left(\frac{2m^2}{M_\tau^2}\right)^{l+1} \left(1 - \frac{l+1}{2}\right), \tag{7.77}$$

while the modified moments for $l = 0$ read

$$M_{k0} = \left(1 - \frac{2m^2}{M_\tau^2}\right)^{k+1} + (k+1)\frac{m^2}{M_\tau^2} + (k+1)\left(1 - \frac{2m^2}{M_\tau^2}\right)^k \frac{m^2}{M_\tau^2}. \tag{7.78}$$

The second term in M_{k0} comes from the pion resonance at $s = 0$. Therefore, the moments M_{k0} will not vanish for increasing values of k but will increase linearly in k. These two features of the experimental k moments, namely the decrease of the vector contribution and the increase of the axial contribution due to the pion (as found in Eqs. (7.76) and (7.78)) are very essential for a successful comparison with the theoretical description of the τ-moments.

	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$
$M_{0l}^{V\ \mathrm{ex}}$	0.98692	0.99021	0.99497	0.99777	0.99909	0.99964
$M_{0l}^{V(3)}$	0.98692	1.0000	1.0000	1.0000	1.0000	1.0000
$M_{0l}^{A\ \mathrm{ex}}$	1.0262	1.0196	1.0110	1.0055	1.0026	1.0011
$M_{0l}^{A(3)}$	1.0262	1.0000	1.0000	1.0000	1.0000	1.0000

	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$M_{k0}^{V\ \mathrm{ex}}$	0.98692	0.95745	0.91306	0.85716	0.79358	0.72591
$M_{k0}^{V(3)}$	0.98692	0.94766	0.86915	0.73830	0.54203	0.26725
$M_{k0}^{A\ \mathrm{ex}}$	1.0262	1.0851	1.1748	1.2902	1.4256	1.5761
$M_{k0}^{A(3)}$	1.0262	1.1047	1.2617	1.5234	1.9159	2.4655

Table 7.3: Moments M_{kl} for $k=0$ (top) and $l=0$ (bottom) for the vector cannel and the axial channel model, using the exact result M_{0l}^{ex} as well as the power series expansion up to the third order term in $1/Q^2$, $M_{0l}^{(3)}$ ($m_A^2 = m_V^2 = m_\rho^2 = (769.3\,MeV)^2$ [127]).

7.3.4 A quantitative analysis

In this subsection a few features of an analysis done in Ref. [51] will be shown. In this analysis it is assumed that the two models give the exact results at any order of the power expansion. Considering the moments, one can find out whether it is enough to keep only the three terms usually available in phenomenology in order to have a reasonable (given) accuracy.

- The upper part of Table 7.3 shows the situation for the l moments. One sees that the perturbation theory contribution dominates the results. For both vector and axial correlators the accuracy of the three-term approximation is better than 2% and improves for large values of l as expected.
- For the k moments shown in the lower part of Table 7.3 one sees the dominance of nonperturbative contributions to the vector and axial correlator results. For $k=4$ the accuracy is already about 10% and deteriorates fast.

The general arguments given here are becoming more transparent if one considers modified moments M_{kl} with non-zero values of l. This is done in Table 7.4 where the ratios

$$\frac{M_{kl}^\tau - M_{kl}^{\tau(3)}}{M_{kl}^\tau} \tag{7.79}$$

for the τ-moments defined by

$$M_{kl}^\tau = N_{kl}^\tau \int_0^{M_\tau^2} \left(1-\frac{s}{M_\tau^2}\right)^{k+2} \left(\frac{s}{M_\tau^2}\right)^l \left(1+\frac{2s}{M_\tau^2}\right) \frac{\rho(s)ds}{M_\tau^2} \tag{7.80}$$

are given (the normalization factor N_{kl}^τ is chosen so that $M_{kl}^\tau = 1$ for $\rho(s)=1$). In this case only τ-moments for large values of l and small values of k show perturbative behaviour. The bottom line in Table 7.4 is drastically different from the rest of the table, the reason being that the massless pion only contributes to moments with $l=0$. This makes the large k moments still reasonably precise using only third-order power corrections.

$l=7$	−0.00	−0.01	−0.02	−0.02	−0.03	−0.03	−0.02	−0.00
$l=6$	−0.01	−0.01	−0.02	−0.02	−0.02	−0.01	+0.00	+0.02
$l=5$	−0.01	−0.02	−0.02	−0.02	−0.01	+0.00	+0.01	+0.02
$l=4$	−0.01	−0.02	−0.02	−0.01	−0.00	+0.00	+0.01	+0.02
$l=3$	−0.01	−0.01	−0.01	−0.01	−0.00	+0.01	+0.04	+0.10
$l=2$	+0.02	+0.06	+0.14	+0.32	+0.66	+1.27	+2.33	+4.11
$l=1$	+0.02	+0.02	−0.02	−0.21	−0.71	−1.88	−4.34	−9.24
$l=0$	−0.01	−0.01	−0.02	−0.02	−0.01	+0.01	+0.00	−0.05
	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$

Table 7.4: Relative deviation of the moments $M_{kl}^{\tau(3)}$ for the power series up to third order from the moments M_{kl}^{τ} for the full result containing the vector and the axial-vector channel model (see Eq. (7.79)). The approximation $m_A^2 = m_V^2 = m_\rho^2 = (769.3\,MeV)^2$ [127] is used.

7.3.5 Modifications of the model for the axial channel

For the axial channel there is a freedom in choosing the spectrum because there are two resonances which are essential at low energies, the pion and the a_1 meson. This brings in four free parameters: the two residues of the resonances, the mass of the a_1 meson, and the starting point for the continuum. Therefore, two requirements for the vanishing of the power corrections $1/Q^2$ and $1/Q^4$ are not sufficient to uniquely fix the low-energy spectrum in a simple way (with only one scale). There can be many different additional requirements all reasonably close to the experimental data. The general expression

$$\rho_A(s) = 4\pi^2 f_\pi^2 \delta(s) + 4\pi^2 f_a^2 \delta(s - 2m_A^2) + \theta(s - s_0) \tag{7.81}$$

has been analyzed in Ref. [51] where the threshold s_0 of the continuum is considered as a free parameter. There are constraints to this parameter, one of the contraints is $s_0 > 2m_A^2$. However, this constraint is quite weak. One can actually admit a resonance as a bump in the continuum, a situation that is almost given for the a_1 meson. The positivity constraint for f_π^2 realized by $s_0 < 4m_A^2$, however, has to be taken seriously. Note that $s_0 = 4m_A^2$ leads to $f_\pi = 0$. Therefore, the pion decouples and does not contribute to the spectrum of the correlator. This is not the case in reality. For some values of s_0 in the interval $2m_A^2 < s_0 < 4m_A^2$ the spectrum can be fixed by taking the experimental value for f_π. Experimentally one has $4\pi^2 f_\pi^2 \approx m_A^2$ which is close to the model without a gap. Nevertheless, the conclusions about moments are valid for these modified models as well.

7.3.6 A note on other models for power corrections

Of course, other models can be used for the spectrum. The *model of narrow resonances* inspired by the 't Hooft model, for instance, was analyzed in QCD on the basis of local duality [249]. The spectrum was studied within the local duality approach where one has

$$m_n^2 = (2n+1)m_\rho^2, \qquad f_n^2 = 2m_\rho^2 \qquad n = 0, 1, \dots \tag{7.82}$$

for the vector channel and where m_ρ is the mass of the the ground state ρ meson. For the axial channel the result of the local duality approach coupled with the $1/N_c$ approximation reads

$$m_n^2 = nm_{a_1}^2, \qquad f_0^2 = 4\pi^2 f_\pi^2 = \frac{1}{2}m_{a_1}^2 = m_\rho^2, \qquad f_n^2 = 2m_\rho^2 = m_{a_1}^2, \qquad n = 1, 2\ldots \tag{7.83}$$

where m_{a_1} is the mass of the ground state a_1 meson. The structure of the spectrum reflects the classical results on chiral symmetry and Weinberg's relations for axial-vector and vector meson masses which is realized if one identifies the scales in both channels, $m_{a_1}^2 = 2m_\rho^2 = 2m^2$. This identification leads to a simplified picture where two chains of resonances are simply shifted by an amount $m_\rho^2 = m^2$. This is the gross structure of the spectrum. Experimental results differ quite a bit from this picture, but such details can be accounted for by using the operator product expansion [150].

Within the model based on the local duality approach the summation of all resonances results in the vector correlator

$$\begin{aligned}
\Pi_V(Q^2) &= \sum_{n=0}^{\infty} \frac{f_n^2}{m_n^2 + Q^2} = \sum_{n=0}^{\infty} \frac{2m_\rho^2}{(2n+1)m_\rho^2 + Q^2} = \\
&= \sum_{n=0}^{\infty} \frac{1}{n + (Q^2 + m_\rho^2)/(2m_\rho^2)} = -\psi\left(\frac{Q^2 + m_\rho^2}{2m_\rho^2}\right) + \text{subtractions} = \\
&= \ln\left(\frac{\mu^2}{Q^2}\right) - \frac{m_\rho^4}{6Q^4} + 0\frac{m_\rho^6}{Q^6} + \frac{7m_\rho^8}{60Q^8} + 0\frac{m_\rho^{10}}{Q^{10}} + \ldots
\end{aligned} \tag{7.84}$$

where the renormalization scale μ comes in through the subtraction term. $\psi(z)$ is the digamma function (cf. Appendix D.3). Note that the expression in Eq. (7.84) is quite different from the result obtained in Eq. (7.65).

In the axial channel the model based on local duality (with parameters from Eq. (7.83)) leads to the expression

$$\begin{aligned}
\Pi_A(Q^2) &= \sum_{n=0}^{\infty} \frac{f_n^2}{m_n^2 + Q^2} = \frac{f_0^2}{Q^2} + \sum_{n=1}^{\infty} \frac{m_{a_1}^2}{m_{a_1}^2 n + Q^2} = \\
&= \frac{m_{a_1}^2}{2Q^2} + \sum_{n=1}^{\infty} \frac{1}{n + Q^2/m_{a_1}^2} = -\frac{m_{a_1}^2}{2Q^2} - \psi\left(\frac{Q^2}{m_{a_1}^2}\right) + \text{subtractions} = \\
&= \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{m_{a_1}^4}{12Q^4} + 0\frac{m_{a_1}^6}{Q^6} - \frac{m_{a_1}^8}{120Q^8} + 0\frac{m_{a_1}^{10}}{Q^{10}} + \ldots
\end{aligned} \tag{7.85}$$

In this case all power corrections of dimension $2(2k+1)$ vanish because of $B_{2k+1} = 0$. This contradicts the results of the operator product expansion since there is no explicit (symmetry) reason for such a vanishing. At moderate orders of n the structure of the expansion in Eq. (7.85) is inconsistent with the asymptotic expansion expected from the operator product expansion and, therefore, is not supported by phenomenology.

The qualitative difference of the models given in Eqs. (7.84) and (7.85) from the previous case with a continuum contribution, however, is the analytic structure of the correlators. In the narrow resonance model one only has a single dimensional parameter m^2 and one would expect the power corrections to behave as $(m^2/Q^2)^n$ with m^2 determining the scale. However, the coefficients of the power corrections grow more than exponentially for large values of k because of the Bernoulli numbers B_{2k} which come in as coefficients of the expansion of the digamma function. The reason for the divergent nature of the series of power corrections is that for the models given in Eqs. (7.84) and (7.85) there are poles located arbitrarily far away from the origin $Q^2 = 0$. In these models, therefore, there is in fact an infinite number of scales nm^2 for positive integers n. This is the reason why perturbation theory does not work even at sufficiently large s. But if one approximates a chain of resonances by a continuum starting from some threshold s_0, one instead gets a finite radius of convergence of the order s_0, $Q^2 > s_0$ and a well working perturbation theory picture at $s > s_0$.

One can conclude that the narrow resonance model for the vector and axial channel does not display the difference between low and high energies. The only criterion of perturbation theory calculability is the length of the averaging interval while its position in terms of energies is almost unimportant. This is a natural feature of the translation invariance of the spectra in these models. The symmetry of the simplified model spectrum is definitely violated in the realistic phenomenological spectrum, making the region of high energy essentially different from the low energy domain. Therefore, the spectrum in terms of an infinite chain of infinitely narrow resonances does not properly incorporate the asymptotic freedom of QCD in the sense that such a spectrum violates scale invariance.

7.3.7 A final remark on the concept of duality

The description of strong interactions based on QCD proves to be very successful for processes at large energies where the coupling constant is small due to the property of asymptotic freedom [250]. This makes perturbation theory computations reliable. At low energies the problem of strong coupling prevents using QCD as an unambiguous theoretical tool for computations of physical observables and various phenomenological models are introduced. These models are inspired by QCD but it is difficult to establish a quantitative relation between the underlying theory and a model used in practice. An example is given by the chiral perturbation theory (ChPT) for Goldstone modes [251]. Chiral perturbation theory is very convenient in describing interactions of pions (as lightest hadrons) with nucleons or resonances at low energy in the small momentum (and mass) expansion [252]. Thus the description of strong interactions at low energies relies on phenomenological models with explicit introduction of elementary hadron fields or on the closely related approaches based on general principles of analyticity, unitarity and symmetry [253].

A general idea of linking this approach for the low energy description of hadrons with QCD is the concept of *duality* which means that the description of inclusive observables which are sensitive to the contribution of many particles is simpler than that of exclusive processes and can be represented by almost free fermions or weakly coupled quarks [254]. This concept works well for infrared soft observables in τ-decays and other sum rules where the limit of massless quarks is nonsingular [155, 255, 150, 256, 53, 257, 258, 50, 49, 259]. For the infrared sensitive observables the realization of the duality concept for the light modes is not quite straightforward since the infrared cutoff explicitly enters the calculation. In such cases the cutoff is usually taken from experiment such as the mass of a real hadron.

7.4 Interpolation of the correlation function

Since the hadronic contribution is sensitive to the details of the strong coupling regime of QCD at low energies and cannot be unambiguously computed in a perturbation theory framework, the theoretical prediction for the anomalous magnetic moment of the muon within the Standard Model depends crucially on how this contribution is estimated [245]. In the absence of a reliable theoretical tool for the computation in this region one turns to experimental data on low-energy hadron interactions for extracting a numerical value, as it was shown in Section 7.2.

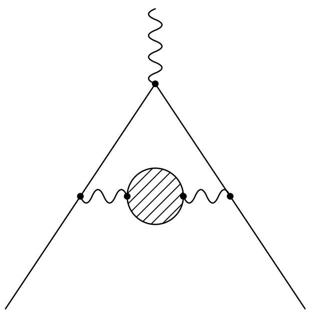


Figure 7.7: The leading order hadronic contribution, to the anomalous magnetic moment of the muon, the shaded bubble indicates the hadronic two-point correlator

In general terms the hadronic contribution to the anomalous magnetic moment of the muon is determined by the correlation functions of electromagnetic currents. Since a source for the electromagnetic current is readily available for a wide range of energies, one tries to extract these functions or some of their characteristics relevant for a particular application from experiment. Without explicit use of QCD the correction $a_\mu^{\rm had}$ in the Standard Model is generated through the electromagnetic interaction $ej_\mu^{\rm had}A^\mu$ with $j_\mu^{\rm had}$ being the hadronic part of the electromagnetic current. The leading contribution comes from the two-point correlator referred to as the hadronic part of the photon vacuum polarization contribution (expressed in terms of the function $K(s)$ introduced in Section 7.2, see Fig. 7.7) while the four-point function first emerges at order α^3, most explicitly as the light-by-light scattering amplitude. These correlators are not calculable perturbatively in the region essential for the determination of the hadronic contributions to the anomalous magnetic moment. To avoid using QCD in the strong coupling mode one can extract the necessary contribution by studying these two correlation functions experimentally without an explicit realization of the hadronic electromagnetic current j_μ^{had} in terms of elementary fields. Another possibility which is close in spirit is to use phenomenological models to saturate these correlators with contributions of real hadrons at low energies [260, 261, 262, 263, 264]. There is also a possibility to use a concept of duality between hadron and quark-gluon descriptions modified for handling IR sensitive observables [226, 265, 243]. In Ref. [266] this last option is discussed.

7.4.1 Hadronic contribution at leading order

At the leading order in α the hadronic contribution is described by the correlator in Eq. (7.1) in terms of a single function $\Pi(q^2) = \Pi^{\rm had}(q^2)$ of one variable q^2. The contribution of $\Pi^{\rm had}(q^2)$ to the muon anomalous magnetic moment (see e.g. Ref. [267]) is given by

$$a_\mu^{\rm had}({\rm LO}) = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{4m_\pi^2}^\infty \frac{ds}{s} K(s)\rho^{\rm had}(s) \tag{7.86}$$

with a one-loop kernel of the form (cf. Eq. (7.37))

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)s/m^2}. \tag{7.87}$$

Here $\rho^{\rm had}(s) = {\rm Disc}\,\Pi^{\rm had}(s)/2\pi i$ and m is the muon mass. The leading order hadronic contribution to the anomalous magnetic moment of the muon is depicted in Fig. 7.7. It is represented by an integral over the hadron spectrum and no specific information about the function $\rho^{\rm had}(s)$ is necessary point-wise. However, a QCD approach based on light quark duality in the massless approximation is not directly applicable as the integral in Eq. (7.86) is IR sensitive and depends strongly on the threshold structure of the function $\Pi^{\rm had}(q^2)$. In most applications the threshold structure is extracted from experiment. To leading order in α the function $\rho^{\rm had}(s)$ can uniquely be identified with data from e^+e^- annihilation into hadrons. Introducing the relative e^+e^- cross section

$$R^{\rm exp}(s) = \frac{\sigma(e^+e^- \to {\rm hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}, \qquad s = (p_{e^+} + p_{e^-})^2 \tag{7.88}$$

and identifying it with the theoretical quantity $\rho^{\rm had}(s)$ one finds

$$a_\mu^{\rm had}({\rm LO}) = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{4m_\pi^2}^\infty \frac{R^{\rm exp}(s)K(s)}{s} ds. \tag{7.89}$$

In order to create a framework for the analysis of hadronic contributions at next-to-leading order (NLO) based on duality arguments one rewrites the leading order (LO) expression for the hadronic contribution to the anomalous magnetic moment given in Eq. (7.86) in a different form. As discussed earlier, a two-point correlator $\Pi(q^2)$ as a function of the complex variable q^2 can have a cut along the positive semiaxis $s > 0$ with a positive discontinuity [268]. This spectral condition plays a crucial role in the analysis of the structure of the two-point correlators and related observables [269, 82]). The dispersion representation with a subtraction at the origin (cf. Eq. (2.42)) for the Euclidean domain reads

$$\Pi^{\rm had}(-q^2) = q^2 \int_{4m_\pi^2}^\infty \frac{\rho^{\rm had}(s)}{s(s-q^2)} ds \tag{7.90}$$

which implies the normalization condition $\Pi^{\rm had}(0) = 0$. Using Eqs. (7.86) and (7.87) one can rewrite the LO contribution to the anomalous magnetic moment as an integral over Euclidean values of q^2 for $\Pi^{\rm had}(-q^2)$,

$$a_\mu^{\rm had}({\rm LO}) = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \left\{-\Pi^{\rm had}(t)\right\} W(t) dt \tag{7.91}$$

with

$$W(t) = \frac{4m^4}{\sqrt{t^2+4m^2t}(t+2m^2+\sqrt{t^2+4m^2t})^2}. \tag{7.92}$$

Such a representation is well-known and is often written as a parametric integral [270, 271].

The representation in Eq. (7.89) is suitable for the evaluation of the hadronic contributions to the anomalous magnetic moment by using experimental data, since it can be rewritten in terms of the hadronic cross section for e^+e^- annihilation. The representation in Eq. (7.91) is more suitable for a theoretical study as perturbation theory should preferably be applied in the Euclidean domain. Integration by parts in Eq. (7.91) results in

$$\int_{4m_\pi^2}^\infty \frac{ds}{s} K(s)\rho^{\rm had}(s) = \int_0^\infty \left(-\frac{d\Pi^{\rm had}(t)}{dt}\right) F(t) dt, \qquad F(t) = \int_t^\infty W(\zeta) d\zeta \tag{7.93}$$

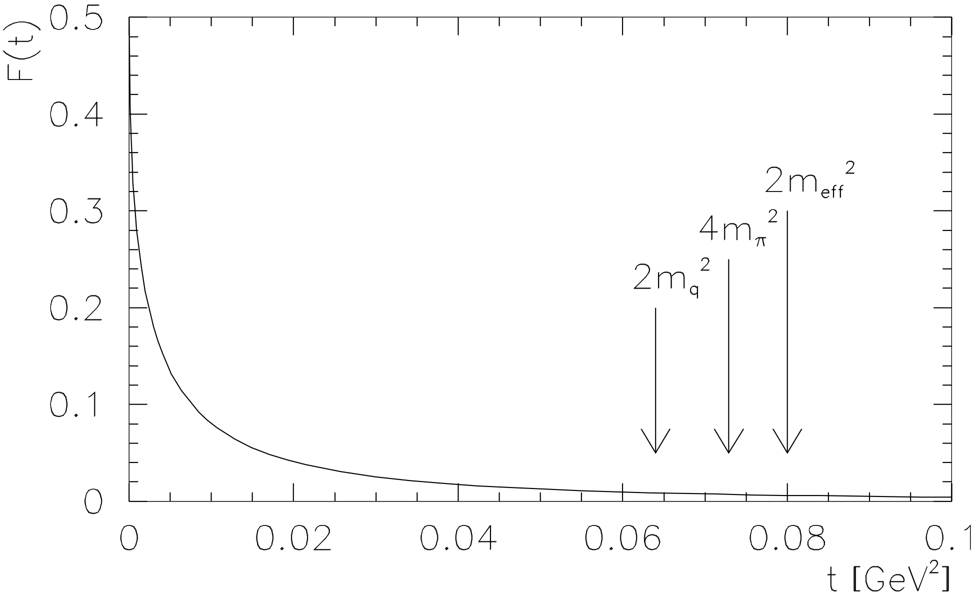


Figure 7.8: The LO Euclidean weight function $F(t)$

with

$$F(t) = \frac{1}{2}\left(\frac{t+2m^2-\sqrt{t^2+4m^2t}}{t+2m^2+\sqrt{t^2+4m^2t}}\right) = \frac{2m^4}{(t+2m^2+\sqrt{t^2+4m^2t})^2}. \tag{7.94}$$

The behaviour of the function $F(t)$ is shown in Fig. 7.8 for small and large t. It reads

$$F(t)|_{t\to 0} = \frac{1}{2} - \frac{\sqrt{t}}{m} + O(t), \qquad F(t)|_{t\to\infty} = \frac{m^4}{2t^2} + O(1/t^3). \tag{7.95}$$

The surface terms of the integration by parts vanish because the integrand in Eq. (7.91) satisfies the conditions $|\Pi^{\rm had}(t)| < C\sqrt{t}$ at small t and $|\Pi^{\rm had}(t)| < C'/t^2$ at large t with some given constants C, C'.

A key physical quantity of the analysis is the derivative of the hadron vacuum polarization function $d\Pi^{\rm had}(t)/dt$ which is closely related to the Adler function (cf. Eq. (2.42))

$$D(t) = -t\frac{d\Pi^{\rm had}(t)}{dt}. \tag{7.96}$$

This quantity can be computed in perturbative QCD with massless quarks for large t,

$$-t\frac{d\Pi^{\rm had}(t)}{dt} = Q_q^2 N_c\left(1+\frac{\alpha_s(t)}{\pi}\right) \tag{7.97}$$

where Q_q is the charge of the quark in units of the elementary electric charge and N_c is the number of colours. Computation at small t in perturbation theory is not possible for light quarks with small masses as the theory enters the regime of strong coupling. The behavior of the function $d\Pi^{\rm had}(t)/dt$ for small t can be extracted from experiment where the lower limit of the spectrum is determined by the finite pion masses. This leads to a finite value for the function $d\Pi^{\rm had}(t)/dt$ at $t = 0$.

7.4.2 A model spectral function

Using the patterns of small and large t behaviour of the function $d\Pi^{\rm had}(t)/dt$ for the light modes, an interpolation function $f(t)$ valid for all t in the form

$$-\frac{d\Pi^{\rm had}(t)}{dt} = Q_q^2 N_c f(t), \qquad f(t) = \frac{1}{t+\Delta} \tag{7.98}$$

is suggested. Writing

$$f(t) = -\frac{dp(t)}{dt} \tag{7.99}$$

one has

$$p(t) = \ln\left(\frac{\Delta}{t+\Delta}\right), \qquad p(0) = 0. \tag{7.100}$$

The analytic properties of the function $p(t)$ are given by the cut along the positive semiaxis starting at $s = \Delta$. The discontinuity across the cut is equal to one,

$$r(s) = \frac{1}{2\pi i}\,{\rm Disc}\,p(s) = \theta(s-\Delta). \tag{7.101}$$

Thus the contribution to the anomalous magnetic moment contains an integral

$$I(\Delta) = \int_0^\infty f(t)F(t)dt \tag{7.102}$$

which is the basic quantity for the theoretical analysis. The analytical expression for $I(\Delta)$ is available but too cumbersome to be presented here. This expression is used in numerical calculations. However, in order to understand the integral in Eq. (7.102) more deeply, in particular, to find where the integral in Eq. (7.102) is saturated or what region of integration is important, an approximation can be useful. The constant approximation for the function $f(t)$,

$$f^{\rm appr}(t) = {\rm const} = f(0) = \frac{1}{\Delta} \tag{7.103}$$

gives

$$I^{\rm appr}(\Delta) = f(0)\int_0^\infty F(t)dt = f(0)\frac{m^2}{3} = \frac{m^2}{3\Delta}. \tag{7.104}$$

This result represents the leading term of the series expansion of $I(\Delta)$ for small m^2. The series expansion of $I(\Delta)$ for small m^2 up to terms of order m^6 is given by

$$I(\Delta) = \frac{1}{3}v + \left(\frac{19}{24}+\frac{1}{2}\ln v\right)v^2 + \left(\frac{77}{30}+2\ln v\right)v^3 + \dots \tag{7.105}$$

with $v = m^2/\Delta$. This series converges nicely for small values of v. By comparing different results for the anomalous magnetic moment of the muon (given in the literature in Ref. [266]), the value

$$\Delta = 4m_{\rm eff}^2, \qquad m_{\rm eff} = 201.0 \pm 1.8\,MeV \tag{7.106}$$

is obtained. The function $r(s)$ in Eq. (7.101) is depicted in Fig. 7.9 for $m_{\rm eff} = 201\,MeV$. This completes the quantitative description of the interpolation function for the two-point correlator of the light modes which can be used for the computation of the hadronic contributions at NLO. This interpolation is named “model 1” in the following.

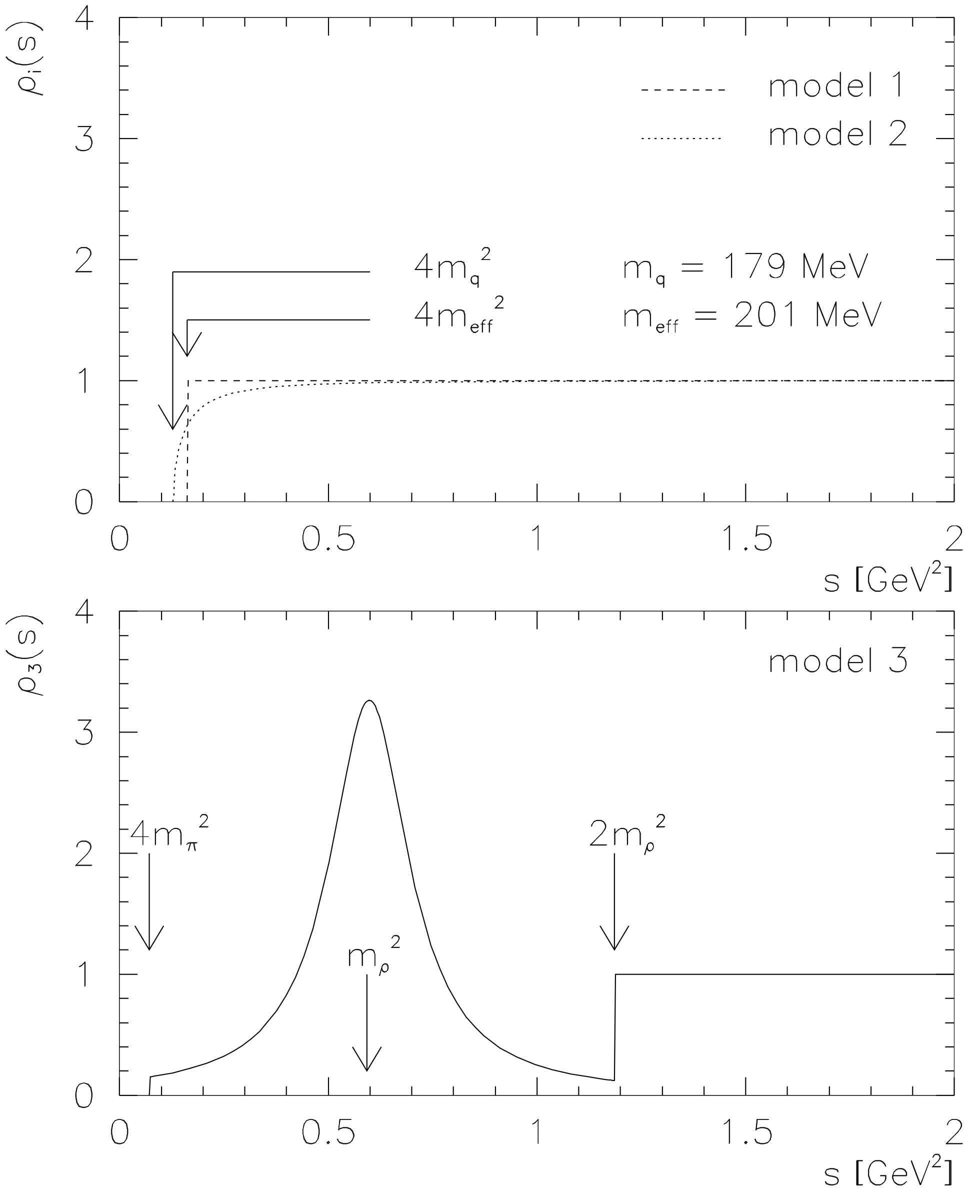


Figure 7.9: s-dependence of the spectral functions $\rho_1(s) = r(s)$ of model 1 in Eq. (7.101) and $\rho_2(s) = 3\rho_q(s)$ of model 2 in Eq. (7.108) (upper diagram), as compared to the spectral function $\rho_3(s) = \rho^{\text{had}}(s)$ for model 3 in Eqs. (7.109) and (7.111). $m_{\text{eff}} = 201\,MeV$, $m_q = 179\,MeV$ and central values $m_\rho = 769.9\,MeV$ and $\Gamma_\rho = 150.2\,MeV$ [127] are used.

7.4.3 Alternatives for the model spectral function

The interpolation function for the two-point correlator of hadronic electromagnetic currents in Eq. (7.98) is very simple. One can use more sophisticated interpolations. A formal criterion for the choice of the interpolation is its consistency with general principles of quantum field theory (analyticity and unitarity in this case). A practical criterion is its simplicity such that analytical calculations become technically feasible. One can turn to free field models in a search for mathematical functions that can be used in the interpolation procedure. For instance, the scalar or fermionic correlation functions with masses as free parameters can be taken as suitable candidates. The fermionic interpolation function was considered in detail in Ref. [243]. It is given by the expression

$$\pi(t, m_q) = \left(\frac{1}{3z} - 1\right)\varphi(z) - \frac{1}{9},$$

$$\varphi(z) = \frac{1}{\sqrt{z}}\operatorname{artanh}(\sqrt{z}) - 1, \quad z = \frac{t}{4m_q^2 + t}. \tag{7.107}$$

The discontinuity across the cut $(4m_q^2, \infty)$ at $t = -s - i0$ is given by the fermionic spectral density of the form

$$\rho_q(s) = \frac{1}{3}\sqrt{1 - \frac{4m_q^2}{s}}\left(1 + \frac{2m_q^2}{s}\right). \tag{7.108}$$

A pictorial representation of $\rho_q(s)$ is shown in Fig. 7.9. The two functions $f(t, m_{\text{eff}})/3$ and $-d\pi(t, m_q)/dt$ coincide within 1% accuracy in the interval $t = (0, m_q^2)$ if the effective parameters are related through $m_{\text{eff}}/m_q = \sqrt{5}/2 \approx 1.12$. The interpolation given by the spectral density in Eq. (7.108) will be referred to as "model 2" in the following.

In order to show that the particular shape of the spectral density is of importance only up to a "smearling" in the low energy region, a more realistic interpolation for the vacuum polarization function in the Euclidean domain can be chosen in the simple form

$$\rho^{\text{had}}(s) = 2m_\rho^2\delta(s - m_\rho^2) + \theta(s - 2m_\rho^2). \tag{7.109}$$

This is the one-scale no-parameter model that was introduced in the last section which satisfies the duality constraints from the operator product expansion. The hadronic scale of the model is given by the ρ-meson mass m_ρ which is eventually fixed from experiment [127]. The spectrum in Eq. (7.109) gives an interpolation function of the form

$$f^{\text{had}}(t) = \frac{2m_\rho^2}{(t + m_\rho^2)^2} + \frac{1}{t + 2m_\rho^2}. \tag{7.110}$$

The value of the interpolation function $f^{\text{had}}(t)$ at the origin $t = 0$ reads $f^{\text{had}}(0) = 5/2m_\rho^2$. However, the approximation of an infinitely narrow resonance in Eq. (7.109) is too rough for computing such an integral. A natural modification of the spectrum is to introduce a finite width for the ρ meson. This is achieved by replacing the function $\delta(s - m_\rho^2)$ by the Breit-Wigner function for the resonance part of the spectrum in Eq. (7.109),

$$\begin{aligned}
\rho_{\text{R}}^{\text{had}}(s) &= \frac{2m_\rho^2}{\pi}\frac{\Gamma_\rho m_\rho}{(s - m_\rho^2 + \Gamma_\rho^2/4)^2 + \Gamma_\rho^2 m_\rho^2}, \\
\rho_{\Gamma}^{\text{had}}(s) &= \theta(s - 4m_\pi^2)\theta(2m_\rho^2 - s)\rho_{\text{R}}^{\text{had}}(s) + \theta(s - 2m_\rho^2).
\end{aligned} \tag{7.111}$$

The interpolation function based on this spectrum will be called "model 3" in the following. Fig. 7.9 shows the s-dependence of $\rho_\Gamma^{\rm had}(s)$. The expression for the resonance part of the spectrum reduces to $2m_\rho^2\delta(s-m_\rho^2)$ in the limit $\Gamma_\rho \to 0$. Using the Breit-Wigner form of the spectrum for the region $4m_\pi^2 < s < 2m_\rho^2$ one finds the contribution of the resonance to the interpolation function in the Euclidean domain

$$f_{\rm R}^{\rm had}(t) = \int_{4m_\pi^2}^{2m_\rho^2} \frac{\rho_{\rm R}^{\rm had}(s)ds}{(s+t)^2}. \tag{7.112}$$

The interpolation function in the Euclidean domain for the spectrum with nonzero width reads

$$f_\Gamma^{\rm had}(t) = f_{\rm R}^{\rm had}(t) + \frac{1}{t+2m_\rho^2}. \tag{7.113}$$

Computing the value of the interpolation function at the origin for $\Gamma_\rho = 150.2\,MeV$ [127] one finds

$$f_\Gamma^{\rm had}(0) = f_{\rm R}^{\rm had}(0) + \frac{1}{2m_\rho^2} = (5.15+0.84)\,GeV^{-2} = 6.0\,GeV^{-2} \tag{7.114}$$

instead of the result obtained before in the infinitely narrow resonance approximation.

7.4.4 Comparison of results for the three models

The spectral functions taken for model 1 from Eq. (7.101), for model 2 from Eq. (7.108), and for model 3 from Eq. (7.111) are shown in Fig. 7.9 in order to allow one to compare these models. Neither model 1 nor model 2 has a discontinuity across the positive semiaxis of the s-plane resembling the experimental spectrum. However, both models result in integrals over the spectrum for the respective kernels which are very close to the result obtained in the experimentally inspired model 3 and, eventually, to the data. From the purely mathematical point of view this is related to the fact that the procedure of analytic continuation is an incorrectly posed problem: small variations of functions in the Euclidean domain can produce big variations on the cut.

The t-dependence of the Euclidean representation by the functions $f(t)$, $-3d\pi(t)/dt$, and $f_\Gamma^{\rm had}(t)$ is shown in Fig. 7.10. A phenomenological interpretation of the situation is given by duality between hadrons and free light fermions with QCD quantum numbers as for the particular application related to the computation of the anomalous magnetic moment. Of course, the main objective of the calculation of the hadronic contribution at the leading order from experiment is to reach a high precision. The use of direct data seems to be superior to a parameterization of the spectrum from indirect observations. However, as soon as the integral over the data is computed, a smooth interpolation function of a simple form can be introduced in the Euclidean domain to be used in higher order calculations. Because this interpolation function is explicit and complies with the general properties of analyticity and unitarity one can find its discontinuity across the positive semiaxis and perform further calculations in the spectral representation as well. The analysis of NLO contributions along these lines shows that the data-based results are accurately reproduced [243]. Numerical values for all these models are compared extensively in Ref. [266], showing agreement up to $O(1\%)$.

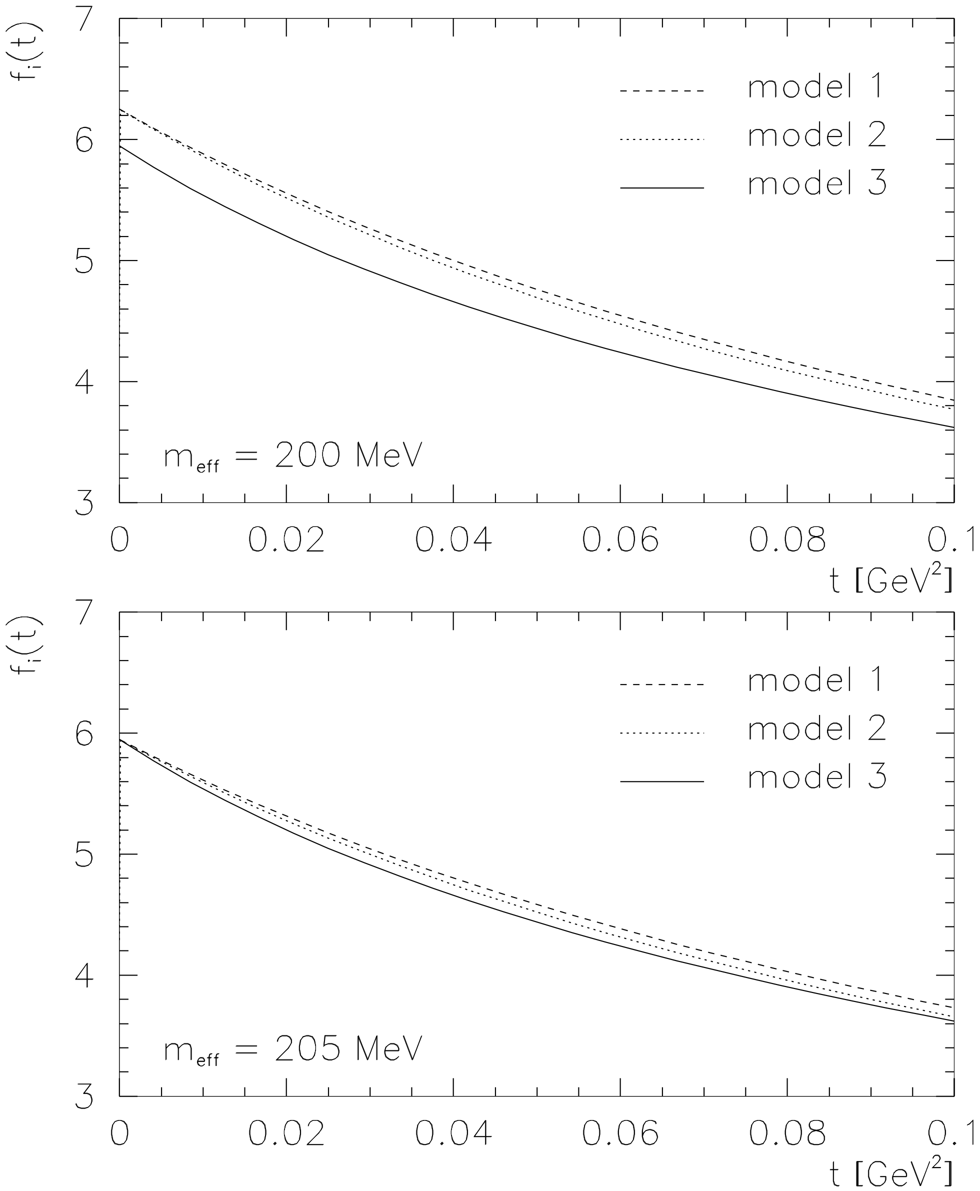
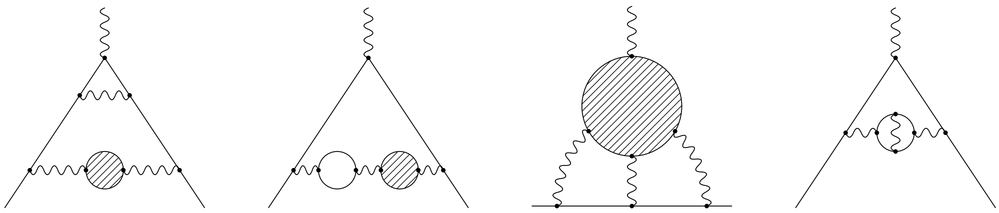


Figure 7.10: The functions $f_i(t)$ for the three different models where $f_1(t) = f(t)$ is given by Eq. (7.98), $f_2(t) = -3d\pi(t)/dt$ is given by Eq. (7.107) and $f_3(t) = f_\Gamma^{\rm had}(t)$ is given by Eq. (7.113). $\Delta = 4m_{\rm eff}^2$ is used where $m_{\rm eff} = 200\,MeV$ is applied for the upper diagram and $m_{\rm eff} = 205\,MeV$ for the lower diagram. The parameter m_q used for $\pi(t)$ is connected to $m_{\rm eff}$ by $m_q = 2m_{\rm eff}/\sqrt{5}$. The values $m_\rho = 769.9\,MeV$ and $\Gamma_\rho = 150.2\,MeV$ used in $f_R^{\rm had}$ are taken from Ref. [127].



(a) (b) (c) (d)

Figure 7.11: NLO contributions to the anomalous magnetic moment of the muon involving the contribution of the hadronic two-point correlator (a), a lepton-hadron type (so-called double bubble) diagram (b), the light-by-light contribution (c), and the two-photon Green function (d)

7.4.5 Hadronic contribution at next-to-leading order

The interpolation given by the function $f(t)$ for the two-point correlator with the numerical value of the phenomenological parameter from Eq. (7.106) is now used at NLO. Two of the NLO diagrams involving the hadronic two-point correlator are shown in Fig. 7.11(a) and (b). The NLO contribution is an integral of Im $\Pi^{\rm had}(s)$ with the two-loop kernel $K^{(2)}(s)$,

$$a_\mu^{\rm had}({\rm NLO}) = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty \frac{ds}{s} K^{(2)}(s)\rho^{\rm had}(s). \tag{7.115}$$

The analytical expression for the kernel $K^{(2)}(s)$ is known [272]. Assuming that the infrared scale M_h of the hadronic spectrum $\rho^{\rm had}(s)$ is larger than m (the infrared scale of the data is given by the explicit cutoff at $\sqrt{s} = 2m_\pi$) one can use an expansion of $K^{(2)}(s)$ in m^2/s under the integration sign in Eq. (7.115) to generate an expansion in m/M_h for the integral. For example, the vertex part of the kernel has an expansion [273]

$$\begin{aligned} K_{\rm ver}^{(2)}(s) &= \frac{m^2}{s}\left(\frac{223}{27} - \frac{2\pi^2}{3} - \frac{23}{18}\ln\left(\frac{s}{m^2}\right)\right) + \\ &+\frac{m^4}{s^2}\left(\frac{8785}{576} - \frac{37\pi^2}{24} - \frac{367}{108}\ln\left(\frac{s}{m^2}\right) + \frac{19}{72}\ln^2\left(\frac{s}{m^2}\right)\right) + \\ &+\frac{m^6}{s^3}\left(\frac{13072841}{216000} - \frac{883\pi^2}{120} - \frac{10079}{1800}\ln\left(\frac{s}{m^2}\right) + \frac{141}{40}\ln^2\left(\frac{s}{m^2}\right)\right) + \ldots. \end{aligned} \tag{7.116}$$

Generally, the terms of the expansion contain powers and logarithms of the variable m^2/s. For pure powers one can use a generating integral representation with a polynomial $P(x)$, given by

$$m^2\int_0^1 \frac{P(x)dx}{m^2x+s} = \frac{m^2}{s}\sum_n a_n \left(\frac{m^2}{s}\right)^n, \qquad a_n = \int_0^1 P(x)(-x)^n dx. \tag{7.117}$$

A given polynomial $P(x)$ restores the pure power expansion of Eq. (7.116). For the logarithmic part the generating integral representation can be chosen with a polynomial $G(x)$ of the form

$$m^2\int_0^1 \frac{G(x)dx}{sx+m^2} = G_1(m^2/s) + G_2(m^2/s)\ln\left(\frac{s}{m^2}\right). \tag{7.118}$$

The polynomial $G(x)$ generates polynomials $G_1(x)$, $G_2(x)$ through Eq. (7.118). The mixture of pure powers due to the polynomial $G_1(x)$ leads to a redefinition of the polynomial $P(x)$ in Eq. (7.117). Using Eqs. (7.115) and (7.117) one finds the expression for the pure power part of the expansion to be

$$\int_0^\infty \frac{ds}{s} K^{(2)}(s)|_{\rm power}\rho^{\rm had}(s) = \int_0^1 \frac{dx}{x} P(x)[-\Pi^{\rm had}(m^2 x)] \tag{7.119}$$

which reduces to derivatives of $\Pi^{\rm had}(t)$ at the origin and gives the analytic part of the expansion in m/M_h. For the logarithmic part one finds the representation

$$\int_0^\infty \frac{ds}{s} K^{(2)}(s)|_{\rm power\&log}\rho^{\rm had}(s) = \int_0^1 dx\, G(x)[-\Pi^{\rm had}(m^2/x)] \tag{7.120}$$

which is sensitive to the entire Euclidean domain and gives the nonanalytic part of the expansion containing $\ln(m/M_h)$. This procedure can be performed up to any finite order in m^2, and the whole calculation can be organized in a way such that only Euclidean values of momenta are necessary for $\Pi^{\rm had}(-q^2)$. Therefore, this procedure can be performed even if the spectral density is not known pointwise. However, if one uses simple model spectral functions like the one in Eq. (7.101) (model 1), even the integration in Minkowskian space can be performed easily.

For the spectral density $\rho_1^{\rm had}(s) = r(s)$ starting at $s = \Delta = 4m_{\rm eff}^2$ (see Fig. 7.9), the basic elements that emerge in Eq. (7.115) are integrals of the form

$$\mathcal{M}_{n,p}(\Delta) = \Delta^n \int_\Delta^\infty \frac{ds}{s^{n+1}} \ln^p\left(\frac{s}{m^2}\right) \tag{7.121}$$

for which the recurrence relation

$$\mathcal{M}_{n,p}(\Delta) = \frac{1}{n}\ln^p(\Delta/m^2) + \frac{p}{n}\mathcal{M}_{n,p-1}(\Delta), \qquad \mathcal{M}_{n,0}(\Delta) = \frac{1}{n} \tag{7.122}$$

can be used. With Eq. (7.116) one finally obtains

$$a_{\mu,\rm ver}^{\rm had}(\rm NLO) = \left(\frac{\alpha}{\pi}\right)^3 V(m^2/\Delta) \tag{7.123}$$

with

$$V(v) = \frac{v}{9}\left(\frac{377}{18} - 2\pi^2 + \frac{23}{6}\ln v\right) + \frac{v^2}{9}\left(\frac{23647}{1152} - \frac{37\pi^2}{16} + \frac{677}{144}\ln v + \frac{19}{48}\ln^2 v\right) + o(v^2) \tag{7.124}$$

where $o(v^2)$ is any function that satisfies $\lim_{v\to 0} o(v^2)/v^2 = 0$. For brevity only two terms of the expansion of the function $V(m^2/\Delta)$ at small m^2/Δ have been presented, resulting from the corresponding expansion of the kernel in Eq. (7.116). This contribution is only one part of the next-to-leading order contributions. In Ref. [266] the so-called light-by-light contribution (see Fig. 7.11(c)) and two-photon Green function (Fig. 7.11(d)) are considered in more detail which is not repeated here. To conclude this section, the numerical values shown in Ref. [266] demonstrate that even with a simple model like the one introduced here the calculation of next-to-leading order contributions can be accomplished – contributions which become important as the designed target accuracy 40×10^{-11} [239] for the measurement of a_μ will be reached.

7.5 Low-energy gluon contributions

Before starting with the construction of sum rules for the determination of e.g. the charm quark mass, as it will be explained in the next section, a further constraint has to be mentioned. Besides the considerations on the "value", i.e. the reliability and usefulness of different kinds of modified moments in Section 7.3, also the use of direct moments is restricted by results presented in Ref. [274] which are dealt with in detail in this section.

A contribution of massless intermediate states to the correlators of heavy quark currents is discussed in this section. For the correlator of the vector currents such a contribution first appears at the $O(\alpha_s^3)$ order of perturbation theory and is given by a *three-gluon state*. This gluon contribution to the correlator has a qualitatively new feature – its absorptive part starts at zero energy in contrast to other contributions where the absorptive parts start at the two-particle threshold. This feature determines the analytic structure of the correlator at small q^2 – at the order $O(\alpha_s^3)$ of perturbation theory a cut along the positive semiaxis emerges. The non-analyticity at the origin resulting from such a cut leads to strong limitations on the observables that can be theoretically constructed for comparison with experimental data. Because the data are most precise near the production threshold, the theoretical analysis should enhance this part of the spectrum. Technically an enhancement of the near-threshold contributions is achieved by considering integrals of the production rate with weight functions which suppress the high-energy tail of the spectrum. The integrals with weight functions $1/s^n$ for different positive integer n (known from the previous considerations as direct moments of the spectral density) are most often used in the sum rule analysis. Theoretically such moments are given by the $q^2 = 0$ derivatives of the vacuum polarization function $\Pi(q^2)$ which is a basic quantity for the analysis of the heavy quark production in the $J^{PC} = 1^{--}$ channel.

7.5.1 The effective action

In order to obtain the relevant three-gluon contributions, an investigation of the special issues of quantum field theory is of order here, given by the gauge group $SU(N_c) \otimes U(1)$ representing both gluons and photons (see e.g. Ref. [88, 275]). The Lagrangian of a heavy fermion field ψ interacting with a gauge field $\mathcal{B}$ of the gauge group reads

$$L = \bar{\psi}\,(i\gamma^\mu\partial_\mu + \gamma^\mu\mathcal{B}_\mu - m)\,\psi \tag{7.125}$$

where $\mathcal{B}_\mu = eA_\mu + g_sB_\mu$. Here A_μ is a gauge field of the $U(1)$ subgroup (photon) with the coupling constant e and B_μ is a gauge field of the $SU(N_c)$ subgroup (gluon) with the coupling constant g_s. The matrix notation for the non-Abelian gauge field potentials is used, $B_\mu = t_aB_\mu^a$, t_a are generators of the gauge group $SU(N_c)$. A generating functional $W[J]$ of connected Green functions is given by a functional integral with the sources J,

$$\begin{aligned} Z[J] &= \exp(iW[J]) = \int[d\bar{\psi}\,d\psi]\exp\left(i\int L^J(x)d^4x\right) = \\ &= \int[d\bar{\psi}\,d\psi]\exp\left(i\int\left(\bar{\psi}(i\gamma^\mu\partial_\mu + \gamma^\mu\mathcal{B}_\mu - m)\psi + J^\mu\mathcal{B}_\mu\right)d^4x\right) \end{aligned} \tag{7.126}$$

where the product $J^\mu\mathcal{B}_\mu$ implies a trace with respect to the representation of combined group $SU(N_c)\otimes U(1)$. A proper gauge fixing is implied as well. The effective action $\Gamma[\bar{\mathcal{B}}]$

for the gauge field is then given by the Legendre transform

$$\Gamma[\bar{\mathcal{B}}] = W[J] - J\bar{\mathcal{B}}, \qquad \bar{\mathcal{B}} = \frac{\delta W[J]}{\delta J}. \tag{7.127}$$

It was shown that this procedure is equivalent to the more direct calculation in external fields (see e.g. Ref. [275]). It is also a generalization of results obtained for constant external fields [276]. Up to leading order in $\hbar$ the effective action constructed with a Legendre transform can also be found through

$$\begin{aligned}\exp(i\Gamma[\mathcal{B}]) &= \int [d\bar{\psi}\, d\psi] \exp\left(i \int \bar{\psi}(i\gamma^\mu\partial_\mu + \gamma^\mu\mathcal{B}_\mu - m)\psi\, d^4x\right) = \\ &= \int \det\left(i\gamma^\mu\partial_\mu + \gamma^\mu\mathcal{B}_\mu - m\right) d^4x \end{aligned}\tag{7.128}$$

where $\mathcal{B}$ is now a classical gauge field. By using the identity $\det M = \exp(\mathrm{Tr}(\ln M))$ for an operator M one continues with

$$i\Gamma[\mathcal{B}] = \int \mathrm{Tr}\Big[\ln\left(i\gamma^\mu\partial_\mu + \gamma^\mu\mathcal{B}_\mu - m\right)\Big] d^4x. \tag{7.129}$$

Using the leading order inverse fermion propagator

$$S_0^{-1} = i\gamma^\mu\partial_\mu - m, \tag{7.130}$$

the effecive action $\Gamma[\mathcal{B}]$ can be expanded in $\mathcal{B}$ to obtain

$$\begin{aligned} i\Gamma[\mathcal{B}] &= \int \Big\{\mathrm{Tr}\left[\ln(S_0^{-1})\right] + \mathrm{Tr}\Big[\ln(1+\gamma^\mu\mathcal{B}_\mu S_0)\Big]\Big\} d^4x = \\ &= \int \Big\{\mathrm{Tr}\left[\ln(S_0^{-1})\right] + \mathrm{Tr}[\gamma^\mu\mathcal{B}_\mu S_0] + \frac{1}{2}\mathrm{Tr}[\gamma^\mu\mathcal{B}_\mu S_0\gamma^\nu\mathcal{B}_\nu S_0] + \\ &\quad + \frac{1}{3}\mathrm{Tr}[\gamma^\mu\mathcal{B}_\mu S_0\gamma^\nu\mathcal{B}_\nu S_0\gamma^\rho\mathcal{B}_\rho S_0] + \frac{1}{4}\mathrm{Tr}[\gamma^\mu\mathcal{B}_\mu S_0\gamma^\nu\mathcal{B}_\nu S_0\gamma^\rho\mathcal{B}_\rho S_0\gamma^\sigma\mathcal{B}_\sigma S_0] + O(\mathcal{B}^5)\Big\} d^4x. \end{aligned}\tag{7.131}$$

The first term can be omitted for it does not depend on $\mathcal{B}$ and therefore will not contribute to the current. Remaining with the electromagnetic part for the moment, the current is given by

$$eJ = -\frac{\delta\Gamma[\mathcal{B}]}{\delta A}, \qquad eJ^\mu = -\frac{\delta\Gamma[\mathcal{B}]}{\delta A_\mu} = \mathrm{Tr}\left[ie\gamma^\mu \frac{1}{i\gamma^{\mu'}\partial_{\mu'} + \gamma^{\mu'}\mathcal{B}_{\mu'} - m}\right]. \tag{7.132}$$

This expression could be calculated in principle, taking a general gauge. This general gauge, however, would take the most complicated form

$$A_\mu(x) = \frac{i}{2}\int \partial_x^\alpha D(x-y)F_{\alpha\mu}(y)dy \tag{7.133}$$

where $F_{\alpha\mu}$ is the field strength tensor. This expression can be seen to be correct by calculating (all derivatives acting with respect to x)

$$\begin{aligned} \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) &= \frac{i}{2}\int \Big(\partial_\mu\partial^\alpha D(x-y)F_{\alpha\nu}(y) - \partial_\nu\partial^\beta D(x-y)F_{\beta\mu}(y)\Big)\, dy = \\ &= \frac{i}{2}\int \Big(-ig_\mu^\alpha\delta(x-y)F_{\alpha\nu}(y) + ig_\nu^\beta\delta(x-y)F_{\beta\mu}(y)\Big)\, dy = \\ &= \frac{1}{2}\int \delta(x-y)\left(F_{\mu\nu}(y) - F_{\nu\mu}(y)\right) dy = F_{\mu\nu}(x). \end{aligned}\tag{7.134}$$

The situation simplifies because for small values of x the special choice

$$A^a_\mu(x) = \frac{1}{2}x^\alpha F^a_{\alpha\mu}(x) \tag{7.135}$$

can be used. The corresponding expression in the general $SU(N_c)\otimes U(1)$ case is given by

$$\mathcal{B}_\mu(x) = \frac{1}{2}x^\alpha \mathcal{G}_{\alpha\mu}(x), \qquad \mathcal{G}_{\alpha\mu} = F_{\alpha\mu} + G^a_{\alpha\mu}t_a. \tag{7.136}$$

At first sight this solution seem to spoil the calculability of an effective action. Indeed, if the current is calculated from the expression of the effective action with this simple solution for the gauge field, the configuration space components x^α acting on the vacuum state result vanish. On the other hand, because of

$$\int e^{iqx}d^4x = (2\pi)^4\delta^{(4)}(q) \quad \to \quad \int d^4x = (2\pi)^4\delta^{(4)}(0) \tag{7.137}$$

for $q \to 0$, one ends up with an indefinite expression $0\cdot\infty$. The way out of this situation is given by "shielding" the vacuum state against the action of x^α. This is done by a formal trick, namely to differentiate the expression with respect to the mass m (which occurs in S_0 only) and to integrate again afterwards. The trick "works" because of the fact that

$$\frac{\partial}{\partial m}S_0 = S_0^2 \tag{7.138}$$

where the trace can be rearranged accordingly so that one of the factors S_0 is standing in front of the whole expression and the other at the end. For the fourth order contribution in Eq. (7.131) one obtains for instance

$$i\frac{\partial}{\partial m}\Gamma_4[\mathcal{B}] = \mathrm{Tr}[S_0\gamma^\mu\mathcal{B}_\mu S_0\gamma^\nu\mathcal{B}_\nu S_0\gamma^\rho\mathcal{B}_\rho S_0\gamma^\sigma\mathcal{B}_\sigma S_0] \tag{7.139}$$

(the factor 4 from the Leibnitz rule cancels the factor from the expansion of the logarithm). Now one can indeed use the simple gauge fixing in Eq. (7.136). The field strength tensor components can be extracted from the trace (only the trace over the representation space of the algebra is kept), and one is left with

$$i\frac{\partial}{\partial m}\Gamma_4[\mathcal{B}] = \frac{1}{16}t(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma)\mathrm{Tr}(\mathcal{G}_{\alpha\mu}\mathcal{G}_{\beta\nu}\mathcal{G}_{\gamma\rho}\mathcal{G}_{\delta\sigma}). \tag{7.140}$$

In momentum space ($x_\mu \to -i\partial/\partial p^\mu$) the coefficient $t(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma)$ is given by

$$t(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma) = \mathrm{Tr}\left(S(p)\gamma^\mu\partial^\alpha S(p)\gamma^\nu\partial^\beta S(p)\gamma^\rho\partial^\gamma S(p)\gamma^\sigma\partial^\delta S(p)\right). \tag{7.141}$$

On the level of Feynman diagrams, this trace can be understood differently as well. Because of

$$\begin{aligned} S(p+k) &= S(p) + k_\mu\partial^\mu S(p) + O(k^2), && (7.142)\\ -i\partial^\mu S(p) &= -i\frac{\partial}{\partial p_\mu}S(p) \;=\; S(p)\gamma^\mu S(p) && (7.143)\end{aligned}$$

and

$$S(p+k) = S(p) + S(p)i\not{k}S(p) + S(p)i\not{k}S(p)i\not{k}S(p) + O(k^3), \tag{7.144}$$

the trace can be understood as an expansion of

$$\mathrm{Tr}\left(S(p)\gamma^\mu S(p-k_0)\gamma^\nu S(p-k_0-k_1)\gamma^\rho S(p-k_0-k_1-k_2)\gamma^\sigma S(p-k_0-k_1-k_2-k_3)\right) \tag{7.145}$$

to first order in the momenta k_0, k_1, k_2, and k_3. Taking $-k_0$ to be the incoming momentum in the diagram in Fig. 7.12, this setting would lead to a diagram where the momentum is not conserved from the very beginning. Only the condition $k_0 + k_1 + k_2 + k_3 = 0$ would lead to momentum conservation. This path will not be followed here. Instead, the expression in Eq. (7.141) is calculated directly. But because there are many technical details involved again, the explicit calculation is presented in Appendix K. At this point only the result is shown,

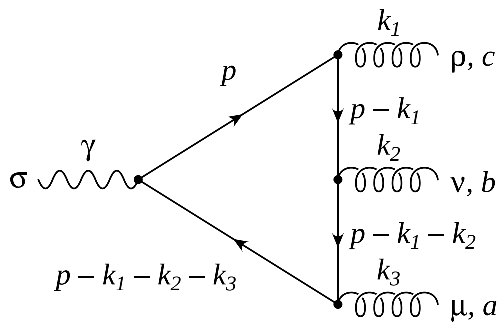


Figure 7.12: 3-gluon–photon diagram

$$\Delta\Gamma_{\mathrm{QCD}} = \frac{eg_s^3 d_{abc}}{180m^4(4\pi)^2}\Big[14\mathrm{Tr}(FG^aG^bG^c) - 5\mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c)\Big] \tag{7.146}$$

where d_{abc} are the totally symmetric $SU(N_c)$ structure constants defined by the relation $d_{abc} = 2\mathrm{Tr}(\{t_a, t_b\}t_c)$. The trace in Eq. (7.146) is understood as a trace with respect to the Lorentz indices of the fields, i.e. one considers the field strength tensors of gauge fields as matrices for which $\mathrm{Tr}(FG^a) = F^{\mu\nu}G^a_{\nu\mu}$. This makes the formulae shorter and more transparent. Note that the two-gluon transitions are forbidden according to the generalization of Furry's theorem to non-Abelian theories [277]. Therefore, the diagram in Fig. 7.12 calculated in the limit of low energies (and, therefore, heavy quarks) is the leading contribution in the calculation of the correlator function. But before proceeding to the calculation of the induced current, the result is compared with a corresponding expression given in QED.

7.5.2 Comparison with QED

The effective action within QED corresponding to Fig. 7.12 with gluons substituted by photons is known as *Euler-Heisenberg Lagrangian* [275],

$$\Delta\Gamma_{\mathrm{QED}} = \frac{2\alpha^2}{45m^4}\Big[(\vec{E}^2 - \vec{H}^2)^2 + 7(\vec{E}\cdot\vec{H})^2\Big], \qquad \alpha = \frac{e^2}{4\pi}. \tag{7.147}$$

This expression can be obtained by a direct calculation in the same way as the result in Eq. (7.146). One can also extract it from Eq. (7.146) by modifying the gauge group factors and taking into account the symmetry of the action with respect to the external gauge fields. Note that Eq. (7.146) is only the term linear in the photon field while the higher order contributions are not explicitly written down because they are redundant for the primary purpose to determine the low-energy structure of the heavy quark correlators. In the following some relations between the fourth-order monomials of the photon field are given to convert the basis of Eq. (7.146) into the traditional QED basis used in Eq. (7.147). With the definitions for the electric field $\vec{E}$ and the magnetic field $\vec{H}$,

$$F^{0j} = -F_{0j} = -E_j, \quad F^{i0} = -F_{i0} = E_i, \quad F^{ij} = F_{ij} = -\epsilon_{ijk}H_k \tag{7.148}$$

one finds

$$\mathrm{Tr}(F^2) = 2(\vec{E}^2 - \vec{H}^2), \qquad \mathrm{Tr}(F^4) = 2(\vec{E}^2 - \vec{H}^2)^2 + 4(\vec{E}\cdot\vec{H})^2. \tag{7.149}$$

Using these results for the traces, the correspondence between Eq. (7.146) and its QED counterpart in Eq. (7.147) can be established easily.

7.5.3 The induced vector current

The induced electromagnetic current J^μ is an effective electromagnetic current for the low-energy effective theory describing the interaction of photons and fermions. As shown before, an expression for this induced current is given by the derivative of the effective action with respect to the external Abelian gauge field, shown in Eq. (7.132). The derivative with respect to A_μ can be replaced by a derivative with respect to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$eJ^\mu(x) = -\frac{\delta F_{\mu'\nu'}}{\delta A_\mu}\frac{\delta\Gamma[\mathcal{B}]}{\delta F_{\mu'\nu'}} = -2\partial_\nu\frac{\delta\Gamma[\mathcal{B}]}{\delta F_{\nu\mu}}. \tag{7.150}$$

Expressed differently, one can say that after having integrated out the heavy quark field in the functional integral, one obtains the relation

$$\langle j^\mu\rangle_\psi = \langle\bar\psi\gamma^\mu\psi\rangle_\psi \equiv J^\mu \tag{7.151}$$

for the electromagnetic current of the heavy quark. With the explicit expression for the effective action given in Eq. (7.146) one obtains

$$J^\mu = \partial_\nu\mathcal{O}^{\mu\nu}, \qquad \mathcal{O}^{\mu\nu} = \frac{-g_s^3 d_{abc}}{90m^4(4\pi)^2}[14(G^aG^bG^c)^{\mu\nu} - 5(G^a)^{\mu\nu}\mathrm{Tr}(G^bG^c)]. \tag{7.152}$$

Note that the current conservation $\partial_\mu J^\mu = 0$ following from the original relation $\partial_\mu j^\mu = 0$ is necessary for a gauge invariant interaction with photons. It is automatically guaranteed because the operator $\mathcal{O}^{\mu\nu}$ is antisymmetric, $\mathcal{O}^{\mu\nu} + \mathcal{O}^{\nu\mu} = 0$. Higher order corrections in the coupling constant α of the $U(1)$ subgroup are omitted. The induced electromagnetic current in Eq. (7.152) is a correction of order $1/m^4$ in the inverse heavy quark mass which vanishes in the limit of an infinitely heavy quark. Corrections in the inverse heavy quark masses are important for tests of the Standard Model at the present level of precision and have been already discussed in various areas of particle phenomenology [89, 278, 279, 280].

7.5.4 The induced tensor current

Expressions for the induced currents with quantum numbers other than that of the electromagnetic current $J^{PC} = 1^{--}$ can be obtained in a similar way. A review was recently presented in Ref. [281]. As an example the calculation of the induced (antisymmetric) tensor current interacting with photons in the context of the effective action is discussed here. A tensor current of the form

$$j^{\mu\nu} = \bar\psi\sigma^{\mu\nu}\psi, \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] \tag{7.153}$$

is considered and its low-energy limit induced by a heavy quark loop is calculated. The properties of this current are rather similar to those of the electromagnetic current. Note that the classical vector mesons (like ρ, ω, ϕ) interact with this current and can thus be created by it. Introducing an interaction

$$\Delta L_T = g_T \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu} \tag{7.154}$$

in the Lagrangian of heavy quarks, one readily finds the effective action for gauge fields induced by such a vertex. The low-energy limit at one-loop order reads

$$\Gamma_T = \frac{-g_T g_s^3 d_{abc}}{6m^3(4\pi)^2}\Big(2\mathrm{Tr}(FG^aG^bG^c) - \mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c)\Big) \tag{7.155}$$

with the same notations as in Eq. (7.146) (cf. Appendix K for details of the calculation). According to the form of the effective interaction in Eq. (7.154) the induced current $J_{\mu\nu}$ is given by a derivative

$$g_T J^{\mu\nu} = -\frac{\delta \Gamma_T}{\delta F_{\mu\nu}} \tag{7.156}$$

and explicitly reads

$$J^{\mu\nu} = \frac{-g_s^3 d_{abc}}{6m^3(4\pi)^2}\left(2(G^aG^bG^c)^{\mu\nu} - (G^a)^{\mu\nu}\mathrm{Tr}(G^bG^c)\right). \tag{7.157}$$

Note the lower power of the heavy quark mass in Eq. (7.157) as compared to Eq. (7.152).

7.5.5 The spectrum for the induced vector current correlator

The consequences of the low-energy contributions to vector and tensor currents just calculated become manifest when the correlator functions are calculated. First the case of the vector current is discussed where the data obtained from e^+e^- annihilation experiments are rather precise. The correlator of the induced vector current J^μ (as integral kernel of the correlator function) has the general form

$$\langle TJ^\mu(x)J^\nu(0)\rangle = -\partial_\alpha \partial_\beta \langle T\mathcal{O}^{\mu\alpha}(x)\mathcal{O}^{\nu\beta}(0)\rangle \tag{7.158}$$

where an explicit representation of the current given by the derivative of the antisymmetric operator $\mathcal{O}^{\mu\nu}$ has been employed. The resulting correlator $\langle T\mathcal{O}^{\mu\alpha}(x)\mathcal{O}^{\nu\beta}(0)\rangle$ in Eq. (7.158) contains only gluonic operators. Such correlators were considered previously in the framework of perturbation theory [71, 282, 283]. In leading order of perturbation theory the correlator in Eq. (7.158) has the topological structure of a sunset diagram as shown in Fig. 7.13(a). Technically, a convenient procedure of computing the sunset-type diagrams is to work in configuration space [107, 108, 112]. One finds

$$\langle TJ_\mu(x)J_\nu(0)\rangle = \frac{-34 d_{abc}d_{abc}}{2025\pi^4 m^8}\left(\frac{\alpha_s}{\pi}\right)^3 \left(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2\right)\frac{1}{x^{12}}. \tag{7.159}$$

A Fourier transform of the correlator in Eq. (7.159) gives the vacuum polarization function in momentum space which reads

$$12\pi^2 i \int \langle TJ_\mu(x)J_\nu(0)\rangle e^{iqx} d^4x = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2) \tag{7.160}$$

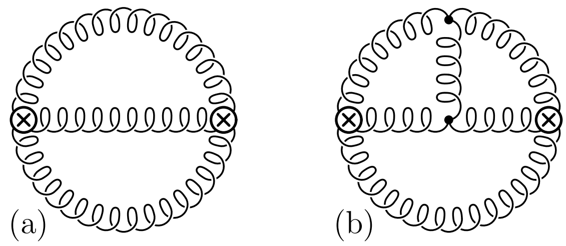


Figure 7.13: Induced massless correlator diagrams

where at small q^2 ($q^2 \ll m^2$)

$$\Pi(q^2)|_{q^2\approx 0} = C_g \left(\frac{q^2}{4m^2}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right), \qquad C_g = \frac{17 d_{abc}d_{abc}}{243000}\left(\frac{\alpha_s}{\pi}\right)^3. \tag{7.161}$$

For QCD with the colour group $SU(3)$ one has $d_{abc}d_{abc} = 40/3$. The spectral density of the vacuum polarization function $\Pi(q^2)$ in Eq. (7.160) at small values for s is given by

$$\rho(s)|_{s\approx 0} = C_g \left(\frac{s}{4m^2}\right)^4. \tag{7.162}$$

Note that the spectral density given in Eq. (7.162) can be found without an explicit calculation of the Fourier transform of the correlator in Eq. (7.160). Instead of computing the Fourier transform one can use a spectral decomposition (dispersion representation) in configuration space which was heavily employed for the analysis of sunset diagrams in Ref. [107, 108, 112]. In this particular instance the spectral representation of the correlator in configuration space reads

$$\frac{i}{x^{12}} = \frac{\pi^2}{2^8\Gamma(6)\Gamma(5)}\int_0^\infty s^4 D(x^2, s)ds \tag{7.163}$$

with $D(x^2, s)$ being the propagator of a scalar particle of mass $\sqrt{s}$,

$$D(x^2, m^2) = \frac{im\sqrt{-x^2}K_1(m\sqrt{-x^2})}{4\pi^2(-x^2)} \tag{7.164}$$

where $K_1(z)$ is the McDonald function (a modified Bessel function of the third kind, see e.g. Appendix D.1.3 and Ref. [85]). $\Gamma(z)$ is Euler's gamma function.

An asymptotic behaviour of the spectral density of the corresponding contribution for large energies (where the limit of massless quarks can be used) enters the expression for the ratio $R(s)$ of e^+e^- annihilation into hadrons and has been known since long ago [233]. This term is usually called light-by-light (lbl) contribution and reads

$$R^{\rm lbl}(s) = \left(\frac{\alpha_s}{\pi}\right)^3 \frac{d_{abc}d_{abc}}{1024}\left(\frac{176}{3} - 128\zeta(3)\right). \tag{7.165}$$

Here $\zeta(z)$ is the Riemann ζ function with $\zeta(3) = 1.20206\ldots$ The contribution to the spectral density given in Eq. (7.165) is negative while the result given in Eq. (7.162) is positive as it should be the case for the spectral density of the electromagnetic current which is a Hermitean operator.

7.5.6 The spectrum for the induced tensor current correlator

The results for the correlator of the tensor current given in Eq. (7.157) are slightly more complicated. The correlator reads

$$\begin{aligned} 12\pi^2 i \int \langle T J_{\mu\nu}(x) J_{\alpha\beta}(0)\rangle e^{iqx} d^4x &= (g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})\Pi_g(q^2) + \\ &+ (g_{\mu\alpha}q_\nu q_\beta - g_{\mu\beta}q_\nu q_\alpha - g_{\nu\alpha}q_\mu q_\beta + g_{\nu\beta}q_\mu q_\alpha)\Pi_q(q^2) \end{aligned} \tag{7.166}$$

with the two scalar amplitudes $\Pi_g(q^2)$ and $\Pi_q(q^2)$. With the explicit expressions for the induced tensor current $J_{\mu\nu}$ in Eq. (7.157) one finds

$$\Pi_g(q^2) = -\frac{q^2}{2}\Pi_q(q^2) = \frac{d_{abc}d_{abc}}{3240}\left(\frac{\alpha_s}{\pi}\right)^3 \frac{-q^2}{4}\left(\frac{q^2}{4m^2}\right)^3 \ln\left(\frac{\mu^2}{-q^2}\right). \tag{7.167}$$

The physical content of the amplitudes $\Pi_g(q^2)$ and $\Pi_q(q^2)$ is related to the contributions of the states with $J^{PC} = 1^{--}$ and $J^{PC} = 1^{+-}$, resp. Note that the sum rule analysis for the mesons with quantum numbers $J^{PC} = 1^{+-}$ has been done in Ref. [284] with quark interpolating currents. From the present results, a possibility is seen also to use gluonic currents as interpolating operators for such mesons. The validity of such a description depends strongly on the strength of the interaction of the meson in question with the corresponding interpolating operator which is difficult to estimate.

Note that there are only two independent gluonic operators available to construct the induced currents under consideration. The electromagnetic current is given by a derivative of a special linear combination of these operators while the tensor current is given by a linear combination of the operators themselves. There is one more current relevant to the situation. It originates from the Gordon decomposition of the electromagnetic current (see e.g. Ref. [275])

$$2m\bar\psi\gamma^\mu\psi = \partial_\nu(\bar\psi\sigma^{\mu\nu}\psi) + \bar\psi\, i\overleftrightarrow{D}^\mu\psi, \qquad \overleftrightarrow{D} = \overrightarrow{D} - \overleftarrow{D}\,. \tag{7.168}$$

This relation holds for the induced currents as well. The left hand side and the right hand side of Eq. (7.168) have different parity since the number of Dirac γ-matrices between spinor fields is different. This is reflected in an additional factor m at the left hand side of Eq. (7.168). In the massless limit the two types of currents are alien and can never mix. At the level of induced currents the Dirac structure of the initial heavy quark currents is reflected in different degrees of suppression by the heavy quark mass m.

7.5.7 The spectrum for the mixed current correlator

Having both vector and tensor induced currents at hand, one can study a mixed correlator of the form

$$12\pi^2 i \int \langle T J_\mu(x) J_{\alpha\beta}(0)\rangle e^{iqx} d^4x = i(g_{\mu\alpha}q_\beta - g_{\mu\beta}q_\alpha)\Pi_M(q^2) \tag{7.169}$$

with a single scalar amplitude $\Pi_M(q^2)$. Such mixed correlators are useful in sum rule applications [285]. One finds

$$\Pi_M(q^2) = \frac{d_{abc}d_{abc}}{8100}\left(\frac{\alpha_s}{\pi}\right)^3 \frac{-q^2}{4m}\left(\frac{q^2}{4m^2}\right)^3 \ln\left(\frac{\mu^2}{-q^2}\right). \tag{7.170}$$

The physical content of the amplitude $\Pi_M(q^2)$ is given by $J^{PC}=1^{--}$ resonances, i.e. by the Υ meson family in case of b quarks for the original currents and by the ρ meson family in case of induced currents at low energies.

7.5.8 Consequences for the moments

The qualitative new feature of the effective currents in Eqs. (7.152) and (7.157) is that they are expressed through massless fields. Therefore, the spectrum of the two-point correlators of these currents start at zero energy (at least for finite orders of perturbation theory). This feature drastically changes the analytic structure of the two-point correlators of these currents and, in particular, their infrared (IR) or small q^2 behaviour because of the branching point (cut) singularity of $\Pi(q^2)$ at the origin $q^2=0$. This new feature of having a nonvanishing spectrum below the formal tree-level two-particle threshold appearing at $O(\alpha_s^3)$ order of perturbation theory for induced current correlators has important phenomenological consequences. Indeed, such a change of the analytic structure of induced current correlators affects strongly the theoretical expressions for some observables usually employed in heavy quark physics for the precision determination of the parameters of heavy quarks and their interactions.

Sum rule expressions for heavy quarks are normally formulated in terms of moments

$$\mathcal{M}_n=\int\frac{\rho(s)ds}{s^{n+1}} \tag{7.171}$$

of the spectral density $\rho(s)$. The moments in Eq. (7.171) are related to the derivatives of the correlator function $\Pi(q^2)$ at the origin,

$$\mathcal{M}_n=\frac{1}{n!}\left(\frac{d}{dq^2}\right)^n\Pi(q^2)\Big|_{q^2=0}. \tag{7.172}$$

Such moments are chosen in order to suppress the high energy part of the spectral density $\rho(s)$ which is not measured accurately in the experiment. Within the sum rule method one assumes that the moments in Eq. (7.171) can be calculated for any n or, equivalently, that the derivatives in Eq. (7.172) exist for any n. The existence of these moments seems to be obvious because one implicitly assumes that the spectral density $\rho(s)$ of the current correlator of the heavy quarks with mass m vanishes below the two-particle threshold at $s=4m^2$. But according to the considerations of this section [274], this assumption does not hold true at $O(\alpha_s^3)$. Caused by the factor $(s/4m^2)^4$ in Eq. (7.162), the moments become infrared singular for $n\geq 4$ in case of the induced vector current. This can already be seen by looking at the factor $1/m^4$ in the induced vector current in Eq. (7.152). For the induced tensor current the corresponding moments start to diverge earlier because of a weaker suppression by the heavy quark mass. The corresponding factor in Eq. (7.157) is $1/m^3$ instead of $1/m^4$. Therefore, in this case the moments become infrared singular already for $n\geq 3$. However, the sum rule analysis of the charmonium ($c\bar{c}$ system) was done for n up to 7 [286, 287], for the bottonium ($b\bar{b}$ system) even up to 20. In view of the result on the low-energy behaviour of the spectral density, one has either to limit the accuracy of theoretical calculations for the standard moments to the $O(\alpha_s^2)$ order of perturbation theory which seems insufficient for a high precision analysis of quarkonium systems (especially for $b\bar{b}$ with the Coulomb resummation performed to all orders) or to use only the first few moments.

7.6 Finite energy sum rules for the charm quark

The mass of the charm quark is a parameter which is very important for phenomenological applications. The charm quark mass lies between the light and heavy mass parameters. The dependence of perturbation theory on the strange quark mass is very weak because it is only a small parameter.[2] Therefore, the uncertainty does not play a crucial role. On the other hand, the numerical value of the bottom quark mass is quite well determined. Therefore, the strong dependence of perturbation theory on the bottom quark mass can be met by a reasonable uncertainty.

Roughly speaking, while the numerical value of the charm quark mass has the same uncertainty as the value for the bottom quark mass at an absolute scale, the relative uncertainty is bigger by a factor of three. On the other hand, the influence of perturbation theory on the charm quark mass is already important. The conclusion is that it is desirable to decrease the uncertainty for the numerical value of the charm quark mass in order to fit the improving accuracy of experimental data. The improved perturbation theory has then to be compared again with the experimental data in order to determine other parameters and to verify (or falsify) the Standard Model of elementary particle physics. The work presented in this section joins in this effort by analyzing *finite energy sum rules* for the charm quark mass.

7.6.1 The path to sum rules

As has been already discussed in previous sections, the charm quark mass is an important ingredient in high precision measurements of parameters such as the running QED coupling α or the anomalous magnetic moment. In order to construct the sum rules, one can start with one of these quantities. The running QED coupling at the energy $\sqrt{q^2}$ is given by (cf. Eq. (7.23))

$$\alpha(-q^2) = \frac{\alpha_0}{1 - 4\pi\alpha_0\Pi(q^2)} \tag{7.173}$$

While $\alpha_0 = 1/137.04$ is a constant, the q^2 dependence of the running coupling is depending on the scheme selected for the renormalization of the singularities occuring in the correlator function $\Pi(q^2)$. However, the quantity

$$\frac{1}{4\pi\alpha(q^2)} + \Pi(q^2) = \frac{1}{4\pi\alpha_0} \tag{7.174}$$

is an invariant for all schemes at all values of q^2. One can therefore use a specific scheme like the $\overline{\text{MS}}$ scheme and at the point $q^2 = 0$ to obtain the running coupling in by using the correlator function for the *same* scheme and at the same point. The aim is thus to calculate this correlator function. As usual, because of the nonperturbative character of the spectral function at thresholds, one cannot calculate the correlator function by integrating the spectral density over the whole range from the first resonances to infinity. Instead one starts with subdividing this huge interval up into a part up to a energy square s_0 for which one takes experimental input data for the spectral density, and a part starting from s_0, assuming that s_0 is large enough to use perturbation theory. The expression then

[2] A still weaker dependence would be given by the u and d quarks if they were taken into account.

reads

$$\Pi(0)=\int_0^{s_0}\frac{\rho(s)}{s}ds+\int_{s_0}^{\infty}\frac{\rho(s)}{s}ds\approx\int_0^{s_0}\frac{\rho^{\rm exp}(s)}{s}ds+\int_{s_0}^{\infty}\frac{\rho^{\rm the}(s)}{s}ds \tag{7.175}$$

where the lower limit $s=0$ can be replaced by the lowest lying resonance in the experimental spectrum. The prediction for the correlator function from theory is given as integral of the corresponding spectral density $\rho^{\rm the}(s)$ starting from the mass threshold $4m^2$ where the mass value should be taken corresponding to the selected renormalization scheme. This threshold need not (and for the $\overline{\rm MS}$ mass actually doesn't) coincide with the experimental threshold. The mass parameter is actually the pole mass which will be determined by the sum rule method. For this purpose one considers the above approximation as an equality, replaces the theory part by a difference and writes

$$\Pi(0)=\int_0^{s_0}\frac{\rho^{\rm exp}(s)}{s}ds-\int_{4m^2}^{s_0}\frac{\rho^{\rm the}(s)}{s}ds+\Pi^{\rm the}(0). \tag{7.176}$$

The interesting part of this expression is the difference of the first two terms, and the interesting question is, how far they cancel each other, in order that $\Pi(0)=\Pi^{\rm the}(0)$. Therefore, the last term will no longer be considered, instead a set of sum rules (corresponding to values $q^2\neq 0$) will be set up for this difference. The sum rules one has to consider for this purpose are those constructed by moments closest to the moments given by the original equation. The two sum rules

$$R_{-1}=\int_0^{s_0}\rho^{\rm exp}(s)ds=\int_{4m^2}^{s_0}\rho^{\rm the}(s)ds,\qquad R_1=\int_0^{s_0}\frac{1}{s^2}\rho^{\rm exp}(s)ds=\int_{4m^2}^{s_0}\frac{1}{s^2}\rho^{\rm the}(s)ds \tag{7.177}$$

suffice to determine the two parameters of the sum rule analysis, namely s_0 and m. But actually a broader range of moments will be considered. It turns out that s_0 obtains a value which lies just above the resonance region. Therefore, the input for the left hand sides are given by resonance contributions – a fact that renders the form of these sum rules, namely to modell the non-perturbative contribution by summing over these modes up to a reasonable energy $\sqrt{s_0}$.

k	M_k $[GeV]$	Γ_k $[10^{-3}\,GeV]$	Γ_k^{ee} $[10^{-6}\,GeV]$	R_{-1} $[GeV^2]$	R_1 $[GeV^{-2}]$
1	3.09687 ± 0.00004	0.087 ± 0.005	5.26 ± 0.37	6.483	0.0705
2	3.68596 ± 0.00009	0.50907 ± 0.00013	2.12 ± 0.18	3.110	0.01685
3	3.7699 ± 0.0025	25.3 ± 2.9	0.26 ± 0.04	0.390	0.001931
4	4.040 ± 0.010	52 ± 10	0.75 ± 0.15	1.206	0.004527
5	4.159 ± 0.020	70 ± 20	0.77 ± 0.23	1.275	0.004260
6	4.415 ± 0.006	43 ± 15	0.47 ± 0.10	0.826	0.002174
				13.29	0.1002

Table 7.5: Resonance contributions for the c quark, as taken from Ref. [127]

The experiment side

Different threshold regions occur for higher and higher center-of-mass energies. Here the contribution of the c quark is considered. For the left hand sides of the sum rules which are called the "experimental sides" one obtains the values shown in Table 7.5 using the resonance positions M_k and the e^+e^- widths Γ_k^{ee} given in Ref. [127] according to

$$
\begin{aligned}
R_{-1}^{\rm exp} &= \int_0^{s_0} \rho^{\rm exp}(s)ds = \frac{9\pi}{\bar\alpha^2 N_c Q_c^2}\sum_{k=1}^{6}\Gamma_k^{ee}M_k,\\
R_{1}^{\rm exp} &= \int_0^{s_0} \frac{\rho^{\rm exp}(s)}{s^2}ds = \frac{9\pi}{\bar\alpha^2 N_c Q_c^2}\sum_{k=1}^{6}\Gamma_k^{ee}(M_k)^{-3}
\end{aligned}
\tag{7.178}
$$

where $\bar\alpha = \alpha(4m_c^2) \approx \alpha_0 = 1/137.04$ is the running QED coupling at the threshold, $N_c = 3$ is the number of colours, $Q_c = 2/3$ is the electric charge of the c quark, and the spectral density is normalized to 1.

The theory side

On the theory side, the spectral density can be considered in different approximations. In the following subsection these expressions will be built up from very simple model spectral densities up to first order expressions. All this can and will be done analytically for the moments from R_{-2} to R_3 (note the restriction for the degree of the moments from the considerations of the last section [274]). In the steps that follow, second order approximations are included numerically, and different resummation techniques are applied.

7.6.2 Theory side spectral densities

The expressions considered here include the leading order contribution in α_s, the first order corrections in a Schwinger parametrization as well as in the exact form. Finally, the first (gluon) condensate contributions are included. In all cases, the spectral density is alternatively taken to be a constant, given by its the value at the upper boundary s_0 (so-called "box approximation"), or the exact function.

Leading order box approximation

To leading order the spectral density is given by

$$
\rho^{(0)}(s) = \sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)
\tag{7.179}
$$

(normalized to 1). The box approximation is obtained by multiplying this function at the point $s = s_0$ with the corresponding integral over 1. Therefore, one obtains

$$
\begin{aligned}
R_{-1}^{[0]} &= \rho^{(0)}(s_0)\int_{4m^2}^{s_0} ds = \rho^{(0)}(s_0)(s_0-4m^2) = s_0\left(1-\frac{4m^2}{s_0}\right)^{3/2}\left(1+\frac{2m^2}{s_0}\right),\\
R_{1}^{[0]} &= \rho^{(0)}(s_0)\int_{4m^2}^{s_0} \frac{ds}{s^2} = \rho^{(0)}(s_0)\left(\frac{1}{4m^2}-\frac{1}{s_0}\right) = \frac{1}{4m^2}\left(1-\frac{4m^2}{s_0}\right)^{3/2}\left(1+\frac{2m^2}{s_0}\right).
\end{aligned}
\tag{7.180}
$$

Leading order exact integration

It is easy in this case to perform the exact integration. With the substitution

$$v=\sqrt{1-\frac{4m^2}{s}}\quad\Rightarrow\quad s=\frac{4m^2}{1-v^2},\quad ds=\frac{8m^2v\,dv}{(1-v^2)^2}\tag{7.181}$$

one obtains

$$\begin{aligned}R_{-1}^{(0)} &= \int_{4m^2}^{s_0}\rho^{(0)}(s)ds \;=\; \int_{4m^2}^{s_0}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)ds \;=\\ &= \int_0^{v_0}v\left(1+\frac{1-v^2}{2}\right)\frac{8m^2v\,dv}{(1-v^2)^2} \;=\; 4m^2\int_0^{v_0}\frac{(3-v^2)v^2}{(1-v^2)^2}dv\end{aligned}\tag{7.182}$$

where $v_0=\sqrt{1-4m^2/s_0}$ is the velocity at the energy square s_0. Partial fractioning

$$\frac{(3-v^2)v^2}{(1-v^2)^2}=\frac{1}{2(1-v)^2}+\frac{1}{2(1+v)^2}-1\tag{7.183}$$

leads to

$$\begin{aligned}R_{-1}^{(0)} &= 4m^2\int_0^{v_0}\left(\frac{1}{2(1-v)^2}+\frac{1}{2(1+v)^2}-1\right)dv \;=\; 4m^2\left[\frac{1}{2(1-v)}-\frac{1}{2(1+v)}-v\right]_0^{v_0}=\\ &= 4m^2\left[\frac{v}{1-v^2}-v\right]_0^{v_0} \;=\; 4m^2\left[\frac{v^3}{1-v^2}\right]_0^{v_0} \;=\; 4m^2\frac{v_0^3}{1-v_0^2} \;=\; s_0\left(1-\frac{4m^2}{s_0}\right)^{3/2}.\end{aligned}\tag{7.184}$$

For the second moment one obtains

$$\begin{aligned}R_1^{(0)} &= \int_{4m^2}^{s_0}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)\frac{ds}{s^2} \;=\; \int_0^{v_0}v\left(1+\frac{1-v^2}{2}\right)\frac{(1-v^2)^2}{(4m^2)^2}\frac{8m^2v\,dv}{(1-v^2)^2} \;=\\ &= \frac{1}{4m^2}\int_0^{v_0}(3-v^2)v^2dv \;=\; \frac{1}{4m^2}\left[v^3-\frac{1}{5}v^5\right]_0^{v_0} \;=\; \frac{1}{4m^2}v_0^3(1-\frac{1}{5}v_0^2) \;=\\ &= \frac{1}{20m^2}\left(1-\frac{4m^2}{s_0}\right)^{3/2}\left(5-1+\frac{4m^2}{s_0}\right) \;=\; \frac{1}{5m^2}\left(1-\frac{4m^2}{s_0}\right)^{3/2}\left(1+\frac{m^2}{s_0}\right).\end{aligned}\tag{7.185}$$

First order contribution from the $\overline{\text{MS}}$-scheme

If a mass different from the pole mass is used, one has to take into account the first order difference. For example, the relation between the pole mass m and the running $\overline{\text{MS}}$-mass $m_{\overline{\text{MS}}}$ is given by

$$m=\bar m\left(1+\frac{4}{3}\left(\frac{\bar\alpha_s}{\pi}\right)\right).\tag{7.186}$$

This leads to a change of the velocity variable v,

$$\begin{aligned}v &= \sqrt{1-\frac{4m^2}{s}} \;=\; \sqrt{1-\frac{4\bar m^2}{s}\left(1+\frac{8}{3}\left(\frac{\bar\alpha_s}{\pi}\right)\right)} \;=\; \bar v\sqrt{1-\frac{8}{3\bar v^2}\left(\frac{\bar\alpha_s}{\pi}\right)\frac{4\bar m^2}{s}} \;=\\ &= \bar v\left(1-\frac{4}{3\bar v^2}\left(\frac{\bar\alpha_s}{\pi}\right)\frac{4\bar m^2}{s}\right) =\bar v-\frac{4}{3\bar v}\left(\frac{\bar\alpha_s}{\pi}\right)\frac{4\bar m^2}{s}\qquad\left(\bar v=\sqrt{1-4\bar m^2/s}\right)\end{aligned}\tag{7.187}$$

where only expressions up to $O(\bar{\alpha}_s)$ have been taken into account. Therefore, the leading order spectral density can be written as

$$\begin{aligned}
\rho^{(0)}(s) &= \sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right) = \frac{1}{2}v(3-v^2) = \\
&= \frac{1}{2}\left(\bar{v}-\frac{4}{3\bar{v}}\left(\frac{\bar{\alpha}_s}{\pi}\right)\frac{4\bar{m}^2}{s}\right)\left(3-\bar{v}^2+\frac{8}{3}\left(\frac{\bar{\alpha}_s}{\pi}\right)\frac{4\bar{m}^2}{s}\right) = \\
&= \frac{1}{2}\bar{v}(3-\bar{v}^2)+\frac{8}{3}\bar{v}\left(\frac{\bar{\alpha}_s}{\pi}\right)\frac{4\bar{m}^2}{s}-\frac{4}{3\bar{v}}(3-\bar{v}^2)\left(\frac{\bar{\alpha}_s}{\pi}\right)\frac{4\bar{m}^2}{s} = \\
&= \frac{1}{2}\bar{v}(3-\bar{v}^2)-\frac{4}{\bar{v}}(1-\bar{v}^2)^2\frac{\bar{\alpha}_s}{\pi} = \bar{\rho}^{(0)}(s)-\frac{4}{\bar{v}}\left(\frac{4\bar{m}^2}{s}\right)^2\frac{\bar{\alpha}_s}{\pi}. \qquad (7.188)
\end{aligned}$$

For the moments there are two changes which have to be taken into account, namely the change of the lower limit and the change of the spectral density,

$$\begin{aligned}
R_n^{(0)} &= \int_{4m^2}^{s_0}\frac{ds}{s^n}\rho^{(0)}(s) = \int_{4\bar{m}^2}^{s_0}\frac{ds}{s^n}\rho^{(0)}(s)-\int_{4\bar{m}^2}^{4m^2}\frac{ds}{s^n}\rho^{(0)}(s) = \\
&= \int_{4\bar{m}^2}^{s_0}\frac{ds}{s^n}\bar{\rho}^{(0)}(s)-4\left(\frac{\bar{\alpha}_s}{\pi}\right)\int_{4\bar{m}^2}^{s_0}\frac{ds}{s^n}\left(\frac{4\bar{m}^2}{s}\right)^2-\int_{4\bar{m}^2}^{4m^2}\frac{ds}{s^n}\bar{\rho}^{(0)}(s) \qquad (7.189)
\end{aligned}$$

Because the change of the lower limit is only small, the box approximation can be used for the third part. But because the spectral density vanishes for $s=4\bar{m}^2$, one obtains

$$\int_{4\bar{m}^2}^{4m^2}\frac{ds}{s^n}\bar{\rho}^{(0)}(s)\approx(4m^2-4\bar{m}^2)\bar{\rho}^{(0)}(4\bar{m}^2)(4\bar{m}^2)^{-n}=0 \qquad (7.190)$$

while for the second part one obtains

$$\begin{aligned}
\int_{4\bar{m}^2}^{s_0}\frac{ds}{s^n}\bar{\rho}^{(0)}(s) &= (4\bar{m}^2)^2\int_{4\bar{m}^2}^{s_0}\frac{ds}{s^{n+2}} = \qquad (7.191)\\
&= \begin{cases}(4\bar{m}^2)^2\ln\left(\dfrac{s_0}{4\bar{m}^2}\right)>0 & \text{for } n=-1\\ -\dfrac{(4\bar{m}^2)^2}{(n+1)s^{n+1}}\Big|_{4\bar{m}^2}^{s_0}=\dfrac{(4\bar{m}^2)^2}{n+1}\left((4\bar{m}^2)^{-n-1}-s_0^{-n-1}\right)>0 & \text{for } n\neq-1\end{cases}
\end{aligned}$$

Naive running of the coupling

Since the strong coupling enters the spectral density in the previous paragraph, one faces the question of whether this coupling should be used at a fixed energy scale or as a running coupling for an energy scale which varies according to the moment degree. A naive scale definition which depends on the degree of the moment is given by the mean value of the integration measure,

$$\begin{aligned}
\int_{4m^2}^{s_0}\frac{ds}{s^n} &= (4m^2)^{1-n}\int_{x_0}^{1}x^{n-2}dx = \frac{(4m^2)^{1-n}}{n-1}\left(1-x_0^{n-1}\right) = \\
&=: (4m^2)^{1-n}\left(\frac{1}{x_0}-1\right)\bar{x}^n=(s_0-4m^2)\bar{s}^{-n}. \qquad (7.192)
\end{aligned}$$

therefore

$$\bar{x}^n=\frac{x_0}{n-1}\,\frac{1-x_0^{n-1}}{1-x_0}. \qquad (7.193)$$

There are two values for the degree n at which the expression must be considered more carefully. For the case $n=1$ one inserts $n=1+\varepsilon$ and expands in ε,

$$\bar{x}^{1+\varepsilon}=\bar{x}+O(\varepsilon)=\frac{x_0}{\varepsilon}\,\frac{-\varepsilon\ln x_0}{1-x_0}+O(\varepsilon)=\frac{x_0}{\varepsilon}\,\frac{1-x_0^\varepsilon}{1-x_0} \tag{7.194}$$

in order to obtain

$$\bar{x}=-\frac{x_0\ln x_0}{1-x_0}\qquad (n=1). \tag{7.195}$$

For $n=0$ one inserts $n=\varepsilon$ and expands again,

$$1+\varepsilon\ln\bar{x}=\frac{x_0}{1-\varepsilon}\,\frac{1-x_0^{-1}(1+\varepsilon\ln x_0)}{1-x_0}=(1+\varepsilon)\left(1+\frac{\varepsilon\ln x_0}{1-x_0}\right) \tag{7.196}$$

to obtain

$$\ln\bar{x}=1+\frac{\ln x_0}{1-x_0}\qquad (n=0). \tag{7.197}$$

Schwinger parametrization in box approximation

Before proceeding to the full first-order correction of the spectral density in α_s, the so-called *Schwinger parametrization* of this first order term is used, in terms of v given by [141]

$$\tilde{\rho}^{(sw)}(v)=v\frac{3-v^2}{2}\left\{1+\frac{4\alpha_s}{3\pi}\left(\frac{\pi^2}{2v}-\frac{3+v}{4}\left(\frac{\pi^2}{2}-\frac{3}{4}\right)\right)\right\}. \tag{7.198}$$

The moments are given by multiplying this spectral density at $v=v_0$ (corresponding to $s=s_0$) with the factors

$$(s_0-4m^2)=s_0\left(1-\frac{4m^2}{s_0}\right)=s_0v_0^2\quad\text{and}\quad\left(\frac{1}{4m^2}-\frac{1}{s_0}\right)=\frac{1}{4m^2}\left(1-\frac{4m^2}{s_0}\right)=\frac{v_0^2}{4m^2}. \tag{7.199}$$

Schwinger parametrization in exact integration

For the purpose of an exact integration the integrals

$$\begin{aligned}
I_{-1,-1} &= \int_0^{v_0}\frac{(3-v^2)v}{(1-v^2)^2}dv = \int_0^{v_0}\frac{1}{2}\left(\frac{dv}{(1-v)^2}-\frac{dv}{(1+v)^2}+\frac{dv}{1-v}-\frac{dv}{1+v}\right)=\\
&= \frac{1}{2}\left[\frac{1}{1-v}+\frac{1}{1+v}-\ln(1-v)-\ln(1+v)\right]_0^{v_0} = \frac{1}{2}\left[\frac{2(1+v^2)}{1-v^2}-\ln(1-v^2)\right]_0^{v_0}=\\
&= \frac{1+v_0^2}{1-v_0^2}-\frac{1}{2}\ln(1-v_0^2)-1 = \frac{v_0^2}{1-v_0^2}-\frac{1}{2}\ln(1-v_0^2),
\end{aligned} \tag{7.200}$$

$$I_{-1,0} = \int_0^{v_0}\frac{(3-v^2)v^2}{(1-v^2)^2}dv = \frac{v_0^3}{1-v_0^2}\qquad\text{(cf. before)} \tag{7.201}$$

$$\begin{aligned}
I_{-1,1} &= \int_0^{v_0}\frac{(3-v^2)v^3}{(1-v^2)^2}dv =\\
&= \int_0^{v_0}\left(\frac{1}{2}\left(\frac{dv}{(1-v)^2}-\frac{dv}{(1+v)^2}-\frac{dv}{1-v}+\frac{dv}{1+v}\right)-v\,dv\right) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{1-v}+\frac{1}{1+v}+\ln(1-v)+\ln(1+v)-v^2\right]_0^{v_0} = \\
&= \frac{1}{2}\left[\frac{2}{1-v^2}-v^2+\ln(1-v^2)\right]_0^{v_0} = \frac{1}{2}\left(\frac{(1+v_0^2)v_0^2}{1-v_0^2}+\ln(1-v_0^2)\right) \qquad (7.202)
\end{aligned}
$$

are calculated in case of the moment R_{-1}, and

$$
\begin{aligned}
I_{1,-1} &= \int_0^{v_0}(3-v^2)v\,dv = \left[\frac{3}{2}v^2-\frac{1}{4}v^4\right]_0^{v_0} = \frac{1}{4}(6-v_0^2)v_0^2, && (7.203)\\
I_{1,0} &= \int_0^{v_0}(3-v^2)v^2dv = \left[v^3-\frac{1}{5}v^5\right]_0^{v_0} = \frac{1}{5}(5-v_0^2)v_0^3, && (7.204)\\
I_{1,1} &= \int_0^{v_0}(3-v^2)v^3dv = \left[\frac{3}{4}v^4-\frac{1}{6}v^6\right]_0^{v_0} = \frac{1}{12}(9-2v_0^2)v_0^4 && (7.205)
\end{aligned}
$$

in case of the moment R_1. Using these integrals one obtains

$$
\begin{aligned}
R_{-1}^{(sw)}-R_{-1}^{(0)} &= 4m^2\frac{4\alpha_s}{3\pi}\left\{\frac{\pi^2}{8}(4I_{-1,-1}-3I_{-1,0}-I_{-1,1})+\frac{3}{16}(3I_{-1,0}+I_{-1,1})\right\} = \\
&= \frac{m^2\alpha_s}{2\pi}\left\{\frac{2\pi^2}{3}\left(\frac{7-6v_0-v_0^2}{1-v_0^2}v_0^2-5\ln(1-v_0^2)\right)+\right.\\
&\qquad\left.+\frac{1+6v_0+v_0^2}{1-v_0^2}v_0^2+\ln(1-v_0^2)\right\}, \qquad (7.206)\\
R_1^{(sw)}-R_1^{(0)} &= \frac{1}{4m^2}\,\frac{4\alpha_s}{3\pi m^2}\left\{\frac{\pi^2}{8}(4I_{1,-1}-3I_{1,0}-I_{1,1})+\frac{3}{16}(3I_{1,0}+I_{1,1})\right\} = \\
&= \frac{\alpha_s v_0^2}{960\pi m^2}\left\{\frac{2\pi^2}{3}\left(360-180v_0-105v_0^2+36v_0^3+10v_0^4\right)+\right.\\
&\qquad\left.+(180+45v_0-36v_0^2-10v_0^3)v_0\right\}. \qquad (7.207)
\end{aligned}
$$

First order in box approximation

The next step is to calculate the first order contribution in box approximation. The "box part" is obvious whereas the exact first order contribution to the spectral density is given by [141]

$$
\tilde\rho^{(1)}(v) = v\frac{3-v^2}{2}\left[1+\frac{4\alpha_s}{3\pi v}\left\{A(v)+\frac{3P_V(v)}{3-v^2}\ln\left(\frac{1+v}{1-v}\right)+\frac{3Q_V(v)}{3-v^2}\right\}\right] \qquad (7.208)
$$

where

$$
\begin{aligned}
A(v) &= (1+v^2)\left\{\mathrm{Li}_2\left(\frac{(1-v)^2}{(1+v)^2}\right)+2\,\mathrm{Li}_2\left(\frac{1-v}{1+v}\right)+\ln\left(\frac{(1+v)^3}{8v^2}\right)\ln\left(\frac{1+v}{1-v}\right)\right\}+\\
&\qquad +3v\ln\left(\frac{1-v^2}{4v}\right)-v\ln v, \qquad (7.209)\\
P_V(v) &= \frac{33}{24}+\frac{22}{24}v^2-\frac{7}{24}v^4, \qquad Q_V(v) = \frac{5}{4}v-\frac{3}{4}v^3. \qquad (7.210)
\end{aligned}
$$

First order in exact integration

Even this expression can be integrated exactly including the different weights. The expressions obtained by using MATHEMATICA for the integration according to

$$\begin{aligned} R_{-1}^{(1)} &= \int_{4m^2}^{s_0} \rho(s) ds \;=\; 4m^2 \int_0^{v_0} \tilde{\rho}(v) \frac{2v\,dv}{(1-v^2)^2}, \\ R_1^{(1)} &= \int_{4m^2}^{s_0} \frac{1}{s^2} \rho(s) ds \;=\; \frac{1}{4m^2} \int_0^{v_0} \tilde{\rho}(v) 2v\,dv \end{aligned} \tag{7.211}$$

and using the dilogarithm identities (see Appendix E)

$$\mathrm{Li}_2(1+v) = -\mathrm{Li}_2(-v) + \frac{\pi^2}{6} - \frac{1}{2}\ln(v^2)\ln(1+v) - i\pi\ln(1+v), \tag{7.212}$$

$$\mathrm{Li}_2(1-v) = -\mathrm{Li}_2(v) + \frac{\pi^2}{6} - \frac{1}{2}\ln(v^2)\ln(1-v), \tag{7.213}$$

$$\mathrm{Li}_2\left(\frac{1+v}{2}\right) = \mathrm{Li}_2\left(-\frac{1-v}{1+v}\right) + \frac{\pi^2}{6} + \frac{1}{2}\ln\left(\frac{2}{1+v}\right)\ln\left(\frac{(1-v)^2}{2(1+v)}\right) \tag{7.214}$$

are given by

$$R_{-1}^{(1)} = R_{-1}^{(0)} + 4m^2 \frac{\alpha_s}{4\pi} A_{-1}, \qquad R_1^{(1)} = R_1^{(0)} + \frac{1}{4m^2}\frac{\alpha_s}{4\pi} A_1 \tag{7.215}$$

where

$$\begin{aligned} A_{-1} &= \frac{1}{3}\Bigg\{ -2v(7-13v^2) + 48v(3+v^2)\left(\ln\left(\frac{1-v}{2}\right) - \frac{2}{3}\ln v\right) + \\ &\quad +(1-v)(55-17v+21v^2-3v^3)\ln\left(\frac{1+v}{1-v}\right) - 20(1-v^2)\pi^2 + \\ &\quad -16(1-v^2)\ln v \ln\left(\frac{1+v}{1-v}\right) + 24(3+v^4)\ln\left(\frac{1+v}{2}\right)\ln\left(\frac{1+v}{1-v}\right) + \\ &\quad +80(1-v^2)\left(\mathrm{Li}_2(v) - \mathrm{Li}_2(-v)\right) - 48(1-v^2)\,\mathrm{Li}_2\left(-\frac{1-v}{1+v}\right) + \\ &\quad +8(4-v^2+v^4)\left(\mathrm{Li}_2\left(\frac{(1-v)^2}{(1+v)^2}\right) + 2\,\mathrm{Li}_2\left(\frac{1-v}{1+v}\right) - 2\ln v \ln\left(\frac{1+v}{1-v}\right)\right)\Bigg\}, \end{aligned} \tag{7.216}$$

$$\begin{aligned} A_1 &= \frac{1}{9}\Bigg\{ \frac{2v}{9}(453+296v^2-93v^4) + 16v(33-16v^2+3v^4)\left(\frac{2}{3}\ln v - \ln\left(\frac{1+v}{2}\right)\right) + \\ &\quad -\frac{1-v}{3}(679-113v-230v^2+154v^3+55v^4-17v^5)\ln\left(\frac{1+v}{1-v}\right) - 8\pi^2 + \\ &\quad -32\ln v\ln\left(\frac{1+v}{1-v}\right) - 24(1-v^2)(11+2v^2-v^4)\ln\left(\frac{1+v}{2}\right)\ln\left(\frac{1+v}{1-v}\right) + \\ &\quad +32\left(\mathrm{Li}_2(v) - \mathrm{Li}_2(-v)\right) - 72\,\mathrm{Li}_2\left(\frac{(1-v)^2}{(1+v)^2}\right) - 240\,\mathrm{Li}_2\left(-\frac{1-v}{1+v}\right) + \\ &\quad +8v^2(9+3v^2-v^4)\left(\mathrm{Li}_2\left(\frac{(1-v)^2}{(1+v)^2}\right) + 2\,\mathrm{Li}_2\left(\frac{1-v}{1+v}\right) - 2\ln v\ln\left(\frac{1+v}{1-v}\right)\right)\Bigg\}. \end{aligned} \tag{7.217}$$

The gluon condensate contribution

Finally, one can include also the gluon condensate contribution. For this one uses the vector correlator given in Ref. [287] (dropping the index "V" and a factor $1/12\pi^2$)

$$\Pi(q^2) = \mathbb{1}\Pi_I + \langle g^2G^2\rangle\Pi_G \tag{7.218}$$

where

$$\Pi_I(q^2) = \int_{4m^2}^{\infty}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)\frac{ds}{s+q^2} \tag{7.219}$$

and

$$\Pi_G(q^2) = \frac{-1}{4(q^2)^2}(1-3\mathcal{J}_2(q^2/m^2)+2\mathcal{J}_3(q^2/m^2)) \tag{7.220}$$

where

$$\mathcal{J}_N(q^2/m^2) = \int_0^1 \frac{dx}{(1+x(1-x)q^2/m^2)^N}. \tag{7.221}$$

As a first step the integrals over spectral densities for the correlator parts $\mathcal{J}_N(q^2/m^2)$, denoted by $\rho_N(-s/m^2)$, are calculated. For this purpose the integration path from $4m^2$ to s_0 closely above resp. below the real axis is replaced by a circle path about the center $s = m^2$ with radius $s_0 - m^2$ which will not include the origin for s_0 but is close enough to the threshold. This circle path is denoted by C. Taking the simplest case, namely $N = 1$ and the pure integration of the spectral density without any weight, one obtains

$$\int_{4m^2}^{s_0}\rho_1(s)ds = \frac{-1}{2\pi i}\oint_C J_1(-s/m^2)ds = \frac{-1}{2\pi i}\oint_C ds\int_0^1\frac{dx}{1-x(1-x)s/m^2}. \tag{7.222}$$

One can change the Feynman parameter x to a better choice,

$$x = \frac{1}{2}(1+v), \qquad dx = \frac{1}{2}dv, \qquad v = 2x-1, \qquad x(1-x) = \frac{1}{4}(1-v^2) \tag{7.223}$$

and furthermore can use the fact that v appears only quadratically to absorb the factor 1/2 of the measure in a convolution of the interval $[-1, 1]$ to $[0, 1]$. Therefore, one finally obtains

$$\int_{4m^2}^{s_0}\rho_1(s)ds = \frac{-1}{2\pi i}\oint_C ds\int_0^1\frac{dv}{1-(1-v^2)s/(4m^2)} = \frac{4m^2}{2\pi i}\int_0^1\frac{dv}{1-v^2}\oint_C\frac{ds}{s-4m^2/(1-v^2)}. \tag{7.224}$$

Now the residue theorem can be used. The circle integral only contributes if the singularity at the point $s_p = 4m^2/(1-v^2)$ lies within the circle, such that

$$s_p - 4m^2 = \frac{4m^2}{1-v^2} - 4m^2 \le s_0 - 4m^2 \quad\Leftrightarrow\quad v^2 \le 1-\frac{4m^2}{s_0} =: v_0^2. \tag{7.225}$$

In this case the value for the residue integral is $2\pi i$, otherwise zero. For this simple case one therefore obtains

$$\int_{4m^2}^{s_0}\rho_1(s)ds = 4m^2\int_0^{v_0}\frac{dv}{1-v^2} = 2m^2\int_0^{v_0}\left(\frac{dv}{1-v}+\frac{dv}{1+v}\right) = 2m^2\ln\left(\frac{1+v_0}{1-v_0}\right). \tag{7.226}$$

If the simple example is extended to higher values in N, the circle integral will result in zero in all cases because of the residue theorem. However, an additional weight such as $1/s^2$ will change the situation drastically. In this case one has to expand this weight function $f(s)$ in a Taylor series around the pole position s_p,

$$f(s)=\sum_{i=0}^{\infty}(s-s_p)^i\frac{1}{i!}f^{(i)}(s_p) \tag{7.227}$$

and take the $i=(N-1)$th term to obtain the residue. This will be done in a more general way separately for positive and negative powers of the weight. For $f(s)=1/s^n$ one obtains

$$f(s)=\frac{1}{s^n},\qquad f^{(i)}(s)=\frac{(-1)^i(n+i-1)!}{(n-1)!s^{n+i}},\qquad f^{(N-1)}(s)=\frac{(-1)^{N-1}(n+N-2)!}{(n-1)!s^{n+N-1}} \tag{7.228}$$

and therefore

$$\begin{aligned}\int_{4m^2}^{s_0}\frac{1}{s^n}\rho_N(s)ds &= \frac{-1}{2\pi i}\oint_C\frac{1}{s^n}J_N(-s/m^2)ds \;=\\ &= -\frac{(4m^2)^N}{2\pi i}\int_0^1 dv\oint\frac{ds}{s^n\left(4m^2-(1-v^2)s\right)^N} \;=\\ &= -\frac{(4m^2)^N}{2\pi i}\int_0^1\frac{dv}{(1-v^2)^N}\oint_C\frac{(-1)^N ds}{s^n\left(s-4m^2/(1-v^2)\right)^N} \;=\\ &= (4m^2)^N\frac{(n+N-2)!}{(n-1)!(N-1)!}\int_0^{v_0}\frac{dv}{(1-v^2)^N}\left(\frac{1-v^2}{4m^2}\right)^{n+N-1} \;=\\ &= \frac{1}{(4m^2)^{n-1}}\frac{(n+N-2)!}{(n-1)!(N-1)!}\int_0^{v_0}(1-v^2)^{n-1}dv.\quad (N\geq 1,n\geq 1)\end{aligned} \tag{7.229}$$

For $f(s)=s^n$ one obtains

$$f(s)=s^n,\qquad f^{(i)}(s)=\frac{n!}{(n-i)!}s^{n-i-1},\qquad f^{(N-1)}(s)=\frac{n!}{(n-N+1)!}s^{n-N} \tag{7.230}$$

and thus

$$\begin{aligned}\int_{4m^2}^{s_0}s^n\rho_N(s)ds &= \frac{-1}{2\pi i}\oint_C s^n J_N(-s/m^2)ds \;=\\ &= -\frac{(4m^2)^N}{2\pi i}\int_0^1 dv\oint_C\frac{s^n ds}{\left(4m^2-(1-v^2)s\right)^N} \;=\\ &= -\frac{(4m^2)^N}{2\pi i}\int_0^1\frac{dv}{(1-v^2)^N}\oint_C\frac{(-1)^N s^n ds}{\left(s-4m^2/(1-v^2)\right)^N} \;=\\ &= (4m^2)^N\frac{(-1)^{N-1}n!}{(n-N+1)!(N-1)!}\int_0^{v_0}\frac{dv}{(1-v^2)^N}\left(\frac{4m^2}{1-v^2}\right)^{n-N} \;=\\ &= (4m^2)^n\frac{(-1)^{N-1}n!}{(n-N+1)!(N-1)!}\int_0^{v_0}\frac{dv}{(1-v^2)^n}.\quad (N\geq 1,n\geq N-1)\end{aligned} \tag{7.231}$$

Integrals with $\rho_0(s)=1$ vanish in all possible cases. If one now takes

$$\rho_G(s)=\frac{1}{4s^2}\left(3\rho_2(-s/m^2)-2\rho_3(-s/m^2)\right), \tag{7.232}$$

one obtains

$$
\begin{aligned}
\int_{4m^2}^{s_0} s^2\rho_G(s)ds &= \frac{1}{4}\int_{4m^2}^{s_0}\Big(3\rho_2(-s/m^2)-2\rho_3(-s/m^2)\Big)\,ds \;=\; 0, &(7.233)\\
\int_{4m^2}^{s_0} s\rho_G(s)ds &= \frac{1}{4}\int_{4m^2}^{s_0}\frac{1}{s}\Big(3\rho_2(-s/m^2)-2\rho_3(-s/m^2)\Big)\,ds \;=\\
&= \frac{1}{4}\left(\frac{3\;1!}{0!\;1!}-\frac{2\;2!}{1!\;1!}\right)\int_0^{v_0}dv \;=\; \frac{-v_0}{4}, &(7.234)\\
\int_{4m^2}^{s_0} \rho_G(s)ds &= \frac{1}{4}\int_{4m^2}^{s_0}\frac{1}{s^2}\Big(3\rho_2(-s/m^2)-2\rho_3(-s/m^2)\Big)\,ds \;=\\
&= \frac{1}{4}\left(\frac{1}{4m^2}\right)\left(\frac{3\;2!}{1!\;1!}-\frac{2\;3!}{1!\;2!}\right)\int_0^{v_0}(1-v^2)dv \;=\; 0, &(7.235)\\
\int_{4m^2}^{s_0} \frac{1}{s}\rho_G(s)ds &= \frac{1}{4}\int_{4m^2}^{s_0}\frac{1}{s^3}\Big(3\rho_2(-s/m^2)-2\rho_3(-s/m^2)\Big)\,ds \;=\\
&= \frac{1}{4}\left(\frac{1}{(4m^2)^2}\right)\left(\frac{3\;3!}{2!\;1!}-\frac{2\;4!}{2!\;2!}\right)\int_0^{v_0}(1-v^2)^2dv \;=\\
&= \frac{-3}{4(4m^2)^2}\left(v_0-\frac{2}{3}v_0^3+\frac{1}{5}v_0^5\right), &(7.236)\\
\int_{4m^2}^{s_0} \frac{1}{s^2}\rho_G(s)ds &= \frac{1}{4}\int_{4m^2}^{s_0}\frac{1}{s^4}\Big(3\rho_2(-s/m^2)-2\rho_3(-s/m^2)\Big)\,ds \;=\\
&= \frac{1}{4}\left(\frac{1}{(4m^2)^3}\right)\left(\frac{3\;4!}{3!\;1!}-\frac{2\;5!}{3!\;2!}\right)\int_0^{v_0}(1-v^2)^3dv \;=\\
&= \frac{-8}{4(4m^2)^3}\left(v_0-v_0^3+\frac{3}{5}v_0^5-\frac{1}{7}v_0^7\right). &(7.237)
\end{aligned}
$$

One can find a general expression (which is be proven to work for all integer values of n),

$$
\begin{aligned}
\int_{m^2}^{s_0} \frac{1}{s^{n+1}}\rho_G(s)ds &= \frac{1}{4}\int_{4m^2}^{s_0}\frac{1}{s^{n+3}}\Big(3\rho_2(-s/m^2)-2\rho_3(-s/m^2)\Big)\,ds \;=\\
&= \frac{1}{4}\left(\frac{1}{(4m^2)^{n+2}}\right)\left(\frac{3(n+3)!}{(n+2)!1!}-\frac{2(n+4)!}{(n+2)!2!}\right)\int_0^{v_0}(1-v^2)^{n+2}dv \;=\\
&= \frac{-(n+1)(n+3)}{4(4m^2)^{n+2}}\int_0^{v_0}(1-v^2)^{n+2}dv. &(7.238)
\end{aligned}
$$

The general result can be found by comparison with the moment provided in Ref. [287]. For this purpose the result in case of $s_0\to\infty$, i.e. $v_0\to 1$ is calculated and

$$
\begin{aligned}
\int_0^1(1-v^2)^{n+2}dv &= \frac{1}{2}\int_0^1 t^{-1/2}(1-t)^{n+2}dt \;=\; \frac{1}{2}B(1/2,n+3) \;=\\
&= \frac{\Gamma(1/2)\Gamma(n+3)}{2\Gamma(n+7/2)} \;=\; \frac{2^{n+2}(n+2)!}{(2n+5)!!} &(7.239)
\end{aligned}
$$

$(t=v^2)$ is used to obtain

$$
\int_{4m^2}^{\infty}\frac{1}{s_{n+1}}\rho_G(s)ds = \frac{-2^{n+2}(n+1)(n+3)(n+2)!}{4(4m^2)^{n+2}(2n+5)!!} = \frac{-2^{n+2}(n+1)(n+3)!}{4(2n+5)!!(4m^2)^{n+2}}. \qquad (7.240)
$$

This checks for the leading order term as well. Here one obtains

$$\begin{aligned}
\int_{4m^2}^{s_0}\frac{1}{s^{n+1}}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)ds &= \\
&= \int_0^{v_0}\left(\frac{1-v^2}{4m^2}\right)^{n+1} v\left(1+\frac{1-v^2}{2}\right)\frac{8m^2v\,dv}{(1-v^2)^2} = \\
&= \frac{1}{(4m^2)^n}\left(2\int_0^{v_0}(1-v^2)^{n-1}v^2dv+\int_0^{v_0}(1-v^2)^n v^2dv\right) \qquad (7.241)
\end{aligned}$$

and for $s\to\infty$ resp. $v_0\to 1$

$$\begin{aligned}
\int_{4m^2}^{\infty}\frac{1}{s^{n+1}}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)ds &= \\
&= \frac{1}{(4m^2)^n}\left(\int_0^1 t^{1/2}(1-t)^{n-1}dt+\frac{1}{2}\int_0^1 t^{1/2}(1-t)^n dt\right) = \\
&= \frac{1}{(4m^2)^n}\left(B(3/2,n)+\frac{1}{2}B(3/2,n+1)\right) = \\
&= \frac{1}{(4m^2)^n}\left(\frac{\Gamma(3/2)\Gamma(n)}{\Gamma(n+3/2)}+\frac{\Gamma(3/2)\Gamma(n+1)}{2\Gamma(n+5/2)}\right) = \\
&= \frac{1}{(4m^2)^n}\left(\frac{\frac{1}{2}\sqrt{\pi}\ (n-1)!2^{n+1}}{(2n+1)!!\sqrt{\pi}}+\frac{\frac{1}{2}\sqrt{\pi}\ n!2^{n+2}}{2(2n+3)!!\sqrt{\pi}}\right) = \\
&= \frac{2^n}{(4m^2)^n}\,\frac{(2n+3+n)(n-1)!}{(2n+3)!!} = 3\frac{2^n(n+1)(n-1)!}{(4m^2)^n(2n+3)!!} \qquad (7.242)
\end{aligned}$$

where

$$\Gamma(n+5/2)=\frac{(2n+3)!!\sqrt{\pi}}{2^{n+2}} \qquad (7.243)$$

is used.

7.6.3 The sum rule pairs

The contributions calculated so far are coded in the file `sumrules.add`. The different kinds of moments are enumerated according to the previous subsections. Therefore,

- `mom1[n_,s_,m_]` is leading order in box approximation
- `mom2[n_,s_,m_]` is leading order in exact integration
- `mom3[n_,s_,m_]` is Schwinger parametrization in box approximation
- `mom4[n_,s_,m_]` is Schwinger parametrization in exact integration
- `mom5[n_,s_,m_]` is first order in box approximation
- `mom6[n_,s_,m_]` is first order in exact integration
- `mom7[n_,s_,m_]` is first order with gluon condensate contribution

	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$	$n_2 = 3$
$n_1 = -2$		1.377059	1.370344	1.369071	1.371422	1.375783
$n_1 = -1$	1.377059		1.367445	1.368029	1.371137	1.375755
$n_1 = 0$	1.370344	1.367445		1.368283	1.371624	1.376026
$n_1 = 1$	1.369074	1.368031	1.368283		1.373075	1.377219
$n_1 = 2$	1.371425	1.371139	1.371624	1.373074		1.378978
$n_1 = 3$	1.375787	1.375758	1.376026	1.377215	1.378969	

Table 7.6: mass values obtained by the sum rule pair (n_1, n_2)

	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$	$n_2 = 3$
$n_1 = -2$		4.196339	4.192714	4.192032	4.193292	4.195646
$n_1 = -1$	4.196339		4.183859	4.184615	4.188644	4.194643
$n_1 = 0$	4.192714	4.183859		4.186418	4.196623	4.210066
$n_1 = 1$	4.192041	4.184621	4.186418		4.220750	4.250832
$n_1 = 2$	4.193302	4.188650	4.196623	4.220747		4.325102
$n_1 = 3$	4.195661	4.194651	4.210066	4.250818	4.325067	

Table 7.7: energy values obtained by the sum rule pair (n_1, n_2)

(the moments `mom8[n_,s_,m_]` and `mom9[n_,s_,m_]` not appearing in the list are designed for the second order Coulomb contribution used in the next subsection). The parameter `n`, `s` and `m` are the moment label, the threshold value s_0 and the mass. Having all these theory ingredients at hand, one can use them in the sum rule analysis.

For this purpose a *pair of sum rules* is constructed, consisting of equations where the left hand side contains the resonance contributions and the right hand side the corresponding theoretical moment according to Eq. (7.177). Free parameters of these equations are the mass m of the c quark and the threshold value s_0. Doing this for a whole range of moment pairs where the first order contribution is taken to be in Schwinger parametrization, for the example `mom7` one obtains the results shown in Table 7.6 for the mass m_c and in Table 7.7 for $\sqrt{s_0}$, both given in *GeV*. The tables are generated automatically by the procedures `masstable[name_,fn_]` and `energytable[name_,fn_]` where `name` is the name of the LaTeX file to be generated and `fn` is the label of the moment function used, so in this case `mom7`. The two procedures themselves use the procedure `rrsolve[fn_,n1_,n2_,n_]` for solving the sum rule pair (n_1, n_2) with n resonances included. $n = 6$ resonances have been taken to obtain a first insight, even though it is obvious that $\sqrt{s_0}$ lies below the fifth resonance. This fact has to be taken into account for further considerations.

The numbers appearing in these tables show a huge degree of stability. While these numbers are nearly independent on whether the gluon condensate contribution ($\langle g^2G^2\rangle = (0.83\,GeV)^4$) is taken into account, the values depend on the other parameters such as the QED coupling $\bar{\alpha}$ (which is assumed here to be roughly $1.03\alpha_0$) and the strong coupling α_s (taken as $\alpha_s = 0.32$). This dependence is the reason why more work is done about the form of the perturbation series in the following subsection.

	LO	NLO	NNLO	+ Gluon
$n=2$	1.417	1.86357	2.11214	2.11217
$n=1$	1.24851	1.66829	1.93996	1.93996
$n=0$	1.18593	1.50412	1.72644	1.72618
$n=-1$	1.18957	1.4229	1.58931	1.5886
$n=-2$	1.2121	1.39412	1.52702	1.52583
$n=-3$	1.23674	1.38733	1.49962	1.49795

Table 7.8: masses obtained by the sum rule for $\sqrt{s_0} = 4.6\,GeV$ including six resonances

7.6.4 Resummation and other rearrangements

As a first step the (unphysical) parameter s_0 is removed from the sum rule analysis. The only remaining parameter m can therefore be determined for each single moment.

Second order Coulomb contribution

In this analysis a second order QCD correction is included which is called *Coulombic* because it consists of a power series in $1/v$. This contribution is given by [118]

$$\Delta\tilde{\rho}^{(2c)}(v) = C_F\left(\frac{\alpha_s}{\pi}\right)^2\tilde{\rho}^{(0)}(v)\left(C_F\rho_A^{(2c)}(v) + C_A\rho_{NA}^{(2c)}(v) + T_RN_L\rho_L^{(2c)}(v) + T_RN_H\rho_H^{(2c)}(v)\right) \tag{7.244}$$

where for the colour group $SU(3)$ one has $C_F = 4/3$, $C_A = 3$, $T_R = 1/2$. $N_L = 3$ indicates the number of light flavours and $N_H = 1$ the number of heavy flavours. The different parts are given by

$$\begin{aligned}
\rho_A^{(2c)}(v) &= \frac{\pi^4}{12v^2} - \frac{2\pi^2}{v} + \frac{\pi^4}{6} + \pi^2\left(-\frac{35}{18} - \frac{2}{3}\ln v + \frac{4}{3}\ln 2\right) + \frac{39}{4} - \zeta(3),\\
\rho_{NA}^{(2c)}(v) &= \frac{\pi^2}{v}\left(\frac{31}{72} - \frac{11}{12}\ln(2v)\right) + \pi^2\left(\frac{179}{72} - \ln v - \frac{8}{3}\ln 2\right) - \frac{151}{36} - \frac{13}{2}\zeta(3),\\
\rho_L^{(2c)}(v) &= \frac{\pi^2}{v}\left(\frac{1}{3}\ln(2v) - \frac{5}{18}\right) + \frac{11}{9}, \qquad \rho_H^{(2c)}(v) = \frac{44}{9} - \frac{4\pi^2}{9}.
\end{aligned} \tag{7.245}$$

In Table 7.8 the leading order, the next-to-leading order, and the next-to-next-to-leading order with or without the gluon condensate contribution are compared for the different moments, taking as an example $s_0 = (4.6\,GeV)^2$ and all six resonances. The table is created by using the procedure `contritable[name_,e_,n_]` where `name` is the name of the generated LaTeX file, `e` is the square root of s_0, and `n` is the number of resonances. This procedure itself makes use of the procedure `eesolve[fn_,e_,n1_,n_]` where `fn` is the label of the moment function and `n1` is the degree of the moment. The following new moments are used in the package `sumrules.add`:

- `mom8[n_,s_,m_]`: second order Coulomb box approximation
- `mom9[n_,s_,m_]`: second order Coulomb exact integration
- `moma[n_,s_,m_]`: the latter including the gluon condensate contribution

From Table 7.8 it is obvious that the addition of the gluon condensate contribution has nearly no effect. Instead, the degree of the moments as well as the order of perturbative expansion plays an important role. One therefore has to think about the errors which are incurred by terminating the perturbation series at a specified order and about whether one can improve the situation by resumming specific terms. These two points are discussed in the following.

Coulomb resummation

The terms $(\alpha_s/v)^k$ in Eq. (7.244) can be *resummed* by using the correlator [120]

$$\Pi_C(q^2)=\frac{3\pi}{4m}\left\{-\sqrt{4m^2+q^2}+C_F\alpha_s m\left(\ln\left(\frac{\mu_f^2}{4m^2+q^2}\right)-2\gamma_E-2\psi\left(1-\frac{C_F\alpha_s m}{\sqrt{4m^2+q^2}}\right)\right)\right\} \tag{7.246}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function (see Appendix D.5) and μ_f is the factorization scale. Because of the expansion

$$\psi(1-z)=-\gamma_E-\sum_{n=1}^{\infty}\zeta(n+1)z^n \tag{7.247}$$

one obtains

$$\gamma_E+\psi\left(1-\frac{C_D\alpha_s m}{\sqrt{4m^2+q^2}}\right)=-\sum_{n=1}^{\infty}\zeta(n+1)\left(\frac{C_F\alpha_s m}{\sqrt{4m^2+q^2}}\right)^n. \tag{7.248}$$

Taking the discontinuity of the whole expression, the even terms of the expansion

$$\Pi_C(q^2)=\frac{3\pi}{4m}\left\{-\sqrt{4m^2+q^2}+C_F\alpha_s m\left(\ln\left(\frac{\mu_f^2}{4m^2+q^2}\right)+2\sum_{n=1}^{\infty}\zeta(n+1)\left(\frac{C_F\alpha_s m}{\sqrt{4m^2+q^2}}\right)^n\right)\right\} \tag{7.249}$$

can be seen to give no contribution. One therefore has to calculate

$$\begin{aligned}
\mathrm{Disc}\left(\sqrt{4m^2-s}\right) &= \sqrt{4m^2+se^{-i\pi}}-\sqrt{4m^2+se^{i\pi}} = \\
&= \sqrt{(s-4m^2)e^{-i\pi}}-\sqrt{(s-4m^2)e^{i\pi}} = \\
&= \sqrt{s-4m^2}e^{-i\pi/2}-\sqrt{s-4m^2}e^{i\pi/2} = \\
&= -2i\sqrt{s-4m^2} = -2iv\sqrt{s}, && (7.250)\\
\mathrm{Disc}\ln\left(\frac{\mu_f^2}{4m^2-s}\right) &= \ln\left(\frac{\mu_f^2}{4m^2+se^{-i\pi}}\right)-\ln\left(\frac{\mu_f^2}{4m^2+se^{i\pi}}\right) = \\
&= \ln\left(\frac{4m^2+se^{i\pi}}{4m^2+se^{-i\pi}}\right) = 2\pi i, && (7.251)\\
\mathrm{Disc}\left(\frac{1}{\sqrt{4m^2-s}}\right) &= \frac{1}{\sqrt{4m^2+se^{-i\pi}}}-\frac{1}{\sqrt{4m^2+se^{i\pi}}} = \\
&= \frac{e^{i\pi/2}}{\sqrt{s-4m^2}}-\frac{e^{-i\pi/2}}{\sqrt{s-4m^2}} = \frac{2i}{\sqrt{s-4m^2}} = \frac{2i}{v\sqrt{s}}, && (7.252)\\
\mathrm{Disc}\left(\frac{1}{(\sqrt{4m^2-s})^3}\right) &= \frac{1}{(\sqrt{4m^2+se^{-i\pi}})^3}-\frac{1}{(\sqrt{4m^2+se^{i\pi}})^3} = && (7.253)\\
&= \frac{e^{3i\pi/2}}{(\sqrt{s-4m^2})^3}-\frac{e^{-3i\pi/2}}{(\sqrt{s-4m^2})^3} = \frac{-2i}{(\sqrt{s-4m^2})^3} = \frac{-2i}{v^3s^{3/2}}.
\end{aligned}$$

The spectral density close to threshold (and therefore $\sqrt{s}\approx 2m$) is given by

$$\begin{aligned}\rho_C(s) &\approx \frac{3\pi}{4m}\left\{\frac{1}{\pi}\sqrt{s-4m^2}+C_F\alpha_s m\left(1+\frac{2\zeta(2)}{\pi}\frac{C_F\alpha_s m}{2mv}\right)\right\} = \\ &\approx \frac{3v}{2}\left\{1+\frac{C_F\alpha_s}{\pi}\frac{\pi^2}{2v}+\left(\frac{C_F\alpha_s}{\pi}\right)^2\frac{\pi^4}{12v^2}\right\} \approx \tilde\rho_C(v). \qquad (7.254)\end{aligned}$$

This reproduces the terms of leading order in $1/v$. One therefore can use $\Pi_C(q^2)$ to resum the Coulomb terms. For the moments one then obtains

$$R_n=\int_{4m^2}^{s_0}\frac{1}{s^{n+1}}\rho_C(s)ds=\frac{-1}{2\pi i}\oint_C\frac{1}{s^{n+1}}\Pi_C(-s)ds \qquad (7.255)$$

where C is a contour containing the cut up to $s=s_0$ drough $s=2m^2$ which can be parametrized by

$$s=m^2+s_0/2+(m^2-s_0/2)e^{i\varphi},\qquad \varphi\in\,]-\pi,\pi]. \qquad (7.256)$$

Note that these calculations are done with a new macro file called `quasico.add`.

To delineate the progress so far, the calculation was started with the expression

$$\begin{aligned}\tilde\rho^{(sc)}(v) &= \tilde\rho^{(sw)}(v)+\Delta\tilde\rho^{(2c)}(v) = \frac{v(3-v^2)}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2\right) = \\ &= \frac{3v}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2\right)+\frac{v^3}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2\right). \qquad (7.257)\end{aligned}$$

While the second part is not Coulombic to this order, one can resum the leading $1/v$ terms according to the previous considerations,

$$\begin{aligned}\tilde\rho^{(sc)}(v) &= \frac{3v}{2}\left(1+C_F\frac{\alpha_s}{\pi}\frac{\pi^2}{2v}+C_F^2\left(\frac{\alpha_s}{\pi}\right)^2\frac{\pi^4}{12v^2}\right)+ \\ &\quad +\frac{3v}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1'+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2'\right)+\frac{v^3}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2\right) = \\ &= \tilde\rho_C(v)+\frac{3v}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1'+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2'\right)+\frac{v^3}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2\right)\end{aligned} \qquad (7.258)$$

where

$$t_1'=t_1-\frac{\pi^2}{2v},\qquad t_2'=t_2-C_F\frac{\pi^4}{12v^2}. \qquad (7.259)$$

The hard correction factor

The resummation described before frees one from the leading $1/v$ terms which become dominant near threshold. However, subleading terms in the parts $\rho_A^{(c)}$, $\rho_{NA}^{(c)}$, and $\rho_L^{(c)}$ remain. The first one can be removed by extracting the so-called *hard correction factor* $(1-4C_F\alpha_s/\pi)$ (cf. e.g. Ref. [118], Eq. (4)) before the Coulomb terms are resummed,

$$\tilde\rho^{(sc)}(v)=\left(1-4C_F\frac{\alpha_s}{\pi}\right)\frac{v(3-v^2)}{2}\left(1+C_F\frac{\alpha_s}{\pi}(t_1+4)+C_F\left(\frac{\alpha_s}{\pi}\right)^2(t_2+4C_F(t_1+4))\right). \qquad (7.260)$$

where now t_1+4 and $t_2+4C_F(t_1+4)$ have to be taken instead of t_1 and t_2 as starting expressions for the contributions of different orders in the resummation procedure. This factorization can be done separately for the part proportional to v which resum to the Coulomb term afterwards. On the other hand, the subleading terms in $1/v$ occuring in $\rho_{NA}^{(c)}$ and $\rho_L^{(c)}$ can be absorbed into an effective coupling constant for the Coulomb term.

The effective coupling

In order to absorb the subleading order terms $1/v$ occuring in $\rho_{NA}^{(c)}$ and $\rho_L^{(c)}$ into an *effective coupling*, one rewrites the decomposition as

$$\begin{aligned}\tilde\rho^{(sc)}(v) &= \frac{3v}{2}\left(1+C_F\frac{\alpha_s}{\pi}\frac{\pi^2}{2v}\left(1+\frac{\alpha_s}{\pi}t_\alpha\right)+C_F^2\left(\frac{\alpha_s}{\pi}\right)^2\frac{\pi^4}{12v^2}\right)+\\&+\frac{3v}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1'+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2''\right)+\frac{v^3}{2}\left(1+C_F\frac{\alpha_s}{\pi}t_1+C_F\left(\frac{\alpha_s}{\pi}\right)^2t_2\right)\end{aligned}\tag{7.261}$$

where

$$t_\alpha = \left(\frac{31}{36}-\frac{11}{6}\ln(2v)\right)C_A-\left(\frac{5}{9}-\frac{2}{3}\ln(2v)\right)T_RN_L,\tag{7.262}$$

$$t_1' = t_1-\frac{\pi^2}{2v},\qquad t_2'' = t_2-C_F\frac{\pi^4}{12v^2}-t_\alpha.\tag{7.263}$$

All leading and subleading $1/v$ terms are removed by redefining t_1 and t_2 through the extraction of the hard correction factor and then going to t_1' and t_2'', all (leading and subleading) $1/v$ terms are removed. The effective coupling is defined by

$$\alpha_C=\alpha_s\left\{1+\frac{\alpha_s}{\pi}\left(\frac{31}{36}C_A-\frac{5}{9}T_RN_L-\left(\frac{11}{6}C_A-\frac{2}{3}T_RN_L\right)\ln(2v)\right)\right\}\tag{7.264}$$

where the coefficient proportional to $\ln(2v)$ is proportional to the coefficient β_0 contained in the renormalization group equation.

The high energy tale coupling

The terms not affected by the Coulomb resummation are called the *high energy tail*. While the Coulomb expression as considered to be the leading order term, the convergence of the perturbation series is crucial and bears witness to whether the resummation is effective or not, depending on the different parameters like the degree of the moment and the parameter s_0. A simple way to improve the first two terms of this perturbation series is to define an effective coupling for the high energy tail. This has to be done for a specific (central) moment and is done by demanding that for this moment the second order contribution should vanish. Starting with

$$R_n=R_n^{(c)}+\frac{\alpha_s}{\pi}R_n^{(1)}+\left(\frac{\alpha_s}{\pi}\right)^2R_n^{(2)},\tag{7.265}$$

the requirement is to find an effective coupling α_{HE} so that (e.g. for the zeroth moment)

$$R_0=R_0^{(c)}+\frac{\alpha_s}{\pi}R_0^{(1)}+\left(\frac{\alpha_s}{\pi}\right)^2R_0^{(2)}=R_0^{(c)}+\frac{\alpha_{HE}}{\pi}R_0^{(1)}+0.\tag{7.266}$$

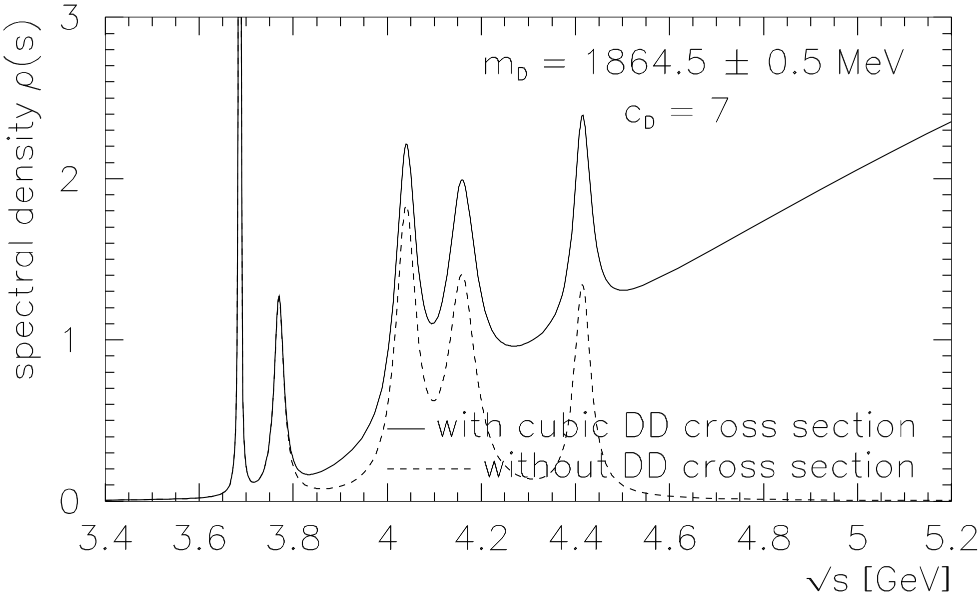


Figure 7.14: Energy dependence of the spectral density with Breit–Wigner resonances with and without a cubic partial $D\bar{D}$ cross section for $c_D = 7$

The effective coupling is therefore given by

$$\alpha_{HE} = \alpha_s\left(1 + \frac{\alpha_s}{\pi}\,\frac{R_0^{(2)}}{R_0^{(1)}}\right) \quad\Rightarrow\quad \alpha_s = \alpha_{HE}\left(1 - \frac{\alpha_{HE}}{\pi}\,\frac{R_0^{(2)}}{R_0^{(1)}}\right) + O(\alpha_{HE}^3). \tag{7.267}$$

Expressed in terms of this effective coupling, for an arbitrary moment one obtains

$$R_n = R_n^{(c)} + \frac{\alpha_{HE}}{\pi}R_n^{(1)} + \left(\frac{\alpha_{HE}}{\pi}\right)^2\left(R_n^{(2)} - \frac{R_n^{(1)}}{R_0^{(1)}}\right). \tag{7.268}$$

By construction, the second order term vanishes for $n = 0$, but the suppression of this term is expected to be also effective for moments close to $n = 0$.

Improvements on the experiment side

There are two possible improvements on the experimental input. First one can replace the narrow resonances by the *Breit–Wigner distribution function*,

$$\rho_k^{\exp}(s) = \frac{9\pi\Gamma_k^{ee}M_k}{\bar{\alpha}^2N_cQ_c^2}\delta(s - M_k^2) \;\to\; \frac{9\pi\Gamma_k^{ee}M_k}{\bar{\alpha}^2N_cQ_c^2}\,\frac{\Gamma_kM_k}{\pi((s - M_k)^2 + \Gamma_k^2M_k^2)}. \tag{7.269}$$

The total widths of the six resonances are again taken from Ref. [127] and listed in Table 7.5. A second improvement is the simulation of the *partial $D\bar{D}$ production cross section.* According to Ref. [288], this cross section can be assumed to be proportional to v_D^3. In this case the spectral density is given by

$$\rho_D(s) = c_Dv_D^3(s) = c_D\left(1 - \frac{4m_D^2}{s}\right)^{3/2} \tag{7.270}$$

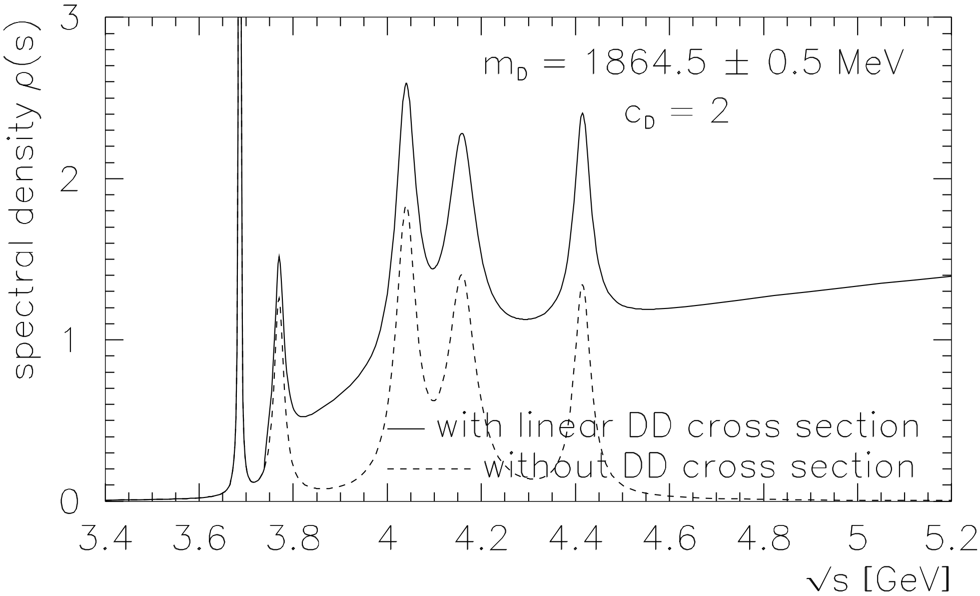


Figure 7.15: Energy dependence of the spectral density with Breit–Wigner resonances with and without a linear partial $D\bar{D}$ cross section for $c_D = 2$

where $m_D = (1864.5 \pm 0.5)\,MeV$ [127] is the mass of the D^0 meson. The combination of both for a sample value of $c_D = 7$ is shown in Fig. 7.14. The direct proportionality to v_D (instead of the cubic one), however, simulates the experimental data more accurately. The influence of the linear $D\bar{D}$ threshold on the resonances is shown in Fig. 7.15 for $c_D = 2$. In any case it is obvious, however, that for a fixed parameter s_0 the introduction of the parameter c_D opens again the window for considering pairs of sum rules in order to determine the parameter pair (m_c, c_D).

The choice of a non-vanishing partial $D\bar{D}$ production cross section is not only optional but also necessary. The reason is that there is no solution for moments with non-negative degree if the partial $D\bar{D}$ cross section is not included. To demonstrate this fact, in Fig. 7.16 the the mass dependence of the theory predictions for the moments (solid curves) with the experiment values including a partial (cubic) $D\bar{D}$ cross section is shown for $c_D = 0$, $c_D = 4$, and $c_D = 8$ (dashed lines, from bottom to top). The solution is given by the intersection of the solid curve with the corresponding dashed curve. The three parts of Fig. 7.16 display the results for the moments R_{-2} (left), R_{-1} (middle), and R_0 (right). As one can see, there is no intersection with the lowest line for negative degree of the momentum. For this reason the the partial $D\bar{D}$ cross section has to be taken into account (note that the interval for the mass is limited to the left by the threshold).

7.6.5 Sum rule analysis

Before starting with the sum rules analysis, the different options for setting up these sum rules are listed. Note though that the resummation of the Coulombic terms and the choice of a corresponding effective coupling α_C is not given as option but is used in all cases. To see the difference emerging from this (non-optional) technique one can compare with the results obtained earlier for the non-resummed theory contribution. The options are the

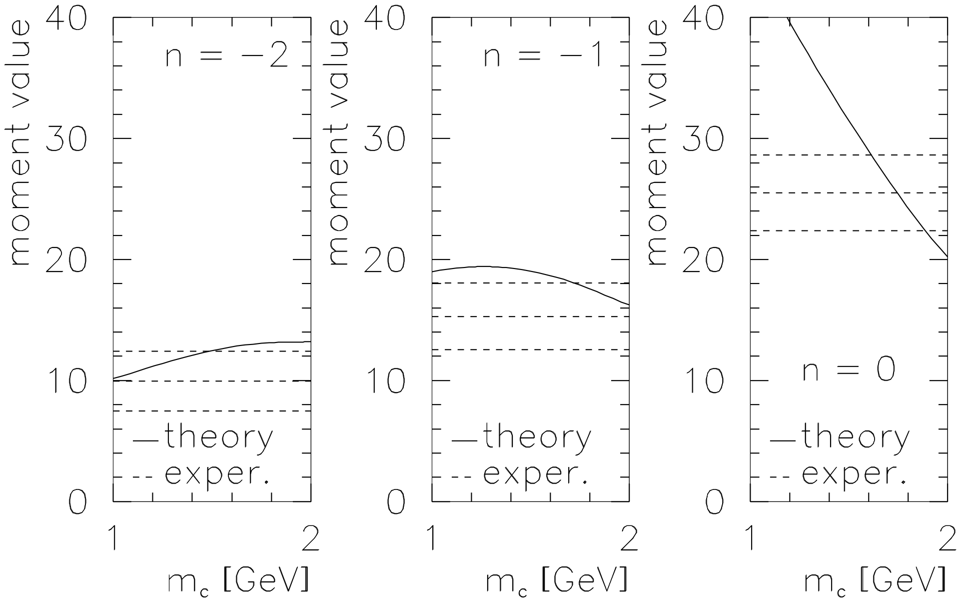


Figure 7.16: mass dependence of the moments R_{-2} (left), R_{-1} (middle), and R_0 (right) as predicted by the theory (solid curves) and by the experiment (dashed curves) for the values $c_D = 0$, $c_D = 4$, and $c_D = 8$ (from bottom to top). The intersection point gives rise to the solution of the corresponding sum rule.

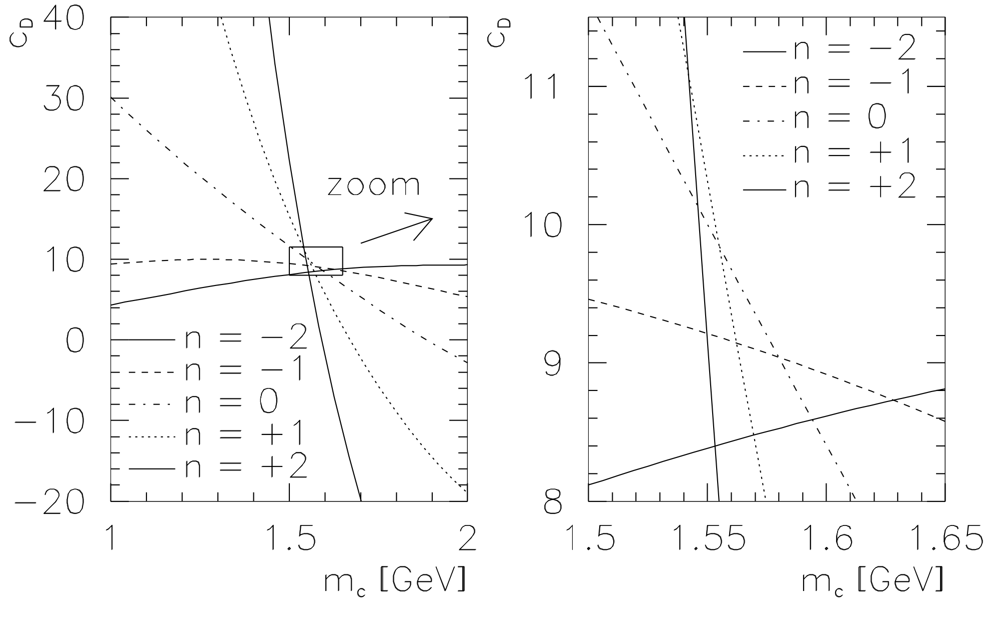


Figure 7.17: dependence of the solution for c_D (cubic partial $D\bar{D}$ cross section) on m_c for different degrees n of the moments R_n at $\sqrt{s_0} = 4.6\,GeV$

following:

1. including the Breit–Wigner shape of the resonances.
2. giving the logarithms occuring in the effective coupling α_C a fixed value at the threshold value $\sqrt{s_0}$ instead of letting them run with the integration variable.
3. working with the effective coupling α_{HE} for the high energy tail.
4. extracting the hard correction factor $(1 - 4C_F\alpha_s/\pi)$ before Coulomb resummation.
5. changing the value of α_s. Note that it is not obvious that α_s should take a constant value (we normally take $\alpha_s(m_\tau) = 0.32$). An (unfortunately not practicable) model would be to use the running coupling on the circle. This is not practicable because the dependence of α_s has to be derived from the corresponding correlator.

The different options and combinations of them are analyzed in the following, both for the mass parameter m_c as well as for c_D, for various pairs (n_1, n_2) of moment degrees around the "central point" $(0, 0)$ and for continuum energy values $\sqrt{s_0} = 4.5\,GeV$, $4.6\,GeV$, and $4.7\,GeV$. In order to visualize the solution pairs (m_c, c_D) obtained, in Fig. 7.17 the dependence of c_D on m_c is shown for different degrees of the moments at $\sqrt{s_0} = 4.6\,GeV$.

The solutions shown in Fig. 7.17 are obtained by using `solveco[fn_,n_,m_,s_]` (a procedure within `quasico.add`) where `fn` is the head of the moment function, `n` is the moment degree, `m` is the quark mass m_c, and `s` is the threshold energy square s_0. The numerical calculation of the intersection points (m_c, c_D) is done by using `masscross[fn_,s_,n1_,n2_]`, this procedure is used in `mdtable[name_,fn_,e_]` to automatically generate the tables.

An argument in favour of the choice of a linear $D\bar{D}$ threshold

Looking at the tables for different options, one finds a stable behaviour of the parameters for different choices of the moment pairs (n_1, n_2) as well as for different choices of the energy $\sqrt{s_0}$. Especially the last possibility gives a criterion whether to chose the linear or cubic $D\bar{D}$ threshold. As mentioned earlier, the threshold linear in v_D in combination with Breit–Wigner resonances (see Fig. 7.15) already visually approximates the actually measured cross section much better. This feature shows up also in the tables. While there is no convergence of the parameter c_D for different energy values $\sqrt{s_0}$, the values converge in case of the linear threshold for increasing energies. Therefore, in the following only the linear threshold will be considered.

Two favourite choices in the final choice

While the inclusion of Breit–Wigner resonances (option 1) or the effective high energy tail (option 3) affects the results only slightly, the other options give rise to noticeable changes. Criterion for the favourite choice of the method is the afore mentioned stability. The choice of a fixed effective coupling as well as the choice of using the hard correction factor both lead to a stabilization for both the quark mass m_c and the threshold parameter c_D, while their combination destabilizes the parameter c_D profoundly. Therefore, either of these choices should be used. Unfortunately, these choices result in different values for the charm quark mass (and the parameter c_D). Therefore, a further criterion is necessary.

The convergence criterion leading to the final estimate

A criterion on whether the fixed effective coupling or the hard correction factor should be used is given by the criterion of convergence. The results for the zeroth momentum up to second order for $m_c = 1.5\,GeV$ and $\sqrt{s_0} = 4.6\,GeV$, for instance, read

$$\begin{aligned} R_0^{(2)} &= 12.96 + 28.75 - 8.60 - 5.02, \\ R_0^{(4)} &= 12.96 + 13.33 - 8.60 + 8.22 \end{aligned} \tag{7.271}$$

where the different contributions in Eq. (7.271) are the leading order contribution, the Coulomb resummed term, the non-Coulombic first and second order contribution, respectively. Because first and second order corrections have to be considered in relation to the combined leading order and Coulomb contribution, the comparison favours option 2 (fixed effective coupling) and disfavours option 4 (hard correction factor). The tables for the fixed effective coupling (with Breit-Wigner resonances and a linear $D\bar{D}$ threshold) are shown in Tables 7.9 and 7.10. The results which can be read off these tables are given by

$$m_c = 1.510 \pm 0.013\,GeV, \qquad c_D = 1.61 \pm 0.05. \tag{7.272}$$

m_c [GeV]	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$
$n_1 = -2$		1.522968	1.514630	1.510111	1.507389
$n_1 = -1$	1.522968		1.511241	1.508545	1.506650
$n_1 = 0$	1.514630	1.511241		1.507386	1.506041
$n_1 = 1$	1.510111	1.508545	1.507386		1.505440
$n_1 = 2$	1.507389	1.506650	1.506041	1.505440	

m_c [GeV]	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$
$n_1 = -2$		1.510094	1.505495	1.503391	1.502222
$n_1 = -1$	1.510094		1.503704	1.502624	1.501877
$n_1 = 0$	1.505495	1.503704		1.502173	1.501646
$n_1 = 1$	1.503391	1.502624	1.502173		1.501415
$n_1 = 2$	1.502222	1.501877	1.501646	1.501415	

m_c [GeV]	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$
$n_1 = -2$		1.498417	1.497313	1.497406	1.497626
$n_1 = -1$	1.498417		1.496900	1.497297	1.497594
$n_1 = 0$	1.497313	1.496900		1.497457	1.497678
$n_1 = 1$	1.497406	1.497297	1.497457		1.497772
$n_1 = 2$	1.497626	1.497594	1.497678	1.497772	

Table 7.9: values for m_c obtained by the pair (n_1, n_2) for $\sqrt{s_0} = 4.5\,GeV$, $4.6\,GeV$, and $4.7\,GeV$ (top to bottom) for Coulomb resummation with options 1 and 2 combined

c_D	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$
$n_1 = -2$		1.576556	1.567129	1.561946	1.558799
$n_1 = -1$	1.576556		1.590757	1.593953	1.596183
$n_1 = 0$	1.567129	1.590757		1.617685	1.627092
$n_1 = 1$	1.561946	1.593953	1.617685		1.657688
$n_1 = 2$	1.558799	1.596183	1.627092	1.657688	

c_D	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$
$n_1 = -2$		1.577719	1.572814	1.570555	1.569296
$n_1 = -1$	1.577719		1.583442	1.584398	1.585055
$n_1 = 0$	1.572814	1.583442		1.592533	1.595664
$n_1 = 1$	1.570555	1.584398	1.592533		1.606222
$n_1 = 2$	1.569296	1.585055	1.595664	1.606222	

c_D	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$
$n_1 = -2$		1.571350	1.570238	1.570331	1.570553
$n_1 = -1$	1.571350		1.572360	1.572097	1.571899
$n_1 = 0$	1.570238	1.572360		1.569496	1.568362
$n_1 = 1$	1.570331	1.572097	1.569496		1.564404
$n_1 = 2$	1.570553	1.571899	1.568362	1.564404	

Table 7.10: values for c_D obtained by the pair (n_1, n_2) for $\sqrt{s_0} = 4.5\,GeV$, $4.6\,GeV$, and $4.7\,GeV$ (top to bottom) for Coulomb resummation with options 1 and 2 combined

Appendix A

The electroweak coupling matrix

The electro-weak coupling coefficients $g_{ij}(q^2)$ used in Chapter 1 are given by

$$
\begin{aligned}
g_{11} &= Q_f^2 - 2Q_f v_e v_f \operatorname{Re}\chi_Z + (v_e^2 + a_e^2)(v_f^2 + a_f^2)|\chi_Z|^2, \\
g_{12} &= Q_f^2 - 2Q_f v_e v_f \operatorname{Re}\chi_Z + (v_e^2 + a_e^2)(v_f^2 - a_f^2)|\chi_Z|^2, \\
g_{13} &= -2Q_f v_e a_f \operatorname{Im}\chi_Z, \\
g_{14} &= 2Q_f v_e a_f \operatorname{Re}\chi_Z - 2(v_e^2 + a_e^2) v_f a_f |\chi_Z|^2,
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
g_{21} &= q_f^2 - 2Q_f v_e v_f \operatorname{Re}\chi_Z + (v_e^2 - a_e^2)(v_f^2 + a_f^2)|\chi_Z|^2, \\
g_{22} &= q_f^2 - 2Q_f v_e v_f \operatorname{Re}\chi_Z + (v_e^2 - a_e^2)(v_f^2 - a_f^2)|\chi_Z|^2, \\
g_{23} &= -2Q_f v_e a_f \operatorname{Im}\chi_Z, \\
g_{24} &= 2Q_f v_e a_f \operatorname{Re}\chi_Z - 2(v_e^2 - a_e^2) v_f a_f |\chi_Z|^2,
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
g_{31} &= -2Q_f a_e v_f \operatorname{Im}\chi_Z, \\
g_{32} &= -2Q_f a_e v_f \operatorname{Im}\chi_Z, \\
g_{33} &= 2Q_f a_e a_f \operatorname{Re}\chi_Z, \\
g_{34} &= 2Q_f a_e a_f \operatorname{Im}\chi_Z,
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
g_{41} &= 2Q_f a_e v_f \operatorname{Re}\chi_Z - 2v_e a_e (v_f^2 + a_f^2)|\chi_Z|^2, \\
g_{42} &= 2Q_f a_e v_f \operatorname{Re}\chi_Z - 2v_e a_e (v_f^2 - a_f^2)|\chi_Z|^2, \\
g_{43} &= 2Q_f a_e a_f \operatorname{Im}\chi_Z, \\
g_{44} &= -2Q_f a_e a_f \operatorname{Re}\chi_Z + 4v_e a_e v_f a_f |\chi_Z|^2
\end{aligned} \tag{A.4}
$$

where $\chi_Z(q^2) = gM_Z^2 q^2/(q^2 - M_Z^2 + iM_Z\Gamma_Z)$, with M_Z and Γ_Z the mass and width of the Z^0 and $g = G_F(8\sqrt{2}\pi\alpha)^{-1} \approx 4.49 \cdot 10^{-5}\text{GeV}^{-2}$. Q_f are the charges of the final state quarks to which the electro-weak currents directly couple; v_e and a_e, v_f and a_f are the electro-weak vector and axial vector coupling constants. For example, in the Weinberg–Salam model, one has $v_e = -1 + 4\sin^2\theta_W$, $a_e = -1$ for leptons, $v_f = 1 - \frac{8}{3}\sin^2\theta_W$, $a_f = 1$ for up-type quarks ($Q_f = \frac{2}{3}$), and $v_f = -1 + \frac{4}{3}\sin^2\theta_W$, $a_f = -1$ for down-type quarks ($Q_f = -\frac{1}{3}$). The left- and right-handed coupling constants are then given by $g_L = v + a$ and $g_R = v - a$, respectively. In the purely electromagnetic case one has $g_{11} = g_{12} = g_{21} = g_{22} = Q_f^2$ and all other $g_{r'r} = 0$. The terms linear in $\operatorname{Re}\chi_Z$ and $\operatorname{Im}\chi_Z$ come from $\gamma - Z^0$ interference, whereas the terms proportional to $|\chi_Z|^2$ originate from Z-exchange.

Note that the generalization to the case where one starts with longitudinally polarized beams is straightforward and amounts to the replacement

$$
\begin{aligned}
g_{1i} &\rightarrow (1 - h^- h^+) g_{1i} + (h^- - h^+) g_{4i} \\
g_{4i} &\rightarrow (1 - h^- h^+) g_{4i} + (h^- - h^+) g_{1i}
\end{aligned}
\tag{A.5}
$$

where h^- and h^+ $(-1 \leq h^\pm \leq +1)$ denote the longitudinal polarization of the electron and the positron beam, respectively. Clearly there is no interaction between the beams when $h^+ = h^- = \pm 1$.

Appendix B

The decay rate terms

It is convenient to define the mass dependent variables $a := 2 + \sqrt{\xi}$, $b := 2 - \sqrt{\xi}$ and $w_0 := \sqrt{(1-\sqrt{\xi})/(1+\sqrt{\xi})}$. The rate functions $t_1, \ldots, t_{12}$ are then given by [8]

$$t_1 := \ln\left(\frac{2\xi\sqrt{\xi}}{b^2(1+\sqrt{\xi})}\right), \quad t_2 := \ln\left(\frac{2\sqrt{\xi}}{1+\sqrt{\xi}}\right) \quad \Rightarrow \quad t_1 - t_2 = \ln\left(\frac{\xi}{b^2}\right) \tag{B.1}$$

$$t_3 := \ln\left(\frac{1+v}{1-v}\right) \tag{B.2}$$

$$t_4 := \mathrm{Li}(w_0) - \mathrm{Li}(-w_0) + \mathrm{Li}(\frac{a}{b}w_0) - \mathrm{Li}(-\frac{a}{b}w_0) \tag{B.3}$$

$$\begin{aligned} t_5 := {} & \frac{1}{2}\ln\left(\frac{a\sqrt{\xi}}{4(1+\sqrt{\xi})}\right)\ln\left(\frac{1+v}{1-v}\right) + \mathrm{Li}\left(\frac{2\sqrt{\xi}}{a(1+w_0)}\right) - \mathrm{Li}\left(\frac{2\sqrt{\xi}}{a(1-w_0)}\right) + \\ & + \mathrm{Li}\left(\frac{1+w_0}{2}\right) - \mathrm{Li}\left(\frac{1-w_0}{2}\right) + \mathrm{Li}\left(\frac{a(1+w_0)}{4}\right) - \mathrm{Li}\left(\frac{a(1-w_0)}{4}\right) \end{aligned} \tag{B.4}$$

$$\begin{aligned} t_6 := {} & \ln^2(1+w_0) + \ln^2(1-w_0) + \ln\left(\frac{a}{8}\right)\ln(1-w_0^2) + \\ & + \mathrm{Li}\left(\frac{2\sqrt{\xi}}{a(1+w_0)}\right) + \mathrm{Li}\left(\frac{2\sqrt{\xi}}{a(1-w_0)}\right) - 2\,\mathrm{Li}\left(\frac{2\sqrt{\xi}}{a}\right) + \\ & + \mathrm{Li}\left(\frac{1+w_0}{2}\right) + \mathrm{Li}\left(\frac{1-w_0}{2}\right) - 2\,\mathrm{Li}\left(\frac{1}{2}\right) + \\ & + \mathrm{Li}\left(\frac{a(1+w_0)}{4}\right) + \mathrm{Li}\left(\frac{a(1-w_0)}{4}\right) - 2\,\mathrm{Li}\left(\frac{a}{4}\right) \end{aligned} \tag{B.5}$$

$$\begin{aligned} t_7 := {} & 2\ln\left(\frac{1-\xi}{2\xi}\right)\ln\left(\frac{1+v}{1-v}\right) - \mathrm{Li}\left(\frac{2v}{(1+v)^2}\right) + \mathrm{Li}\left(-\frac{2v}{(1-v)^2}\right) + \\ & -\frac{1}{2}\,\mathrm{Li}\left(-\left(\frac{1+v}{1-v}\right)^2\right) + \frac{1}{2}\,\mathrm{Li}\left(-\left(\frac{1-v}{1+v}\right)^2\right) + \\ & + \mathrm{Li}\left(\frac{2w_0}{1+w_0}\right) - \mathrm{Li}\left(-\frac{2w_0}{1-w_0}\right) - 2\,\mathrm{Li}\left(\frac{w_0}{1+w_0}\right) + 2\,\mathrm{Li}\left(-\frac{w_0}{1-w_0}\right) + \\ & + \mathrm{Li}\left(\frac{2aw_0}{b+aw_0}\right) - \mathrm{Li}\left(-\frac{2aw_0}{b-aw_0}\right) - 2\,\mathrm{Li}\left(\frac{aw_0}{b+aw_0}\right) + 2\,\mathrm{Li}\left(-\frac{aw_0}{b-aw_0}\right) \end{aligned} \tag{B.6}$$

$$t_8 := \ln\left(\frac{\xi}{4}\right)\ln\left(\frac{1+v}{1-v}\right) + \mathrm{Li}\left(\frac{2v}{1+v}\right) - \mathrm{Li}\left(-\frac{2v}{1-v}\right) - \pi^2 \tag{B.7}$$

$$t_9 := 2\ln\left(\frac{2(1-\xi)}{\sqrt{\xi}}\right)\ln\left(\frac{1+v}{1-v}\right)+2\left(\mathrm{Li}\left(\frac{1+v}{2}\right)-\mathrm{Li}\left(\frac{1-v}{2}\right)\right)+$$
$$+3\left(\mathrm{Li}\left(-\frac{2v}{1-v}\right)-\mathrm{Li}\left(\frac{2v}{1+v}\right)\right) \tag{B.8}$$
$$t_{10} := \ln\left(\frac{4}{\xi}\right),\quad t_{11} := \ln\left(\frac{4(1-\sqrt{\xi})^2}{\xi}\right),\quad t_{12} := \ln\left(\frac{4(1-\xi)}{\xi}\right) \tag{B.9}$$

Decay rate terms which were published in Ref. [18] and which are needed for the longitudinal spin-spin correlation without polar angle dependence are given by

$$t_{13} := \ln\left(\frac{1+v}{2-\sqrt{\xi}}\right),$$
$$t_{14} := \ln\left(\frac{4}{\xi}\right)\ln\left(\frac{1+v}{2-\sqrt{\xi}}\right)+$$
$$+2\,\mathrm{Li}\left(\frac{2-\sqrt{\xi}}{2}\right)-2\,\mathrm{Li}\left(\frac{\sqrt{\xi}}{2}\right)+\mathrm{Li}\left(\frac{1-v}{2}\right)-\mathrm{Li}\left(\frac{1+v}{2}\right),$$
$$t_{15} := \left(\ln\left(\frac{1+v}{1-v}\right)+\ln\left(\frac{\sqrt{\xi}}{2-\sqrt{\xi}}\right)\right)^2+$$
$$-4\,\mathrm{Li}\left(\sqrt{\frac{1-v}{1+v}}\right)+2\,\mathrm{Li}\left(\frac{2-\sqrt{\xi}}{1+v}\right)+2\,\mathrm{Li}\left(\frac{1-v}{2-\sqrt{\xi}}\right),$$
$$t_{16} := \ln\left(\frac{1+v}{1-v}\right)\ln\left(\frac{4v^4}{\xi(1+v)^2}\right)-\mathrm{Li}\left(\frac{2v}{(1+v)^2}\right)+\mathrm{Li}\left(\frac{-2v}{(1-v)^2}\right)+$$
$$+\frac{1}{2}\,\mathrm{Li}\left(-\frac{(1-v)^2}{(1+v)^2}\right)-\frac{1}{2}\,\mathrm{Li}\left(-\frac{(1+v)^2}{(1-v)^2}\right). \tag{B.10}$$

Finally, for the polar angle dependent spin-spin correlation the relevant decay rate terms can be found in Ref. [24],

$$t_{17} := \frac{5}{2}\ln\left(\frac{1+v}{2-\sqrt{\xi}}\right)\ln\left(\frac{2-\sqrt{\xi}}{2}\right)+\frac{1}{2}\ln\left(\frac{1-v}{2-\sqrt{\xi}}\right)\ln\left(\frac{1+v}{2}\right)$$
$$+\mathrm{Li}_2(1)-\mathrm{Li}_2\left(\sqrt{\frac{1-v}{1+v}}\right)-2\,\mathrm{Li}_2\left(\frac{\xi}{(2-\sqrt{\xi})^2}\right)+$$
$$+2\,\mathrm{Li}_2\left(\frac{\sqrt{\xi}}{2-\sqrt{\xi}}\right)+\mathrm{Li}_2\left(\frac{(1-v)^2}{\sqrt{\xi}(2-\sqrt{\xi})}\right)-\mathrm{Li}_2\left(\frac{2-\sqrt{\xi}}{1+v}\right), \tag{B.11}$$
$$t_{18} := \ln\left(\frac{1+v}{2-\sqrt{\xi}}\right)\ln\left(\frac{2}{\sqrt{\xi}}\right)+\ln\left(\frac{\sqrt{\xi}(1+v)}{(2-\sqrt{\xi})^2}\right)\ln\left(\frac{2}{2+\sqrt{\xi}}\right)+$$
$$+\mathrm{Li}_2(-1)+\mathrm{Li}_2\left(\frac{2-\sqrt{\xi}}{2+\sqrt{\xi}}\right)-\mathrm{Li}_2\left(-\frac{2-\sqrt{\xi}}{2+\sqrt{\xi}}\right)+$$
$$-2\,\mathrm{Li}_2\left(\frac{-\sqrt{\xi}}{2-\sqrt{\xi}}\right)+2\,\mathrm{Li}_2\left(\frac{-\xi}{4-\xi}\right)-\mathrm{Li}_2\left(\frac{-(1-v)^2}{\sqrt{\xi}(2+\sqrt{\xi})}\right)+$$
$$-\mathrm{Li}_2\left(\frac{1+v}{2+\sqrt{\xi}}\right)+\mathrm{Li}_2\left(-\sqrt{\frac{1-v}{1+v}}\right), \tag{B.12}$$

$$
\begin{aligned}
t_{19} &:= \ln\left(\frac{1+v}{1-v}\right)\ln\left(\frac{1+v}{2-\sqrt{\xi}}\right)+\ln^2\left(\frac{1+v}{2-\sqrt{\xi}}\right)+ \\
&\quad +2\,\mathrm{Li}_2\left(\frac{1-v}{2-\sqrt{\xi}}\right)+2\,\mathrm{Li}_2\left(\frac{2-\sqrt{\xi}}{1+v}\right)-4\,\mathrm{Li}_2(1), && \text{(B.13)} \\
t_{20} &:= \ln\left(\frac{1+v}{1-v}\right)\ln\left(\frac{2-\sqrt{\xi}}{1+v}\right)+2\,\mathrm{Li}_2(-w_0)-2\,\mathrm{Li}_2(w_0)\,+ \\
&\quad -2\,\mathrm{Li}_2\left(-\frac{1-v}{2-\sqrt{\xi}}w_0\right)+2\,\mathrm{Li}_2\left(\frac{1+v}{2-\sqrt{\xi}}w_0\right), && \text{(B.14)} \\
t_{21} &:= 3\left(\frac{1}{2}\ln^2\left(\frac{1+v}{1-v}\right)+\ln\left(\frac{4}{\xi}\sqrt{\frac{1-v}{1+v}}\right)\ln\left(\frac{\sqrt{\xi}}{2-\sqrt{\xi}}\right)\right)+ \\
&\quad +3\Bigg(\mathrm{Li}_2\left(\frac{2-\sqrt{\xi}}{1+v}\right)-\mathrm{Li}_2\left(\frac{2-\sqrt{\xi}}{1+v}\sqrt{\frac{1-v}{1+v}}\right)+ \\
&\quad +\mathrm{Li}_2\left(\frac{1-v}{2-\sqrt{\xi}}\sqrt{\frac{1-v}{1+v}}\right)-\mathrm{Li}_2\left(\frac{1-v}{2-\sqrt{\xi}}\right)\Bigg)+ \\
&\quad +\mathrm{Li}_2\left(\frac{4v}{(1+v)^2}\right)+\mathrm{Li}_2\left(\frac{-2v}{1-v}\right)-\mathrm{Li}_2\left(\frac{2v}{1+v}\right), && \text{(B.15)} \\
t_{22} &:= 2\ln\left(\frac{1+v}{2-\sqrt{\xi}}\right)\ln\left(\frac{2(1+\sqrt{\xi})(2-\sqrt{\xi})}{(1+v)^2}\right)+ \\
&\quad +4\,\mathrm{Li}_2\left(\frac{\sqrt{\xi}-1+v}{2v}\right)-4\,\mathrm{Li}_2\left(\frac{(1-v)(\sqrt{\xi}-1+v)}{2v(2-\sqrt{\xi})}\right) && \text{(B.16)}
\end{aligned}
$$

In the massless limit (i.e. for $\xi \to 0$ and $v \to 1$) the decay rate terms are given by

$$
\begin{array}{llll}
t_1 \to \ln 4 - \frac{3}{2}\ln\left(\frac{4}{\xi}\right) & t_2 \to \ln 4 - \frac{1}{2}\ln\left(\frac{4}{\xi}\right) & t_3 \to \ln\left(\frac{4}{\xi}\right) & t_4 \to \frac{\pi^2}{2} \\
t_5 \to \frac{\pi^2}{6} - \frac{1}{4}\ln^2\left(\frac{4}{\xi}\right) & t_6 \to \frac{\pi^2}{6} + \frac{1}{4}\ln^2\left(\frac{4}{\xi}\right) & t_7 \to -\frac{\pi^2}{2} - \frac{1}{4}\ln^2\left(\frac{4}{\xi}\right) & \\
t_8 \to -\frac{2\pi^2}{3} - \frac{1}{2}\ln^2\left(\frac{4}{\xi}\right) & t_9 \to -\frac{2\pi^2}{3} - \frac{1}{2}\ln^2\left(\frac{4}{\xi}\right) & t_{10} \to \ln\left(\frac{4}{\xi}\right) & \\
t_{11} \to \ln\left(\frac{4}{\xi}\right) & t_{12} \to \ln\left(\frac{4}{\xi}\right) & t_{13} \to 0 & \\
t_{14} \to \frac{\pi^2}{6} & t_{15} \to \frac{\pi^2}{3} + \frac{1}{4}\ln^2\left(\frac{4}{\xi}\right) & t_{16} \to -\frac{\pi^2}{6} & \\
t_{17} \to 0 & t_{18} \to 0 & t_{19} \to -\frac{\pi^2}{3} & \\
t_{20} \to -\frac{\pi^2}{6} & t_{21} \to \frac{\pi^2}{3} + \frac{1}{4}\ln^2\left(\frac{4}{\xi}\right) & t_{22} \to 0 &
\end{array}
\qquad \text{(B.17)}
$$

Appendix C

Gluon energy cut decay rate terms

C.1 Steps for the gluon energy cut calculations

This appendix contains the calculations for the two-fold integrals which are necessary to obtain the unpolarized and polarized structure functions in Sec. 1.1. The calculation is done in a "cascade" of four levels which is detailed in the first section. The sections that follow deal with the new decay rate terms which are relevant for these calculations.

C.1.1 The first step: formal integration

The first step is the formal integration over the phase space variable z where because of $z_\pm = (\mathcal{A} \pm \mathcal{B})/\mathcal{C}$ the result of this integration are expressed by

$$\mathcal{A} = 2y - 2y^2 - \xi y, \quad \mathcal{B} = 2y\sqrt{(1-y)^2 - \xi}, \quad \mathcal{C} = 4y + \xi, \tag{C.1}$$

and λ. One obtains

$$\begin{aligned} \int_0^{y_1} \int_{z_-}^{z_+} y^l z^m dy\, dz &= \tilde{I}_o(l,m), \\ \int_{y_1}^{y_2} \int_{z_-}^{2\lambda - y} y^l z^m dy\, dz &= \tilde{I}_\lambda(l,m) - \frac{1}{2}\tilde{I}_{ab}(l,m) + \frac{1}{2}\tilde{I}_{ba}(l,m). \end{aligned} \tag{C.2}$$

The principle of this splitting procedure is shown by an example. In case of $m = 2$ the z-integration for the first integral results in

$$\int_{z_-}^{z_+} z^2 dz = \frac{(\mathcal{A}+\mathcal{B})^3}{3\mathcal{C}^3} - \frac{(\mathcal{A}-\mathcal{B})^3}{3\mathcal{C}^3} = 2\frac{3\mathcal{A}^2 + \mathcal{B}^2}{3\mathcal{C}^3}\mathcal{B}, \tag{C.3}$$

giving a contribution to $\tilde{I}_o$. In contrast to this, the splitting of the second integral is somehow more involved. By separating the parts symmetric and antisymmetric in $\mathcal{B}$ one obtains three parts which shall be presented in the same order as in the above displayed classification,

$$\int_{z_-}^{2\lambda - y} z^2 dz = \frac{1}{3}(2\lambda - y)^3 - \frac{(\mathcal{A}-\mathcal{B})^3}{3\mathcal{C}^3} = \frac{1}{3}(2\lambda - y)^3 - \mathcal{A}\frac{\mathcal{A}^2 + 3\mathcal{B}^2}{3\mathcal{C}^3} + \frac{3\mathcal{A}^2 + \mathcal{B}^2}{3\mathcal{C}^3}\mathcal{B}. \tag{C.4}$$

It is easy to see that the third part is equal to the one obtained in the z-integration of the first integral except for a factor 1/2. However, one has to keep in mind that the limits of the y-integration have changed. The characteristic for this part for all orders of m is the appearance of $\sqrt{(1-y)^2-\xi}$. This square root which also separates the integral classes themselves does not appear in the second part. Therefore, both parts lead to similar contributions in neighbouring integral classes. For this reason a further integral class $\tilde{K}$ shall be introduced which is classified behind $\tilde{T}$. In contrast to this, the first part is new. Collecting all parts, one ends up with

$$\tilde{I}(l,m)=\frac{1}{2}\tilde{I}_{ba}(l,m)+\tilde{I}_o(l,m)-\frac{1}{2}\tilde{I}_{ab}(l,m)+\tilde{I}_\lambda(l,m). \tag{C.5}$$

C.1.2 The second step: global substitution

According to the polynomial function in the numerator of the integrands, the integral parts can be split up into integrals with simple powers of y. In case of $m=-1$ the z-integration leads to a logarithmic factor which necessitates a separated classification. These logarithmic integrals are denoted by raised indices. One obtains

$$\begin{aligned}
\tilde{I}_{ba}(l,2) &= \frac{4}{3}\Big((16-16\xi+3\xi^2)\hat{I}_{ab}(3,l+3)-4(8-3\xi)\hat{I}_{ab}(3,l+4)+16\hat{I}_{ab}(3,l+5)\Big),\\
\tilde{I}_{ba}(l,1) &= 4\Big((2-\xi)\hat{I}_{ab}(2,l+2)-2\hat{I}_{ab}(2,l+3)\Big),\\
\tilde{I}_{ba}(l,0) &= 4\hat{I}_{ab}(1,l+1),\\
\tilde{I}_{ba}(l,-1) &= \hat{I}^{ba}(l),\\
\tilde{I}_{ba}(l,-2) &= \frac{4}{\xi}\hat{I}_{ab}(0,l-1),
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
\tilde{I}_o(l,2) &= \frac{4}{3}\Big((16-16\xi+3\xi^2)\hat{I}_o(3,l+3)-4(8-3\xi)\hat{I}_o(3,l+4)+16\hat{I}_o(3,l+5)\Big),\\
\tilde{I}_o(l,1) &= 4\Big((2-\xi)\hat{I}_o(2,l+2)-2\hat{I}_o(2,l+3)\Big),\\
\tilde{I}_o(l,0) &= 4\hat{I}_o(1,l+1),\\
\tilde{I}_o(l,-1) &= \hat{I}_o^{ba}(l),\\
\tilde{I}_o(l,-2) &= \frac{4}{\xi}\hat{I}_o(0,l-1),
\end{aligned} \tag{C.7}$$

$$\begin{aligned}
\tilde{I}_{ab}(l,2) &= \frac{2}{3}\Big((16-16\xi+\xi^2)(2-\xi)\hat{S}_{ab}(3,l+3)-6(16-12\xi+\xi^2)\hat{S}_{ab}(3,l+4)\\
&\quad +24(4-\xi)\hat{S}_{ab}(3,l+5)-32\hat{S}_{ab}(3,l+6)\Big),\\
\tilde{I}_{ab}(l,1) &= (8-8\xi+\xi^2)\hat{S}_{ab}(2,l+2)-4(4-\xi)\hat{S}_{ab}(2,l+3)+8\hat{S}_{ab}(2,l+4),\\
\tilde{I}_{ab}(l,0) &= 2\Big((2-\xi)\hat{S}_{ab}(1,l+1)-2\hat{S}_{ab}(1,l+2)\Big),\\
\tilde{I}_{ab}(l,-1) &= \ln\left(\xi(1+\sqrt{\xi})^2\right)\hat{S}_{ab}(0,l)+2\hat{I}^z(l)-\hat{I}^{ab}(l),\\
\tilde{I}_{ab}(l,-2) &= -\frac{2}{\xi}\Big((2-\xi)\hat{S}_{ab}(0,l-1)-2\hat{S}_{ab}(0,l)\Big),
\end{aligned} \tag{C.8}$$

$$\tilde{I}_\lambda(l,2) = \frac{1}{3}\hat{S}_\lambda(-3,l),$$

$$\begin{aligned}
\tilde{I}_\lambda(l,1) &= \frac{1}{2}\hat{S}_\lambda(-2,l),\\
\tilde{I}_\lambda(l,0) &= \hat{S}_\lambda(-1,l),\\
\tilde{I}_\lambda(l,-1) &= \ln\left(1-2\lambda+\sqrt{\xi}\right)\hat{S}_\lambda(0,l)+\hat{I}^\lambda(l),\\
\tilde{I}_\lambda(l,-2) &= -\hat{S}_\lambda(1,l).
\end{aligned} \tag{C.9}$$

The notation with hats stands for

$$\begin{aligned}
\hat{S}_o(k,l) &= \int_0^{y_1}\frac{y^l dy}{(4y+\xi)^k} = (1+\sqrt{\xi})^l\int_{w_1}^{w_0}\left(\frac{1-w^2}{b^2-a^2w^2}\right)^k\left(\frac{w_0^2-w^2}{1-w^2}\right)^l\frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
\hat{S}_{ab}(k,l) &= \int_{y_1}^{y_2}\frac{y^l dy}{(4y+\xi)^k} = (1+\sqrt{\xi})^l\int_{w_2}^{w_1}\left(\frac{1-w^2}{b^2-a^2w^2}\right)^k\left(\frac{w_0^2-w^2}{1-w^2}\right)^l\frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
\hat{S}_\lambda(k,l) &= \int_{y_1}^{y_2}\frac{y^l dy}{(2\lambda-y)^k} = (1+\sqrt{\xi})^l\int_{w_2}^{w_1}\left(\frac{1-w^2}{w^2-w_\lambda^2}\right)^k\left(\frac{w_0^2-w^2}{1-w^2}\right)^l\frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2}
\end{aligned} \tag{C.10}$$

and similarly for the other integral classes. To calculate the second form of the integrals the global substitution

$$y = 1-\sqrt{\xi}\frac{1+w^2}{1-w^2} \tag{C.11}$$

is introduced, leading to $v=\sqrt{1-\xi}$, $v_i=\sqrt{(1-y_i)^2-\xi}$, $v_\lambda=\sqrt{(1-2\lambda)^2-\xi}$,

$$w_0=\sqrt{\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}},\quad w_i=\sqrt{\frac{1-y_i-\sqrt{\xi}}{1-y_i+\sqrt{\xi}}},\quad w_\lambda=\sqrt{\frac{1-2\lambda-\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}} \tag{C.12}$$

as notations as well as $a=2+\sqrt{\xi}$ and $b=2-\sqrt{\xi}$ for limits and constants. The logarithmic integrals are given by

$$\begin{aligned}
\hat{I}^{ba}(l) &= \int_{y_1}^{y_2}\ln\left(\frac{z_+(y)}{z_-(y)}\right)y^l dy\\
&= (1+\sqrt{\xi})^l\int_{w_2}^{w_1}\ln\left(\frac{(1+w)(b+aw)}{(1-w)(b-aw)}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l\frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
\hat{I}^{ba}_o(l) &= \int_0^{y_1}\ln\left(\frac{z_+(y)}{z_-(y)}\right)y^l dy\\
&= (1+\sqrt{\xi})^l\int_{w_1}^{w_0}\ln\left(\frac{(1+w)(b+aw)}{(1-w)(b-aw)}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l\frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
\hat{I}^{z}(l) &= \int_{y_1}^{y_2}\ln\left(\frac{y}{1+\sqrt{\xi}}\right)y^l dy\\
&= (1+\sqrt{\xi})^l\int_{w_2}^{w_1}\ln\left(\frac{w_0^2-w^2}{1-w^2}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l\frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
\hat{I}^{ab}(l) &= \int_{y_1}^{y_2}\ln(4y+\xi)\,y^l dy
\end{aligned}$$

$$
\begin{aligned}
&= (1+\sqrt{\xi})^l \int_{w_2}^{w_1} \ln\left(\frac{b^2-a^2w^2}{1-w^2}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l \frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
\hat{I}^\lambda(l) &= \int_{y_1}^{y_2} \ln\left(\frac{2\lambda-y}{1-2\lambda+\sqrt{\xi}}\right) y^l dy\\
&= (1+\sqrt{\xi})^l \int_{w_2}^{w_1} \ln\left(\frac{w^2-w_\lambda^2}{1-w^2}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l \frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2}.
\end{aligned}
\tag{C.13}
$$

C.1.3 The third step: indefinite integration

All integrals presented in the previous paragraph can be expressed as differences of indefinite integrals. In this context the non-logarithmic integrals with subscripts o and ab can be treated as the same type of integral. One has

$$
\begin{aligned}
\hat{S}_o(k,l) &= S_{ab}(k,l,w_0) - S_{ab}(k,l,w_1),\\
\hat{S}_{ab}(k,l) &= S_{ab}(k,l,w_1) - S_{ab}(k,l,w_2),\\
\hat{S}_\lambda(k,l) &= S_\lambda(k,l,w_1) - S_\lambda(k,l,w_2),\\
\hat{I}^{ba}(l) &= I^{ba}(l,w_1) - I^{ba}(l,w_2),\\
\hat{I}_o^{ba}(l) &= I^{ba}(l,w_0) - I^{ba}(l,w_1),\\
\hat{I}^z(l) &= I^z(l,w_1) - I^z(l,w_2),\\
\hat{I}^{ab}(l) &= I^{ab}(l,w_1) - I^{ab}(l,w_2),\\
\hat{I}^\lambda(l) &= I^\lambda(l,w_1) - I^\lambda(l,w_2).
\end{aligned}
\tag{C.14}
$$

the indefinite integrals are given by

$$
\begin{aligned}
S_{ab}(k,l,w) &= (1+\sqrt{\xi})^l \int \left(\frac{1-w^2}{b^2-a^2w^2}\right)^k \left(\frac{w_0^2-w^2}{1-w^2}\right)^l \frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
S_\lambda(k,l,w) &= \frac{(1+\sqrt{\xi})^l}{(1-2\lambda+\sqrt{\xi})^k} \int \left(\frac{1-w^2}{w^2-w_\lambda^2}\right)^k \left(\frac{w_0^2-w^2}{1-w^2}\right)^l \frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
I^{ba}(l,w) &= (1+\sqrt{\xi})^l \int \ln\left(\frac{(1+w)(b+aw)}{(1-w)(b-aw)}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l \frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
I^z(l,w) &= (1+\sqrt{\xi})^l \int \ln\left(\frac{w_0^2-w^2}{1-w^2}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l \frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
I^{ab}(l,w) &= (1+\sqrt{\xi})^l \int \ln\left(\frac{b^2-a^2w^2}{1-w^2}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l \frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},\\
I^\lambda(l,w) &= (1+\sqrt{\xi})^l \int \ln\left(\frac{w^2-w_\lambda^2}{1-w^2}\right)\left(\frac{w_0^2-w^2}{1-w^2}\right)^l \frac{4\sqrt{\xi}w\,dw}{(1-w^2)^2},
\end{aligned}
\tag{C.15}
$$

and an increase of the order of the integral class (e.g. from $\tilde{I}$ to $\tilde{S}$) leads to an additional factor

$$
\frac{1}{\sqrt{(1-y)^2-\xi}} = \frac{1-w^2}{2\sqrt{\xi}w}
\tag{C.16}
$$

in the integrand. Nevertheless, not all integrals have to be calculated. Instead of this, there exist recurrence relations

$$\begin{aligned} S_{ab}(k,l,w) &= 4S_{ab}(k+1,l+1,w)+\xi S_{ab}(k+1,l,w), \\ S_\lambda(k,l,w) &= 2\lambda S_\lambda(k+1,l,w)-S_\lambda(k+1,l+1,w) \end{aligned} \tag{C.17}$$

for the non-logarithmic integrals resulting from multiplying both denominator and numerator by $4y+\xi$ or $2\lambda-y$, resp., can be used to reduce the number of integrals to a limited number of "border integrals" which denotes integrals with $k=0$ or $l=0$.

C.1.4 The fourth step: partial fractioning

The w-integration can be performed after having done a partial fractioning. This will lead to the final step of the cascade. In the selected example one obtains

$$\begin{aligned} S_{ab}(3,0) &= \frac{4}{(4-\xi)^2}(I^3_{ab-}-I^3_{ab+})+\frac{4-2\sqrt{\xi}+\xi}{(4-\xi)^3}(I^2_{ab-}-I^2_{ab+}), \\ S_{ab}(2,0) &= \frac{1}{4-\xi}(I^2_{ab-}-I^2_{ab+}), \\ S_{ab}(1,0) &= \frac{1}{4}(I^1_{ab-}-I^1_{ab+})-\frac{1}{4}(I^1_{1-}-I^1_{1+}), \\ S_{ab}(0,-2) &= \frac{1}{v}(I^2_{0-}-I^2_{0+}), \\ S_{ab}(0,-1) &= I^1_{0-}-I^1_{0+}-I^1_{1-}+I^1_{1+}, \\ S_{ab}(0,0) &= I^2_{1-}-I^2_{1+} \\ S_{ab}(0,1) &= -I^3_{1-}+I^3_{1+}+\frac{1}{2}(2+\sqrt{\xi})(I^2_{1-}-I^2_{1+}), \\ S_{ab}(0,2) &= I^4_{1-}-I^4_{1+}-(2+\sqrt{\xi})(I^3_{1-}-I^3_{1+})+\frac{1}{2}(2+2\sqrt{\xi}+\xi)(I^2_{1-}-I^2_{1+}), \\ S_{ab}(0,3) &= -I^5_{1-}+I^5_{1+}+\frac{3}{2}(2+\sqrt{\xi})(I^4_{1-}-I^4_{1+})-\frac{1}{4}(12+12\sqrt{\xi}+5\xi)(I^3_{1-}-I^3_{1+})+ \\ &\quad +\frac{1}{8}(8+12\sqrt{\xi}+12\xi+3\xi\sqrt{\xi})(I^2_{1-}-I^2_{1+}), \end{aligned} \tag{C.18}$$

$$\begin{aligned} S_\lambda(1,0) &= I^1_{1-}-I^1_{1+}-I^1_{\lambda-}+I^1_{\lambda+}, \\ S_\lambda(0,l) &= S_{ab}(0,l), \end{aligned} \tag{C.19}$$

$$\begin{aligned} I^\beta(-1) &= I^{\beta1}_{0-}-I^{\alpha1}_{0+}-I^{\beta1}_{1-}+I^{\beta1}_{1+}, \\ I^\beta(0) &= I^{\beta2}_{1-}-I^{\beta2}_{1+}, \\ I^\beta(1) &= -I^{\beta3}_{1-}+I^{\beta3}_{1+}+\frac{1}{2}(2+\sqrt{\xi})(I^{\beta2}_{1-}-I^{\beta2}_{1+}), \\ I^\beta(2) &= I^{\beta4}_{1-}-I^{\beta4}_{1+}-(2+\sqrt{\xi})(I^{\beta3}_{1-}-I^{\beta3}_{1+})+\frac{1}{2}(2+2\sqrt{\xi}+\xi)(I^{\beta2}_{1-}-I^{\beta2}_{1+}), \\ I^\beta(3) &= -I^{\beta5}_{1-}+I^{\beta5}_{1+}+\frac{3}{2}(2+\sqrt{\xi})(I^{\beta4}_{1-}-I^{\beta4}_{1+})-\frac{1}{4}(12+12\sqrt{\xi}+5\xi)(I^{\beta3}_{1-}-I^{\beta3}_{1+})+ \\ &\quad +\frac{1}{8}(8-12\sqrt{\xi}+12\xi+3\xi\sqrt{\xi})(I^{\beta2}_{1-}-I^{\beta2}_{1+}) \end{aligned} \tag{C.20}$$

where $\beta \in \{ba, z, ab, \lambda\}$ and the dependence on w is suppressed. Clearly, the integrals $S_\lambda(0,l)$ coincide with $S_{ab}(0,l)$ since the difference given by the term $2\lambda - y$ or $4y + \xi$ resp. is lost. Remarkable is also the similarity between the splittings for the integrals $S_{ab}(0,l)$ and $I^\alpha(l)$. This splitting is a consequence of the fact that the same term has to be partially fractionized. Before going to this point one has to include a remark. In the presented formulas it seems as if there appear always a difference $I^n_{\alpha-} - I^n_{\alpha+}$ ($\alpha \in \{1, 0, ab, \lambda\}$). Therefore, it would be more convenient to calculate the difference instead of the single parts. But this is only half of the story because in other integral classes there appear sums $I^n_{\alpha-} + I^n_{\alpha-}$ of the same terms. Therefore, the parts should be calculated separately.

$$\begin{aligned}
I^n_w &:= \int \frac{dw}{w^n} \qquad \text{(only appearing for the integral classes } T \text{ and } K\text{)} \\
I^n_{1\pm} &:= (\sqrt{\xi})^{n-1} \int \frac{dw}{(1 \pm w)^n}, \qquad I^n_{ab\pm} := \left(\frac{\sqrt{\xi}}{2+\sqrt{\xi}}\right)^{n-1} \int \frac{a\,dw}{(b \pm aw)^n}, \qquad \text{(C.21)} \\
I^n_{0\pm} &:= \left(\frac{\sqrt{\xi}}{1+\sqrt{\xi}}\right)^{n-1} \int \frac{dw}{(w_0 \pm w)^n}, \qquad I^n_{\lambda\pm} := \left(\frac{\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}\right)^{n-1} \int \frac{dw}{(w_\lambda \pm w)^n}
\end{aligned}$$

are the non-logarithmic terms. The logarithmic terms carry an additional upper index and in combination with this an additional logarithmic factor. This factor is given

$$\begin{aligned}
&\text{for } I^{ba\,n}_{\alpha(\pm)} \text{ by } \ln\left(\frac{(1+w)(b+aw)}{(1-w)(b-aw)}\right), \qquad \text{for } I^{z\,n}_{\alpha(\pm)} \text{ by } \ln\left(\frac{w_0^2 - w^2}{1-w^2}\right), \\
&\text{for } I^{ab\,n}_{\alpha(\pm)} \text{ by } \ln\left(\frac{b^2 - a^2 w^2}{1-w^2}\right), \qquad \text{for } I^{\lambda\,n}_{\alpha(\pm)} \text{ by } \ln\left(\frac{w^2 - w_\lambda^2}{1-w^2}\right). \qquad \text{(C.22)}
\end{aligned}$$

where $\alpha \in \{w, 1, 0\}$ and the paranthesis in the index indicates that in the case $\alpha = w$ the "$\pm$" argument disappears. Logarithmic integrals to $\alpha = ba$ and $\alpha = \lambda$ do not contribute.

C.1.5 Newly classified decay rate terms

All integrals of the last step which lead to polylogarithms shall be used in closed form. These are those logarithmic integrals with a power $n = 1$ of the denominator. Therefore, one defines

$$t^\beta_{\alpha(\pm)}(w) := I^{\beta\,1}_{\alpha(\pm)}(w). \qquad \text{(C.23)}$$

where the new decay rate terms are given in the following section. Using these decay rate terms, the cascade can be followed upwards and will lead to the final result. It is a matter of organization to obtain a short form for it. Appropriate abbreviations have been found. Logarithmic expressions with the same prefactor figure out to be always combined in a few fixed kinds. These terms are denoted by ℓ and found in Appendix C.2.1. The terms ℓ_{m+} ($m \in \{4, 5, 6, 7, 8, 9\}$) occur in the hadron tensor components with even parity and the terms ℓ_{m-} in the components with odd parity. Also the decay rate terms combine almost in the same way. For this reason one defines the combinations t_w, $t_{0\pm}$, and $t_{1\pm}$ which are given in Appendix C.2.2 as well.

C.2 Decay rate terms for the exact gluon energy cut

As already mentioned in the main part of the text, the global substitution

$$y = 1 - \sqrt{\xi}\frac{1+w^2}{1-w^2} \tag{C.24}$$

has been used, leading to $v = \sqrt{1-\xi}$, $v_i = \sqrt{(1-y_i)^2 - \xi}$, $v_\lambda = \sqrt{(1-2\lambda)^2 - \xi}$,

$$w_0 = \sqrt{\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}}, \quad w_i = \sqrt{\frac{1-y_i-\sqrt{\xi}}{1-y_i+\sqrt{\xi}}}, \quad w_\lambda = \sqrt{\frac{1-2\lambda-\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}} \tag{C.25}$$

as well as $a = 2+\sqrt{\xi}$ and $b = 2-\sqrt{\xi}$ for limits and constants.

C.2.1 Logarithmic decay rate terms

The logarithmic decay rate terms ℓ_i are given by

$$\begin{aligned}
\ell_1 &= \ln\left(\frac{w_1^2-w_\lambda^2}{w_0^2-w_1^2}\right) - \ln\left(\frac{1+w_1}{b-aw_1}\right) - \ln\left(\frac{(1+\sqrt{\xi})\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}\right) && \text{(C.26)}\\
\ell_2 &= \ln\left(\frac{w_2^2-w_\lambda^2}{w_0^2-w_2^2}\right) + \ln\left(\frac{b+aw_2}{1-w_2}\right) - \ln\left(\frac{(1+\sqrt{\xi})\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}\right) && \text{(C.27)}\\
\ell_3 &= \ln\left(\frac{w_2}{w_1}\right) && \text{(C.28)}\\
\ell_{4+} &= -\frac{\lambda\xi}{y_1} + \frac{\lambda\xi}{y_2} + 2v\left[4 - 2\ln\left(\frac{4w_0y_1}{\sqrt{\xi}}\right) + \ln\left(\frac{w_0+w_1}{w_0-w_1}\right) + \ln\left(\frac{w_0+w_2}{w_0-w_2}\right)\right] + && \text{(C.29)}\\
&\quad + \left(2v - (2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right)\left[\ln\left(\frac{\xi\Lambda}{v^2}\right) + 2\ln\left(\frac{w_0^2-w_1^2}{1-w_1^2}\right) - 1\right]\\
\ell_{4-} &= 2v\left[2 - 2\ln\left(\frac{2\sqrt{\xi}y_1}{v}\right) + \ln\left(\frac{(1+w_1)(b-aw_1)}{w_0^2-w_1^2}\right) + \ln\left(\frac{(b+aw_2)(1-w_2)}{w_0^2-w_2^2}\right)\right] + \\
&\quad + \left(2v - (2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right)\left[\ln\left(\frac{\xi\Lambda}{v^2}\right) + 2\ln\left(\frac{w_0^2-w_1^2}{1-w_1^2}\right) - 1\right] && \text{(C.30)}\\
\ell_{5+} &= \ln\left(\frac{1-w_2}{1-w_0}\right) - \ln\left(\frac{1+w_1}{1+w_0}\right), \qquad \ell_{5-} = 2\ln\left(\frac{1+v}{1-v}\right) && \text{(C.31)}\\
\ell_{6+} &= 2\ln\left(\frac{1+v}{1-v}\right) - \ln\left(\frac{1+w_1}{b-aw_1}\right) - \ln\left(\frac{b+aw_2}{1-w_2}\right) && \text{(C.32)}\\
\ell_{6-} &= \ln\xi + \ln\left(\frac{1+w_1}{b-aw_1}\right) - \ln\left(\frac{b+aw_2}{1-w_2}\right) && \text{(C.33)}\\
\ell_{7+} &= \ln\left(\frac{w_2^2-w_\lambda^2}{w_1^2-w_\lambda^2}\right), \qquad \ell_{7-} = \ln\left(\frac{w_2-w_\lambda}{w_1-w_\lambda}\right) - \ln\left(\frac{w_2+w_\lambda}{w_1+w_\lambda}\right) && \text{(C.34)}\\
\ell_{8+} &= \ln\left(\frac{w_0^2-w_2^2}{w_0^2-w_1^2}\right), \qquad \ell_{8-} = \ln\left(\frac{w_0-w_2}{w_0-w_1}\right) - \ln\left(\frac{w_0+w_2}{w_0+w_1}\right) && \text{(C.35)}\\
\ell_{9+} &= \ln\left(\frac{1-w_2^2}{1-w_1^2}\right), \qquad \ell_{9-} = \ln\left(\frac{1-w_2}{1-w_1}\right) - \ln\left(\frac{1+w_2}{1+w_1}\right). && \text{(C.36)}
\end{aligned}$$

C.2.2 Dilogarithmic decay rate terms

For the double logarithmic and dilogarithmic decay rate terms one obtains

$$\begin{aligned}
t_w &= \frac{1}{2}(2t_w^{ba}(w_0) - t_w^{ba}(w_1) - t_w^{ba}(w_2)) + (t_w^z(w_2) - t_w^z(w_1)) + \\
&-\frac{1}{2}(t_w^{ab}(w_2) - t_w^{ab}(w_1)) - (t_w^\lambda(w_2) - t_w^\lambda(w_1)) + \ln\left(\frac{(1+\sqrt{\xi})\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}\right)\ln\left(\frac{w_2}{w_1}\right) \quad (C.37) \\
t_{0\pm} &= \frac{1}{2}(2t_{0\pm}^{ba}(w_0) - t_{0\pm}^{ba}(w_1) - t_{0\pm}^{ba}(w_2)) + (t_{0\pm}^z(w_2) - t_{0\pm}^z(w_1)) + \quad (C.38) \\
&-\frac{1}{2}(t_{0\pm}^{ab}(w_2) - t_{0\pm}^{ab}(w_1)) - (t_{0\pm}^\lambda(w_2) - t_{0\pm}^\lambda(w_1)) \pm \ln\left(\frac{(1+\sqrt{\xi})\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}\right)\ln\left(\frac{w_0 \pm w_2}{w_0 \pm w_1}\right) \\
t_{1\pm} &= \frac{1}{2}(2t_{1\pm}^{ba}(w_0) - t_{1\pm}^{ba}(w_1) - t_{1\pm}^{ba}(w_2)) + (t_{1\pm}^z(w_2) - t_{1\pm}^z(w_1)) + \quad (C.39) \\
&-\frac{1}{2}(t_{1\pm}^{ab}(w_2) - t_{1\pm}^{ab}(w_1)) - (t_{1\pm}^\lambda(w_2) - t_{1\pm}^\lambda(w_1)) \pm \ln\left(\frac{(1+\sqrt{\xi})\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}\right)\ln\left(\frac{1 \pm w_2}{1 \pm w_1}\right)
\end{aligned}$$

where

$$\begin{aligned}
t_w^{ba}(w) &= \mathrm{Li}_2(w) - \mathrm{Li}_2(-w) + \mathrm{Li}_2\left(\frac{aw}{b}\right) - \mathrm{Li}_2\left(\frac{-aw}{b}\right), \\
t_w^z(w) &= 2\ln(w_0)\ln(w) + \mathrm{Li}_2(w) - \mathrm{Li}_2(-w) - \mathrm{Li}_2\left(\frac{w}{w_0}\right) - \mathrm{Li}_2\left(\frac{-w}{w_0}\right), \\
t_w^{ab}(w) &= 2\ln(b)\ln(w) + \mathrm{Li}_2(w) - \mathrm{Li}_2(-w) - \mathrm{Li}_2\left(\frac{aw}{b}\right) - \mathrm{Li}_2\left(\frac{-aw}{b}\right), \\
t_w^\lambda(w) &= \ln^2(w) + \mathrm{Li}_2(w) + \mathrm{Li}_2(-w) + \mathrm{Li}_2\left(\frac{w_\lambda}{w}\right) + \mathrm{Li}_2\left(\frac{-w_\lambda}{w}\right), \quad (C.40)
\end{aligned}$$

$$\begin{aligned}
t_{0-}^{ba}(w) &= -2\ln\left(\frac{1+v}{1-v}\right)\ln(w_0 - w) + \\
&+\mathrm{Li}_2\left(\frac{w_0-w}{w_0+1}\right) - \mathrm{Li}_2\left(\frac{w_0-w}{w_0-1}\right) + \mathrm{Li}_2\left(\frac{a(w_0-w)}{aw_0+b}\right) - \mathrm{Li}_2\left(\frac{a(w_0-w)}{aw_0-b}\right), \\
t_{0-}^{ba}(w_0) &= 2\ln\left(\frac{y_1}{\sqrt{\xi}}\right)\ln\left(\frac{1+v}{1-v}\right) - \mathrm{Li}_2\left(\frac{2v}{(1+v)^2}\right) + \mathrm{Li}_2\left(\frac{-2v}{(1-v)^2}\right) + \\
&+\frac{1}{2}\mathrm{Li}_2\left(-\frac{(1-v)^2}{(1+v)^2}\right) - \frac{1}{2}\mathrm{Li}_2\left(-\frac{(1+v)^2}{(1-v)^2}\right), \\
t_{0+}^{ba}(w) &= -2\ln\left(\frac{1+v}{1-v}\right)\ln(w_0 + w) + \\
&+\mathrm{Li}_2\left(\frac{w_0+w}{w_0+1}\right) - \mathrm{Li}_2\left(\frac{w_0+w}{w_0-1}\right) + \mathrm{Li}_2\left(\frac{a(w_0+w)}{aw_0+b}\right) - \mathrm{Li}_2\left(\frac{a(w_0+w)}{aw_0-b}\right), \\
t_{0-}^z(w) &= \frac{1}{2}\ln\left(\frac{\xi}{1-\xi}\right)\ln(w_0 - w) - \frac{1}{2}\ln^2(w_0 - w) + \\
&+\mathrm{Li}_2\left(\frac{w_0-w}{2w_0}\right) - \mathrm{Li}_2\left(\frac{w_0-w}{w_0-1}\right) - \mathrm{Li}_2\left(\frac{w_0-w}{w_0+1}\right), \\
t_{0+}^z(w) &= -\frac{1}{2}\ln\left(\frac{\xi}{1-\xi}\right)\ln(w_0 + w) + \frac{1}{2}\ln^2(w_0 + w) +
\end{aligned}$$

$$
\begin{aligned}
& - \mathrm{Li}_2\left(\frac{w_0+w}{2w_0}\right) + \mathrm{Li}_2\left(\frac{w_0+w}{w_0-1}\right) + \mathrm{Li}_2\left(\frac{w_0+w}{w_0+1}\right), \\
t_{0-}^{ab}(w) &= -\ln\xi \ln(w_0-w) + \\
& + \mathrm{Li}_2\left(\frac{a(w_0-w)}{aw_0-b}\right) + \mathrm{Li}_2\left(\frac{a(w_0-w)}{aw_0+b}\right) - \mathrm{Li}_2\left(\frac{w_0-w}{w_0-1}\right) - \mathrm{Li}_2\left(\frac{w_0-w}{w_0+1}\right), \\
t_{0+}^{ab}(w) &= \ln\xi \ln(w_0+w) + \\
& - \mathrm{Li}_2\left(\frac{a(w_0+w)}{aw_0-b}\right) - \mathrm{Li}_2\left(\frac{a(w_0+w)}{aw_0+b}\right) + \mathrm{Li}_2\left(\frac{w_0+w}{w_0-1}\right) + \mathrm{Li}_2\left(\frac{w_0+w}{w_0+1}\right), \\
t_{0-}^{\lambda}(w) &= -\ln\left(\frac{2\lambda}{1-2\lambda+\sqrt{\xi}}\right)\ln(w_0-w) + \\
& + \mathrm{Li}_2\left(\frac{w_0-w}{w_0-w_\lambda}\right) + \mathrm{Li}_2\left(\frac{w_0-w}{w_0+w_\lambda}\right) - \mathrm{Li}_2\left(\frac{w_0-w}{w_0-1}\right) - \mathrm{Li}_2\left(\frac{w_0-w}{w_0+1}\right), \\
t_{0+}^{\lambda}(w) &= \ln^2(w_0+w) - \ln(1-w_0^2)\ln(w_0+w) + \\
& + \mathrm{Li}_2\left(\frac{w_0-w_\lambda}{w_0+w}\right) + \mathrm{Li}_2\left(\frac{w_0+w_\lambda}{w_0+w}\right) + \mathrm{Li}_2\left(\frac{w_0+w}{w_0-1}\right) + \mathrm{Li}_2\left(\frac{w_0+w}{w_0+1}\right),
\end{aligned}
\tag{C.41}
$$

$$
\begin{aligned}
t_{1-}^{ba}(w) &= \ln^2(1-w) + \ln\left(\frac{a}{8}\right)\ln(1-w) + \\
& + \mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{a(1-w)}\right) + \mathrm{Li}_2\left(\frac{a(1-w)}{4}\right) + \mathrm{Li}_2\left(\frac{1-w}{2}\right), \\
t_{1+}^{ba}(w) &= \ln^2(1+w) + \ln\left(\frac{a}{8}\right)\ln(1+w) + \\
& + \mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{a(1+w)}\right) + \mathrm{Li}_2\left(\frac{a(1+w)}{4}\right) + \mathrm{Li}_2\left(\frac{1+w}{2}\right), \\
t_{1-}^{z}(w) &= -\ln\left(\frac{1+w_0}{2}\right)\ln(1-w) - \mathrm{Li}_2\left(\frac{1-w_0}{1-w}\right) + \mathrm{Li}_2\left(\frac{1-w}{1+w_0}\right) - \mathrm{Li}_2\left(\frac{1-w}{2}\right), \\
t_{1+}^{z}(w) &= \ln\left(\frac{1+w_0}{2}\right)\ln(1+w) - \mathrm{Li}_2\left(\frac{1+w}{1-w_0}\right) + \mathrm{Li}_2\left(\frac{1+w_0}{1+w}\right) + \mathrm{Li}_2\left(\frac{1+w}{2}\right), \\
t_{1-}^{ab}(w) &= -\ln(2a)\ln(1-w) - \mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{a(1-w)}\right) + \mathrm{Li}_2\left(\frac{a(1-w)}{4}\right) - \mathrm{Li}_2\left(\frac{1-w}{2}\right), \\
t_{1+}^{ab}(w) &= \ln(2a)\ln(1+w) + \mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{a(1+w)}\right) - \mathrm{Li}_2\left(\frac{a(1+w)}{4}\right) + \mathrm{Li}_2\left(\frac{1+w}{2}\right), \\
t_{1-}^{\lambda}(w) &= \frac{1}{2}\ln^2(1-w) + \ln 2\ln(1-w) - \ln(1-w_\lambda^2)\ln(1-w) + \\
& + \mathrm{Li}_2\left(\frac{1-w}{1-w_\lambda}\right) + \mathrm{Li}_2\left(\frac{1-w}{1+w_\lambda}\right) - \mathrm{Li}_2\left(\frac{1-w}{2}\right), \\
t_{1+}^{\lambda}(w) &= \frac{1}{2}\ln^2(1+w) - \ln 2\ln(1+w) +
\end{aligned}
$$

$$+\,\mathrm{Li}_2\left(\frac{1-w_\lambda}{1+w}\right)+\mathrm{Li}_2\left(\frac{1+w_\lambda}{1+w}\right)+\mathrm{Li}_2\left(\frac{1+w}{2}\right). \tag{C.42}$$

C.3 Decay rate terms in the limit $\lambda \to 0$

The limit $\lambda \to 0$ is the soft gluon limit. One can take this limit and compare with the results obtained in the main text. The result must be the Born result multiplying a universal factor. Actually, the result has this form. The most basic expressions are the parameters y_i. They satisfy

$$y_1 \to (1-v)\lambda, \qquad y_2 \to (1+v)\lambda. \tag{C.43}$$

One next calculates the limiting values of the w_i,

$$w_i=\sqrt{\frac{1-y_i-\sqrt{\xi}}{1-y_i+\sqrt{\xi}}}\approx w_0\sqrt{1-\frac{y_i}{1-\sqrt{\xi}}+\frac{y_i}{1+\sqrt{\xi}}}=w_0\sqrt{1-\frac{2y_i\sqrt{\xi}}{1-\xi}}\approx w_0\left(1-\frac{y_i\sqrt{\xi}}{v^2}\right) \tag{C.44}$$

and similarly

$$w_i^2 \approx w_0^2\left(1-\frac{2y_i\sqrt{\xi}}{v^2}\right). \tag{C.45}$$

These equations can also be applied to $y_\lambda = 2\lambda$. In addition one uses

$$\begin{aligned}
\ln(1-w_0^2) &= \ln\left(\frac{2\sqrt{\xi}}{1+\sqrt{\xi}}\right), & \ln\left(\frac{1+w_0}{1-w_0}\right) &= \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right),\\
\ln(b^2-a^2w_0^2) &= \ln\left(\frac{2\xi\sqrt{\xi}}{1+\sqrt{\xi}}\right), & \ln\left(\frac{b+aw_0}{b-aw_0}\right) &= \frac{3}{2}\ln\left(\frac{1+v}{1-v}\right).
\end{aligned} \tag{C.46}$$

together with the simplifying identities

$$\begin{aligned}
\ln(1\pm w_0) &= \frac{1}{2}\left(\ln(1-w_0^2)\pm\ln\left(\frac{1+w_0}{1-w_0}\right)\right), & \ln(1\pm v) &= \frac{1}{2}\left(\ln\xi\pm\ln\left(\frac{1+v}{1-v}\right)\right),\\
\ln(b\pm aw_0) &= \frac{1}{2}\left(\ln(b^2-a^2w_0^2)\pm\ln\left(\frac{b+aw_0}{b-aw_0}\right)\right), & \ln w_0 &= \ln v-\ln(1+\sqrt{\xi}),\\
\frac{b}{b\pm aw_0} &= \frac{1}{\xi\sqrt{\xi}}((1+\sqrt{\xi})(2-\sqrt{\xi})^2\mp(4-\xi)v), & \frac{\xi}{1\pm v} &= 1\mp v.
\end{aligned} \tag{C.47}$$

C.3.1 Logarithmic decay rate terms for $\lambda \to 0$

$$\begin{aligned}
\ell_1 &= \ln\left(\frac{w_1^2-w_\lambda^2}{w_0^2-w_1^2}\right)-\ln\left(\frac{1+w_1}{b-aw_1}\right)-\ln\left(\frac{(1+\sqrt{\xi})\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}\right) = \\
&\approx \ln\left(\frac{4\lambda-2y_1}{2y_1}\right)-\ln\left(\frac{1+w_0}{b-aw_0}\right)-\ln\sqrt{\xi} \approx \ln\left(\frac{1+v}{1-v}\right)-\ln\left(\frac{\sqrt{\xi}(1+w_0)}{b-aw_0}\right),
\end{aligned} \tag{C.48}$$

$$\begin{aligned}
\ell_2 &= \ln\left(\frac{w_2^2-w_\lambda^2}{w_0^2-w_2^2}\right)+\ln\left(\frac{b+aw_2}{1-w_2}\right)-\ln\left(\frac{(1+\sqrt{\xi})\sqrt{\xi}}{1-2\lambda+\sqrt{\xi}}\right) = \\
&\approx \ln\left(\frac{4\lambda-2y_2}{2y_2}\right)+\ln\left(\frac{b+aw_0}{1-w_0}\right)-\ln\sqrt{\xi} \approx -\ln\left(\frac{1+v}{1-v}\right)+\ln\left(\frac{b+aw_0}{\sqrt{\xi}(1-w_0)}\right),
\end{aligned} \tag{C.49}$$

$$\ell_3 \;=\; \ln\left(\frac{w_2}{w_1}\right) \;\approx\; \ln\left(\frac{w_0}{w_0}\right) \;=\; 0, \tag{C.50}$$

$$\begin{aligned}
\ell_{4+} \;=\;& -\frac{\lambda\xi}{y_1} + \frac{\lambda\xi}{y_2} + \\
& +2v\left[4 - 2\ln\left(\frac{4w_0 y_1}{\sqrt{\xi}}\right) + \ln\left(\frac{w_0+w_1}{w_0-w_1}\right) + \ln\left(\frac{w_0+w_2}{w_0-w_2}\right)\right] + \\
& + \left(2v - (2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right)\left[\ln\left(\frac{\xi\Lambda}{v^2}\right) + 2\ln\left(\frac{w_0^2-w_1^2}{1-w_1^2}\right) - 1\right] \;=
\end{aligned} \tag{C.51}$$

$$\begin{aligned}
\approx\;& 2v\left[3 - 4\ln(2\lambda) + 2\ln 2 + 2\ln(1+\sqrt{\xi}) - 2\ln\xi + 2\ln v + \ln\left(\frac{1+v}{1-v}\right)\right] + \\
& + \left(2v - (2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right) \;\times \\
& \times\; \left[\ln\Lambda + 2\ln(2\lambda) - 2\ln 2 - 2\ln(1+\sqrt{\xi}) + 2\ln\xi - 2\ln v - \ln\left(\frac{1+v}{1-v}\right) - 1\right],
\end{aligned} \tag{C.52}$$

$$\begin{aligned}
\ell_{4-} \;=\;& 2v\left[2 - 2\ln\left(\frac{2\sqrt{\xi}y_1}{v}\right) + \right. \\
& \left. + \ln\left(\frac{(1+w_1)(b-aw_1)}{w_0^2-w_1^2}\right) + \ln\left(\frac{(b+aw_2)(1-w_2)}{w_0^2-w_2^2}\right)\right] + \\
& + \left(2v - (2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right)\left[\ln\left(\frac{\xi\Lambda}{v^2}\right) + 2\ln\left(\frac{w_0^2-w_1^2}{1-w_1^2}\right) - 1\right] \;=
\end{aligned} \tag{C.53}$$

$$\begin{aligned}
\approx\;& 2v\left[2 - 4\ln(2\lambda) + 2\ln 2 + 2\ln(1+\sqrt{\xi}) - 2\ln\xi + 2\ln v + \ln\left(\frac{1+v}{1-v}\right)\right] + \\
& + \left(2v - (2-\xi)\ln\left(\frac{1+v}{1-v}\right)\right) \;\times \\
& \times\; \left[\ln\Lambda + 2\ln(2\lambda) - 2\ln 2 - 2\ln(1+\sqrt{\xi}) + 2\ln\xi - 2\ln v - \ln\left(\frac{1+v}{1-v}\right) - 1\right],
\end{aligned} \tag{C.54}$$

$$\ell_{5+} \;=\; \ln\left(\frac{1-w_2}{1-w_0}\right) - \ln\left(\frac{1+w_1}{1+w_0}\right) \;\approx\; \ln\left(\frac{(1-w_0)(1+w_0)}{(1-w_0)(1+w_0)}\right) \;=\; 0, \tag{C.55}$$

$$\ell_{5-} \;=\; 2\ln\left(\frac{1+v}{1-v}\right), \tag{C.56}$$

$$\begin{aligned}
\ell_{6+} \;=\;& 2\ln\left(\frac{1+v}{1-v}\right) - \ln\left(\frac{1+w_1}{b-aw_1}\right) - \ln\left(\frac{b+aw_2}{1-w_2}\right) \;= \\
\approx\;& 2\ln\left(\frac{1+v}{1-v}\right) - \ln\left(\frac{(1+w_0)(b+aw_0)}{(1-w_0)(b-aw_0)}\right) \;=\; 0,
\end{aligned} \tag{C.57}$$

$$\begin{aligned}
\ell_{6-} \;=\;& \ln\xi + \ln\left(\frac{1+w_1}{b-aw_1}\right) - \ln\left(\frac{b+aw_2}{1-w_2}\right) \;= \\
\approx\;& \ln\xi + \ln\left(\frac{(1+w_0)(1-w_0)}{(b-aw_0)(b+aw_0)}\right) \;=\; \ln\xi - \ln\xi \;=\; 0,
\end{aligned} \tag{C.58}$$

$$\ell_{7+} \;=\; \ln\left(\frac{w_2^2-w_\lambda^2}{w_1^2-w_\lambda^2}\right) \;\approx\; \ln\left(\frac{4\lambda-2(1+v)\lambda}{4\lambda-2(1-v)\lambda}\right) \;=\; -\ln\left(\frac{1+v}{1-v}\right), \tag{C.59}$$

$$\begin{aligned}
\ell_{7-} &= \ln\left(\frac{w_2-w_\lambda}{w_1-w_\lambda}\right)-\ln\left(\frac{w_2+w_\lambda}{w_1+w_\lambda}\right) \approx \ln\left(\frac{2\lambda-(1+v)\lambda}{2\lambda-(1-v)\lambda}\right) = -\ln\left(\frac{1+v}{1-v}\right), &\text{(C.60)}\\
\ell_{8+} &= \ln\left(\frac{w_0^2-w_2^2}{w_0^2-w_1^2}\right) \approx \ln\left(\frac{2(1+v)\lambda}{2(1-v)\lambda}\right) = \ln\left(\frac{1+v}{1-v}\right), &\text{(C.61)}\\
\ell_{8-} &= \ln\left(\frac{w_0-w_2}{w_0-w_1}\right)-\ln\left(\frac{w_0+w_2}{w_0+w_1}\right) \approx \ln\left(\frac{2(1+v)\lambda}{2(1-v)\lambda}\right) = \ln\left(\frac{1+v}{1-v}\right), &\text{(C.62)}\\
\ell_{9+} &= \ln\left(\frac{1-w_2^2}{1-w_1^2}\right) \approx \ln\left(\frac{1-w_0^2}{1-w_0^2}\right) = 0, &\text{(C.63)}\\
\ell_{9-} &= \ln\left(\frac{1-w_2}{1-w_1}\right)-\ln\left(\frac{1+w_2}{1+w_1}\right) \approx \ln\left(\frac{1-w_0}{1-w_0}\right)-\ln\left(\frac{1+w_0}{1+w_0}\right) = 0. &\text{(C.64)}
\end{aligned}$$

C.3.2 Dilogarithmic decay rate terms for $\lambda \to 0$

It is too involved to go into all details here. However, it should be noted here again that one has to use $t_{0-}^{ba\,\prime}(w_0)$ instead of $t_{0-}^{ba}(w_0)$. The final limiting results for the decay rate terms are

$$\begin{aligned}
t_w &\approx 0, \qquad t_{1+} \approx 0, \qquad t_{1-} \approx 0, \qquad t_{0+} \approx 0, &\text{(C.65)}\\
t_{0-} &\approx \left(4\ln(2\lambda)-2\ln 2+2\ln\xi-2\ln v-2\ln(1+\sqrt{\xi})-\ln\left(\frac{1+v}{1-v}\right)\right)\ln\left(\frac{1+v}{1-v}\right)-t_0
\end{aligned}$$

where

$$\begin{aligned}
t_0 &:= \ln\left(\frac{1+v}{1-v}\right)\ln(2\sqrt{\xi}) + \\
&+\mathrm{Li}_2\left(\frac{1+v}{2}\right)-\mathrm{Li}_2\left(\frac{1-v}{2}\right)+\mathrm{Li}_2\left(\frac{2v}{(1+v)^2}\right)-\mathrm{Li}_2\left(\frac{-2v}{(1-v)^2}\right)+\\
&+\frac{1}{2}\mathrm{Li}_2\left(-\frac{(1+v)^2}{(1-v)^2}\right)-\frac{1}{2}\mathrm{Li}_2\left(-\frac{(1-v)^2}{(1+v)^2}\right). &\text{(C.66)}
\end{aligned}$$

It is possible to show that the decay rate term t_0 can be expressed in a more closed way, namely

$$t_0 = 2\,\mathrm{Li}_2\left(\frac{2v}{1+v}\right)+\frac{1}{2}\ln^2\left(\frac{1+v}{1-v}\right). \tag{C.67}$$

C.4 The full phase space integration limit

In order to calculate the contributions for the full phase space (where $\lambda = (1-\xi)/2$ is the maximal value for the gluon energy divided by $\sqrt{q^2}$). An additional phase space region has to be calculated. The corrected logarithmic and dilogarithmic decay rate terms including these additions are given in the following.

C.4.1 Logarithmic decay rate terms

$$\ell_2^c = \ln\left(\frac{1+w_2}{1-w_2}\right)+\ln\left(\frac{b+aw_2}{b-aw_2}\right),$$

$$
\begin{aligned}
\ell^c_{4-} &= \ln(1-w_2^2)+\ln(b^2-a^2w_2^2), & \ell^c_{4+} &= \ln\left(\frac{w_0+w_2}{w_0-w_2}\right),\\
\ell^c_{5-} &= \ln b \;=\; \ln(2-\sqrt{\xi}), & \ell^c_{5+} &= \ln\left(\frac{1+w_2}{1-w_2}\right),\\
\ell^c_{6-} &= \ln(1-w_2^2)-\ln(b^2-a^2w_2^2), & \ell^c_{7-} &= \ln\left(\frac{w_0^2}{w_0^2-w_2^2}\right).
\end{aligned}
\tag{C.68}
$$

C.4.2 Dilogarithmic decay rate terms

For the additional phase space contribution one obtains

$$
t^c_w = t^{ba}_w(w_2)-t^{ba}_w(0),\qquad t^c_{0\pm} = t^{ba}_{0\pm}(w_2)-t^{ba}_{0\pm}(0),\qquad t^c_{1\pm} = t^{ba}_{1\pm}(w_2)-t^{ba}_{1\pm}(0) \tag{C.69}
$$

where the different parts are given as before.

Appendix D

Special function families

D.1 Bessel functions

Bessel functions Z_λ are solutions of the differential equation

$$\frac{d^2}{dz^2}Z_\lambda + \frac{1}{z}\frac{d}{dz}Z_\lambda + \left(1 - \frac{\lambda^2}{z^2}\right) Z_\lambda = 0. \tag{D.1}$$

Special cases are the Bessel functions of the first kind (simply called Bessel functions) $J_\lambda(z)$, the Bessel functions of the second kind (Neumann or Weber functions) $N_\lambda(z)$, and the Bessel functions of the third kind (Hankel functions) $H_\lambda^{(+)}(z)$ and $H_\lambda^{(-)}(z)$. In addition, there are the modified Bessel functions $I_\lambda(z)$ and the so-called McDonald functions $K_\lambda(z)$ related to the Hankel functions. Only a few features of these functions can be presented in this Appendix. For a more general review see Refs. [85, 101, 126].

D.1.1 Bessel and Neumann function

The general solution of Eq. (D.1) is given by

$$Z_\lambda(z) = c_1 J_\lambda(z) + c_2 J_{-\lambda}(z), \tag{D.2}$$

if λ is non-integer, and

$$Z_\lambda(z) = c_1 J_\lambda(z) + c_2 N_\lambda(z), \tag{D.3}$$

if λ is an integer. In this case $J_{-\lambda}(z) = (-1)^\lambda J_\lambda(z)$ and $N_{-\lambda}(z) = (-1)^\lambda N_\lambda(z)$. The (ordinary) *Bessel function* $J_\lambda(z)$ is regular at the origin while the *Neumann function* $N_\lambda(z)$ is singular. The series expansion of the two *trigonometric Bessel functions* is given by

$$J_\lambda(z) = (z/2)^\lambda \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\Gamma(\lambda + k + 1)} \tag{D.4}$$

while

$$N_\lambda(z) = \frac{1}{\sin(\pi\lambda)} \left(\cos(\pi\lambda) J_\lambda(z) - J_{-\lambda}(z)\right) \qquad (|\arg z| < \pi) \tag{D.5}$$

where the case of integer $\lambda = n$ can be obtained by taking the limit $\lambda \to n$.

D.1.2 Hankel functions

The *Hankel functions (complex valued exponential Bessel functions)* are given by

$$H_\lambda^{(\pm)}(z) = J_\lambda(z) \pm i N_\lambda(z) \tag{D.6}$$

where the compact notation $H_\lambda^{(\pm)}(z)$ is favoured instead of $H_\lambda^{(1)}(z)$ and $H_\lambda^{(2)}(z)$ as used in the literature. The relation $\bar{H}_\lambda^{(\pm)}(z) = H_{\bar{\lambda}}^{(\mp)}(\bar{z})$ is obvious.

D.1.3 Modified Bessel functions

The *modified Bessel functions (real valued exponential Bessel functions)* are given by

$$I_\lambda(z) = \begin{cases} e^{-i\pi\lambda/2} J_\lambda(e^{i\pi/2} z) & \text{for } -\pi < \arg z \leq \pi/2 \\ e^{+3i\pi\lambda/2} J_\lambda(e^{-3i\pi/2} z) & \text{for } \pi/2 < \arg z \leq \pi. \end{cases} \tag{D.7}$$

For integer $\lambda = n$ one obtains $I_n(z) = i^{-n} J_n(iz)$ using a short notation. Finally, the *McDonald functions* are defined via the Hankel functions,

$$K_\lambda(z) = \frac{i\pi}{2} e^{i\pi\lambda/2} H_\lambda^{(1)}(e^{i\pi/2} z) = \frac{i\pi}{2} e^{-i\pi\lambda/2} H_{-\lambda}^{(1)}(e^{i\pi/2} z). \tag{D.8}$$

D.1.4 Asymptotic expansions

The asymptotic expansions for the trigonometric Bessel functions are given by

$$\begin{aligned} J_{\pm\lambda}(z) \;=\; & \sqrt{\frac{2}{\pi z}} \Bigg\{ \cos\left(z \mp \frac{\pi}{2}\lambda - \frac{\pi}{4}\right) \sum_{k=0} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma(\lambda + 2k + 1/2)}{\Gamma(\lambda - 2k + 1/2)(2k)!} + \\ & - \sin\left(z \mp \frac{\pi}{2}\lambda - \frac{\pi}{4}\right) \sum_{k=0} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma(\lambda + 2k + 3/2)}{\Gamma(\lambda - 2k - 1/2)(2k+1)!} \Bigg\}, \end{aligned} \tag{D.9}$$

$$\begin{aligned} N_{\pm\lambda}(z) \;=\; & \sqrt{\frac{2}{\pi z}} \Bigg\{ \sin\left(z \mp \frac{\pi}{2}\lambda - \frac{\pi}{4}\right) \sum_{k=0} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma(\lambda + 2k + 1/2)}{\Gamma(\lambda - 2k + 1/2)(2k)!} + \\ & + \cos\left(z \mp \frac{\pi}{2}\lambda - \frac{\pi}{4}\right) \sum_{k=0} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma(\lambda + 2k + 3/2)}{\Gamma(\lambda - 2k - 1/2)(2k+1)!} \Bigg\}, \end{aligned} \tag{D.10}$$

both for $|\arg z| < \pi$. The upper limit of the sums are not specified and should be replaced by an appropiate limit. For the complex valued exponential Bessel functions one finds

$$H_\lambda^{(\pm)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left(\pm i\left(z - \frac{\pi}{2}\lambda - \frac{\pi}{4}\right)\right) \sum_{k=0} \frac{(\mp 1)^k}{(2iz)^k} \frac{\Gamma(\lambda + k + 1/2)}{\Gamma(\lambda - k + 1/2)k!} \tag{D.11}$$

for $\mathrm{Re}\,\nu > -1/2$ and $|\arg z| < \pi$. The real valued exponential Bessel functions, finally, have the asymptotics

$$\begin{aligned} I_\lambda(z) \;=\; & \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0} \frac{(-1)^k}{(2z)^k} \frac{\Gamma(\lambda + k + 1/2)}{\Gamma(\lambda - k + 1/2)k!} + \\ & + \frac{e^{-z \pm (\lambda + 1/2)\pi i}}{\sqrt{2\pi z}} \sum_{k=0} \frac{1}{(2z)^k} \frac{\Gamma(\lambda + k + 1/2)}{\Gamma(\lambda - k + 1/2)k!}, \end{aligned} \tag{D.12}$$

where the plus sign is valid for $-\pi/2 < \arg z < 3\pi/2$ while the minus sign has to be taken for $-3\pi/2 < \arg z < \pi/2$, and

$$K_\lambda(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0} \frac{1}{(2z)^k} \frac{\Gamma(\lambda + k + 1/2)}{\Gamma(\lambda - k + 1/2) k!}. \tag{D.13}$$

The apparent contradiction in the ranges of definition for $I_\lambda(z)$ is explained by *Stokes' phenomenon*. One can immediately see that the combination $K_\lambda(z) - e^{\pm i\pi(\lambda+1/2)} I_\lambda(z)$ is a purely exponentially increasing function in the upper resp. lower complex half plane.

D.1.5 Bessel functions with half integer index

For $\lambda = n + 1/2$ where n is an integer, the asymptotic expansions for the Bessel functions simplify essentially, they read

$$\begin{aligned} J_{n+1/2}(z) &= \sqrt{\frac{2}{\pi z}} \Bigg\{ \sin(z - \pi n/2) \sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!(2z)^{2k}} + \\ &\quad + \cos(z - \pi n/2) \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}} \Bigg\} = \\ &= (-1)^n z^{n+1/2} \sqrt{\frac{2}{\pi}} \left(\frac{d}{z\,dz} \right)^n \frac{\sin z}{z}, \end{aligned} \tag{D.14}$$

$$\begin{aligned} J_{-n-1/2}(z) &= \sqrt{\frac{2}{\pi z}} \Bigg\{ \cos(z + \pi n/2) \sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!(2z)^{2k}} + \\ &\quad - \sin(z + \pi n/2) \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}} \Bigg\} = \\ &= z^{n+1/2} \sqrt{\frac{2}{\pi}} \left(\frac{d}{z\,dz} \right)^n \frac{\cos z}{z} \end{aligned} \tag{D.15}$$

($[x]$ is the largest integer less or equal to x). For the Neumann functions one obtains

$$N_{n+1/2} = (-1)^{n-1} J_{-n-1/2}(z), \qquad N_{-n-1/2} = (-1)^n J_{n+1/2}(z). \tag{D.16}$$

For the Hankel functions (complex valued exponential Bessel functions) one obtains

$$H^{(\pm)}_{n-1/2}(z) = \sqrt{\frac{2}{\pi z}} e^{\pm i(z - \pi n/2)} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-k-1)!} \left(\frac{\mp 1}{2iz} \right)^k. \tag{D.17}$$

The real valued exponential Bessel functions for half integer index finally read

$$\begin{aligned} I_{\pm(n+1/2)}(z) &= \frac{1}{\sqrt{2\pi z}} \Bigg\{ e^z \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!(n-k)!(2z)^k} + (-1)^{n+1} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2z)^k} \Bigg\}, \\ K_{n+1/2}(z) &= \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2k)^k}. \end{aligned} \tag{D.18}$$

D.1.6 Bessel functions as multi-valued functions

The Bessel functions, except for $J_\lambda(z)$, are multi-valued starting with the branch point $z = 0$. The branches of the functions on different sides of the cut along the negative real axis are related by

$$\begin{aligned}
J_\lambda(e^{i\pi\mu}z) &= e^{i\pi\mu\lambda}J_\lambda(z), \\
N_\lambda(e^{i\pi\mu}z) &= e^{-i\pi\mu\lambda}N_\lambda(z) + 2i\sin(\pi\mu\lambda)\cot(\pi\lambda)J_\lambda(z), \\
H_\lambda^{(\pm)}(e^{i\pi\mu}z) &= e^{-i\pi\mu\lambda}H_\lambda^{(\pm)}(z) \mp 2e^{\mp i\pi\lambda}\frac{\sin(\pi\mu\lambda)}{\sin(\pi\lambda)}J_\lambda(z), \\
I_\lambda(e^{i\pi\mu}z) &= e^{i\pi\mu\lambda}I_\lambda(z), \\
K_\lambda(e^{i\pi\mu}z) &= e^{-i\pi\mu\lambda}K_\lambda(z) - i\pi\frac{\sin(\pi\mu\lambda)}{\sin(\pi\lambda)}I_\lambda(z).
\end{aligned} \tag{D.19}$$

For special cases one has useful relations following from thes general formulas. They are

$$H_\lambda^{(\pm)}(e^{\pm i\pi}z) = -H_{-\lambda}^{(\mp)}(z) = -e^{\mp i\pi\lambda}H_\lambda^{(\mp)}(z) \tag{D.20}$$

and

$$J_\lambda(z)N_{\lambda+1}(z) - J_{\lambda+1}(z)N_\lambda(z) = -\frac{2}{\pi z}, \tag{D.21}$$

$$I_\lambda(z)K_{\lambda+1}(z) + I_{\lambda+1}(z)K_\lambda(z) = \frac{1}{z}. \tag{D.22}$$

D.1.7 Functional equations

Functional equations are of help in order to relate Bessel functions of different degree. In the following the notation $Z_\lambda(z)$ can be used for $J_\lambda(z)$, $N_\lambda(z)$, and $H_\lambda^{(\pm)}(z)$ if no special choice is specified. The fundamental recursion formulas are given by

$$zZ_{\lambda-1}(z) + zZ_{\lambda+1}(z) = 2\lambda Z_\lambda(z), \qquad Z_{\lambda-1}(z) - Z_{\lambda+1}(z) = 2\frac{d}{dz}Z_\lambda(z). \tag{D.23}$$

As a consequence, recursion formulas for the real valued exponential Bessel functions (not included in the notation $Z_\lambda(z)$) can be derived as well,

$$\begin{aligned}
zI_{\lambda+1}(z) - zI_{\lambda+1}(z) = 2\lambda I_\lambda(z), &\qquad I_{\lambda-1}(z) + I_{\lambda+1}(z) = 2\frac{d}{dz}I_\lambda(z), \\
zK_{\lambda+1}(z) - zK_{\lambda+1}(z) = -2\lambda K_\lambda(z), &\qquad K_{\lambda-1}(z) + K_{\lambda+1}(z) = -2\frac{d}{dz}K_\lambda(z),
\end{aligned} \tag{D.24}$$

The recursion equations can be reformulated as

$$z\frac{d}{dz}Z_\lambda(z) + \lambda Z_\lambda(z) = zZ_{\lambda-1}(z), \qquad z\frac{d}{dz}Z_\lambda(z) - \lambda Z_\lambda(z) = -zZ_{\lambda+1}(z). \tag{D.25}$$

and accordingly

$$\begin{aligned}
z\frac{d}{dz}I_\lambda(z) + \lambda I_\lambda(z) = zI_{\lambda-1}(z), &\qquad z\frac{d}{dz}I_\lambda(z) - \lambda I_\lambda(z) = zI_{\lambda+1}(z), \\
z\frac{d}{dz}K_\lambda(z) + \lambda K_\lambda(z) = -zK_{\lambda-1}(z), &\qquad z\frac{d}{dz}K_\lambda(z) - \lambda K_\lambda(z) = -zK_{\lambda+1}(z).
\end{aligned} \tag{D.26}$$

D.2 Hypergeometric functions

Hypergeometric functions are solutions of the *hypergeometric differential equation*

$$z(1-z)\frac{d^2}{dz^2}F + (c-(a+b+1)z)\frac{d}{dz}F - abF = 0. \tag{D.27}$$

The series expansion of the hypergeometric function $F = F(a, b; c; z)$ is given by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k \tag{D.28}$$

where

$$(a)_0 = 1, \qquad (a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{D.29}$$

while the integral representation for $c > a > 0$ reads

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{a-1}(1-x)^{c-a-1}(1-zx)^{-b}dx \tag{D.30}$$

These hypergeometric functions are also often written as ${}_2F_1(a, b; c; z)$ as a special case of the *generalized hypergeometric functions*

$${}_pF_q(a_1, \ldots, a_p; c_1, \ldots, c_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{k!(c_1)_k \cdots (c_q)_k} z^k. \tag{D.31}$$

The hypergeometric functions appear for instance in integrals of Bessel functions. At this point an integral representation will be derived which is useful for the expansion near the production threshold in Sec. 3.5.

D.2.1 Spectral density of the hypergeometric function

It is useful to calculate the discontinuity for the hypergeometric function. This can actually be done for the integral representation before the integration is performed. The starting point is the observation that for $xt > 1$

$$\begin{aligned}\mathrm{Disc}(1-zx)^{-b} &= (1-zxe^{i0})^{-b} - (1-zxe^{-i0})^{-b} = \left((zx-1)e^{-i\pi}\right)^{-b} - \left((zx-1)e^{i\pi}\right)^{-b} = \\ &= (zx-1)^{-b}e^{i\pi b} - (zx-1)^{-b}e^{-i\pi b} = 2i\sin(\pi b)(zx-1)^{-b}\end{aligned} \tag{D.32}$$

while for $xt \leq 1$ the discontinuity vanishes. Therefore, one obtains

$$\begin{aligned}\rho_F(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \frac{\sin(\pi b)}{\pi} \int_{1/z}^1 t^{a-1}(1-x)^{c-a-1}(zx-1)^{-b}dx = \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)\Gamma(b)\Gamma(1-b)} \int_{1/z}^1 x^{a-1}(1-x)^{c-a-1}(zx-1)^{-b}dx\end{aligned} \tag{D.33}$$

(note that $\Gamma(1-b)\Gamma(b) = \pi/\sin(\pi b)$). Using the substitution

$$x = 1 - x' \Rightarrow dx = -dx', \qquad x' = \left(1 - \frac{1}{z}\right)x'' \Rightarrow dx' = \left(1 - \frac{1}{z}\right)dx'' \tag{D.34}$$

one obtains

$$\begin{aligned}
&\int_{1/z}^{1} x^{a-1}(1-x)^{c-a-1}(zx-1)^{-b}dx \;= \\
&= \int_0^{1-1/z}(1-x')^{a-1}x'^{c-a-1}(z-1-zx')^{-b}dx' \;= \\
&= \int_0^1\left(1-\left(1-\frac{1}{z}\right)x''\right)^{a-1}\left(1-\frac{1}{z}\right)^{c-a-1}x''^{c-a-1}(z-1)^{-b}(1-x'')^{-b}\left(1-\frac{1}{z}\right)dx'' \;= \\
&= \left(1-\frac{1}{z}\right)^{c-a}(z-1)^{-b}\int_0^1 x''^{c-a-1}(1-x'')^{-b}\left(1-\left(1-\frac{1}{z}\right)x''\right)^{a-1}dx'' \;= \\
&= z^{-b}\left(1-\frac{1}{z}\right)^{c-a-b}\frac{\Gamma(c-a)\Gamma(1-b)}{\Gamma(c-a-b+1)}F\left(c-a,1-a;c-a-b+1;1-\frac{1}{z}\right) \qquad \text{(D.35)}
\end{aligned}$$

and therefore

$$\rho_F(a,b;c;z)=\frac{\Gamma(c)x^{-b}(1-1/z)^{c-a-b}}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}F\left(c-a,1-a;c-a-b+1;1-\frac{1}{z}\right). \qquad \text{(D.36)}$$

D.2.2 Spectral density of a Bessel function integral

The starting point is the integral given by Eq. (6.621.3) in Ref. [101],

$$\begin{aligned}
&\int_0^\infty x^{\mu-1}e^{-\alpha x}K_\nu(\beta x)dx \;= \\
&= \frac{\sqrt{\pi}(2\beta)^\nu}{(\alpha+\beta)^{\mu+\nu}}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+1/2)}F\left(\mu+\nu,\nu+\frac{1}{2};\mu+\frac{1}{2};\frac{\alpha-\beta}{\alpha+\beta}\right) \qquad \text{(D.37)}
\end{aligned}$$

which is valid for $\operatorname{Re}\mu > |\operatorname{Re}\nu|$ and $\operatorname{Re}(\alpha+\beta)>0$. At first sight, this integral representation seems to be useful. However, one leaves the range of validity if one continues to the complex plane in order to calculate the spectral density of this expression. In addition there can also be discontinuities in the prefactor. Finally,

$$\alpha=\Delta-m_0,\quad \beta=m_0 \quad\Rightarrow\quad \frac{\alpha-\beta}{\alpha+\beta}=1-\frac{2m_0}{\Delta} \qquad \text{(D.38)}$$

means that the the latter ratio takes values in the interval $[-1,1]$ only for $0<\Delta<m_0$. However, one can find a better representation because, as stated in Ref. [126] for the three numbers $\pm(1-c)$, $\pm(a-b)$, and $\pm(a+b-c)$, there is a quadratic transformation available if either two of these numbers are equal or one of them is equal to 1. In the present case one has $a-c=-(a-b)$ or $c=a-b+1$ which gives rise to the possibilities (Eqs. (15.3.26–28) in Ref. [126])

$$\begin{aligned}
&F(a,b;a-b+1;z) \;= \\
&= (1+z)^{-a}F\left(\frac{a}{2},\frac{a}{2}+\frac{1}{2};a-b+1;\frac{4z}{(1+z)^2}\right) \qquad \text{(D.39)} \\
&= \left(1\pm\sqrt{z}\right)^{-2a}F\left(a,a-b+\frac{1}{2};2a-2b+1;\frac{\pm4\sqrt{z}}{(1\pm\sqrt{z})^2}\right) \qquad \text{(D.40)} \\
&= (1-z)^{-a}F\left(\frac{a}{2},\frac{a}{2}-b+\frac{1}{2};a-b+1;\frac{-4z}{(1-z)^2}\right). \qquad \text{(D.41)}
\end{aligned}$$

One can take the quadratic transformation in Eq. (D.39). Then the argument of the hypergeometric function reads

$$4\frac{\alpha-\beta}{\alpha+\beta}\left(\frac{\alpha+\beta}{2\alpha}\right)^2=\frac{\alpha^2-\beta^2}{\alpha^2}=1-\frac{\beta^2}{\alpha^2}. \tag{D.42}$$

Using this one obtains

$$\begin{aligned}\int_0^\infty x^{\mu-1}e^{-\alpha x}K_\nu(\beta x)dx &= \\ &= \frac{\sqrt{\pi}(2\beta)^\nu}{(2\alpha)^{\mu+\nu}}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+1/2)}F\left(\frac{\mu+\nu}{2},\frac{\mu+\nu+1}{2};\mu+\frac{1}{2};1-\frac{\beta^2}{\alpha^2}\right)\end{aligned} \tag{D.43}$$

where the prefactor is now regular unless Δ is greater than m_0. But still, this is not the expression which is most appropriate for the purpose on hand. Instead, one uses a further relation (see Ref. [126], Eq. (15.3.8))

$$\begin{aligned}F(a,b;c;z) &= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}F\left(a,c-b;a-b+1;\frac{1}{1-z}\right)+ \\ &+(1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}F\left(b,c-a;b-a+1;\frac{1}{1-z}\right).\end{aligned} \tag{D.44}$$

With this one obtains

$$\begin{aligned}&\int_0^\infty x^{\mu-1}e^{-\alpha x}K_\nu(\beta x)dx = \\ &= \frac{\sqrt{\pi}(2\beta)^\nu}{(2\alpha)^{\mu+\nu}}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+1/2)}F\left(\frac{\mu+\nu}{2},\frac{\mu+\nu+1}{2};\mu+\frac{1}{2};1-\frac{\beta^2}{\alpha^2}\right) = \\ &= \frac{\sqrt{\pi}(2\beta)^\nu}{(2\alpha)^{\mu+\nu}}\times \\ &\times\left[\frac{\alpha^{\mu+\nu}}{\beta^{\mu+\nu}}\frac{\Gamma(\mu+1/2)\Gamma(1/2)}{\Gamma((\mu+\nu+1)/2)\Gamma((\mu-\nu+1)/2)}F\left(\frac{\mu+\nu}{2},\frac{\mu+\nu}{2};\frac{1}{2};\frac{\alpha^2}{\beta^2}\right)+\right. \\ &\left.+\frac{\alpha^{\mu+\nu+1}}{\beta^{\mu+\nu+1}}\frac{\gamma(\mu+1/2)\Gamma(-1/2)}{\Gamma((\mu+\nu)/2)\Gamma((\mu-\nu)/2)}F\left(\frac{\mu+\nu+1}{2},\frac{\mu-\nu+1}{2};\frac{3}{2};\frac{\alpha^2}{\beta^2}\right)\right] = \\ &= \frac{\sqrt{\pi}\Gamma(1/2)\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{(2\beta)^\mu\Gamma((\mu+\nu+1)/2)\Gamma((\mu-\nu+1)/2)}F\left(\frac{\mu+\nu}{2},\frac{\mu-\nu}{2};\frac{1}{2};\frac{\alpha^2}{\beta^2}\right)+ \\ &+\frac{\sqrt{\pi}\Gamma(-1/2)\alpha\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{(2\beta)^\mu\beta\Gamma((\mu+\nu)/2)\Gamma((\mu-\nu)/2)}F\left(\frac{\mu+\nu+1}{2},\frac{\mu-\nu+1}{2};\frac{3}{2};\frac{\alpha^2}{\beta^2}\right) = \\ &= \frac{\pi}{(2\beta)^\mu}\left[\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma((\mu+\nu+1)/2)\Gamma((\mu-\nu+1)/2)}F\left(\frac{\mu+\nu}{2},\frac{\mu-\nu}{2};\frac{1}{2};\frac{\alpha^2}{\beta^2}\right)+\right. \\ &\left.-\frac{2\alpha}{\beta}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma((\mu+\nu)/2)\Gamma((\mu-\nu)/2)}F\left(\frac{\mu+\nu+1}{2},\frac{\mu-\nu+1}{2};\frac{3}{2};\frac{\alpha^2}{\beta^2}\right)\right] = \\ &= \frac{\pi}{(2\beta)^\mu}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma((\mu+\nu+1)/2)\Gamma((\mu+\nu)/2)\Gamma((\mu-\nu+1)/2)\Gamma((\mu-\nu)/2)}\times \\ &\times\left[\Gamma\left(\frac{\mu+\nu}{2}\right)\Gamma\mu-\nu 2F\left(\frac{\mu+\nu}{2},\frac{\mu-\nu}{2};\frac{1}{2};\frac{\alpha^2}{\beta^2}\right)+\right. \\ &\left.-\frac{2\alpha}{\beta}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\mu-\nu+12F\left(\frac{\mu+\nu+1}{2},\frac{\mu-\nu+1}{2};\frac{3}{2};\frac{\alpha^2}{\beta^2}\right)\right].\end{aligned} \tag{D.45}$$

One can now use

$$\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\mu}{2}\right)=\frac{(\mu-1)!}{2^{\mu-1}}\sqrt{\pi},\qquad \Gamma(\mu)=(\mu-1)! \tag{D.46}$$

to obtain

$$\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma((\mu+\nu+1)/2)\Gamma((\mu+\nu)/2)\Gamma((\mu-\nu+1)/2)\Gamma((\mu-\nu)/2)}=\frac{2^{2\mu-2}}{\pi} \tag{D.47}$$

and therefore

$$\begin{aligned}\int_0^\infty x^{\mu-1}e^{-\alpha x}K_\nu(\beta x)dx\ &= \\ =\ \frac{2^{\mu-2}}{\beta^\mu}\Bigg[&\Gamma\left(\frac{\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)F\left(\frac{\mu+\nu}{2},\frac{\mu-\nu}{2};\frac{1}{2};\frac{\alpha^2}{\beta^2}\right)+ \\ &-\frac{2\alpha}{\beta}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)F\left(\frac{\mu+\nu+1}{2},\frac{\mu-\nu+1}{2};\frac{3}{2};\frac{\alpha^2}{\beta^2}\right)\Bigg].\end{aligned} \tag{D.48}$$

This expression is much more convenient because it contains no outer dicontinuity at all and also only common Gamma factors depending only on the first two arguments of the hypergeometric functions. For this expression one now can determine the spectral density by using the transformations worked out in Eq. (D.36). These transformations apply to the two parts and results in the *same* hypergeometric function,

$$\begin{aligned}\rho_F\left(\frac{\mu+\nu}{2},\frac{\mu-\nu}{2};\frac{1}{2};\frac{\alpha^2}{\beta^2}\right)\ &=\ \left(\frac{|\beta|}{|\alpha|}\right)^{\mu-\nu}\left(1-\frac{\beta^2}{\alpha^2}\right)^{1/2-\mu}\ \times \\ &\times\ \frac{\Gamma(1/2)}{\Gamma((\mu+\nu)/2)\Gamma((\mu-\nu)/2)\Gamma(3/2-\mu)}\ \times \\ &\times\ F\left(\frac{1-\mu-\nu}{2},\frac{2-\mu-\nu}{2};\frac{3}{2}-\mu;1-\frac{\beta^2}{\alpha^2}\right),\end{aligned} \tag{D.49}$$

$$\begin{aligned}\rho_F\left(\frac{\mu+\nu+1}{2},\frac{\mu-\nu+1}{2};\frac{3}{2};\frac{\alpha^2}{\beta^2}\right)\ &=\ \left(\frac{|\beta|}{|\alpha|}\right)^{\mu-\nu+1}\left(1-\frac{\beta^2}{\alpha^2}\right)^{1/2-\mu}\ \times \\ &\times\ \frac{\Gamma(3/2)}{\Gamma((\mu+\nu+1)/2)\Gamma((\mu-\nu+1)/2)\Gamma(3/2-\mu)}\ \times \\ &\times\ F\left(\frac{2-\mu-\nu}{2},\frac{1-\mu-\nu}{2};\frac{3}{2}-\mu;1-\frac{\beta^2}{\alpha^2}\right).\end{aligned} \tag{D.50}$$

Therefore, one ends up with

$$\begin{aligned}\frac{1}{2\pi i}\,\mathrm{Disc}\int_0^\infty x^{\mu-1}e^{-\alpha x}K_\nu(\beta x)dx\ &= \\ =\ \frac{2^{\mu-2}}{\beta^\mu}\left(\frac{|\beta|}{|\alpha|}\right)^{\mu-\nu}&\left(1-\frac{\beta^2}{\alpha^2}\right)^{1/2-\mu}\ \times \\ \times\ \Bigg[&\frac{\Gamma(1/2)}{\Gamma(3/2-\mu)}F\left(\frac{1-\mu-\nu}{2},\frac{2-\mu-\nu}{2};\frac{3}{2}-\mu;1-\frac{\beta^2}{\alpha^2}\right)\ + \\ &-\frac{2\alpha|\beta|}{\beta|\alpha|}\frac{\Gamma(3/2)}{\Gamma(3/2-\mu)}F\left(\frac{2-\mu-\nu}{2},\frac{1-\mu-\nu}{2};\frac{3}{2}-\mu;1-\frac{\beta^2}{\alpha^2}\right)\Bigg]\ =\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{\mu-2}}{\beta^\mu}\left(\frac{|\beta|}{|\alpha|}\right)^{\mu-\nu}\left(1-\frac{\beta^2}{\alpha^2}\right)^{1/2-\mu}\frac{\Gamma(1/2)}{\Gamma(3/2-\mu)}\times \\
&\quad\times\left[F\left(\frac{1-\mu-\nu}{2},\frac{2-\mu-\nu}{2};\frac{3}{2}-\mu;1-\frac{\beta^2}{\alpha^2}\right)+\right. \\
&\qquad\left.+F\left(\frac{2-\mu-\nu}{2},\frac{1-\mu-\nu}{2};\frac{3}{2}-\mu;1-\frac{\beta^2}{\alpha^2}\right)\right] = \qquad\qquad (D.51) \\
&= \frac{2^{\mu-1}}{\beta^\mu}\left(\frac{\beta}{-\alpha}\right)^{\mu-\nu}\left(1-\frac{\beta^2}{\alpha^2}\right)^{1/2-\mu}\frac{\Gamma(1/2)}{\Gamma(3/2-\mu)}F\left(\frac{1-\mu-\nu}{2},\frac{2-\mu-\nu}{2};\frac{3}{2}-\mu;1-\frac{\beta^2}{\alpha^2}\right).
\end{aligned}
$$

Note that on the right hand side one has $\alpha=-E-m_0$ so that $|\alpha|=E+m_0=-\alpha$. This final expression solves the problem completely. For integer values of μ there are no Gamma functions with negative integer argument. Therefore, there are no singularities at all, so that one can set $\varepsilon=0$ in this expression. Thus, a direct transition from the integrand to the spectral density in terms of only one hypergeometric function is found. There is no need to use the recurrence relations for the hypergeometric functions.

D.2.3 Explicit series for hypergeometric functions

Hypergeometric functions did not only occur in Chapter 3 but also in the HQET calculations of Section 5.5. Besides the two-point HQET integrals $I(a,b,c,p,q)$ one encounters also three-point modifications of these, denoted by $I'(a,b,c,p,q;\omega'/\omega)$. The integrals for $a=0$ or $b=0$ were expressed in terms of hypergeometric functions, and the aim of this part of the appendix is to extract the singular part. The result obtained for the integrals $I'(0,b,c,p,q;\omega'/\omega)$ and $I'(a,0,c,p,q;\omega'/\omega)$ in Eqs. (5.211) and (5.213) in Section 5.5 can be written in a more closed form by using the identity

$$
F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x) \qquad (D.52)
$$

for hypergeometric functions. In the case considered here one obtains

$$
\begin{aligned}
&(-2\omega)^{D-2c}(-2\omega')^{D-2b}\left(\frac{\omega}{\omega'}\right)^{2c+p-D}\times \\
&\qquad\times F\left(p+q+2b+2c-D,p+2c-D;p+q+2c-D;1-\frac{\omega}{\omega'}\right) = \\
&= (-2\omega)^{D-2c}(-2\omega')^{D-2b}\left(\frac{\omega}{\omega'}\right)^{2c+p-D}\left(\frac{\omega}{\omega'}\right)^{-p-2b-2c+2D}\times \\
&\qquad\times F\left(D-2b,q;p+q+2c-D;1-\frac{\omega}{\omega'}\right) = \\
&= (-2\omega)^{2D-2b-2c}F\left(D-2b,q;p+q+2c-D;1-\frac{\omega}{\omega'}\right). \qquad (D.53)
\end{aligned}
$$

There are two improvements. The second entry has become an integer number and the prefactor no longer depends on ω'. The hypergeometric function itself will now be worked on. Inserting $D=4-2\varepsilon$, one obtains functions such as $F(n_1-2\varepsilon,n_2;n_3+2\varepsilon;x)$ where n_1, n_2 and n_3 are integers. One has to look at ten different cases to obtain explicit series expressions up to $O(\varepsilon^0)$.

The case $n_2 \leq 0$

For $n_2 = -m_2 \leq 0$, the Taylor series breaks down even in the case where one does not restrict to $O(\varepsilon^0)$. One obtains

$$\begin{aligned}
F(n_1-2\varepsilon, n_2; n_3+2\varepsilon; x) \; &= \\
= \;\; & 1 + \frac{(n_1-2\varepsilon)n_2}{n_3+2\varepsilon}x + \frac{(n_1-2\varepsilon)(n_1+1-2\varepsilon)n_2(n_2+1)}{2(n_3+2\varepsilon)(n_3+1+2\varepsilon)}x^2 + \ldots \\
& \ldots + \frac{(n_1-2\varepsilon)_{m_2}(n_2)_{m_2}}{m_2!(n_3+2\varepsilon)_{m_3}}x^{m_2} \; = \; \sum_{s=0}^{-n_2}\frac{(n_1-2\varepsilon)_s(n_2)_s}{s!(n_3+2\varepsilon)_s}x^s \; = \\
= \;\; & 1 + \frac{(n_1-2\varepsilon)n_2}{n_3+2\varepsilon}x\left(1 + \frac{(n_1+1-2\varepsilon)(n_2+1)}{2(n_3+1+2\varepsilon)}x + \ldots\right) \; = \\
=: \;\; & F^{-n_2}(n_1-2\varepsilon, n_2; n_3+2\varepsilon; x). \qquad \text{(D.54)}
\end{aligned}$$

The case $n_2 > 0$, $n_1 = n_3 < 0$

With $m_1 = -n_1 = -n_3$ one has $n_1 + m_1 = 0$, $(n_1 - 2\varepsilon)_{m_1+1} \sim 2\varepsilon$. One therefore obtains

$$\begin{aligned}
& 1 + \frac{(n_1-2\varepsilon)n_2}{n_1+2\varepsilon}x + \ldots + \frac{(n_1-2\varepsilon)m_1(n_2)_{m_1}}{m_1!(n_1+2\varepsilon)m_1}x^{m_1} + \\
& \quad + \frac{(n_1-2\varepsilon)_{n_1+1}(n_2)_{m_1+1}}{(m_1+1)!(n_1+2\varepsilon)_{m_1+1}}x^{m_1+1}\left(1 + \frac{(n_1+m_1+1-2\varepsilon)(n_2+m_1+1)}{(m_1+2)(n_1+m_1+1+2\varepsilon)}x + \ldots\right) \; = \\
& = \; F^{-n_1}(n_1-2\varepsilon, n_2; n_1+2\varepsilon; x) \; + \\
& \quad + \frac{(n_1-2\varepsilon)_{1-n_1}(n_2)_{1-n_1}}{(1-n_1)!(n_1+2\varepsilon)_{1-n_1}}x^{1-n_1}{}_3F_2(1-2\varepsilon, n_2-n_1+1, 1; 2-n_1, 1+2\varepsilon; x) \qquad \text{(D.55)}
\end{aligned}$$

The case $n_2 > 0$, $n_1 < n_3 < 0$

If one choses $m_1 = -n_1$ and $m_3 = -n_3$, one obtains two critical points in the series,

$$\begin{aligned}
& 1 + \frac{(n_1-2\varepsilon)n_2}{n_3+2\varepsilon}x + \ldots + \frac{(n_1-2\varepsilon)_{m_3}(n_2)_{m_3}}{m_3!(n_3+2\varepsilon)_{m_3}}x^{m_3} + \\
& \quad + \frac{(n_1-2\varepsilon)_{m_3+1}(n_2)_{m_3+1}}{(m_3+1)!(n_3+2\varepsilon)_{m_3+1}}x^{m_3+1}\left(1 + \frac{(n_1+m_3+1-2\varepsilon)(n_2+m_3+1)}{(m_3+2)(n_3+m+3+1+2\varepsilon)}x + \ldots\right. \\
& \qquad \left.\ldots + \frac{(n_1+m_3+1-2\varepsilon)_{m_1-m_3-1}(n_2+m_3+1)_{m_1-m_3-1}}{(m_3+2)_{m_1-m_3-1}(1+2\varepsilon)_{m_1-m_3-1}}\right) + \\
& \quad + \frac{(n_1-2\varepsilon)_{m_1+1}(n_2)_{m_1+1}}{(m_1+1)!(n_3+2\varepsilon)_{m_1+1}}x^{m_1+1}\left(1 + \frac{(n_1+m+1+1-2\varepsilon)(n_2+m_1+1)}{(m_1+2)(n_3+m_1+1+2\varepsilon)}x + \ldots\right) \; = \\
& = \; F^{-n_3}(n_1-2\varepsilon, n_2; n_3+2\varepsilon; x) \; + \\
& \quad + \frac{(n_1-2\varepsilon)_{1-n_3}(n_2)_{1-n_3}}{(1-n_3)!(n_3+2\varepsilon)_{1-n_3}}x^{1-n_3} \; \times \\
& \qquad \times \; {}_3F_2{}^{n_3-n_1-1}(n_1-n_3+1-2\varepsilon, n_2-n_3+1, 1; 2-n_3, 1+2\varepsilon; x) \; + \\
& \quad + \frac{(n_1-2\varepsilon)_{1-n_1}(n_2)_{1-n_1}}{(1-n_1)!(n_3+2\varepsilon)_{1-n_1}}x^{1-n_1}{}_3F_2(1-2\varepsilon, n_2-n_1+1, 1; 2-n_1, n_3-n_1+1+2\varepsilon; x).
\end{aligned}$$

(D.56)

The case $n_2 > 0$, $n_3 < n_1 < 0$

In a similar fashion one obtains

$$\begin{aligned}
& 1+\frac{(n_1-2\varepsilon)n_2}{n_3+2\varepsilon}x+\ \ldots\ +\frac{(n_1-2\varepsilon)_{m_1}(n_2)_{m_1}}{m!(n_3+2\varepsilon)_{m_1}}x^{m_1}+ \\
& \quad +\frac{(n_1-2\varepsilon)_{m_1+1}(n_2)_{m_1+1}}{(m_1+1)!(n_3+2\varepsilon)_{m_1+1}}x^{m_1+1}\Bigg(1+\frac{(n_1+m_1+1-2\varepsilon)(n_2+m_1+1)}{(m_2+2)(n_3+m_1+1+2\varepsilon)}x+\ \ldots \\
& \qquad +\frac{(1-2\varepsilon)_{m_3-m_1-1}(n_2+m_1+1)_{m_3-m_1-1}}{(m_1+2)_{m_3-m_1-1}(n_3+m_1+1+2\varepsilon)_{m_3-m_1-1}}x^{m_3-m_1-1}\Bigg)+ \\
& \quad +\frac{(n_1-2\varepsilon)_{m_3+1}(n_2)_{m_3+1}}{(m_3+1)!(n_3+2\varepsilon)_{m_3+1}}x^{m_3+1}\left(1+\frac{(n_1+m_3+1-2\varepsilon)(n_2+m_3+1)}{(m_3+2)(n_2+m_3+1+2\varepsilon)}x+\ \ldots\right)= \\
= \;\; & F^{-n_1}(n_1-2\varepsilon,n_2;n_3+2\varepsilon;x)+ \\
& \quad +\frac{(n_1-2\varepsilon)_{1-n_1}(n_2)_{1-n_1}}{(1-n_1)!(n_3+2\varepsilon)_{1-n_1}}x^{1-n_1}\times \\
& \qquad \times\ {}_3F_2{}^{n_1-n_3-1}(1-2\varepsilon,n_2-n_1+1,1;2-n_1,n_3-n_1+1+2\varepsilon;x)+ \\
& \quad +\frac{(n_1-2\varepsilon)_{1-n_3}(n_2)_{1-n_3}}{(1-n_3)!(n_3+2\varepsilon)_{1-n_3}}x^{1-n_3}{}_3F_2(n_1-n_3+1-2\varepsilon,n_2-n_3+1,1;2-n_3,1+2\varepsilon;x).
\end{aligned} \tag{D.57}$$

The case $n_2 > 0$, $n_1 = n_2 = 0$

Here one simply obtains

$$\begin{aligned}
& 1+\frac{(-2\varepsilon)n_2}{2\varepsilon}x\left(1+\frac{(1-2\varepsilon)(n_2+1)}{2(1+2\varepsilon)}+\ \ldots\right)= \\
= \;\; & 1-n_2x\ {}_3F_2(1-2\varepsilon,n_2+1,1;2,1+2\varepsilon;x).
\end{aligned} \tag{D.58}$$

The case $n_3 > 0$, $n_1 < n_3 = 0$

With $m_1 = -n_1$ one has

$$\begin{aligned}
& 1+\frac{(n_1-2\varepsilon)n_2}{2\varepsilon}x\Bigg(1+\frac{(n_1+1-2\varepsilon)(n_2+1)}{2(1+2\varepsilon)}x+\ \ldots \\
& \qquad \ldots\ +\frac{(n_1+1-2\varepsilon)_{m_1-1}(n_2+1)_{m_1-1}}{(2)_{m_1-1}(1+2\varepsilon)_{m_1-1}}x^{m_1-1}\Bigg)+ \\
& \quad +\frac{(n_1-2\varepsilon)_{m_1+1}(n_2)_{m_1+1}}{(m_1+1)!(2\varepsilon)_{m_1+1}}x^{m_1+1}\left(1+\frac{(n_1+m_1+1-2\varepsilon)(n_2+m_1+1)}{(m_1+2)(m_1+1+2\varepsilon)}x+\ \ldots\right)= \\
= \;\; & 1+\frac{(n_1-2\varepsilon)n_2}{2\varepsilon}x\ {}_3F_2{}^{-1-n_1}(n_1+1-2\varepsilon,n_2+1,1;2,1+2\varepsilon;x)+ \\
& \quad +\frac{(n_1-2\varepsilon)_{1-n_1}(n_2)_{1-n_1}}{(1-n_1)!(2\varepsilon)_{1-n_1}}x^{1-n_1}\ {}_3F_2(1-2\varepsilon,n_2-n_1+1,1;2-n_1,1-n_1+2\varepsilon;x).
\end{aligned} \tag{D.59}$$

The case $n_2 > 0$, $n_3 < n_1 = 0$

In using $m_3 = -n_3$ one obtains

$$\begin{aligned}
&1 + \frac{(-2\varepsilon)n_2}{n_3 + 2\varepsilon} x \Bigg(1 + \frac{(1-2\varepsilon)(n_2+1)}{2(n_3+1+2\varepsilon)} x \; \ldots \\
&\qquad \ldots + \frac{(1-2\varepsilon)_{m_3-1}(n_2+1)_{m_3-1}}{(2)_{m_3-1}(n_3+1+2\varepsilon)_{m_3-1}} x^{m_3-1} \Bigg) + \\
&\quad + \frac{(-2\varepsilon)_{m_3+1}(n_2)_{m_3+1}}{(m_3+1)!(n_3+2\varepsilon)_{m_3+1}} x^{m_3+1} \left(1 + \frac{(m_3+1-2\varepsilon)(n_2+m_3+1)}{(m_3+2)(n_3+m_3+1+2\varepsilon)} x + \; \ldots \right) = \\
&= \; 1 - \frac{2\varepsilon n_2}{n_3+2\varepsilon} x \; {}_3F_2{}^{-1-n_3}(1-2\varepsilon, n_2+1, 1; 2, n_3+1+2\varepsilon, x) \; + \\
&\quad + \frac{(-2\varepsilon)_{1-n_3}(n_2)_{1-n_3}}{(1-n_3)!(n_3+2\varepsilon)_{1-n_3}} x^{1-n_3} \; {}_3F_2(1-n_3-2\varepsilon, n_2-n_3+1, 1; 2-n_3, 1+2\varepsilon; x).
\end{aligned} \tag{D.60}$$

The case $n_2 > 0$, $n_3 > n_1 = 0$

$$\begin{aligned}
&1 + \frac{(-2\varepsilon)n_2}{n_3 + 2\varepsilon} x \left(1 + \frac{(1-2\varepsilon)(n_2+1)}{2(n_3+1+2\varepsilon)} x + \; \ldots \right) = \\
&= \; 1 - \frac{2\varepsilon n_2}{n_3+2\varepsilon} x \; {}_3F_2(1-2\varepsilon, n_2+1, 1; 2, n_3+1+2\varepsilon; x)
\end{aligned} \tag{D.61}$$

The case $n_2 > 0$, $n_1 > n_3 = 0$

$$\begin{aligned}
&1 + \frac{n_1 - 2\varepsilon)n_2}{2\varepsilon} x \left(1 + \frac{(n_1+1-2\varepsilon)(n_2+1)}{2(1+2\varepsilon)} x + \; \ldots \right) = \\
&= \; 1 + \frac{(n_1-2\varepsilon)n_2}{2\varepsilon} x \; {}_3F_2(n_1+1-2\varepsilon, n_2+1, 1; 2, 1+2\varepsilon; x)
\end{aligned} \tag{D.62}$$

The case $n_2 > 0$, $n_1 > 0$, $n_3 > 0$

In this case there are no critical points in the series, the result is simply the starting expression $F(n_1 - 2\varepsilon, n_2; n_3 + 2\varepsilon; x)$.

D.3 Integrals containing Bessel functions

Two examples for integrals containing three Bessel functions are cited here. The first example is given by [91]

$$\begin{aligned}
&\int_0^\infty x^{\alpha-1} K_\mu(mx) K_\nu(mx) dx \; = \\
&= \; \frac{2^{\alpha-3}}{m^\alpha \Gamma(\alpha)} \Gamma\left(\frac{\alpha+\mu+\nu}{2}\right) \Gamma\left(\frac{\alpha+\mu-\nu}{2}\right) \Gamma\left(\frac{\alpha-\mu+\nu}{2}\right) \Gamma\left(\frac{\alpha-\mu-\nu}{2}\right).
\end{aligned} \tag{D.63}$$

The second example is used to calculate Eq. (3.88) and is found in Eq. (2.16.42) of Ref. [91],

$$\begin{aligned}\int_0^\infty x^{\alpha-1} I_\lambda(ax) I_\mu(bx) K_\nu(cx) dx \ &= \\ = \ &\frac{2^{\alpha-1}a^\lambda b^\mu}{c^{\alpha+\lambda+\mu}} \Gamma\left[\begin{matrix}(\alpha+\lambda+\mu-\nu)/2, \ (\alpha+\lambda+\mu+\nu)/2 \\ \lambda+1, \ \mu+1\end{matrix}\right] \times \\ &\times F_4\left(\frac{\alpha+\lambda+\mu-\nu}{2}, \frac{\alpha+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right),\end{aligned} \tag{D.64}$$

the elements are

$$\Gamma\left[\begin{matrix}a_1, \ldots, a_m \\ b_1, \ldots, b_n\end{matrix}\right] = \frac{\prod_{k=1}^m \Gamma(a_k)}{\prod_{l=1}^n \Gamma(b_l)}, \qquad F_4(a, b; c, c'; z, \zeta) = \sum_{k,l=0}^\infty \frac{(a)_{k+l}(b)_{k+l}}{(c)_k (c')_l} \frac{z^k \zeta^l}{k! l!}, \tag{D.65}$$

valid for $\sqrt{|z|} + \sqrt{|\zeta|} < 1$. For $\alpha = 2$, $\lambda = \mu = \nu = 0$ and $a = m_1$, $b = m_2$ and $c = m$ one obtains the integral needed,

$$\int_0^\infty r K_0(mr) I_0(m_1 r) I_0(m_2 r) dr = \frac{2}{m^2} \sum_{k,l=0}^\infty \frac{((k+l)!)^2}{(k!)^2 (l!)^2} \left(\frac{m_1^2}{m^2}\right)^k \left(\frac{m_2^2}{m^2}\right)^l. \tag{D.66}$$

What remains to be calculated are integrals containing four Bessel functions such as

$$\int_0^\infty x^3 K_0^4(x) dx = -\frac{3}{16} + \frac{7}{32}\zeta(3). \tag{D.67}$$

D.4 Gegenbauer polynomials

A few relations for the *Gegenbauer polynomials* are listed here, an extensive treatment of these polynomials can be found in Refs. [61, 87, 101]. The Gegenbauer polynomials satisfy the orthogonality relations ($\hat{x}_i = x_i/|x_i|$)

$$\int C_m^\lambda(\hat{x}_1 \cdot \hat{x}_2) C_n^\lambda(\hat{x}_2 \cdot \hat{x}_3) d\Omega_2 = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} \frac{\lambda \delta_{mn}}{\lambda+n} C_n^\lambda(\hat{x}_1 \cdot \hat{x}_3), \quad \int d\Omega_2 = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}, \tag{D.68}$$

or, normalized differently,

$$\int C_m^\lambda(\hat{x}_1 \cdot \hat{x}_2) C_n^\lambda(\hat{x}_2 \cdot \hat{x}_3) d\hat{x}_2 = \frac{\lambda \delta_{mn}}{\lambda+n} C_n^\lambda(\hat{x}_1 \cdot \hat{x}_3) \qquad \left(\int d\hat{x}_2 = 1\right) \tag{D.69}$$

Especially, one has $C_0^\lambda(x) = 1$, $C_1^\lambda(x) = 2\lambda x$, and

$$(j+1) C_{j+1}^\lambda(x) = 2(j+\lambda) x C_j^\lambda(x) - (j+2\lambda-1) C_{j-1}^\lambda(x). \tag{D.70}$$

Moreover

$$C_j^\lambda(1) = \frac{\Gamma(j+2\lambda)}{j! \Gamma(2\lambda)}. \tag{D.71}$$

The corresponding characteristic polynomial is given by

$$(t^2 - 2tx + 1)^{-\lambda} = \sum_{j=0}^\infty t^j C_j^\lambda(x). \tag{D.72}$$

D.5 Gamma and polygamma function relations

The calculations presented here were done in the context of arriving at a closed form expression for the δ_n part of the moments in Section 4.7. Starting from

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \quad \Leftrightarrow \quad \sin(\pi x)\Gamma(x)\Gamma(1-x) = \pi \tag{D.73}$$

one can derive relations for the *polygamma functions*

$$\psi^{(0)}(x) := \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}, \qquad \psi^{(n)}(x) := \psi^{(n-1)\prime}(x) \tag{D.74}$$

by simply calculating derivatives of it. Taking the first derivative, one obtains

$$0 = \pi\cos(\pi x)\Gamma(x)\Gamma(1-x) + \sin(\pi x)\Gamma'(x)\Gamma(1-x) - \sin(\pi x)\Gamma(x)\Gamma'(1-x) \tag{D.75}$$

and by dividing this by $\sin(\pi x)\Gamma(x)\Gamma(1-x)$ finally

$$0 = \pi\cot(\pi x) + \frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(1-x)}{\Gamma(1-x)} \quad \Leftrightarrow \quad \psi(x) - \psi(1-x) = -\pi\cot(\pi x). \tag{D.76}$$

Note that the derivative denoted by the prime acts on the argument of the function. Note further that one can use

$$\cot(\pi(x+1)) = \frac{\cos(\pi x)\cos\pi - \sin(\pi x)\sin\pi}{\sin(\pi x)\cos\pi + \cos(\pi x)\sin\pi} = \frac{\cos(\pi x)}{\sin(\pi x)} = \cot(\pi x) \tag{D.77}$$

to extend the validity of the starting equation from the interval $[0,1]$ for x to arbitrary (real) values. The second derivative results in

$$\begin{aligned} 0 \;=\; & -\pi^2\sin(\pi x)\Gamma(x)\Gamma(1-x) + \pi\cos(\pi x)\Gamma'(x)\Gamma(1-x) \;+ \\ & -\pi\cos(\pi x)\Gamma(x)\Gamma'(1-x) + \pi\cos(\pi x)\Gamma'(x)\Gamma(1-x) \;+ \\ & +\sin(\pi x)\Gamma''(x)\Gamma(1-x) - \sin(\pi x)\Gamma'(x)\Gamma'(1-x) \;+ \\ & -\pi\cos(\pi x)\Gamma(x)\Gamma'(1-x) - \sin(\pi x)\Gamma'(x)\Gamma'(1-x) \;+ \\ & +\sin(\pi x)\Gamma(x)\Gamma''(1-x). \end{aligned} \tag{D.78}$$

One now uses

$$\psi'(x) = \left(\frac{\Gamma'(x)}{\Gamma(x)}\right)' = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)}\right)^2 \quad \Rightarrow \quad \frac{\Gamma''(x)}{\Gamma(x)} = \psi'(x) + \psi(x)^2 \tag{D.79}$$

to obtain

$$\begin{aligned} 0 \;=\; & -\pi^2 + 2\pi\cot(\pi x)\psi(x) - 2\pi\cot(\pi x)\psi(1-x) \;+ \\ & \quad +\frac{\Gamma''(x)}{\Gamma(x)} - 2\psi(x)\psi(1-x) + \frac{\Gamma''(1-x)}{\Gamma(1-x)} \;= \\ =\; & -\pi^2 + 2\pi\cot(\pi x)\left(\psi(x) - \psi(1-x)\right) \;+ \\ & \quad + \left(\psi(x) - \psi(1-x)\right)^2 + \psi'(x) + \psi'(1-x) \end{aligned} \tag{D.80}$$

and therefore

$$\begin{aligned} \psi'(x)+\psi'(1-x) &= \pi^2 - 2\pi\cot(\pi x)\left(\psi(x)-\psi(1-x)\right) - \left(\psi(x)-\psi(1-x)\right)^2 = \\ &= \pi^2 + 2\pi^2\cot^2(\pi x) - \pi^2\cot^2(\pi x) = \pi^2\left(1+\cot^2(\pi x)\right). \end{aligned} \tag{D.81}$$

This can be used to replace polygamma functions with negative arguments by the cotangens function and polygamma functions with positive arguments,

$$\begin{aligned} \psi(-4-n) &= \psi(n+5) + \pi\cot(\pi n), \\ \psi(-3-n) &= \psi(n+4) + \pi\cot(\pi n), \\ \psi(-2-n) &= \psi(n+3) + \pi\cot(\pi n), \\ \psi(-1-n) &= \psi(n+2) + \pi\cot(\pi n), \\ \psi(-n) &= \psi(n+1) + \pi\cot(\pi n), \\ \psi'(-1-n) &= -\psi'(n+2) + \pi^2\left(1+\cot^2(\pi n)\right), \\ \psi'(-2-n) &= -\psi'(n+3) + \pi^2\left(1+\cot^2(\pi n)\right). \end{aligned} \tag{D.82}$$

In general, the asymptotic expansion

$$\begin{aligned} \psi^{(n)}(z) &= (-1)^{n-1}\left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k}\frac{(2k+n-1)!}{(2k)!z^{2k+n}}\right], \\ \psi(z) &= \ln z - \frac{1}{2z} - \sum_{k=1}^{\infty}\frac{B_{2k}}{2kz^{2k}} \qquad (B_k \text{ are Bernoulli's numbers}) \end{aligned} \tag{D.83}$$

holds, as well as $\psi^{(n)}(1) = (-1)^{n+1}n!\zeta(n+1)$, $\psi(1) = -\gamma_E$. The most useful relation, however, is

$$\psi(n+1) = \psi(n) + \frac{1}{n}, \qquad \psi'(n+1) = \psi'(n) - \frac{1}{n^2} \tag{D.84}$$

which allows for a collection of the polygamma functions. All these relations have been numerically checked and implemented in the module `moments.add`. Using these replacements, one finally obtains the results used in Section 4.7,

$$\begin{aligned} \frac{M_n^{m(1)}}{M_n^{m(0)}} &= \frac{1}{3(n-1)(n-2)(n-3)}\Bigg[(n-1)(n-5)(5n-14) + \\ &\quad +4(3n^3-21n^2+45n-29)(\psi(n-1)+\gamma_E) + \\ &\quad -4(n-1)(n-2)(n-3)\left(\psi'(n-1)-\frac{\pi^2}{3}\right)\Bigg], \end{aligned} \tag{D.85}$$

$$\begin{aligned} \frac{M_n^{q(1)}}{M_n^{q(0)}} &= \frac{1}{3(n-1)(n-2)(n-3)}\Bigg[(n-1)(5n^2-33n+55) + \\ &\quad +4(3n^3-21n^2+45n-29)(\psi(n-1)+\gamma_E) + \\ &\quad -4(n-1)(n-2)(n-3)\left(\psi'(n-1)-\frac{\pi^2}{3}\right)\Bigg] \end{aligned} \tag{D.86}$$

for the relative moments.

Appendix E

Polylogarithms and their relations

Besides the shuffling methods explained in Appendix F, only two relations are used for the *dilogarithm function* defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{dz}{z}\ln(1-z), \tag{E.1}$$

namely [289]

$$\begin{aligned} \mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) &= -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z) \qquad \text{for } z \notin [0,1[, & \text{(E.2)} \\ \mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) &= \frac{\pi^2}{6} - \ln z \ln(1-z) & \text{(E.3)} \end{aligned}$$

These relations are used to obtain an argument $z \leq 1$ for the dilogarithm function. If the initial expression was real valued, the consequent application of these rules leads to real valued dilogarithms while possible imaginary parts cancel among the double logarithmic expressions. Because of this complete cancellation, nothing has to be done in order to determine the Riemann sheet for the complex continuation of the logarithms and dilogarithms.

E.1 Indefinite dilogarithms

Dilogarithms occur in most of the cases before the limits of the integration are inserted. Therefore, only indefinite expressions (without constant terms) are necessary here. One can define an *indefinite dilogarithm function*

$$\mathrm{li}_2(z) = -\int \frac{dz}{z}\ln(1-z) \tag{E.4}$$

and can use the dilogarithm identities in Eq. (E.2) and (E.3) without the constant $\pm\pi^2/6$.

E.2 "Zig-zag" shuffles for dilogarithms

In applying Eq. (E.2) and (E.3) in turn, one is led to a "zig-zag" shuffle of the arguments. Typical examples used in practise are shown in the following scheme. Only the arguments of the dilogarithms are displayed. The (trivial) double logarithms are not mentioned but can be identified easily.

$$
\begin{aligned}
\boxed{-\frac{\sqrt{\xi}-1+v}{1-v}} &\leftrightarrow \frac{1-v+\sqrt{\xi}-1+v}{1-v} = \frac{\sqrt{\xi}}{1-v} \leftrightarrow \\
\boxed{\boxed{\frac{1-v}{\sqrt{\xi}}}} &\leftrightarrow 1-\frac{1-v}{\sqrt{\xi}} = \boxed{\frac{\sqrt{\xi}-1+v}{\sqrt{\xi}}} \leftrightarrow \\
\frac{\sqrt{\xi}(1+v)}{2(\sqrt{\xi}-1+v)} &\leftrightarrow \frac{\sqrt{\xi}-1+v-\sqrt{\xi}}{\sqrt{\xi}-1+v} = \frac{-(1-v)}{\sqrt{\xi}-1+v} \leftrightarrow
\end{aligned}
\tag{E.5}
$$

$$
\begin{aligned}
\boxed{\frac{\sqrt{\xi}-1+v}{1+v}} &\leftrightarrow \frac{1+v-\sqrt{\xi}+1-v}{1+v} = \boxed{\boxed{\frac{2-\sqrt{\xi}}{1+v}}} \leftrightarrow \\
\frac{1+v}{2-\sqrt{\xi}} &\leftrightarrow \frac{2-\sqrt{\xi}-1-v}{2-\sqrt{\xi}} = \boxed{-\frac{\sqrt{\xi}-1+v}{2-\sqrt{\xi}}} \leftrightarrow \\
\frac{-(2-\sqrt{\xi})}{\sqrt{\xi}-1+v} &\leftrightarrow \frac{\sqrt{\xi}-1+v+2-\sqrt{\xi}}{\sqrt{\xi}-1+v} = \frac{1+v}{\sqrt{\xi}-1+v} \leftrightarrow
\end{aligned}
\tag{E.6}
$$

$$
\begin{aligned}
\boxed{-\frac{2(\sqrt{\xi}-1+v)}{(1-v)^2}} &\leftrightarrow \frac{1-2v+v^2+2\sqrt{\xi}-2+2v}{(1-v)^2} = \frac{\sqrt{\xi}(2-\sqrt{\xi})}{(1-v)^2} \leftrightarrow \\
\boxed{\boxed{\frac{(1-v)^2}{\sqrt{\xi}(2-\sqrt{\xi})}}} &\leftrightarrow \frac{2\sqrt{\xi}-1+v^2-1+2v-v^2}{\sqrt{\xi}(2-\sqrt{\xi})} = \boxed{\frac{2(\sqrt{\xi}-1+v)}{\sqrt{\xi}(2-\sqrt{\xi})}} \leftrightarrow \\
\frac{\sqrt{\xi}(2-\sqrt{\xi})}{2(\sqrt{\xi}-1+v)} &\leftrightarrow \frac{2\sqrt{\xi}-2+2v-2\sqrt{\xi}+1-v^2}{2(\sqrt{\xi}-1+v)} = \frac{-(1-v)^2}{2(\sqrt{\xi}-1+v)} \leftrightarrow
\end{aligned}
\tag{E.7}
$$

$$
\begin{aligned}
\boxed{-\frac{2(\sqrt{\xi}-1+v)}{(2-\sqrt{\xi})(1-v)}} &\leftrightarrow \frac{(2-\sqrt{\xi})(1-v)+2\sqrt{\xi}-2+2v}{(2-\sqrt{\xi})(1-v)} = \frac{\sqrt{\xi}(1+v)}{(2-\sqrt{\xi})(1-v)} \leftrightarrow \\
\boxed{\boxed{\frac{(2-\sqrt{\xi})(1-v)}{\sqrt{\xi}(1+v)}}} &\leftrightarrow \frac{\sqrt{\xi}+v\sqrt{\xi}-2+2v+\sqrt{\xi}-v\sqrt{\xi}}{\sqrt{\xi}(1+v)} = \boxed{\frac{2(\sqrt{\xi}-1+v)}{\sqrt{\xi}(1+v)}} \leftrightarrow \\
\frac{\sqrt{\xi}(1+v)}{2(\sqrt{\xi}-1+v)} &\leftrightarrow \frac{2\sqrt{\xi}-2+2v-\sqrt{\xi}-v\sqrt{\xi}}{2(\sqrt{\xi}-1+v)} = -\frac{(2-\sqrt{\xi})(1-v)}{2(\sqrt{\xi}-1+v)} \leftrightarrow
\end{aligned}
\tag{E.8}
$$

$$
\begin{aligned}
\boxed{\frac{\sqrt{\xi}(\sqrt{\xi}-1+v)}{(2-\sqrt{\xi})(1-v)}} &\leftrightarrow \frac{2-2v-\sqrt{\xi}+v\sqrt{\xi}-1+v^2+\sqrt{\xi}-v\sqrt{\xi}}{(2-\sqrt{\xi})(1-v)} = \boxed{\boxed{\frac{1-v}{2-\sqrt{\xi}}}} \leftrightarrow \\
\frac{2-\sqrt{\xi}}{1-v} &\leftrightarrow \frac{1-v-2+\sqrt{\xi}}{1-v} = \frac{\sqrt{\xi}-1-v}{1-v} = \boxed{-\frac{\sqrt{\xi}(\sqrt{\xi}-1+v)}{(1-v)^2}} \leftrightarrow \\
\frac{-(1-v)^2}{\sqrt{\xi}(\sqrt{\xi}-1+v)} &\leftrightarrow \frac{1-v^2-\sqrt{\xi}+v\sqrt{\xi}+1-2v+v^2}{\sqrt{\xi}(\sqrt{\xi}-1+v)} = \frac{(2-\sqrt{\xi})(1-v)}{\sqrt{\xi}-1+v} \leftrightarrow
\end{aligned}
\tag{E.9}
$$

In this scheme elements are indicated by a box which should be unified by the shuffle, while the double box indicates the desired argument with a value less than 1.

E.3 Representations for polylogarithms

The dilogarithm function is only one member of the family of *polylogarithm functions*. The polylogarithms $\mathrm{Li}_p(z)$ are in general defined by the series representation

$$\mathrm{Li}_p(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p}. \tag{E.10}$$

It is easy to see that $z\,\mathrm{Li}'_p(z) = \mathrm{Li}_{p-1}(z)$. This relation leads to the iterative integral representation

$$\mathrm{Li}_1(z) = -\ln(1-z), \qquad \mathrm{Li}_p(z) = \int \frac{dz}{z}\,\mathrm{Li}_{p-1}(z) \quad (p>1). \tag{E.11}$$

Note that for $z=1$ the polylogarithms reduce to *Euler's zeta function*,

$$\mathrm{Li}_p(1) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \zeta(p). \tag{E.12}$$

For further explicit representations see Appendix F.

E.4 The Clausen polylogarithms

For arguments of the polylogarithm function lying on the unit circle one can use the *Clausen polylogarithm* defined as

$$\mathrm{Cl}_p(x) := \mathrm{Im}\,\mathrm{Li}_p(e^{ix}). \tag{E.13}$$

These polylogarithms are used in Appendix F.

E.5 A trilogarithm relation

In integrating the dilogarithm identity in Eq. (E.2) it is possible to obtain a *trilogarithm identity*. This identity reads

$$\mathrm{Li}_3(z) - \mathrm{Li}_3\left(\frac{1}{z}\right) = -\frac{\pi^2}{3}\ln z + \frac{i\pi}{2}\ln^2 z + \frac{1}{6}\ln^3 z \qquad (z>1). \tag{E.14}$$

Further relations can be found by using the shuffle methods in Appendix F.

Appendix F

The shuffle algebra of nested integrals

In connection with the calculation of master integrals for the three-loop bubble diagrams of the water melon and spectacle topology (see Sec. 3.4). Much work has been invested in the calculation of integrals containing logarithms and polylogarithms. These integrals can be represented by nested integrals (or, when expanded into their arguments, as nested sums). Integration-by-parts then allows one to "shuffle" these integrals in order to transform them into a standard form. The principles of the algebra of such shuffling methods can be found in Ref. [111] while the explicit calculations for the cases relevant for the present work are found in this Appendix. They do not appear in the main text because the *shuffle algebra of nested integrals* itself is a pure mathematical concept without any relation to the concrete physical problem.

F.1 Foundations of the shuffle method

There are in general three steps one has to consider, namely

1. how to enter the shuffle algebra from the side of the physical problem
2. how to organize the shuffling in a straightforward way
3. how to interprete the result in terms of basic elements

The following subsections are dedicated to these three steps.

F.1.1 Getting into the shuffle

In order to enter the shuffle algebra which will then enshuffle the given integral expression, the integral representations

$$\ln(z') = \int_1^{z'} \frac{dz}{z}, \qquad \ln^n(z_2) - \ln^n(z_1) = n \int_{z_1}^{z_2} \frac{dz}{z} \ln^{n-1}(z) \tag{F.1}$$

and

$$\mathrm{Li}_1(z') := -\ln(1-z') = \int_0^{z'} \frac{dz}{1-z}, \qquad \mathrm{Li}_n(z_2) - \mathrm{Li}_n(z_1) = \int_{z_1}^{z_2} \frac{dz}{z} \mathrm{Li}_{n-1}(z) \tag{F.2}$$

(with the integer $n > 1$) are of importance here. Using these relations, every integral containing logarithms, polylogarithms and single pole functions can finally be represented by a more or less nested integral chain. To give a fairly complicated example, the integral from $z = 0$ to 1 over $\ln^2(z)\,\mathrm{Li}_3(z)$, divided by $1 + z$, can be represented as

$$\begin{aligned}\int_0^1 \frac{\ln^2(z)\,\mathrm{Li}_3(z)}{1+z}dz &= \int_0^1 \frac{dz}{1+z}\left(2\int_1^z \frac{dz_1}{z_1}\ln(z_1)\right)\left(\int_0^z \frac{dz_3}{z_3}\mathrm{Li}_2(z_3)\right) = \\ &= \int_0^1 \frac{dz}{1+z}\left(2\int_1^z \frac{dz_1}{z_1}\int_1^{z_1}\frac{dz_2}{z_2}\right)\left(\int_0^z \frac{dz_3}{z_3}\int_0^{z_3}\frac{dz_4}{z_4}\mathrm{Li}_1(z_4)\right) = \\ &= \int_0^1 \frac{dz}{1+z}\left(2\int_1^z \frac{dz_1}{z_1}\int_1^{z_1}\frac{dz_2}{z_2}\right)\left(\int_0^z \frac{dz_3}{z_3}\int_0^{z_3}\frac{dz_4}{z_4}\int_0^{z_4}\frac{dz_5}{1-z_5}\right).\end{aligned} \tag{F.3}$$

F.1.2 Doing the shuffle

The previous example shows two features of the starting expression which will be handled by the shuffle method. First, shuffled integrals by definition always have a common lower boundary (normally, this limit is given by 0). For this reason, integrals not including this common limit have to be changed. This is done by the identity

$$\int_{z_1}^{z_2} dz\,\chi(z) = \int_0^{z_2} dz\,\chi(z) - \int_0^{z_1} dz\,\chi(z) \tag{F.4}$$

where $\chi(z)$ can contain a nested integral expression. For the first part in parentheses in the above example, one has $z_1 = 1$ and $z_2 = z$. The second feature of the nested integrals of the shuffle is that they are totally nested. This is not the case for the example as well. Therefore, a procedure has to be applied which reminds one of knitting in the sense that not threads but integrals have to be interchanged,

$$\int_{z_0}^{z} dz_1\chi_1(z_1)\int_{z_1}^{z} dz_2\chi_2(z_2) = \int_{z_0}^{z} dz_2\chi_2(z_2)\int_{z_0}^{z_2} dz_1\chi_1(z_1). \tag{F.5}$$

Using these two methods interchangingly, the goal of totally nested (enshuffled) integrals can be reached. The last method, by the way, can of course be used in the opposite direction to deshuffle (i.e. un-nest) a nested integral expression. Speaking in terms of the knitting technique again, this can be called "taking off the thread".

F.1.3 Interpreting the shuffle

In the above example the denominator of the starting integrand took the form $(1 + z)$. Generally, the whole concept is based on denominator factors of the kind z or $(\lambda^{-p} - z)$ where λ^{-p} with $\lambda = e^{i\pi/3}$ are the *sixth roots of unity* (so $\zeta = -1$ is one example, cf. Section 3.4). Following Ref. [111], the final result can be read as a sum ("sentence") of so-called "words" (representing the nested integrals) which themselves are built up by "letters". More formally, these "letters" are one-forms taken from the set ("alphabet") $\mathcal{A}_\lambda := \{\Omega, \omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ where $\Omega := dz/z$, $\omega_p := dz/(\lambda^{-p} - z)$, and $\omega_{p+6} = \omega_p$. Examples for such "words" will not be given here because they follow én masse afterwards.

F.2 The shuffle for $M(\lambda^p)$

Starting point is the already nested integral $M(\lambda^p)$ of Eq. (3.182) for $\lambda = e^{i\pi/3}$,

$$\begin{aligned} M(\lambda^p) &= \int_0^1 \frac{dt}{\lambda^{-p}-t}\ln t \int_0^t \frac{du}{1+u}\ln u = \int_0^1 \frac{du}{1+u}\ln u \int_u^1 \frac{dt}{\lambda^{-p}-t}\ln t = \\ &= \int_0^1 \frac{du}{1+u}\ln u \int_0^1 \frac{dt}{\lambda^{-p}-t}\ln t - \int_0^1 \frac{du}{1+u}\ln u \int_0^u \frac{dt}{\lambda^{-p}-t}\ln t = M_0 + M_1. \end{aligned} \tag{F.6}$$

The first part M_0 is simple and results in the product of two integrals or words,

$$\int_0^1 \frac{dz}{\lambda^{-p}-z}\ln z = -\int_0^1 \frac{dz_1}{\lambda^{-p}-z_1}\int_{z_1}^1 \frac{dz_2}{z_2} = -\int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{\lambda^{-p}-z_1} =: -\zeta(\Omega\omega_p). \tag{F.7}$$

and (as special case of this)

$$\int_0^1 \frac{dz}{1+z}\ln z = -\int_0^1 \frac{dz}{\lambda^{-3}-z}\ln z = \zeta(\Omega\omega_3) \tag{F.8}$$

One therefore continues with

$$\begin{aligned} M_1 &= -\int_0^1 \frac{du}{1+u}\ln u \int_0^u \frac{dt}{\lambda^{-p}-t}\ln t = -\int_0^1 \frac{dz_1}{1+z_1}\ln(z_1)\chi_1(z_1) = \\ &= -\int_0^1 \frac{dz_1}{1+z_1}\chi_1(z_1)\left(\ln 1 - \int_{z_1}^1 \frac{dz_2}{z_2}\right) = \\ &\qquad \text{(the “}\ln 1\text{” is only written down to show the general procedure)} \\ &= \int_0^1 \frac{dz_1}{1+z_1}\chi_1(z_1)\int_{z_1}^1 \frac{dz_2}{z_2} = \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\chi_1(z_1) = \\ &= \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\int_0^{z_1} \frac{dz_3}{\lambda^{-p}-z_3}\ln(z_3) = \\ &= \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\int_0^{z_1} \frac{dz_3}{\lambda^{-p}-z_3}\left(\ln(z_1) - \int_{z_3}^{z_1} \frac{dz_4}{z_4}\right) = \\ &= \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\ln(z_1)\int_0^{z_1} \frac{dz_3}{\lambda^{-p}-z_3} - \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\int_0^{z_1} \frac{dz_3}{\lambda^{-p}-z_3}\int_{z_3}^{z_1} \frac{dz_4}{z_4} = \\ &= \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\ln(z_1)\int_0^{z_1} \frac{dz_3}{\lambda^{-p}-z_3} - \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\int_0^{z_1} \frac{dz_4}{z_4}\int_0^{z_4} \frac{dz_3}{\lambda^{-p}-z_3}. \end{aligned} \tag{F.9}$$

The second part is already enshuffled, resulting in a word $\zeta(\Omega\omega_3\Omega\omega_p)$. Therefore, one looks on the first term which will be called M_2 henceforth. There was some $\ln(z_1)$ flown back which have to be resettled. One obtains

$$\begin{aligned} M_2 &= \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\ln(z_1)\int_0^{z_1} \frac{dz_3}{\lambda^{-p}-z_3} = \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\ln(z_1)\chi_2(z_1) = \\ &= \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\chi_2(z_1)\left(\ln(z_2) - \int_{z_1}^{z_2} \frac{dz_4}{z_4}\right) = \\ &= \int_0^1 \frac{dz_2}{z_2}\ln(z_2)\int_0^{z_2} \frac{dz_1}{1+z_1}\chi_2(z_1) - \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_1}{1+z_1}\chi_2(z_1)\int_{z_1}^{z_2} \frac{dz_4}{z_4} = \\ &= \int_0^1 \frac{dz_2}{z_2}\ln(z_2)\int_0^{z_2} \frac{dz_1}{1+z_1}\chi_2(z_1) - \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_4}{z_4}\int_0^{z_4} \frac{dz_1}{1+z_1}\chi_2(z_1) = \\ &= \int_0^1 \frac{dz_2}{z_2}\ln(z_2)\int_0^{z_2} \frac{dz_1}{1+z_1}\int_0^{z_1} \frac{dz_3}{\lambda^{-p}-z_3} - \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{dz_4}{z_4}\int_0^{z_4} \frac{dz_1}{1+z_1}\int_0^{z_1} \frac{dz_3}{\lambda^{-p}-z_3}. \end{aligned} \tag{F.10}$$

Again, the second part is enshuffled, resulting in the word $\zeta(\Omega^2\omega_3\omega_p)$ while the first (called M_3) has to be treated again separately,

$$\begin{aligned}
M_3 &= \int_0^1 \frac{dz_2}{z_2}\ln(z_2)\int_0^{z_2}\frac{dz_1}{1+z_1}\int_0^{z_1}\frac{dz_3}{\lambda^{-p}-z_3} \;=\; \int_0^1\frac{dz_2}{z_2}\ln(z_2)\chi_3(z_2) \;=\\
&= \int_0^1\frac{dz_2}{z_2}\chi_3(z_2)\left(\ln 1-\int_{z_2}^1\frac{dz_4}{z_4}\right) \;= \qquad\qquad (F.11)\\
&= -\int_0^1\frac{dz_2}{z_2}\chi_3(z_2)\int_{z_2}^1\frac{dz_4}{z_4} \;=\; -\int_0^1\frac{dz_4}{z_4}\int_0^{z_4}\frac{dz_2}{z_2}\chi_3(z_2) \;=\\
&= -\int_0^1\frac{dz_4}{z_4}\int_0^{z_4}\frac{dz_2}{z_2}\int_0^{z_2}\frac{dz_1}{1+z_1}\int_0^{z_1}\frac{dz_3}{\lambda^{-p}-z_3}
\end{aligned}$$

which is again the word $\zeta(\Omega^2\omega_3\omega_p)$. Finally, one ends up with the sentence

$$\begin{aligned}
M(\lambda^p) &= M_0+M_1 \;=\; -\zeta(\Omega\omega_3)\zeta(\Omega\omega_p)+\zeta(\Omega\omega_3\Omega\omega_p)+M_2 \;=\\
&= -\zeta(\Omega\omega_3)\zeta(\Omega\omega_p)+\zeta(\Omega\omega_3\Omega\omega_p)+\zeta(\Omega^2\omega_3\omega_p)+M_3 \;=\\
&= -\zeta(\Omega\omega_3)\zeta(\Omega\omega_p)+\zeta(\Omega\omega_3\Omega\omega_p)+2\zeta(\Omega^2\omega_3\omega_p). \qquad (F.12)
\end{aligned}$$

Note that this procedure is straightforward. This is essential for an implementation in a computer program. Furthermore, note that the functions $\chi_1(z)$, $\chi_2(z)$ and $\chi_3(z)$ are only intermediate replacements. Therefore, they will be no longer distinguished explicitly in the following, such that $\chi(z)$ just stands for the intermediate replacement.

F.3 Words of depth one and two and a formalization

Words of depth one are words $\zeta(\Omega^n\omega_p)$ with only one single letter ω_p. All of those are easily seen to be polylogarithms with the argument λ^p,

$$\begin{aligned}
\zeta(\omega_p) &= \int_0^1\frac{dz_1}{\lambda^{-p}-z_1} \;=\; \int_0^{\lambda^p}\frac{dz_1}{1-z_1} \;=\\
&= \Big[-\ln(1-z_1)\Big]_0^{\lambda^p} \;=\; -\ln(1-\lambda^p) \;=\; \mathrm{Li}_1(\lambda^p), \qquad (F.13)\\
\zeta(\Omega\omega_p) &= \int_0^1\frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{\lambda^{-p}-z_2} \;=\; -\int_0^1\frac{dz_1}{z_1}\int_0^{z_1\lambda^p}\frac{dz_2}{1-z_2} \;=\\
&= \int_0^1\frac{dz_1}{z_1}\mathrm{Li}_1(z_1\lambda^p) \;=\; \int_0^{\lambda^p}\frac{dz_1}{z_1}\mathrm{Li}_1(z_1) \;=\; \Big[\mathrm{Li}_2(z_1)\Big]_0^{\lambda^p} \;=\; \mathrm{Li}_2(\lambda^p), \qquad (F.14)\\
\zeta(\Omega^2\omega_p) &= \int_0^1\frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\int_0^{z_2}\frac{dz_3}{\lambda^{-p}-z_3} \;=\;\ldots\;=\; \mathrm{Li}_3(\lambda^p) \quad \text{and} \qquad (F.15)\\
\zeta(\Omega^3\omega_p) &= \int_0^1\frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\int_0^{z_2}\frac{dz_3}{z_3}\int_0^{z_3}\frac{dz_4}{\lambda^{-p}-z_4} \;=\;\ldots\;=\; \mathrm{Li}_4(\lambda^p). \qquad (F.16)
\end{aligned}$$

In general, the *depth of a word* indicates the number of elements ω_p. There are in total three different types of words of depth two, namely

$$\zeta(\Omega^2\omega_p\omega_q), \qquad \zeta(\Omega\omega_p\Omega\omega_q) \quad \text{and} \quad \zeta(\omega_p\Omega^2\omega_q) \qquad (F.17)$$

(the last one is not relevant for reducing the expression $M(\lambda^p)$ but it is included for completeness). And there are ways to reduce them to words of depth one and a limited set of words of depth two. But before one is able to understand the formalized procedures called depth-length shuffle and weight-length shuffle in Ref. [111], one has to switch to a different representation of words, namely the representation by nested sums.

F.3.1 Nested sum representation of words

The transition from nested integral to nested sums is done by expanding the integrands. Taking for example the trilogarithm $\mathrm{Li}_3(\lambda^p)$, one obtains

$$\begin{aligned}
\mathrm{Li}_3(\lambda^p) &= \zeta(\Omega^2\omega_p) = \int_0^1 \frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\int_0^{z_2}\frac{dz_3}{\lambda^{-p}-z_3} = \\
&= \int_0^1 \frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\int_0^{z_2\lambda^p}\frac{dz_3}{1-z_3} = \int_0^1 \frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\int_0^{z_2\lambda^p} dz_3\sum_{n=0}^{\infty} z_3^n = \\
&= \int_0^1 \frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\sum_{m=0}^{\infty}\frac{1}{n+1}(z_2\lambda^p)^{n+1} = \int_0^1 \frac{dz_1}{z_1}\int_0^{z_1} dz_2\sum_{n+0}^{\infty}\frac{(\lambda^p)^{n+1}}{n+1}z_2^n = \\
&= \int_0^1 \frac{dz_1}{z_1}\sum_{n=0}^{\infty}\frac{(\lambda^p)^{n+1}}{(n+1)^2}z_1^{n+1} = \int_0^1 dz_1\sum_{n=0}^{\infty}\frac{(\lambda^p)^{n+1}}{(n+1)^2}z_1^m = \\
&= \sum_{n=0}^{\infty}\frac{(\lambda^p)^{n+1}}{(n+1)^3} = \sum_{n>0}\frac{(\lambda^p)^n}{n^3}. \qquad \text{(F.18)}
\end{aligned}$$

This last expression is known as the Taylor series representation of the trilogarithm. To give an example for a word of depth two, the expression

$$\begin{aligned}
U_{3,1} &= \zeta(\Omega^2\omega_3\omega_0) = -\int_0^1 \frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_1}\int_0^{z_2}\frac{dz_3}{1+z_3}\int_0^{z_3}\frac{dz_4}{1-z_4} = \\
&= -\int_0^1 \frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\int_0^{z_2}\frac{dz_3}{1+z_3}\sum_{n=0}^{\infty}\frac{z_3^{n+1}}{n+1} = \\
&= -\int_0^1 \frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\int_0^{z_2} dz_3\sum_{m,n=0}^{\infty}\frac{(-1)^m}{n+1}z_3^{m+n+1} = \\
&= -\int_0^1 \frac{dz_1}{z_1}\int_0^{z_1}\frac{dz_2}{z_2}\sum_{m,n=0}^{\infty}\frac{(-1)^m z_2^{m+n+2}}{(m+n+2)(n+1)} = \\
&= -\int_0^1 \frac{dz_1}{z_1}\sum_{m,n=0}^{\infty}\frac{(-1)^m z_1^{m+n+2}}{(m+n+2)^2(n+1)} = -\sum_{m,n=0}^{\infty}\frac{(-1)^m}{(m+n+2)^3(n+1)} = \\
&= \sum_{m,n=1}^{\infty}\frac{(-1)^m}{(m+n)^3 n} = \sum_{n=1}^{\infty}\sum_{m'=n+1}^{\infty}\frac{(-1)^{m'-n}}{(m')^3 n} = \sum_{m>n>0}\frac{(-1)^{m+n}}{m^3 n} \qquad \text{(F.19)}
\end{aligned}$$

is shown. In general one defines

$$\zeta\begin{pmatrix} s_1 & \cdots & s_k \\ \lambda^{p_1} & \cdots & \lambda^{p_k}\end{pmatrix} := \sum_{n_i>n_{i+1}>0}\prod_{i=1}^{k}\frac{\lambda^{p_i n_i}}{n_i^{s_i}}, \qquad \text{(F.20)}$$

It is not difficult to find the translation between the words and the nested sums,

$$\zeta\left(\Omega^{s_1-1}\omega_{p_1}\Omega^{s_2-1}\omega_{p_2}\cdots\Omega^{s_k-1}\omega_{p_k}\right) := \zeta\begin{pmatrix} s_1 & s_2 & \cdots & s_k \\ \lambda^{p_1} & \lambda^{p_2-p_1} & \cdots & \lambda^{p_k-p_{k-1}}\end{pmatrix}. \qquad \text{(F.21)}$$

The *depth of the word* in general is given by the number of letters ω_p. In this case it is given by k. On the other hand, the *weight of a word* is given by the total number of letters which in terms of the nested sums is equivalent to the total power of the numerator factor. In this case one has $s_1 + \ldots + s_k$. The weight is conserved in the shuffles that follow.

F.3.2 Depth-length shuffles

If one multiplies two infinite sums with summation indices n_1 and n_2, they can be written as the sum of two nested sums (for $n_1 > n_2$ and $n_2 > n_1$, resp.) plus a residual term (for $n_1 = n_2$) containing only one sum,

$$\begin{aligned}
\zeta\begin{pmatrix} s_1 \\ \lambda^{p_1} \end{pmatrix} \zeta\begin{pmatrix} s_2 \\ \lambda^{p_2} \end{pmatrix} &= \sum_{n_1>0} \frac{\lambda^{p_1 n_1}}{n_1^{s_1}} \sum_{n_2>0} \frac{\lambda^{p_2 n_2}}{n_2^{s_2}} = \\
&= \sum_{n_1>n_2>0} \frac{\lambda^{p_1 n_1}\lambda^{p_2 n_2}}{n_1^{s_1} n_2^{s_2}} + \sum_{n_2>n_1>0} \frac{\lambda^{p_1 n_1}\lambda^{p_2 n_2}}{n_1^{s_1} n_2^{s_2}} + \sum_{n_1>0} \frac{\lambda^{(p_1+p_2)n_1}}{n_1^{s_1+s_2}} = \\
&= \zeta\begin{pmatrix} s_1 & s_2 \\ \lambda^{p_1} & \lambda^{p_2} \end{pmatrix} + \zeta\begin{pmatrix} s_2 & s_1 \\ \lambda^{p_2} & \lambda^{p_1} \end{pmatrix} + \zeta\begin{pmatrix} s_1+s_2 \\ \lambda^{p_1+p_2} \end{pmatrix}. \qquad \text{(F.22)}
\end{aligned}$$

This can of course be extended also to products including words of depth two,

$$\begin{aligned}
\zeta\begin{pmatrix} s_1 \\ \lambda^{p_1} \end{pmatrix} \zeta\begin{pmatrix} s_2 & s_3 \\ \lambda^{p_2} & \lambda^{p_3} \end{pmatrix} &= \sum_{n_1>0} \frac{\lambda^{p_1 n_1}}{n_1^{s_1}} \sum_{n_2>n_3>0} \frac{\lambda^{p_2 n_2}\lambda^{p_3 n_3}}{n_2^{s_2} n_3^{s_3}} = \\
= \sum_{n_1>n_2>n_3>0} \frac{\lambda^{p_1 n_1}\lambda^{p_2 n_2}\lambda^{p_3 n_3}}{n_1^{s_1} n_2^{s_2} n_3^{s_3}} &+ \sum_{n_2>n_3>0} \frac{\lambda^{(p_1+p_2)n_2}\lambda^{p_3 n_3}}{n_2^{s_1+s_2} n_3^{s_3}} + \\
+ \sum_{n_2>n_1>n_3>0} \frac{\lambda^{p_2 n_2}\lambda^{p_1 n_1}\lambda^{p_3 n_3}}{n_2^{s_2} n_1^{s_1} n_3^{s_3}} &+ \sum_{n_2>n_3>0} \frac{\lambda^{p_2 n_2}\lambda^{(p_1+p_3)n_3}}{n_2^{s_2} n_3^{s_1+s_3}} + \sum_{n_2>n_3>n_1>0} \frac{\lambda^{p_2 n_2}\lambda^{p_3 n_3}\lambda^{p_1 n_1}}{n_2^{s_2} n_3^{s_3} n_1^{s_1}} = \\
= \zeta\begin{pmatrix} s_1 & s_2 & s_3 \\ \lambda^{p_1} & \lambda^{p_2} & \lambda^{p_3} \end{pmatrix} &+ \zeta\begin{pmatrix} s_1+s_2 & s_3 \\ \lambda^{p_1+p_2} & \lambda^{p_3} \end{pmatrix} + \\
+\zeta\begin{pmatrix} s_2 & s_1 & s_3 \\ \lambda^{p_2} & \lambda^{p_1} & \lambda^{p_3} \end{pmatrix} &+ \zeta\begin{pmatrix} s_2 & s_1+s_3 \\ \lambda^{p_2} & \lambda^{p_1+p_3} \end{pmatrix} + \zeta\begin{pmatrix} s_2 & s_3 & s_1 \\ \lambda^{p_2} & \lambda^{p_3} & \lambda^{p_1} \end{pmatrix}. \qquad \text{(F.23)}
\end{aligned}$$

Relations such as Eq. (F.22) are called *depth-length shuffles*. The final terms have the same weight but different depth. In terms of words, for instance, the depth-length shuffle of the word $\zeta(\omega_p)$ with the word $\zeta(\Omega^2\omega_q)$ is given by

$$\zeta(\omega_p)\zeta(\Omega^2\omega_q) = \zeta(\omega_p\Omega^2\omega_{p+q}) + \zeta(\Omega^2\omega_q\omega_{p+q}) + \zeta(\Omega^3\omega_{p+q}) \qquad \text{(F.24)}$$

F.3.3 Weight-length shuffles

The second kind of shuffles is the analogue of the depth-length shuffle on the nested integral level. The *weight-length shuffle* can best be defined at the level of words,

$$\zeta(W_1)\zeta(W_2) = \sum_{W_i \in S_{1,2}} \zeta(W_i) \qquad \text{(F.25)}$$

where $S_{1,2}$ obtains all shufflings of W_1 and W_2 by preserving the order of each of these words. Two examples are in order to illustrate the main procedure,

$$\begin{aligned}
\zeta(\Omega\omega_p)\zeta(\Omega'\omega_q) &= \zeta(\Omega\omega_p\Omega'\omega_q) + \zeta(\Omega\Omega'\omega_p\omega_q) + \zeta(\Omega\Omega'\omega_q\omega_p) + \\
&\quad +\zeta(\Omega'\Omega\omega_p\omega_q) + \zeta(\Omega'\Omega\omega_q\omega_p) + \zeta(\Omega'\omega_q\Omega\omega_p) = \\
&= \zeta(\Omega\omega_p\Omega\omega_q) + \zeta(\Omega\omega_q\Omega\omega_p) + 2\zeta(\Omega^2\omega_p\omega_q) + 2\zeta(\Omega^2\omega_q\omega_p) \qquad \text{(F.26)}
\end{aligned}$$

(the prime is used only to indicate the position) and

$$\zeta(\omega_p)\zeta(\Omega^2\omega_{p+q}) = \zeta(\omega_p\Omega^2\omega_{p+q}) + \zeta(\Omega\omega_p\Omega\omega_{p+q}) + \zeta(\Omega^2\omega_p\omega_{p+q}) + \zeta(\Omega^2\omega_{p+q}\omega_p). \qquad \text{(F.27)}$$

F.3.4 The reduction procedure

Both shuffle methods, the depth-length and the weight-length shuffle, are implemented in a procedure to reduce depth-two words down to depth-one words and some basic elements of depth two. The MATHEMATICA package `shuffle.add` will be explained in the following. The set of words of depth one is already given in Eq. (F.13), they are denoted by `zeta[p]`, `zeta[Om,p]`, `zeta[Om^2,p]`, and `zeta[Om^3,p]`, respectively. As was mentioned earlier, there are three kinds of depth-two words, given in Eq. (F.17). They are denoted by `zeta1[p,q]`, `zeta2[p,q]`, and `zeta3[p,q]`, respectively.

Now the weight-length shuffle in Eq. (F.27) can be used to reduce `zeta2[p,q]` to `zeta3[p,q]` and `zeta1[p,q]`, while the depth-length shuffle in Eq. (F.24) can be used to reduce `zeta3[p,q]` to `zeta1[p,q]`. These reductions are subsequently implemented in the package, where the final results of these are obtained by calling `zeta2sol` resp. `zeta3sol`. Therefore, the systems of equations for 36 equations with 36 unknowns can be solved fully and one is left with depth-two words of the kind $\zeta(\Omega^2\omega_p\omega_q)$.

These 36 depth-two words can be calculated by using the depth-length and weight-length shuffle of $\zeta(\Omega\omega_p)$ and $\zeta(\Omega\omega_q)$. This is done by the procedure `zeta1sol`. Even though the system of 72 equations containing 36 unknown quantities seems to be highly over-determined, one cannot express all the quantities by depth-one words. Instead one is left with seven terms, a number which cannot be obtained by any counting method. But there are also some relations *between the depth-one words themselves* which had to be checked. They make the programming of this reduction a quite complicated task because formally contradictionary equations can "kill" the reduction obtained up to this point. Therefore, one has to be very careful in ruling out the equations which are "deadly" for the whole system. This problem does *not* appear when all depth-one words are (artifically) set to zero, as one can see by running the procedure `zeta1pur`.

In a first step `zeta1sol` reduces the words to a basis of seven words which MATHEMATICA selects by its own choice. Simultaneously, the procedure writes a file called `shuffle.m` which is executable and contains the relations between the depth-one words which can be tested numerically by simply executing this file under MATHEMATICA (all `rest` terms have to vanish). Finally, the procedure `result` expresses the `zeta2` and `zeta3` functions by depth-one words and the depth-two word basis which is given by

$$\begin{gathered}\zeta(\Omega^2\omega_1\omega_0)=\mathtt{zeta1}[1,0],\quad \zeta(\Omega^2\omega_1\omega_3)=\mathtt{zeta1}[1,3],\\ \zeta(\Omega^2\omega_2\omega_0)=\mathtt{zeta1}[2,0],\quad \zeta(\Omega^2\omega_2\omega_3)=\mathtt{zeta1}[2,3],\\ \zeta(\Omega^2\omega_3\omega_0)=\mathtt{zeta1}[3,0],\quad \zeta(\Omega^2\omega_3\omega_1)=\mathtt{zeta1}[3,1],\quad \zeta(\Omega^2\omega_3\omega_5)=\mathtt{zeta1}[3,5].\end{gathered}\tag{F.28}$$

The final result is written to the file `shuffle.dat` (or `shuffle6.dat`, see below). In the file `shuffle6.con` this final result is used in order to express the integrals $M(\lambda^p)$ and thereby the water melon and the spectacle integral in terms of the primitives.

Finally, note that there is a further parameter `num` introduced in all procedures in `shuffle.add` which indicates the order of the root of unity. Therefore, the procedures are applicable for instance also to the fourth roots of unity. The different sets of results are found in `shuffle1.dat` to `shuffle6.dat`.

F.4 Back to the integrals

At this point one can return to the integrals which are to be calculated using the concept of the shuffle algebra. These are the (generalized) water melon contribution

$$M = \int_0^1 \frac{2f(t)\ln t}{1-t^2} = S(1) - S(-1) \tag{F.29}$$

and the spectacle contribution

$$S = \int_0^1 \frac{2tf(t)\ln t\,dt}{(1-t^2)(\lambda-t)(\lambda^{-1}-t)} = S(1) + \frac{1}{3}S(-1) - \frac{2}{3}S(\lambda) - \frac{2}{3}S(\lambda^{-1}) \tag{F.30}$$

where

$$S(\lambda^p) = \int_0^1 \frac{f(t)\ln t}{\lambda^{-p}-t}dt = 2M(\lambda^p) + 3\,\mathrm{Li}_4(\lambda^p) - \zeta(2)\,\mathrm{Li}_2(\lambda^p) \tag{F.31}$$

and

$$M(\lambda^p) = \int_0^1 \frac{dt}{\lambda^{-p}-t}\ln t \int_0^t \frac{du}{1+u}\ln u. \tag{F.32}$$

The function $f(t)$ of the integrand is given by

$$f(t) = 2\,\mathrm{Li}_2(-t) + 2\ln t\ln(1+t) - \frac{1}{2}\ln^2 t + \zeta(2). \tag{F.33}$$

According to the package `shuffle6.con` there are three steps which finally lead to the results for M and S. These are dealt with in the following subsections.

F.4.1 Reduction to depth-one words

The values for the seven basic elements for the depth-two words (cf. Ref. [111]) read

$$\begin{aligned}
\mathrm{Re}\left(\zeta(\Omega^2\omega_1\omega_0)\right) &= \frac{\pi^4}{3240},\\
\mathrm{Re}\left(\zeta(\Omega^2\omega_1\omega_3)\right) &= \frac{29\pi^4}{9720} - \frac{7}{9}\zeta(3)\ln(2) + \frac{1}{3}\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) + \frac{4}{9}U_{3,1} - V_{3,1},\\
\mathrm{Re}\left(\zeta(\Omega^2\omega_2\omega_0)\right) &= \frac{127\pi^4}{29160} - \frac{4}{9}\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) + \frac{4}{3}V_{3,1},\\
\mathrm{Re}\left(\zeta(\Omega^2\omega_2\omega_3)\right) &= -\frac{17\pi^4}{2592} + \frac{7}{9}\zeta(3)\ln(2) - \frac{4}{9}U_{3,1},\\
\mathrm{Re}\left(\zeta(\Omega^2\omega_3\omega_0)\right) &= U_{3,1}, \qquad \mathrm{Re}\left(\zeta(\Omega^2\omega_3\omega_1)\right) = V_{3,1},
\end{aligned} \tag{F.34}$$

and

$$\begin{aligned}
2i\,\mathrm{Im}\left(\zeta(\Omega^2\omega_1\omega_3)\right) &= \zeta(\Omega^2\omega_1\omega_3) - \zeta(\Omega^2\omega_5\omega_3) = \\
&= 2\zeta(\Omega\omega_2)\zeta(\Omega\omega_3) + \zeta(\omega_2)\left(\zeta(\Omega^2\omega_1) - \zeta(\Omega^2\omega_5)\right) - \zeta(\Omega^3\omega_2) - \zeta(\Omega^3\omega_3) - 3\zeta(\Omega^3\omega_5),
\end{aligned} \tag{F.35}$$

$$\begin{aligned}
2i\,\mathrm{Im}\left(\zeta(\Omega^2\omega_2\omega_3)\right) &= \zeta(\Omega^2\omega_2\omega_3) - \zeta(\Omega^2\omega_4\omega_3) = \\
&= 2\zeta(\Omega\omega_1)\zeta(\Omega\omega_3) + \zeta(\omega_1)\left(\zeta(\Omega^2\omega_2) - \zeta(\Omega^2\omega_4)\right) - \zeta(\Omega^3\omega_1) - \zeta(\Omega^3\omega_3) - 3\zeta(\Omega^3\omega_4),
\end{aligned} \tag{F.36}$$

$$\begin{aligned}
2i\,\mathrm{Im}\left(\zeta(\Omega^2\omega_3\omega_1)\right) &= \zeta(\Omega^2\omega_3\omega_1) - \zeta(\Omega^2\omega_3\omega_5) = \\
&= -\zeta(\Omega\omega_2)\left(\zeta(\Omega\omega_1) + \zeta(\Omega\omega_5)\right) - \zeta(\omega_2)\left(\zeta(\Omega^2\omega_1) - \zeta(\Omega^2\omega_5)\right) + \\
&\quad - \left(\zeta(\omega_1) - \zeta(\omega_5)\right)\left(\zeta(\Omega^2\omega_2) - \zeta(\Omega^2\omega_3)\right) + \zeta(\Omega^3\omega_2) + 3\zeta(\Omega^3\omega_3) + \zeta(\Omega^3\omega_5).
\end{aligned} \tag{F.37}$$

The results in `shuffle6.dat` can be used to reduce the rest of the expressions for $M(\lambda^p)$ to depth-one words (the results have already been used to calculate the imaginary parts just shown). The results read

$$\begin{aligned}
M(1) &= 2U_{3,1}-\zeta(\Omega\omega_0)\zeta(\Omega\omega_3)+\frac{1}{2}\Big(\zeta(\Omega\omega_3)^2-\zeta(\Omega^3\omega_0)\Big),\\
M(\lambda) &= \frac{139\pi^4}{38880}-\frac{1}{3}\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)+2V_{3,1}\ +\\
&\quad +\zeta(\omega_1)\Big(\zeta(\Omega^2\omega_2)-\zeta(\Omega^2\omega_3)\Big)-\frac{1}{2}\zeta(\omega_1)\Big(\zeta(\Omega^2\omega_2)-\zeta(\Omega^2\omega_4)\Big)\ +\\
&\quad -\zeta(\omega_2)\Big(\zeta(\Omega^2\omega_1)-\zeta(\Omega^2\omega_5)\Big)-\frac{1}{2}\left(\zeta(\omega_1)-\zeta(\omega_5)\right)\Big(\zeta(\Omega^2\omega_2)-\zeta(\Omega^2\omega_3)\Big)\ +\\
&\quad -\zeta(\Omega\omega_3)\left(\zeta(\Omega\omega_1)+\zeta(\Omega\omega_2)\right)-\frac{1}{2}\zeta(\Omega\omega_2)\left(\zeta(\Omega\omega_1)+\zeta(\Omega\omega_5)\right)\ +\\
&\quad +\frac{1}{2}\zeta(\Omega^3\omega_1)+\zeta(\Omega^3\omega_2)+\frac{3}{2}\zeta(\Omega^3\omega_3)+\frac{3}{2}\zeta(\Omega^3\omega_4)+2\zeta(\Omega^3\omega_5),\\
M(\lambda^2) &= \frac{139\pi^4}{38880}-\frac{1}{3}\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\ +\\
&\quad -\frac{1}{2}\left(\zeta(\Omega\omega_1)-\zeta(\Omega\omega_5)\right)\zeta(\Omega\omega_2)-\left(\zeta(\Omega\omega_1)+\zeta(\Omega\omega_2)\right)\zeta(\Omega\omega_3)\ +\\
&\quad -\zeta(\omega_1)\zeta(\Omega^2\omega_2)-\frac{1}{2}\zeta(\omega_5)\zeta(\Omega^2\omega_2)+\frac{1}{2}\zeta(\omega_1)\zeta(\Omega^2\omega_3)+\frac{1}{2}\zeta(\omega_5)\zeta(\Omega^2\omega_3)\ +\\
&\quad +\frac{1}{2}\zeta(\omega_1)\zeta(\Omega^2\omega_4)+\frac{1}{2}\zeta(\Omega^3\omega_1)+\frac{3}{2}\zeta(\Omega^3\omega_3)+\frac{3}{2}\zeta(\Omega^3\omega_4)+\zeta(\Omega^3\omega_5),\\
M(\lambda^3) &= -\frac{3}{2}\zeta(\Omega\omega_3)^2+\frac{3}{2}\zeta(\Omega^3\omega_0)+\zeta(\Omega^3\omega_3).
\end{aligned}\tag{F.38}$$

F.4.2 Insertion of the depth-one words

The depth-one words are known as polylogarithms. However, one has to deal with the fact that the arguments are given by the roots of unity which take complex values in general. The reduction to polylogarithms with real arguments is done by considering the Taylor series expansion

$$\mathrm{Li}_p(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^p}.\tag{F.39}$$

The sixth roots of unity are made explicit,

$$\begin{aligned}
\lambda &= e^{i\pi/3} = \frac{1}{2}+i\sqrt{\frac{3}{4}}, & \lambda^2 &= e^{2i\pi/3} = -\frac{1}{2}+i\sqrt{\frac{3}{4}}, & \lambda^3 &= e^{i\pi} = -1,\\
\lambda^4 &= e^{4i\pi/3} = -\frac{1}{2}-i\sqrt{\frac{3}{4}}, & \lambda^5 &= e^{5i\pi/3} = \frac{1}{2}-i\sqrt{\frac{3}{4}}, & \lambda^6 &= e^{2i\pi} = 1.
\end{aligned}\tag{F.40}$$

For the argument λx one obtains

$$\mathrm{Li}_p(\lambda x) = \sum_{n=0}^{\infty}\frac{\lambda x^{6n+1}}{(6n+1)^p}+\sum_{n=0}^{\infty}\frac{\lambda^2 x^{6n+2}}{(6n+2)^p}+\sum_{n=0}^{\infty}\frac{\lambda^3 x^{6n+3}}{(6n+3)^p}\ +$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}\frac{\lambda^4 x^{6n+4}}{(6n+4)^p}+\sum_{n=0}^{\infty}\frac{\lambda^5 x^{6n+5}}{(6n+5)^p}+\sum_{n=0}^{\infty}\frac{\lambda^6 x^{6n+6}}{(6n+6)^p} \;=\\
=\;& \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^{6n+1}}{(6n+1)^p}-\sum_{n=0}^{\infty}\frac{x^{6n+2}}{(6n+2)^p}-\sum_{n=0}^{\infty}\frac{x^{6n+4}}{(6n+4)^p}+\sum_{n=0}^{\infty}\frac{x^{6n+5}}{(6n+5)^p}\right)+\\
&-\sum_{n=0}^{\infty}\frac{x^{6n+3}}{(6n+3)^p}+\sum_{n=0}^{\infty}\frac{x^{6n+6}}{(6n+6)^p}\;+ \qquad\qquad \text{(F.41)}\\
&+i\sqrt{\frac{3}{4}}\left(\sum_{n=0}^{\infty}\frac{x^{6n+1}}{(6n+1)^p}+\sum_{n=0}^{\infty}\frac{x^{6n+2}}{(6n+2)^p}-\sum_{n=0}^{\infty}\frac{x^{6n+4}}{(6n+4)^p}-\sum_{n=0}^{\infty}\frac{x^{6n+5}}{(6n+5)^p}\right).
\end{aligned}
$$

The imaginary part cannot be converted into polylogarithms again, but for the real part one can use a supplementary procedure explained later to obtain

$$
\begin{aligned}
\mathrm{Re}\left(\mathrm{Li}_p(\lambda x)\right) \;=\;& \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^p}-2\sum_{n=0}^{\infty}\frac{x^{6n+2}}{(6n+2)^p}-2\sum_{n=0}^{\infty}\frac{x^{6n+4}}{(6n+4)^p}\;+\right.\\
&\left.-3\sum_{n=0}^{\infty}\frac{x^{6n+3}}{(6n+3)^p}+\sum_{n=0}^{\infty}\frac{x^{6n+6}}{(6n+6)^p}\right) \;=\\
=\;& \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^p}-2\sum_{n=1}^{\infty}\frac{(x^2)^n}{2^p n^p}-3\sum_{n=0}^{\infty}\frac{x^{6n+3}}{(6n+3)^p}+3\sum_{n=0}^{\infty}\frac{x^{6n+6}}{(6n+6)^p}\right) \;=\\
=\;& \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^p}-2\sum_{n=1}^{\infty}\frac{(x^2)^n}{2^p n^p}-3\sum_{n=1}^{\infty}\frac{(x^3)^n}{3^p n^p}+6\sum_{n=0}^{\infty}\frac{x^{6n+6}}{(6n+6)^p}\right) \;=\\
=\;& \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^p}-2\sum_{n=1}^{\infty}\frac{(x^2)^n}{2^p n^p}-3\sum_{n=1}^{\infty}\frac{(x^3)^n}{3^p n^p}+6\sum_{n=1}^{\infty}\frac{(x^6)^n}{6^p n^p}\right. \;=\\
=\;& \frac{1}{2}\left(6^{1-p}\,\mathrm{Li}_p(x^6)-3^{1-p}\,\mathrm{Li}_p(x^3)-2^{1-p}\,\mathrm{Li}_p(x^2)+\mathrm{Li}_p(x)\right). \qquad \text{(F.42)}
\end{aligned}
$$

For the argument $z=\lambda^2 x$ one ends up with

$$
\begin{aligned}
\mathrm{Li}_p(\lambda^2 x) \;=\;& \sum_{n=0}^{\infty}\frac{\lambda^2 x^{3n+1}}{(3n+1)^p}+\sum_{n=0}^{\infty}\frac{\lambda^4 x^{3n+2}}{(3n+2)^p}+\sum_{n=0}^{\infty}\frac{\lambda^6 x^{3n+3}}{(3n+3)^p} \;=\\
=\;& -\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^{3n+1}}{(3n+1)^p}+\sum_{n=0}^{\infty}\frac{x^{3n+2}}{(3n+2)^p}\right)+\sum_{n=0}^{\infty}\frac{x^{3n+3}}{(3n+3)^p}\;+\\
&+i\sqrt{\frac{3}{4}}\left(\sum_{n=0}^{\infty}\frac{x^{3n+1}}{(3n+1)^p}-\sum_{n=0}^{\infty}\frac{x^{3n+2}}{(3n+2)^p}\right) \qquad \text{(F.43)}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathrm{Re}\Big(\mathrm{Li}_p(\lambda^2 x)\Big) \;=\;& -\frac{1}{2}\sum_{n=1}^{\infty}\frac{x^n}{n^p}+\frac{3}{2}\sum_{n=0}^{\infty}\frac{x^{3n+3}}{(3n+3)^p} \;=\; -\frac{1}{2}\sum_{n=1}^{\infty}\frac{x^n}{n^p}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{(x^3)^n}{3^p n^p} \;=\\
=\;& \frac{1}{2}\Big(3^{1-p}\,\mathrm{Li}_p(x^3)-\mathrm{Li}_p(x)\Big). \qquad \text{(F.44)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathrm{Li}_p(\lambda^3 x) \;=\;& \sum_{n=0}^{\infty}\frac{\lambda^3 x^{2n+1}}{(2n+1)^p}+\sum_{n=0}^{\infty}\frac{\lambda^6 x^{2n+2}}{(2n+2)^p} \;=\; -\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)^p}+\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+2)^p} \;=\\
=\;& -\sum_{n=1}^{\infty}\frac{x^n}{n^p}+2\sum_{n=1}^{\infty}\frac{(x^2)^n}{2^p n^p} \;=\; 2^{1-p}\,\mathrm{Li}_p(x^2)-\mathrm{Li}_p(x). \qquad \text{(F.45)}
\end{aligned}
$$

The supplementary procedure mentioned earlier is best understood by looking at the last calculation. In the expression

$$-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)^p}+\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+2)^p} \tag{F.46}$$

the second part can already been written as a simpler sum,

$$\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+2)^p}=\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)^p}=\sum_{n=1}^{\infty}\frac{(x^2)^n}{2^p n^p} \tag{F.47}$$

while one has to find the "missing" parts of the first sum,

$$-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)^p}=-\frac{x}{1^p}-\frac{x^3}{3^p}-\frac{x^5}{5^p}+\ldots=-\sum_{n=1}^{\infty}\frac{x^n}{n^p}+\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)^p}. \tag{F.48}$$

Using these expressions for the polylogarithms as well as the relations

$$\begin{aligned}
\operatorname{Im}\zeta(\Omega^2\omega_1) &= \frac{(\operatorname{Re}\zeta(\Omega\omega_0)-\operatorname{Re}\zeta(\Omega\omega_1))(\operatorname{Re}\zeta(\Omega\omega_0)+2\operatorname{Re}\zeta(\Omega\omega_1))}{3\operatorname{Im}\zeta(\omega_1)},\\
\operatorname{Im}\zeta(\Omega^2\omega_2) &= \frac{(\operatorname{Re}\zeta(\Omega\omega_0)-\operatorname{Re}\zeta(\Omega\omega_2))(\operatorname{Re}\zeta(\Omega\omega_0)-4\operatorname{Re}\zeta(\Omega\omega_2))}{21\operatorname{Im}\zeta(\omega_2)},\\
\operatorname{Re}\zeta(\Omega^3\omega_0) &= \frac{2}{5}\left(\operatorname{Re}\zeta(\Omega\omega_0)\right)^2,\\
\operatorname{Re}\zeta(\Omega^3\omega_1) &= \frac{1}{3}\left(\frac{1}{5}\left(\operatorname{Re}\zeta(\Omega\omega_0)\right)^2+2\operatorname{Re}\zeta(\Omega\omega_0)\operatorname{Re}\zeta(\Omega\omega_1)-\left(\operatorname{Re}\zeta(\Omega\omega_1)\right)^2\right),\\
\operatorname{Re}\zeta(\Omega^3\omega_2) &= -\frac{2}{21}\left(\frac{4}{5}\left(\operatorname{Re}\zeta(\Omega\omega_0)\right)^2-4\operatorname{Re}\zeta(\Omega\omega_0)\operatorname{Re}\zeta(\Omega\omega_2)-\left(\operatorname{Re}\zeta(\Omega\omega_2)\right)^2\right),\\
\operatorname{Re}\zeta(\Omega^3\omega_3) &= -\frac{1}{2}\operatorname{Re}\zeta(\Omega\omega_0)\left(\frac{1}{5}\operatorname{Re}\zeta(\Omega\omega_0)-\operatorname{Re}\zeta(\Omega\omega_3)\right)
\end{aligned} \tag{F.49}$$

between depth-one words which are found in `shuffle6.m`, one ends up with

$$\begin{aligned}
\operatorname{Re}\zeta(\omega_0) &= \mathrm{Li}_1(1)=-\ln(1-1)=\infty,\\
\operatorname{Re}\zeta(\omega_1) &= \operatorname{Re}(\mathrm{Li}_1(\lambda)) = \operatorname{Re}(-\ln(1-\lambda)) = \ln(1) = 0,\\
\operatorname{Re}\zeta(\omega_2) &= \operatorname{Re}\left(\mathrm{Li}_1(\lambda^2)\right) = \operatorname{Re}\left(-\ln(1-\lambda^2)\right) = \ln(\sqrt{3}) = \frac{1}{2}\ln(3),\\
\operatorname{Re}\zeta(\omega_3) &= \mathrm{Li}_1(\lambda^3) = -\ln(1+1) = -\ln(2),\\
\operatorname{Re}\zeta(\Omega\omega_0) &= \mathrm{Li}_2(1) = \zeta(2),\\
\operatorname{Re}\zeta(\Omega\omega_1) &= \operatorname{Re}(\mathrm{Li}_2(\lambda)) = \frac{1}{2}\left(\mathrm{Li}_2(\lambda^1)+\mathrm{Li}_2(\lambda^5)\right) =\\
&= \frac{1}{2}\left(\frac{1}{6}\mathrm{Li}_2(1)-\frac{1}{3}\mathrm{Li}_2(1)-\frac{1}{2}\mathrm{Li}_2(1)+\mathrm{Li}_2(1)\right) = \frac{1}{6}\mathrm{Li}_2(1) = \frac{1}{6}\zeta(2),\\
\operatorname{Re}\zeta(\Omega\omega_2) &= \operatorname{Re}\left(\mathrm{Li}_2(\lambda^2)\right) = \frac{1}{2}\left(\mathrm{Li}_2(\lambda^2)+\mathrm{Li}_2(\lambda^4)\right) =\\
&= \frac{1}{2}\left(\frac{1}{3}\mathrm{Li}_2(1)-\mathrm{Li}_2(1)\right) = -\frac{1}{3}\mathrm{Li}_2(1) = -\frac{1}{3}\zeta(2),\\
\operatorname{Re}\zeta(\Omega\omega_3) &= \mathrm{Li}_2(-1) = \frac{1}{2}\mathrm{Li}_2(1)-\mathrm{Li}_2(1) = -\frac{1}{2}\mathrm{Li}_2(1) = -\frac{1}{2}\zeta(2),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}\,\zeta(\Omega^2\omega_0) &= \mathrm{Li}_3(1) \;=\; \zeta(3), \\
\mathrm{Re}\,\zeta(\Omega^2\omega_1) &= \mathrm{Re}\,(\mathrm{Li}_3(\lambda)) \;=\; \frac{1}{2}\left(\frac{1}{36}\,\mathrm{Li}_3(1) - \frac{1}{9}\,\mathrm{Li}_3(1) - \frac{1}{4}\,\mathrm{Li}_3(1) + \mathrm{Li}_3(1)\right) \;=\; \frac{1}{3}\zeta(2), \\
\mathrm{Re}\,\zeta(\Omega^2\omega_2) &= \mathrm{Re}\left(\mathrm{Li}_3(\lambda^2)\right) \;=\; \frac{1}{2}\left(\frac{1}{9}\,\mathrm{Li}_3(1) - \mathrm{Li}_3(1)\right) \;=\; -\frac{4}{9}\zeta(3), \\
\mathrm{Re}\,\zeta(\Omega^2\omega_3) &= \mathrm{Li}_3(\lambda^3) \;=\; \frac{1}{4}\,\mathrm{Li}_3(1) - \mathrm{Li}_3(1) \;=\; -\frac{3}{2}\zeta(3), \\
\mathrm{Re}\,\zeta(\Omega^3\omega_0) &= \mathrm{Li}_4(1) \;=\; \zeta(4) \;=\; \frac{2}{5}\zeta(2)^2 \;=\; \frac{\pi^4}{90}, \\
\mathrm{Re}\,\zeta(\Omega^3\omega_1) &= \frac{1}{3}\left(\frac{1}{5}\zeta(2)^2 + \frac{1}{3}\zeta(2)^2 - \frac{1}{36}\zeta(2)^2\right) \;=\; \frac{91}{540}\zeta(2)^2 \;=\; \frac{91\pi^4}{19440}, \\
\mathrm{Re}\,\zeta(\Omega^3\omega_2) &= -\frac{2}{21}\left(\frac{4}{5}\zeta(2)^2 + \frac{4}{3}\zeta(2)^2 - \frac{1}{9}\zeta(2)^2\right) \;=\; -\frac{26}{135}\zeta(2)^2 \;=\; -\frac{13\pi^4}{2430}, \\
\mathrm{Re}\,\zeta(\Omega^3\omega_3) &= \mathrm{Li}_4(-1) \;=\; -\frac{1}{2}\zeta(2)\left(\frac{1}{5}\zeta(2) + \frac{1}{2}\zeta(2)\right) \;=\; -\frac{7}{20}\zeta(2)^2 \;=\; -\frac{7\pi^4}{720}
\end{aligned} \qquad \text{(F.50)}$$

and

$$\begin{aligned}
\mathrm{Im}\,\zeta(\omega_1) &= \mathrm{Cl}_1\left(\frac{\pi}{3}\right) \;=\; \mathrm{Im}\,(\mathrm{Li}_1(\lambda)) \;=\; \mathrm{Im}\,(-\ln(1-\lambda)) \;=\; \frac{\pi}{3}, \\
\mathrm{Im}\,\zeta(\omega_2) &= \mathrm{Cl}_1\left(\frac{2\pi}{3}\right) \;=\; \mathrm{Im}\left(\mathrm{Li}_1(\lambda^2)\right) \;=\; \mathrm{Im}\left(-\ln(1-\lambda^2)\right) \;=\; \frac{\pi}{6}, \\
\mathrm{Im}\,\zeta(\Omega\omega_1) &= \mathrm{Im}\,(\mathrm{Li}_2(\lambda)) \;=\; \mathrm{Cl}_2\left(\frac{\pi}{3}\right), \\
\mathrm{Im}\,\zeta(\Omega\omega_2) &= \mathrm{Im}\left(\mathrm{Li}_2(\lambda^2)\right) \;=\; \mathrm{Cl}_2\left(\frac{2\pi}{3}\right), \\
\mathrm{Im}\,\zeta(\Omega^2\omega_1) &= \mathrm{Cl}_3\left(\frac{\pi}{3}\right) \;=\; \frac{1}{\pi}\left(\zeta(2) - \frac{1}{6}\zeta(2)\right)\left(\zeta(2) + \frac{1}{3}\zeta(2)\right) \;=\; \frac{10}{9\pi}\zeta(2)^2 \;=\; \frac{5\pi^3}{162}, \\
\mathrm{Im}\,\zeta(\Omega^2\omega_2) &= \mathrm{Cl}_3\left(\frac{2\pi}{3}\right) \;=\; \frac{2}{7\pi}\left(\zeta(2) + \frac{1}{3}\zeta(2)\right)\left(\zeta(2) + \frac{4}{3}\zeta(2)\right) \;=\; \frac{8}{9\pi}\zeta(2)^2 \;=\; \frac{2\pi^3}{81}, \\
\mathrm{Im}\,\zeta(\Omega^3\omega_1) &= \mathrm{Im}\,(\mathrm{Li}_4(\lambda)) \;=\; \mathrm{Cl}_4\left(\frac{\pi}{3}\right), \\
\mathrm{Im}\,\zeta(\Omega^3\omega_2) &= \mathrm{Im}\left(\mathrm{Li}_4(\lambda^2)\right) \;=\; \mathrm{Cl}_4\left(\frac{2\pi}{3}\right).
\end{aligned} \qquad \text{(F.51)}$$

The relations

$$1-\lambda = \frac{1}{2} - i\sqrt{\frac{3}{4}} = e^{-i\pi/3}, \qquad 1-\lambda^2 = \frac{3}{2} - i\sqrt{\frac{3}{4}} = \sqrt{3}\,e^{-i\pi/6}, \qquad \text{(F.52)}$$

have been used. Fortunately, $\zeta(\omega_0)$ does not occur at all in the final contributions. The *Clausen polylogarithms* (see Appendix E.4) with even index are related by

$$\mathrm{Cl}_{2m}\left(\frac{\pi}{3}\right) = \left(1 + \frac{1}{2^{2m-1}}\right)\mathrm{Cl}_{2m}\left(\frac{2\pi}{3}\right). \qquad \text{(F.53)}$$

Using all these results, one obtains the results shown in Sec. 3.4.12.

Appendix G

Effective vertex integrals

This appendix contains integrals that are used in Appendix H to calculate contributions to the soft part of the self energy of the quark. First of all, two basic integral identities are needed. The first one holds for the tensorial integral. It reads

$$\int \frac{d^D l}{(2\pi)^D} l_\mu l_\nu g(l^2) = \frac{1}{D} g_{\mu\nu} \int \frac{d^D l}{(2\pi)^D} l^2 g(l^2), \tag{G.1}$$

no matter whether the function g still depends on an additional outer momentum p or not. If this expression would also contribute to terms for instance such as $p_\mu p_\nu I_{pp}^{v1}$, the inspection of the case $\mu \neq \nu$ would let I_{pp}^{v1} vanish. Next the result for the general integral given by

$$I(\alpha, \beta) = \int \frac{d^D k}{(2\pi)^D} \frac{(-k^2)^\beta}{(-k^2 + m^2)^\alpha} \tag{G.2}$$

has to be found. The first step is to perform a Wick rotation for k_0. In the second step one integrates over the angular components which results in $2\pi^{D/2}/\Gamma(D/2)$. Proceeding in this way one obtains

$$I(\alpha, \beta) = i \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^\beta}{(k^2 + m^2)^\alpha} = \frac{2i}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty \frac{k^{2\beta} k^{D-1} dk}{(k^2 + m^2)^\alpha}. \tag{G.3}$$

Now one substitutes the dimensionless variable

$$y = \frac{k^2}{k^2 + m^2}, \qquad dy = \frac{2km^2 dk}{(k^2 + m^2)^2}, \qquad k^2 = \frac{m^2 y}{1 - y}, \qquad k^2 + m^2 = \frac{m^2}{1 - y} \tag{G.4}$$

where the limits are given by $y = 0$ and $y = 1$ and obtains

$$\begin{aligned} I(\alpha, \beta) &= \frac{i}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty \frac{2k\, dk}{(k^2 + m^2)^2} k^{D+2\beta-2} (k^2 + m^2)^{2-\alpha} = \\ &= \frac{i}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^1 \frac{dy}{m^2} \left(\frac{m^2 y}{1 - y}\right)^{D/2+\beta-1} \left(\frac{m^2}{1 - y}\right)^{2-\alpha} = \\ &= \frac{i(m^2)^{D/2+\beta-\alpha+1-1}}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^1 y^{D/2+\beta-1} (1 - y)^{\alpha-\beta-D/2-1} dy = \\ &= \frac{i\Gamma(D/2 + \beta)\Gamma(\alpha - \beta - D/2)}{(4\pi)^{D/2}\Gamma(D/2)\Gamma(\alpha)} (m^2)^{D/2-\alpha+\beta}. \end{aligned} \tag{G.5}$$

For the case $\beta = 0$ this integral simplifies to

$$I(\alpha) = I(\alpha,0) = \frac{i\Gamma(\alpha - D/2)}{(4\pi)^{D/2}\Gamma(\alpha)}(m^2)^{D/2-\alpha}, \tag{G.6}$$

while for $\beta = 1$ one can combine this result with Eq. (G.1) to obtain

$$\begin{aligned}\int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{(-k^2+m^2)^\alpha} &= \int \frac{d^Dk}{(2\pi)^D}\frac{g_{\mu\nu}k^2/D}{(-k^2+m^2)^\alpha} = \\ &= -\frac{i\Gamma(D/2+1)\Gamma(\alpha-1-D/2)}{(4\pi)^{D/2}\Gamma(D/2)\Gamma(\alpha)D}g_{\mu\nu}(m^2)^{D/2-\alpha+1} = \\ &= -\frac{i(D/2)\Gamma(D/2)\Gamma(\alpha-D/2)}{(4\pi)^{D/2}\Gamma(D/2)\Gamma(\alpha)(\alpha-1-D/2)D}g_{\mu\nu}(m^2)^{D/2-\alpha+1} = \\ &= \frac{i\Gamma(\alpha-D/2)}{(4\pi)^{D/2}\Gamma(\alpha)(D-2\alpha+2)}g_{\mu\nu}(m^2)^{D/2-\alpha+1}.\end{aligned} \tag{G.7}$$

Comparing this result with Eq. (G.6), the effective replacement rule

$$k_\mu k_\nu \to \frac{m^2 g_{\mu\nu}}{D-2\alpha+2} \tag{G.8}$$

is obtained which turns out to be quite helpful in the following.

G.1 Integral class for the abelian diagram

The integrals I^{v1}, I^{v1}_μ, and $I^{v1}_{\mu\nu}$ in Sec. H.1.1 can be subsummed by the generic integral

$$I^{v1}_f = \int \frac{d^Dl}{(2\pi)^D}\frac{f(l)}{((p_1+l)^2-m^2)((p_2+l)^2-m^2)l^2} \tag{G.9}$$

where $f(l)$ is a scalar, vectorial, or tensorial expression in l. Now one has to use the Feynman parametrization with the three parameters x_1, x_2, and $1-x_1-x_2$,

$$\frac{1}{ABC} = \frac{\Gamma(3)}{\Gamma(1)^3}\int_0^1 dx_1\int_0^{1-x_1}dx_2\frac{1}{(x_1A+x_2B+(1-x_1-x_2)C)^3} \tag{G.10}$$

to obtain

$$\begin{aligned}I^{v1}_f &= \int\frac{d^Dl}{(2\pi)^D}\frac{f(l)}{((p_1+l)^2-m^2)((p_2+l)^2-m^2)l^2} = \\ &= \frac{\Gamma(3)}{\Gamma(1)^3}\int_0^1 dx_1\int_0^{1-x_1}dx_2 \;\times \\ &\quad\times \int\frac{d^Dl}{(2\pi)^D}\frac{1}{(x_1((p_1+l)^2-m^2)+x_2((p_2+l)^2-m^2)+(1-x_1-x_2)l^2)^3} = \\ &= 2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{f(l)}{(l^2+2x_1p_1l+2x_2p_2l+x_1(p_1^2-m^2)+x_2(p_2^2-m^2))^3} = \\ &= -2\int_0^1 dx_1\int_0^{1-x_1}dx_2 \;\times \\ &\quad\times \int\frac{d^Dl}{(2\pi)^D}\frac{f(l)}{(-(l+x_1p_1+x_2p_2)^2+(x_1p_1+x_2p_2)^2-x_1(p_1^2-m^2)-x_2(p^2-m^2))^3} = \\ &= -2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{f(l-x_1p_1-x_2p_2)}{(-l^2+(x_1p_1+x_2p_2)^2-x_1(p_1^2-m^2)-x_2(p_2^2-m^2))^3}.\end{aligned} \tag{G.11}$$

The second term in the denominator is the square of the effective mass m_x. At this point one inserts $p_1 = p + k$ and $p_2 = p$ and neglects terms proportional to k^2 to obtain

$$\begin{aligned}
m_x^2 &= (x_1p_1 + x_2p_2)^2 - x_1(p_1^2 - m^2) - x_2(p_2^2 - m^2) \;= \\
&= x_1^2p_1^2 + 2x_1x_2(p_1p_2) + x_2^2p_2^2 - x_1(p_1^2 - m^2) - x_2(p_2^2 - m^2) \;= \\
&\approx x_1^2(p^2 + 2pk) + 2x_1x_2(p^2 + pk) + x_2^2p^2 + x_1(m^2 - p^2 - 2pk) + x_2(m^2 - p^2) \;= \\
&= (x_1 + x_2)^2p^2 + (x_1 + x_2)(m^2 - p^2) - 2x_1(1 - x_1 - x_2)pk. \qquad \text{(G.12)}
\end{aligned}$$

For matter of convenience one denotes $m^2 - p^2 =: \omega p^2$ and $2pk =: -\eta p^2$. One can also replace the parameter x_2 by $z = x_1 + x_2$ with $x_1 < z < 1$. The argument of f changes accordingly to $l - x_1p_1 - x_2p_2 = l - zp - x_1k$. For the squared effective mass one obtains

$$m_x^2 = p^2\left(z^2 + z\omega + x_1(1 - z)\eta\right). \qquad \text{(G.13)}$$

Finally, for the applications it is sufficient to consider the case $\omega = 0$, therefore

$$m_x^2 = p^2\left(z^2 + x_1(1 - z)\eta\right). \qquad \text{(G.14)}$$

For the general integral

$$\begin{aligned}
I_f^{v1} &= -2\int_0^1 dx_1 \int_{x_1}^1 dz \int \frac{d^Dl}{(2\pi)^D}\frac{f(l - zp - x_1k)}{(-l^2 + p^2(z^2 + x_1(1 - z)\eta))^3} \;= \\
&= -2\int_0^1 dz \int_0^z dx_1 \int \frac{d^Dl}{(2\pi)^D}\frac{f(l - x_1k - zp)}{(-l^2 + p^2(x_1(1 - z)\eta + z^2))^3} \qquad \text{(G.15)}
\end{aligned}$$

one now has to discuss the different cases according to the specific form of f.

G.1.1 The scalar integral I^{v1}

The case $f(l) = 1$ defines the scalar integral. Using Eq. (G.6) for $\alpha = 3$ one obtains

$$\begin{aligned}
I^{v1} &= \frac{-2i}{(4\pi)^{D/2}}\frac{\Gamma(3 - D/2)}{\Gamma(3)}\int_0^1 dz \int_0^z dx_1 \left(p^2(x_1(1 - z)\eta + z^2)\right)^{D/2-3} \;= \\
&= \frac{-i\Gamma(1 + \varepsilon)}{(4\pi)^{2-\varepsilon}}(p^2)^{-1-\varepsilon}\int_0^1 dz \int_0^z dx_1 \left(x_1(1 - z)\eta + z^2\right)^{-1-\varepsilon}. \qquad \text{(G.16)}
\end{aligned}$$

For this integral one can set $\varepsilon = 0$ to end up with

$$I^{v1} = \frac{-i}{(4\pi)^2p^2}\int_0^1 dz \int_0^z \frac{dx_1}{x_1(1 - z)\eta + z^2}. \qquad \text{(G.17)}$$

One can even derive an exact formula for this integral. Starting with

$$\begin{aligned}
\int_0^z \frac{dx_1}{x_1(1 - z)\eta + z^2} &= \frac{1}{(1 - z)\eta}\int_0^z \frac{dx_1}{x_1 + z^2/((1 - z)\eta)} \;= \\
&= \frac{1}{(1 - z)\eta}\ln\left(x_1 + \frac{z^2}{(1 - z)\eta}\right)\Bigg|_0^z \;= \\
&= \frac{1}{(1 - z)\eta}\left(\ln\left(z + \frac{z^2}{(1 - z)\eta}\right) - \ln\left(\frac{z^2}{(1 - z)\eta}\right)\right) \;= \\
&= \frac{1}{(1 - z)\eta}\ln\left(\frac{(1 - z)\eta}{z} + 1\right) \;=\; \frac{1}{(1 - z)\eta}\ln\left(\frac{\eta + (1 - \eta)z}{z}\right) \qquad \text{(G.18)}
\end{aligned}$$

the integration over z can be done, resulting in

$$\begin{aligned}\int_0^1 \frac{dz}{(1-z)\eta} \ln\left(\frac{\eta+(1-\eta)z}{z}\right) &= \int_0^1 \frac{dz}{\eta z} \ln\left(\frac{1-(1-\eta)z}{1-z}\right) = \\ = \frac{1}{\eta}\left(\int_0^1 \frac{dz}{z} \ln(1-(1-\eta)z) - \int_0^1 \frac{dz}{z} \ln(1-z)\right) &= \frac{1}{\eta}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(1-\eta)\right)\end{aligned} \tag{G.19}$$

and therefore

$$I^{v1} = \frac{-i}{(4\pi)^2 p^2 \eta}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(1-\eta)\right) = \frac{-i}{(4\pi)^2 p^2}\left\{1 - \ln\eta + \left(\frac{1}{4} - \frac{1}{2}\ln\eta\right)\eta + O(\eta^2)\right\}. \tag{G.20}$$

G.1.2 The vector integral I_μ^{v1}

The vector integral I_μ^{v1} is given by

$$I_\mu^{v1} = \frac{i}{(4\pi)^2 p^2}\int_0^1 dz \int_0^z dx_1 \frac{x_1 k_\mu + z p_\mu}{x_1(1-z)\eta + z^2} = I_k^{v1} k_\mu + I_p^{v1} p_\mu \tag{G.21}$$

where

$$I_k^{v1} = \frac{i}{(4\pi)^2 p^2}\int_0^1 dz \int_0^z \frac{x_1 dx_1}{x_1(1-z)\eta + z^2}, \tag{G.22}$$

$$I_p^{v1} = \frac{i}{(4\pi)^2 p^2}\int_0^1 z\, dz \int_0^z \frac{dx_1}{x_1(1-z)\eta + z^2}. \tag{G.23}$$

In order to evaluate I_k^{v1} one first calculates

$$\begin{aligned}\int_0^z \frac{x_1 dx_1}{x_1(1-z)\eta + z^2} &= \frac{1}{(1-z)\eta}\int_0^z \frac{x_1 dx_1}{x_1 + z^2/((1-z)\eta)} = \\ &= \frac{1}{(1-z)\eta}\left(\int_0^z dx_1 - \frac{z^2}{(1-z)\eta}\int_0^z \frac{dx_1}{x_1 + z^2/((1-z)\eta)}\right) = \\ &= \frac{1}{(1-z)\eta}\left(z - \frac{z^2}{(1-z)\eta}\ln\left(\frac{\eta+(1-\eta)z}{z}\right)\right).\end{aligned} \tag{G.24}$$

Therefore, the next step is to calculate

$$\begin{aligned}I_k^{v1} &= \frac{i}{(4\pi)^2 p^2}\int_0^1 \frac{dz}{(1-z)\eta}\left(z - \frac{z^2}{(1-z)\eta}\ln\left(\frac{\eta+(1-\eta)z}{z}\right)\right) = \\ &= \frac{i}{(4\pi)^2 p^2}\int_0^1 \frac{dz}{\eta z}\left((1-z) - \frac{(1-z)^2}{\eta z}\ln\left(\frac{1-(1-\eta)z}{1-z}\right)\right).\end{aligned} \tag{G.25}$$

The different parts of this integral will be determined as indefinite integrals because they contain divergences which cancel among the contributions. The contributions are given by

$$\int \frac{dz}{z}(1-z) = \ln z - z,$$

$$\begin{aligned}
\int \frac{dz}{z^2} \ln\left(\frac{1-(1-\eta)z}{1-z}\right) &= \\
&= -(1-\eta)\ln z - \left(\frac{1}{z}-(1-\eta)\right)\ln(1-(1-\eta)z) + \ln z + \left(\frac{1}{z}-1\right)\ln(1-z) = \\
&= \eta \ln z - \frac{1}{z}\ln\left(\frac{(1-(1-\eta)z}{1-z}\right) + (1-\eta)\ln(1-(1-\eta)z) - \ln(1-z), \\
\int \frac{dz}{z} \ln\left(\frac{1-(1-\eta)z}{1-z}\right) &= \mathrm{Li}_2(z) - \mathrm{Li}_2((1-\eta)z), \\
\int \ln\left(\frac{1-(1-\eta)z}{1-z}\right) dz &= \\
&= \left(z - \frac{1}{1-\eta}\right)\ln(1-(1-\eta)z) - z - (z-1)\ln(1-z) + z = \\
&= z\ln\left(\frac{1-(1-\eta)z}{1-z}\right) - \frac{1}{1-\eta}\ln(1-(1-\eta)z) + \ln(1-z).
\end{aligned} \tag{G.26}$$

One therefore obtains

$$\begin{aligned}
\int \frac{dz}{\eta z}\left((1-z) - \frac{(1-z)^2}{\eta z}\ln\left(\frac{1-(1-\eta)z}{1-z}\right)\right) &= \\
= \frac{1}{\eta^2}\Bigg\{ &\eta \ln z - \eta z + 2\left(\mathrm{Li}_2(z) - \mathrm{Li}_2((1-\eta)z)\right) + \\
&-\eta\ln z + \frac{1}{z}\ln\left(\frac{1-(1-\eta)z}{1-z}\right) - (1-\eta)\ln(1-(1-\eta)z) + \ln(1-z) + \\
&-z\ln\left(\frac{1-(1-\eta)z}{1-z}\right) + \frac{1}{1-\eta}\ln(1-(1-\eta)z) - \ln(1-z)\Bigg\} = \\
= \frac{1}{\eta}\Bigg\{ &\frac{2}{\eta}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2((1-\eta)z)\right) + \\
&+\frac{1-z^2}{\eta z}\ln\left(\frac{1-(1-\eta)z}{1-z}\right) + \frac{2-\eta}{1-\eta}\ln(1-(1-\eta)z) - \eta z\Bigg\}.
\end{aligned} \tag{G.27}$$

Using

$$\ln\left(\frac{1-(1-\eta)z}{1-z}\right) \to -(1-\eta)z + z = \eta z \quad \text{for} \quad z \to 0 \tag{G.28}$$

one can finally insert the limits and obtains

$$\begin{aligned}
\int_0^1 \frac{dz}{\eta z}\left((1-z) - \frac{(1-z)^2}{\eta z}\ln\left(\frac{1-(1-\eta)z}{1-z}\right)\right) &= \\
&= \frac{1}{\eta}\left\{\frac{2}{\eta}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(1-\eta)\right) + \frac{2-\eta}{1-\eta}\ln\eta - 1 - 1\right\} = \\
&= \frac{2}{\eta^2}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(1-\eta)\right) + \frac{2-\eta}{(1-\eta)\eta}\ln\eta - \frac{2}{\eta}.
\end{aligned} \tag{G.29}$$

Therefore

$$I_k^{v1} = \frac{i}{(4\pi)^2 p^2 \eta}\left\{\frac{2}{\eta}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(1-\eta)\right) + \frac{2-\eta}{1-\eta}\ln\eta - 2\right\} \tag{G.30}$$

which again can be expanded in η (see below). For I_p^{v1} one finally obtains

$$\begin{aligned}
I_p^{v1} &= \frac{i}{(4\pi)^2 p^2}\int_0^1 z\,dz\int_0^z \frac{dx_1}{x_1(1-z)\eta+z^2} = \\
&= \frac{i}{(4\pi)^2 p^2}\int_0^1 \frac{z\,dz}{(1-z)\eta}\ln\left(\frac{\eta+(1-\eta)z}{z}\right) = \\
&= \frac{i}{(4\pi)^2 p^2}\int_0^1 \frac{(1-z)dz}{\eta z}\ln\left(\frac{1-(1-\eta)z}{1-z}\right) = \\
&= \frac{i}{(4\pi)^2 p^2\eta}\Bigg[(\mathrm{Li}_2(z)-\mathrm{Li}_2((1-\eta)z)) - z\ln\left(\frac{1-(1-\eta)z}{1-z}\right) + \\
&\qquad + \frac{1}{1-\eta}\ln(1-(1-\eta)z) - \ln(1-z)\Bigg]_0^1 = \\
&= \frac{i}{(4\pi)^2 p^2\eta}\left\{\mathrm{Li}_2(1)-\mathrm{Li}_2(1-\eta)-\ln\eta+\frac{1}{1-\eta}\ln\eta\right\} = \\
&= \frac{i}{(4\pi)^2 p^2}\left\{\frac{1}{\eta}(\mathrm{Li}_2(1)-\mathrm{Li}_2(1-\eta))+\frac{1}{1-\eta}\ln\eta\right\}.
\end{aligned} \tag{G.31}$$

The expansion in η is given by

$$\begin{aligned}
I_k^{v1} &= \frac{i}{(4\pi)^2 p^2}\left\{\frac{1}{2}+\left(\frac{2}{9}+\frac{1}{3}\ln\eta\right)\eta+O(\eta^2)\right\}, \\
I_p^{v1} &= \frac{i}{(4\pi)^2 p^2}\left\{1+\left(\frac{1}{4}+\frac{1}{2}\ln\eta\right)\eta+O(\eta^2)\right\}.
\end{aligned} \tag{G.32}$$

G.1.3 The tensor integral $I_{\mu\nu}^{v1}$

Because odd powers of l vanish, $f(l-x_1k-zp)$ is effectively given by

$$l_\mu l_\nu + (x_1k_\mu+zp_\mu)(x_1k_\nu+zp_\nu) \to \frac{m^2 g_{\mu\nu}}{D-4} + (x_1k_\mu+zp_\mu)(x_1k_\nu+zp_\nu), \tag{G.33}$$

one therefore obtains

$$\begin{aligned}
I_{\mu\nu}^{v1} &= I_g^{v1} g_{\mu\nu} + I_{\mu\nu}^{v1\prime} \qquad \text{with} \\
I_{\mu\nu}^{v1\prime} &= \frac{-i}{(4\pi)^2 p^2}\int_0^1 dz\int_0^z dx_1 \frac{(x_1k_\mu+zp_\mu)(x_1k_\nu+zp_\nu)}{x_1(1-z)\eta+z^2} = \\
&= I_{kk}k_\mu k_\nu + I_{kp}(k_\mu p_\nu + p_\mu k_\nu) + I_{pp}p_\mu p_\nu \quad \text{where}
\end{aligned} \tag{G.34}$$

$$\begin{aligned}
I_{kk}^{v1} &= \frac{-i}{(4\pi)^2 p^2}\int_0^1 dz\int_0^z \frac{x_1^2 dx_1}{x_1(1-z)\eta+z^2}, \\
I_{kp}^{v1} &= \frac{-i}{(4\pi)^2 p^2}\int_0^1 z\,dz\int_0^z \frac{x_1 dx_1}{x_1(1-z)\eta+z^2}, \\
I_{pp}^{v1} &= \frac{-i}{(4\pi)^2 p^2}\int_0^1 z^2 dz\int_0^z \frac{dx_1}{x_1(1-z)\eta+z^2}, \\
I_g^{v1} &= \frac{-2i\Gamma(3-D/2)}{(4\pi)^{D/2}\Gamma(3)(D-4)}(p^2)^{D/2-2}\int_0^1 dz\int_0^z dx_1\left(x_1(1-z)\eta+z^2\right)^{D/2-2}.
\end{aligned} \tag{G.35}$$

For the first three integrals the results read

$$
\begin{aligned}
I_{kk}^{v1} &= \frac{-i}{(4\pi)^2 p^2 \eta^2}\bigg\{\frac{6}{\eta}\left(\mathrm{Li}_2(1)-\mathrm{Li}_2(1-\eta)\right) + \\
&\qquad +(2-\eta)\frac{6-6\eta-\eta^2}{2(1-\eta)^2}\ln\eta - \frac{12-9\eta-2\eta^2}{2(1-\eta)}\bigg\}, \\
I_{kp}^{v1} &= \frac{-i}{(4\pi)^2 p^2 \eta}\bigg\{\frac{3}{\eta}\left(\mathrm{Li}_2(1)-\mathrm{Li}_2(1-\eta)\right) + \frac{6-9\eta+2\eta^2}{2(1-\eta)^2}\ln\eta - \frac{6-5\eta}{2(1-\eta)}\bigg\}, \\
I_{pp}^{v1} &= \frac{-i}{(4\pi)^2 p^2}\bigg\{\frac{1}{\eta}\left(\mathrm{Li}_2(1)-\mathrm{Li}_2(1-\eta)\right) + \frac{2-3\eta}{2(1-\eta)^2}\ln\eta - \frac{1}{2(1-\eta)}\bigg\}.
\end{aligned} \tag{G.36}
$$

For the series expansions one obtains

$$
\begin{aligned}
I_{kk}^{v1} &= \frac{-i}{(4\pi)^2 p^2}\left\{\frac{1}{6}-\frac{1}{8}\eta+O(\eta^2)\right\}, \\
I_{kp}^{v1} &= \frac{-i}{(4\pi)^2 p^2}\left\{\frac{1}{4}-\frac{1}{6}\eta+O(\eta^2)\right\}, \\
I_{pp}^{v1} &= \frac{-i}{(4\pi)^2 p^2}\left\{\frac{1}{2}-\frac{1}{4}\eta+O(\eta^2)\right\},
\end{aligned} \tag{G.37}
$$

so that in the actual combination of these terms there are no $\ln\eta$ contributions. Finally, I_g^{v1} is given by

$$
\begin{aligned}
I_g^{v1} &= \frac{i\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}\varepsilon}(p^2)^{-\varepsilon}\int_0^1 dz\int_0^z dx_1\left(x_1(1-z)\eta+z^2\right)^{-\varepsilon} = \\
&= \frac{i\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}\varepsilon}(p^2)^{-\varepsilon}\int_0^1 dz\int_0^z dx_1\left(1-\varepsilon\ln(x_1(1-z)\eta+z^2)+O(\varepsilon^2)\right) = \\
&= \frac{i\Gamma(1+\varepsilon)}{4(4\pi)^{2-\varepsilon}\varepsilon}(p^2)^{-\varepsilon} - \frac{i}{2(4\pi)^2}\int_0^1 dz\int_0^z dx_1\ln\left(x_1(1-z)\eta+z^2\right)+O(\varepsilon)
\end{aligned} \tag{G.38}
$$

where

$$
\int_0^1 dz\int_0^z dx_1 = \int_0^1 z\,dz = \frac{1}{2} \tag{G.39}
$$

has been used. One can calculate the finite part by using $x_1' = x_1(1-z)\eta+z^2$ to obtain

$$
\int_0^1 dz\int_0^z \ln(x_1(1-z)\eta+z^2)dx_1 = \int_0^1 \frac{dz}{(1-z)\eta}\int_{z^2}^{z(1-z)\eta+z^2}\ln x_1' dx_1' \;= \tag{G.40}
$$

$$
\begin{aligned}
&= \int_0^1 \frac{dz}{(1-z)\eta}\Big[x_1'\ln x_1' - x_1'\Big]_{z^2}^{z(1-z)\eta+z^2} = \\
&= \int_0^1 \Big[\left(z(1-z)\eta+z^2\right)\ln\left(z(1-z)\eta+z^2\right) - z(1-z)\eta - z^2 - z^2\ln z^2 + z^2\Big]\frac{dz}{(1-z)\eta} = \\
&= \int_0^1 \left[z(1-z)\eta\left(\ln\left(z(1-z)\eta+z^2\right)-1\right) + z^2\ln\left(\frac{z(1-z)\eta+z^2}{z^2}\right)\right]\frac{dz}{(1-z)\eta} = \\
&= \int_0^1 \left(\ln z - 1 + \ln\left((1-z)\eta+z\right)\right)z\,dz + \int_0^1 \ln\left(\frac{(1-z)\eta+z}{z}\right)\frac{z^2 dz}{(1-z)\eta} = \\
&= \int_0^1 (\ln z-1)z\,dz + \int_0^1 \ln\left(1-(1-\eta)z\right)(1-z)dz + \int_0^1 \ln\left(\frac{1-(1-\eta)z}{1-z}\right)\frac{(1-z)^2 dz}{\eta z}
\end{aligned}
$$

where in the last step the substitution $z \to 1-z$ was used for the last two integrals. Now

$$\begin{aligned}
\int_0^1 (\ln z - 1) z\, dz \;&=\; \left[\frac{1}{2}z^2(\ln z - 1)\right]_0^1 - \frac{1}{2}\int_0^1 z\, dz \;= \\
&= \left[\frac{1}{2}z^2(\ln z - 1) - \frac{1}{4}z^2\right]_0^1 \;=\; -\frac{1}{2} - \frac{1}{4} \;=\; -\frac{3}{4}, \qquad \text{(G.41)}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 &(1-(1-\eta)z)\,(1-z)dz \;= \\
&= \Bigg[z\ln(1-(1-\eta)z) - \frac{1}{1-\eta}\ln(1-(1-\eta)z) - z\; + \\
&\qquad -\frac{1}{2}z^2\ln(1-(1-\eta)z) + \frac{1}{2(1-\eta)^2}\ln(1-(1-\eta)z) + \frac{1}{4}z^2 + \frac{z}{2(1-\eta)}\Bigg]_0^1 \;= \\
&= \ln\eta - \frac{1}{1-\eta}\ln\eta - 1 - \frac{1}{2}\ln\eta + \frac{1}{2(1-\eta)^2}\ln\eta + \frac{1}{4} + \frac{1}{2(1-\eta)} \;= \\
&= \left(\frac{1}{2} - \frac{1}{1-\eta} + \frac{1}{2(1-\eta)^2}\right)\ln\eta + \frac{1}{2(1-\eta)} - \frac{3}{4} \;= \qquad \text{(G.42)} \\
&= \frac{1-2\eta+\eta^2-2+2\eta+1}{2(1-\eta)^2}\ln\eta + \frac{2-3(1-\eta)}{4(1-\eta)} \;=\; \frac{\eta^2}{2(1-\eta)^2}\ln\eta - \frac{1-3\eta}{4(1-\eta)},
\end{aligned}$$

$$\begin{aligned}
\int_0^1 &\ln\left(\frac{1-(1-\eta)z}{1-z}\right)\frac{(1-z)^2dz}{\eta z} \;= \\
&= \frac{1}{\eta}\Bigg[-\mathrm{Li}_2\left((1-\eta)z\right)\mathrm{Li}_2(z) - 2z\ln(1-(1-\eta)z) + \frac{2}{1-\eta}\ln(1-(1-\eta)z)\; + \\
&\qquad +2z + 2z\ln(1-z) - 2\ln(1-z) - 2z\; + \\
&\qquad +\frac{1}{2}z^2\ln(1-(1-\eta)z) - \frac{1}{2(1-\eta)^2}\ln(1-(1-\eta)z) - \frac{1}{4}z^2 - \frac{z}{2(1-\eta)}\; + \\
&\qquad -\frac{1}{2}z^2\ln(1-z) + \frac{1}{2}\ln(1-z) + \frac{1}{4}z^2 + \frac{1}{2}z\Bigg]_0^1 \;= \\
&= \frac{1}{\eta}\Bigg(-\mathrm{Li}_2(1-\eta) + \mathrm{Li}_2(1) - 2\ln\eta + \frac{2}{1-\eta}\ln\eta + 2 - 2\; + \\
&\qquad +\frac{1}{2}\ln\eta - \frac{1}{2(1-\eta)^2}\ln\eta - \frac{1}{4} - \frac{1}{2(1-\eta)} + \frac{1}{4} + \frac{1}{2}\Bigg) \;= \\
&= \frac{1}{\eta}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(1-\eta) + \left(-\frac{3}{2} + \frac{2}{1-\eta} - \frac{1}{2(1-\eta)^2}\right)\ln\eta + \frac{1-\eta-1}{2(1-\eta)}\right) \;= \\
&= \frac{1}{\eta}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(1-\eta) + \frac{-3+6\eta-3\eta^2+4-4\eta-1}{2(1-\eta)^2}\ln\eta - \frac{\eta}{2(1-\eta)}\right) \;= \\
&= \frac{1}{\eta}\left(\mathrm{Li}_2(1) - \mathrm{Li}_2(1-\eta)\right) + \frac{2-3\eta}{2(1-\eta)^2}\ln\eta - \frac{1}{2(1-\eta)}. \qquad \text{(G.43)}
\end{aligned}$$

In adding all these parts one obtains

$$\int_0^1 dz \int_0^z \ln(x_1(1-z)\eta + z^2)dx_1 \;=$$

$$\begin{aligned}
&= -\frac{3}{4}+\frac{\eta^2}{2(1-\eta)^2}\ln\eta-\frac{1-3\eta}{4(1-\eta)}\ + \\
&\qquad +\frac{1}{\eta}\left(\mathrm{Li}_2(1)-\mathrm{Li}_2(1-\eta)\right)+\frac{2-3\eta}{2(1-\eta)^2}\ln\eta-\frac{1}{2(1-\eta)}\ = \\
&= \frac{1}{\eta}\left(\mathrm{Li}_2(1)-\mathrm{Li}_2(1-\eta)\right)+\frac{2-3\eta+\eta^2}{2(1-\eta)^2}\ln\eta-\frac{1-3\eta+3(1-\eta)+2}{4(1-\eta)}\ = \\
&= \frac{1}{\eta}\left(\mathrm{Li}_2(1)-\mathrm{Li}_2(1-\eta)\right)+\frac{2-\eta}{2(1-\eta)}\ln\eta-\frac{3}{2} \qquad \text{(G.44)}
\end{aligned}$$

which can also be obtained by using MATHEMATICA. The series expansion in η results in

$$I_g^{v1}=\frac{i\Gamma(1+\varepsilon)}{4(4\pi)^{2-\varepsilon}\varepsilon}(p^2)^{-\varepsilon}+\frac{i}{(4\pi)^2}\left(\frac{1}{4}-\frac{1}{8}\eta+O(\eta^2)\right). \qquad \text{(G.45)}$$

G.2 Integral class for the non-abelian diagram

The generic integral in Sec. H.1.2 is given by

$$\begin{aligned}
I_f^{v2} &= \int\frac{d^Dl}{(2\pi)^D}\frac{f(l)}{((p+l)^2-m^2)l^2(k-l)^2}\ = \\
&= 2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{f(l)}{(x_1((p+l)^2-m^2)+x_2(k-l)^2+(1-x_1-x_2)l^2)^2}\ = \\
&= 2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{f(l)}{(l^2+2x_1pl+x_1(p^2-m^2)-2x_2kl+x_2k^2)^3}\ = \\
&= 2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{f(l)}{((l+x_1p-x_2k)^2-(x_1p-x_2k)^2)^3}\ = \\
&= -2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{f(l-x_1p+x_2k)}{(-l^2+(x_1p-x_2k)^2)^3} \qquad \text{(G.46)}
\end{aligned}$$

where (p^2-m^2) and k^2 are neglected and the inner momentum is shifted. $f(l)$ could be 1, l_μ, or $l_\mu l_\nu$. Therefore one obtains

$$I^{v2} = -2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(-l^2+(x_1p-x_2k)^2)^3}, \qquad \text{(G.47)}$$

$$\begin{aligned}
I_\mu^{v2} &= -2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{l_\mu-x_1p_\mu+x_2k_\mu}{(-l^2+(x_1p-x_2k)^2)^3}\ = \\
&= -2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{x_2k_\mu-x_1p_\mu}{(-l^2+(x_1p-x_2k)^2)^3}, \qquad \text{(G.48)}
\end{aligned}$$

$$\begin{aligned}
I_{\mu\nu}^{v2} &= -2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{(l_\mu-x_1p_\mu+x_2k_\mu)(l_\nu-x_1p_\nu+x_2k_\nu)}{(-l^2+(x_1p-x_2k)^2)^3}\ = \\
&= -2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{l_\mu l_\nu+(x_1p_\mu-x_2k_\mu)(x_1p_\nu-x_2k_\nu)}{(-l^2+(x_1p-x_2k)^2)^3}\ = \\
&= -2\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dl}{(2\pi)^D}\frac{l^2g_{\mu\nu}/D+(x_2k_\mu-x_1p_\mu)(x_2k_\nu-x_1p_\nu)}{(-l^2+(x_1p-x_2k)^2)^3}, \qquad \text{(G.49)}
\end{aligned}$$

thus

$$\begin{aligned}
I^{v2} &= 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D l}{(2\pi)^D} \frac{-1}{(-l^2+(x_1p-x_2k)^2)^3}, \\
I_k^{v2} &= 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D l}{(2\pi)^D} \frac{-x_2}{(-l^2+(x_1p-x_2k)^2)^3}, \\
I_p^{v2} &= 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D l}{(2\pi)^D} \frac{x_1}{(-l^2+(x_1p-x_2k)^2)^3}, \\
I_{kk}^{v2} &= 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D l}{(2\pi)^D} \frac{-x_2^2}{(-l^2+(x_1p-x_2k)^2)^3}, \\
I_{kp}^{v2} &= 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D l}{(2\pi)^D} \frac{x_1x_2}{(-l^2+(x_1p-x_2k)^2)^3}, \\
I_{pp}^{v2} &= 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D l}{(2\pi)^D} \frac{-x_1^2}{(-l^2+(x_1p-x_2k)^2)^3}, \\
I_g^{v2} &= \frac{2}{D}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D l}{(2\pi)^D} \frac{-l^2}{(-l^2+(x_1p-x_2k)^2)^3}.
\end{aligned} \tag{G.50}$$

Next one uses Eq. (G.5) and $(x_2k - x_1p)^2 = x_1p^2(x_1 + x_2\eta)$ (as well as the interchange of the integrations) to obtain

$$\begin{aligned}
I^{v2} &= \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{-1}{x_1(x_1+x_2\eta)}, \\
I_k^{v2} &= \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{-x_2}{x_1(x_1+x_2\eta)}, \\
I_p^{v2} &= \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{x_1}{x_1(x_1+x_2\eta)}, \\
I_{kk}^{v2} &= \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{-x_2^2}{x_1(x_1+x_2\eta)}, \\
I_{kp}^{v2} &= \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{x_1x_2}{x_1(x_1+x_2\eta)}, \\
I_{pp}^{v2} &= \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \frac{-x_1^2}{x_1(x_1+x_2\eta)},
\end{aligned} \tag{G.51}$$

where $\varepsilon = 0$ can be used, and

$$\begin{aligned}
I_g^{v2} &= \frac{2i\Gamma(D/2-1)\Gamma(2-D/2)}{(4\pi)^{D/2}D\Gamma(D/2)\Gamma(3)}(p^2)^{D/2-2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \left(x_1(x_1+x_2\eta)\right)^{D/2-2} = \\
&= \frac{i\Gamma(3-\varepsilon)\Gamma(\varepsilon)}{(4\pi)^{D/2}D\Gamma(2-\varepsilon)}(p^2)^{-\varepsilon}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \left(x_1(x_1+x_2\eta)\right)^{-\varepsilon} = \\
&= \frac{i(2-\varepsilon)\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}(4-2\varepsilon)\varepsilon}(p^2)^{-\varepsilon}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \Big(1 - \varepsilon\ln\left(x_1(x_1+x_2\eta)\right) + O(\varepsilon^2)\Big) = \\
&= \frac{i\Gamma(1+\varepsilon)}{4(4\pi)^{2-\varepsilon}\varepsilon}(p^2)^{-\varepsilon} - \frac{i}{2(4\pi)^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \ln\left(x_1(x_1+x_2\eta)\right) + O(\varepsilon)
\end{aligned} \tag{G.52}$$

where

$$\int_0^1 dx_2 \int_0^{1-x_2} dx_1 = \frac{1}{2} \tag{G.53}$$

is used. The integrals I^{v2}, I_k^{v2}, and I_{kk}^{v2}, however, cannot be regularized by keeping $\varepsilon \neq 0$. In the cases with at least one p the critical x_1 factor in the denominator cancels out, but this does not happen in the other cases. The partial fractioning of the integrand, therefore, results in a term $1/x_1$, and the integration diverges at the lower boundary without being tempered by some other term. One is lucky in realizing that these terms do not occur.

G.2.1 The scalar integral I_p^{v2}

For the first scalar integral one obtains

$$\begin{aligned}
I_p^{v2} &= \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 \int_0^{1-x_2} \frac{dx_1}{x_1+x_2\eta} = \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 \Big[\ln(x_1+x_2\eta)\Big]_0^{1-x_2} = \\
&= \frac{i}{(4\pi)^2p^2}\int_0^1 \ln\left(\frac{1-x_2+x_2\eta}{x_2\eta}\right) dx_2 = \frac{i}{(4\pi)^2p^2}\int_0^1 \ln\left(\frac{1-(1-\eta)x_2}{x_2\eta}\right) dx_2 = \\
&= \frac{i}{(4\pi)^2p^2}\Bigg[x_2 \ln\left(1-(1-\eta)x_2\right) + \\
&\qquad -\frac{1}{1-\eta}\ln\left(1-(1-\eta)x_2\right) - x_2 + \\
&\qquad -x_2\ln x_2 + x_2 - x_2 \ln\eta\Bigg]_0^1 = \\
&= \frac{i}{(4\pi)^2p^2}\left[\ln\eta - \frac{1}{1-\eta}\ln\eta - \ln\eta\right] = \frac{-i}{(4\pi)^2p^2}\frac{1}{1-\eta}\ln\eta. \qquad \text{(G.54)}
\end{aligned}$$

G.2.2 The scalar integral I_{kp}^{v2}

Next one calculates

$$\begin{aligned}
I_{kp}^{v2} &= \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 x_2 \int_0^{1-x_2} \frac{dx_1}{x_1+x_2\eta} = \frac{i}{(4\pi)^2p^2}\int_0^1 dx_2 x_2\Big[\ln(x_1+x_2\eta)\Big]_0^{1-x_2} = \\
&= \frac{i}{(4\pi)^2p^2}\int_0^1 x_2\ln\left(\frac{1-x_2+x_2\eta}{x_2\eta}\right) dx_2 = \frac{i}{(4\pi)^2p^2}\int_0^1 x_2\ln\left(\frac{1-(1-\eta)x_2}{x_2\eta}\right) dx_2 = \\
&= \frac{i}{(4\pi)^2p^2}\Bigg[\frac{1}{2}x_2^2\ln\left(1-(1-\eta)x_2\right) - \frac{1}{2(1-\eta)^2}\ln\left(1-(1-\eta)x_2\right) + \\
&\qquad -\frac{1}{4}x_2^2 - \frac{1}{2(1-\eta)}x_2 - \frac{1}{2}x_2^2\ln x_2 + \frac{1}{4}x_2^2 - \frac{1}{2}x_2^2\ln\eta\Bigg]_0^1 = \\
&= \frac{i}{(4\pi)^2p^2}\left(\frac{1}{2}\ln\eta - \frac{1}{2(1-\eta)^2}\ln\eta - \frac{1}{4} - \frac{1}{2(1-\eta)} + \frac{1}{4} - \frac{1}{2}\ln\eta\right) = \\
&= \frac{-i}{(4\pi)^2p^2}\left(\frac{1}{2(1-\eta)} + \frac{1}{2(1-\eta)^2}\ln\eta\right). \qquad \text{(G.55)}
\end{aligned}$$

G.2.3 The scalar integral I_{pp}^{v2}

Now one has

$$I_{pp}^{v2} = \frac{-i}{(4\pi)^2p^2}\int_0^1 dx_2\int_0^{1-x_2}\frac{x_1 dx_1}{x_1+x_2\eta} = \frac{-i}{(4\pi)^2p^2}\int_0^1 dx_2\int_0^{1-x_2}\left(1-\frac{x_2\eta}{x_1+x_2\eta}\right)dx_1 =$$

$$
\begin{aligned}
&= \frac{-i}{(4\pi)^2p^2}\int_0^1 dx_2\Big[x_1 - x_2\eta\ln(x_1+x_2\eta)\Big]_0^{1-x_2} = \\
&= \frac{-i}{(4\pi)^2p^2}\int_0^1 \left(1-x_2-x_2\eta\ln\left(1-(1-\eta)x_2\right)+x_2\eta\ln(x_2\eta)\right)dx_2 = \\
&= \frac{-i}{(4\pi)^2p^2}\Bigg[x_1-\frac{1}{2}x_2^2-\frac{1}{2}\eta x_2^2\ln\left(1-(1-\eta)x_2\right)+\frac{\eta}{2(1-\eta)^2}\ln\left(1-(1-\eta)x_2\right) + \\
&\qquad +\frac{1}{4}\eta x_2^2+\frac{\eta x_2}{2(1-\eta)}+\frac{1}{2}\eta x_2^2\ln x_2-\frac{1}{4}\eta x_2^2+\frac{1}{2}\eta x_2^2\ln\eta\Bigg]_0^1 = \qquad (G.56)\\
&= \frac{-i}{(4\pi)^2p^2}\left(1-\frac{1}{2}-\frac{1}{2}\eta\ln\eta+\frac{\eta}{2(1-\eta)^2}\ln\eta+\frac{1}{4}\eta+\frac{\eta}{2(1-\eta)}-\frac{1}{4}\eta+\frac{1}{2}\eta\ln\eta\right) = \\
&= \frac{-i}{(4\pi)^2p^2}\left(\frac{1}{2}+\frac{\eta}{2(1-\eta)}+\frac{\eta}{2(1-\eta)^2}\ln\eta\right) = \frac{-i}{(4\pi)^2p^2}\left(\frac{1}{2(1-\eta)}+\frac{\eta}{2(1-\eta)^2}\ln\eta\right).
\end{aligned}
$$

G.2.4 The scalar integral I_g^{v2}

For the last integral one begins with

$$
\begin{aligned}
\int_0^1 dx_2\int_0^{1-x_2}\ln x_1 dx_1 &= \int_0^1\Big[x_1\ln x_1 - x_1\Big]_0^{1-x_2}dx_2 = \\
&= \int_0^1\left((1-x_2)\ln(1-x_2)-(1-x_2)\right)dx_2 = \int_0^1\left(x_2\ln x_2 - x_2\right)dx_2 = \\
&= \left[\frac{1}{2}x_2^2\ln x_2-\frac{1}{4}x_2^2-\frac{1}{2}x_2^2\right]_0^1 = -\frac{1}{4}-\frac{1}{2} = -\frac{3}{4} \qquad (G.57)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 dx_2\int_0^{1-x_2}\ln(x_1+x_2\eta)dx_1 = \int_0^1 dx_2\Big[(x_1+x_2\eta)\ln(x_1+x_2\eta)-(x_1+x_2\eta)\Big]_0^{1-x_2} = \\
&= \int_0^1\Big(\left(1-(1-\eta)x_2\right)\ln\left(1-(1-\eta)x_2\right)-1+(1-\eta)x_2-x_2\eta\ln(x_2\eta)+x_2\eta\Big)dx_2 = \\
&= \Bigg[x_2\ln\left(1-(1-\eta)x_2\right)-\frac{1}{1-\eta}\ln\left(1-(1-\eta)x_2\right)-x_2 + \\
&\qquad -\frac{1}{2}(1-\eta)x_2^2\ln\left(1-(1-\eta)x_2\right)+\frac{1}{2(1-\eta)}\ln\left(1-(1-\eta)x_2\right) + \\
&\qquad\quad +\frac{1}{4}(1-\eta)x_2^2+\frac{1}{2}x_2-x_2+\frac{1}{2}(1-\eta)x_2^2-\frac{1}{2}\eta x_2^2\ln x_2 + \\
&\qquad\qquad +\frac{1}{4}\eta x_2^2-\frac{1}{2}\eta x_2^2\ln\eta+\frac{1}{2}\eta x_2^2\Bigg]_0^1 = \\
&= \ln\eta-\frac{1}{1-\eta}\ln\eta-1-\frac{1}{2}(1-\eta)\ln\eta+\frac{1}{2(1-\eta)}\ln\eta + \\
&\quad +\frac{1}{4}(1-\eta)+\frac{1}{2}-1+\frac{1}{2}(1-\eta)+\frac{1}{4}\eta-\frac{1}{2}\eta\ln\eta+\frac{1}{2}\eta = \qquad (G.58)\\
&= \frac{1}{2(1-\eta)}\Big(1+2(1-\eta)-2-(1-\eta)^2-(1-\eta)\eta\Big)\ln\eta-\frac{3}{4} = \\
&= \frac{1}{2(1-\eta)}\Big(1+2-2\eta-2-1+2\eta-\eta^2-\eta+\eta^2\Big)\ln\eta-\frac{3}{4} = \frac{-\eta}{2(1-\eta)}\ln\eta-\frac{3}{4}.
\end{aligned}
$$

One therefore obtains

$$\begin{aligned} I_g^{v2} &= \frac{i\Gamma(1+\varepsilon)}{4(4\pi)^{2-\varepsilon}\varepsilon}(p^2)^{-\varepsilon} - \frac{i}{2(4\pi)^2}\int_0^1 dx_2 \int_0^{1-x_2} dx_1 \ln\left(x_1(x_1+x_2\eta)\right) = \\ &= \frac{i\Gamma(1+\varepsilon)}{4(4\pi)^{2-\varepsilon}\varepsilon}(p^2)^{-\varepsilon} + \frac{i}{(4\pi)^2}\left(\frac{3}{4}+\frac{\eta}{4(1-\eta)}\ln\eta\right). \end{aligned} \tag{G.59}$$

For an expansion in η, finally, the integrals (for $p^2 = m^2$) result in

$$\begin{aligned} I_p^{v2} &= \frac{-i}{(4\pi)^2 m^2}\left\{\ln\eta + \eta\ln\eta + O(\eta^2)\right\}, \\ I_{kp}^{v2} &= \frac{-i}{(4\pi)^2 m^2}\left\{\left(\frac{1}{2}+\frac{1}{2}\ln\eta\right) + \left(\frac{1}{2}+\ln\eta\right)\eta + O(\eta^2)\right\}, \\ I_{pp}^{v2} &= \frac{-i}{(4\pi)^2 m^2}\left\{\frac{1}{2} + \left(\frac{1}{2}+\frac{1}{2}\ln\eta\right)\eta + O(\eta^2)\right\}, \\ I_g^{v2} &= \frac{i\Gamma(1+\varepsilon)\mu^{-2\varepsilon}}{4(4\pi)^{2-\varepsilon}}c_s + \frac{i}{(4\pi)^2}\left\{\frac{3}{4}+\frac{1}{4}\eta\ln\eta + O(\eta^2)\right\}. \end{aligned} \tag{G.60}$$

G.3 Integral class for the gluon self energy diagrams

The self energy correction at next-to-leading order consists of four different contributions: the massive and massless quark loops, the gluon loop and the ghost loop (for the Feynman gauge). The integral classes needed for the calculation of these contributions are presented in the following.

G.3.1 Integral class for the massive quark loop diagram

One begins with the generic integral in Sec. H.2.1

$$\begin{aligned} L_{1f} &= \int \frac{d^D l}{(2\pi)^D}\frac{f(l)}{((k+l)^2-m^2)(l^2-m^2)} = \\ &= \int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{f(l)}{(l^2+2xkl+xk^2-m^2)^2} = \\ &= \int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{f(l)}{((l+xk)^2+x(1-x)k^2-m^2)^2} = \\ &= \int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{f(l-xk)}{(-l^2+m^2-x(1-x)k^2)^2}. \end{aligned} \tag{G.61}$$

The scalar integral L_1 and the integrals given in the covariant representations

$$L_{1\alpha} = L_{1k}k_\alpha, \qquad L_{1\alpha\beta} = L_{1g}g_{\alpha\beta} + L_{1kk}k_\alpha k_\beta \tag{G.62}$$

will be calculated in the following, using the integration rule

$$\int_0^1 x^n dx = \frac{1}{n+1}, \qquad \int_0^1 x^n(1-x)dx = \frac{1}{(n+1)(n+2)}. \tag{G.63}$$

and the expansion in k^2/m^2 at the end. For the basic scalar integral (with $f(l) = 1$) one obtains

$$\begin{aligned}
L_1 &= \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{1}{(-l^2 + m^2 - x(1-x)k^2)^2} \;= \\
&= \frac{i\Gamma(2-D/2)}{(4\pi)^{D/2}\Gamma(2)} \int_0^1 (m^2 - x(1-x)k^2)^{D/2-2} \;= \\
&= \frac{i\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} (m^2)^{-\varepsilon} \int_0^1 \left(1 - \varepsilon \ln\left(1 - \frac{x(1-x)k^2}{m^2}\right)\right) dx \;= \\
&\approx \frac{i\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} (m^2)^{-\varepsilon} \int_0^1 \left(1 + \varepsilon \frac{x(1-x)k^2}{m^2}\right) dx \;= \\
&= \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}} (m^2)^{-\varepsilon} \left(\frac{1}{\varepsilon} + \frac{k^2}{6m^2}\right) + O\left(\frac{k^4}{m^4}\right).
\end{aligned} \tag{G.64}$$

The next integral is given by

$$\begin{aligned}
L_{1k} &= \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{-x}{(-l^2 + m^2 - x(1-x)k^2)^2} \;= \\
&= -\frac{i\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} (m^2)^{-\varepsilon} \int_0^1 \left(1 - \varepsilon \ln\left(1 - \frac{x(1-x)k^2}{m^2}\right)\right) x\, dx \;= \\
&= -\frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}} (m^2)^{-\varepsilon} \left(\frac{1}{2\varepsilon} + \frac{k^2}{12m^2}\right) + O\left(\frac{k^4}{m^4}\right).
\end{aligned} \tag{G.65}$$

Next one obtains

$$\begin{aligned}
L_{1kk} &= \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{x^2}{(-l^2 + m^2 - x(1-x)k^2)^2} \;= \\
&= \frac{i\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} (m^2)^{-\varepsilon} \int_0^1 \left(1 - \varepsilon \ln\left(1 - \frac{x(1-x)k^2}{m^2}\right)\right) x^2 dx \;= \\
&= \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}} (m^2)^{-\varepsilon} \left(\frac{1}{3\varepsilon} + \frac{k^2}{20m^2}\right) + O\left(\frac{k^4}{m^4}\right).
\end{aligned} \tag{G.66}$$

Finally, one obtains

$$\begin{aligned}
L_{1g} &= \frac{1}{D} \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{l^2}{(-l^2 + m^2 - x(1-x)k^2)^2} \;= \\
&= -\frac{i\Gamma(D/2+1)\Gamma(1-D/2)}{(4\pi)^{D/2}\Gamma(D/2)D} (m^2)^{D/2-1} \int_0^1 dx \left(1 - \frac{x(1-x)k^2}{m^2}\right)^{D/2-1} \;= \\
&= -\frac{i\Gamma(3-\varepsilon)\Gamma(\varepsilon-1)}{(4\pi)^{2-\varepsilon}\Gamma(2-\varepsilon)(4-2\varepsilon)} (m^2)^{1-\varepsilon} \int_0^1 dx \left(1 - \frac{x(1-x)k^2}{m^2}\right)^{1-\varepsilon}.
\end{aligned} \tag{G.67}$$

For the integrand one can use the expansion

$$\left(1 - \frac{x(1-x)k^2}{m^2}\right)^{1-\varepsilon} \approx 1 - (1-\varepsilon)\frac{x(1-x)k^2}{m^2}, \tag{G.68}$$

and with $\Gamma(\varepsilon-1) = -\Gamma(1+\varepsilon)/(\varepsilon(1-\varepsilon))$

$$\begin{aligned}
L_{1g} &= \frac{-i\Gamma(\varepsilon-1)}{2(4\pi)^{2-\varepsilon}}(m^2)^{1-\varepsilon}\left(1-(1-\varepsilon)\frac{k^2}{6m^2}\right)+O\left(\frac{k^4}{m^4}\right) = \\
&= \frac{i\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(m^2)^{1-\varepsilon}\left(\frac{1}{\varepsilon}+1\right)\left(1-\frac{k^2}{6m^2}+\varepsilon\frac{k^2}{6m^2}\right)+O\left(\frac{k^4}{m^4}\right) = \\
&= \frac{i\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(m^2)^{1-\varepsilon}\left\{\left(1-\frac{k^2}{6m^2}\right)\frac{1}{\varepsilon}+1\right\}+O\left(\frac{k^4}{m^4}\right). \qquad \text{(G.69)}
\end{aligned}$$

G.3.2 Integral class for the massless quark loop diagram

The generic integral in Sec. H.2.2 is given by

$$L^0_{1f} = \int_0^1 dx \int \frac{d^Dl}{(2\pi)^D}\frac{f(l-xk)}{(-l^2-x(1-x)k^2)^2}. \qquad \text{(G.70)}$$

For the different special cases one obtains

$$\begin{aligned}
L^0_1 &= \int_0^1 dx \int \frac{d^Dl}{(2\pi)^D}\frac{1}{(-l^2-x(1-x)k^2)^2} = \frac{i\Gamma(2-D/2)}{(4\pi)^{D/2}\Gamma(2)}\int_0^1(-x(1-x)k^2)^{D/2-2}dx = \\
&= \frac{i\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\int_0^1 x^{-\varepsilon}(1-x)^{-\varepsilon}dx = \frac{i\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} = \\
&= \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\left\{\frac{1}{\varepsilon}+2\right\} \qquad \text{(G.71)}
\end{aligned}$$

and in the same manner

$$\begin{aligned}
L^0_{1k} &= -\frac{i\Gamma(1+\varepsilon)(-k^2)^{-\varepsilon}}{(4\pi)^{2-\varepsilon}\varepsilon}\frac{\Gamma(2-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(3-2\varepsilon)} = -\frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\left\{\frac{1}{2\varepsilon}+1\right\}, \\
L^0_{1kk} &= \frac{i\Gamma(1+\varepsilon)(-k^2)^{-\varepsilon}}{(4\pi)^{2-\varepsilon}\varepsilon}\frac{\Gamma(3-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(4-2\varepsilon)} = \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\left\{\frac{1}{3\varepsilon}+\frac{13}{18}\right\}, \\
L^0_{1k}+L^0_{1kk} &= -\frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\left\{\frac{1}{6\varepsilon}+\frac{5}{18}\right\}. \qquad \text{(G.72)}
\end{aligned}$$

For L^0_{1g} one obtains

$$\begin{aligned}
L^0_{1g} &= \frac{1}{D}\int_0^1 dx \int \frac{d^Dl}{(2\pi)^D}\frac{l^2}{(-l^2-x(1-x)k^2)^2} = \\
&= -\frac{i\Gamma(D/2+1)\Gamma(1-D/2)}{(4\pi)^{D/2}\Gamma(D/2)D}\int_0^1(-x(1-x)k^2)^{D/2-1}dx = \\
&= -\frac{i\Gamma(3-\varepsilon)\Gamma(\varepsilon-1)}{(4\pi)^{2-\varepsilon}\Gamma(2-\varepsilon)(4-2\varepsilon)}(-k^2)^{1-\varepsilon}\int_0^1 x^{1-\varepsilon}(1-x)^{1-\varepsilon}dx = \\
&= -\frac{\Gamma(\varepsilon-1)(-k^2)^{1-\varepsilon}}{2(4\pi)^{2-\varepsilon}}\frac{\Gamma(2-\varepsilon)^2}{\Gamma(4-2\varepsilon)} = \frac{i\Gamma(1+\varepsilon)(-k^2)^{1-\varepsilon}}{2(4\pi)^{2-\varepsilon}(1-\varepsilon)\varepsilon}\frac{\Gamma(2-\varepsilon)^2}{\Gamma(4-2\varepsilon)} = \\
&= \frac{i\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(-k^2)^{1-\varepsilon}\left\{\frac{1}{6\varepsilon}+\frac{4}{9}\right\}. \qquad \text{(G.73)}
\end{aligned}$$

G.3.3 Integral class for the gluon loop diagram

One applies the Passarino–Veltman method. In making the ansatz

$$L_{2\alpha\beta} =: L_{2g}k^2 g_{\alpha\beta} + L_{2kk}k_\alpha k_\beta \tag{G.74}$$

one solves for L_{2g} and L_{2kk} by contracting with $g^{\alpha\beta}$ and $k^\alpha k^\beta$, respectively,

$$\begin{aligned} Dk^2 L_{2g} + k^2 L_{2kk} &= g^{\alpha\beta}L_{2\alpha\beta} =: k^2 L_2^g, & \text{(G.75)}\\ k^4 L_{2g} + k^4 L_{2kk} &= k^\alpha k^\beta L_{2\alpha\beta} =: k^4 L_2^{kk}. & \text{(G.76)} \end{aligned}$$

This results in

$$L_{2g} = \frac{L_2^g - L_2^{kk}}{D-1}, \qquad L_{2kk} = \frac{DL_2^{kk} - L_2^g}{D-1}. \tag{G.77}$$

For the first contracted integral one obtains

$$\begin{aligned} k^2 L_2^g &= g_s^2 C_A \delta_{ab} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2} k^2 (k+l)^2 \Big[D\left(4k^2 + l^2 + (k+l)^2\right) + \\ &\qquad + (D-6)k^2 + 2(2D-3)kl + 2(2D-3)l^2 \Big] = \\ &= g_s^2 C_A \delta_{ab} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2} k^2 (k+l)^2 \Big[D\left(4k^2 + l^2 + (k+l)^2\right) + \\ &\qquad -(D+3)k^2 + (2D-3)l^2 + (2D-3)(k+l)^2 \Big] = \\ &= g_s^2 C_A \delta_{ab} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2} k^2 (k+l)^2 \Big[3(D-1)k^2 + 3(D-1)l^2 + 3(D-1)(k+l)^2 \Big] = \\ &= 3(D-1) g_s^2 C_A \delta_{ab} \int \frac{d^D l}{(2\pi)^D} \frac{k^2 + l^2 + (k+l)^2}{l^2 (k+l)^2} = \\ &= \frac{3i(D-1)g_s^2}{(4\pi)^{D/2}} (-k^2)^{D/2-2} C_A \delta_{ab} G(1,1) k^2 \end{aligned} \tag{G.78}$$

where $G(n_1, n_2)$ is the standard massless integral with denominator powers n_1 and n_2 (cf. Sec. 4.1.1). $G(1,1)$ is the only surviving integral while $G(1,0)$ and $G(0,1)$ vanish. For the second contracted integral one obtains

$$\begin{aligned} k^4 L_2^{kk} &= g_s^2 C_A \delta_{ab} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (k+l)^2} \Big[\left(4k^2 + l^2 + (k+l)^2\right) k^2 + \\ &\qquad + (D-6)k^4 + 2(2D-3)(kl)k^2 + 2(2D-3)(kl)^2 \Big] = \\ &= \frac{1}{2} g_s^2 C_A \delta_{ab} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (k+l)^2} \Big[8k^4 + 2k^2 l^2 + 2k^2 (k+l)^2 + \\ &\qquad + 2(D-6)k^4 + 2(2D-3)k^2 (k+l)^2 - 2(2D-3)k^4 - 2(2D-3)k^2 l^2 + \\ &\qquad + (2D-3)(k+l)^4 - 2(2D-3)k^2 (k+l)^2 - 2(2D-3) l^2 (k+l)^2 + \\ &\qquad + (2D-3)k^4 + 2(2D-3)k^2 l^2 + (2D-3) l^4 \Big] = \\ &= \frac{1}{2} g_s^2 C_A \delta_{ab} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (k+l)^2} \times \\ &\qquad \times \Big[-k^4 + 2k^2 l^2 + 2k^2 (k+l)^2 + (2D-3)\left(l^2 - 2l^2 (k+l)^2 + l^4\right) \Big] = \\ &= -\frac{i g_s^2 C_A}{2(4\pi)^{D/2}} (-k^2)^{D/2-2} \delta_{ab} G(1,1) k^4. \end{aligned} \tag{G.79}$$

With

$$G(1,1) = \frac{-2Q_1}{(D-4)(D-3)}, \qquad Q_1 = \Gamma(1+\varepsilon)\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)} \tag{G.80}$$

one obtains

$$\begin{aligned} L_{2g} &= \frac{ig_s^2 C_A}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\left(\frac{6(D-1)+1}{D-1}\right)G(1,1) \;= \\ &= \frac{ig_s^2 C_A}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\frac{6D-5}{D-1}G(1,1) \;= \\ &= \frac{ig_s^2 C_A\Gamma(1+\varepsilon)}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\frac{-2(6D-5)}{(D-4)(D-3)(D-1)}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)} \;= \\ &= \frac{ig_s^2 C_A\Gamma(1+\varepsilon)}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\left\{\frac{19}{3\varepsilon}+\frac{116}{9}+O(\varepsilon)\right\} \end{aligned} \tag{G.81}$$

and

$$\begin{aligned} L_{2kk} &= -\frac{ig_s^2 C_A}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\left(\frac{D+6(D-1)}{D-1}\right)G(1,1) \;= \\ &= -\frac{ig_s^2 C_A}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\frac{7D-6}{D-1}G(1,1) \;= \\ &= \frac{ig_s^2 C_A\Gamma(1+\varepsilon)}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\frac{2(7D-6)}{(D-4)(D-3)(D-1)}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(1-2\varepsilon)} \;= \\ &= \frac{ig_s^2 C_A\Gamma(1+\varepsilon)}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\left\{-\frac{22}{3\varepsilon}-\frac{134}{9}+O(\varepsilon)\right\}. \end{aligned} \tag{G.82}$$

G.3.4 Integral class for the ghost loop diagram

Again, the Passarino–Veltman method is employed, calculating the coefficient functions in

$$L_{3\alpha\beta} = L_{3g}k^2 g_{\alpha\beta} + L_{3kk}k_\alpha k_\beta \tag{G.83}$$

again by contraction. The result reads

$$\begin{aligned} k^2 L_3^g &= g_s^2 C_A\delta_{ab}\int\frac{d^D l}{(2\pi)^D}\frac{(kl+l^2)}{l^2(k+l)^2} \;=\; \frac{1}{2}g_s^2 C_A\delta_{ab}\int\frac{d^D l}{(2\pi)^D}\frac{((k+l)^2-k^2+l^2)}{l^2(k+l)^2} \;= \\ &= -\frac{ig_s^2 C_A}{(4\pi)^{D/2}}\delta_{ab}(-k^2)^{D/2-2}G(1,1)k^2, \end{aligned} \tag{G.84}$$

$$\begin{aligned} k^4 L_3^{kk} &= g_s^2 C_A\delta_{ab}\int\frac{d^D l}{(2\pi)^D}\frac{(kl)(k^2+kl)}{l^2(k+l)^2} \;= \\ &= \frac{1}{4}g_s^2 C_A\delta_{ab}\int\frac{d^D l}{(2\pi)^D}\frac{((k+l)^2-k^2-l^2)((k+l)^2+k^2-l^2)}{l^2(k+l)^2} \;= \\ &= -\frac{ig_s^2 C_A}{4(4\pi)^{D/2}}\delta_{ab}(-k^2)^{D/2-2}G(1,1)k^4. \end{aligned} \tag{G.85}$$

One obtains

$$L_{3g} \;=\; \frac{L_3^g - L_3^{kk}}{D-1} \;=\; -\frac{ig_s^2 C_A}{4(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\frac{1}{D-1}G(1,1) \;=$$

$$= -\frac{ig_s^2 C_A \Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\delta_{ab}\left\{\frac{1}{6\varepsilon}+\frac{4}{9}+O(\varepsilon)\right\}, \quad \text{(G.86)}$$

$$L_{3kk} = \frac{DL_3^{kk}-L_3^g}{D-1} = -\frac{ig_s^2 C_A}{2(4\pi)^{D/2}}(-k^2)^{D/2-2}\delta_{ab}\frac{D-2}{D-1}G(1,1) =$$
$$= -\frac{ig_s^2 C_A \Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\delta_{ab}\left\{\frac{1}{3\varepsilon}+\frac{5}{9}+O(\varepsilon)\right\}. \quad \text{(G.87)}$$

Like in the case of the gluon loop diagram, one can also use the direct method starting from Eq. (H.52) to obtain

$$L_3 = g_s^2 C_A \delta_{ab}\int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{l_\alpha (k+l)_\beta}{(l^2+2xkl+xk^2)^2} =$$
$$= g_s^2 C_A \delta_{ab}\int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{(l-xk)_\alpha (l+(1-x)k)_\beta}{(-l^2-x(1-x)k^2)^2} =$$
$$= g_s^2 C_A \delta_{ab}\int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{-x(1-x)k_\alpha k_\beta + l^2 g_{\alpha\beta}/D}{(-l^2-x(1-x)k^2)^2} =$$
$$= g_s^2 C_A \delta_{ab}\int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{-x(1-x)k_\alpha k_\beta - x(1-x)k^2 g_{\alpha\beta}/(D-2)}{(-l^2-x(1-x)k^2)^2} =$$
$$= -\frac{ig_s^2\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\left(2k_\alpha k_\beta + \frac{k^2 g_{\alpha\beta}}{1-\varepsilon}\right)\int_0^1 (x(1-x))^{1-\varepsilon}dx =$$
$$= -\frac{ig_s^2\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\left(2k_\alpha k_\beta + \frac{k^2 g_{\alpha\beta}}{1-\varepsilon}\right)\frac{\Gamma(2-\varepsilon)^2}{\Gamma(4-2\varepsilon)} \quad \text{(G.88)}$$

G.4 Integral class for the quark self energy diagram

The generic integral for the quark self energy diagram in Sec. H.3 with quark momentum p is given by

$$S_f(p) = \int \frac{d^D l}{(2\pi)^D}\frac{f(l)}{((p+l)^2-m^2)l^2} = \int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{f(l)}{(l^2+2xpl+xp^2-xm^2)^2} =$$
$$= \int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{f(l)}{((l+xp)^2+x(1-x)p^2-xm^2)^2} =$$
$$= \int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{f(l-xp)}{(-l^2+xm^2-x(1-x)p^2)^2}. \quad \text{(G.89)}$$

If one changes from p to $p+k$, one can expand in k to obtain

$$m_x = xm^2 - x(1-x)(p+k)^2 \approx xm^2 - x(1-x)p^2 - 2x(1-x)pk =$$
$$= x\omega p^2 + x^2p^2 - x(1-x)\eta p^2 = x\left((1-\eta)x+\omega+\eta\right)p^2. \quad \text{(G.90)}$$

One now defines pure integral expressions $\sigma(\eta,\omega)$ and $\sigma_p(\eta,\omega)$ by

$$S(p+k) = \int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{1}{(-l^2+x((1-\eta)x+\omega+\eta)p^2)^2} =: \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(p^2)^{-\varepsilon}\sigma(\eta,\omega),$$
$$S_p(p+k) = \int_0^1 dx \int \frac{d^D l}{(2\pi)^D}\frac{-x}{(-l^2+x((1-\eta)x+\omega+\eta)p^2)^2} =: -\frac{i\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(p^2)^{-\varepsilon}\sigma_p(\eta,\omega).$$
$$\text{(G.91)}$$

One now proceeds to calculate the two functions $\sigma(\eta,\omega)$ and $\sigma_p(\eta,\omega)$. Introducing the substitutions

$$\tilde{\omega} := \frac{\omega+\eta}{1-\eta}, \quad 1+\tilde{\omega} = \frac{1+\omega}{1-\eta} \tag{G.92}$$

and

$$(1-\eta)x+\omega+\eta = (1-\eta)(x+\tilde{\omega}) \tag{G.93}$$

one obtains

$$\begin{aligned}
\sigma(\eta,\omega) &= \frac{1}{\varepsilon}\int_0^1 x^{-\varepsilon}\left((1-\eta)(x+\tilde{\omega})\right)^{-\varepsilon}dx = \\
&= (1-\eta)^{-\varepsilon}\frac{1}{\varepsilon}\int_0^1 x^{-\varepsilon}\left(1-\varepsilon\ln(x+\tilde{\omega})\right)dx = \\
&= (1-\eta)^{-\varepsilon}\left[\frac{1}{\varepsilon}\int_0^1 x^{-\varepsilon}dx - \int_0^1 \ln(x+\tilde{\omega})dx\right] = \\
&= (1-\eta)^{-\varepsilon}\left[\frac{1}{\varepsilon(1-\varepsilon)} - (1+\tilde{\omega})\ln(1+\tilde{\omega}) + (1+\tilde{\omega}) + \tilde{\omega}\ln\tilde{\omega} - \tilde{\omega}\right] = \\
&= (1-\eta)^{-\varepsilon}\left[\frac{1}{\varepsilon} + 1 + \frac{1}{1-\eta}\left((\eta+\omega)\ln\left(\frac{\eta+\omega}{1-\eta}\right) - (1+\omega)\left(\frac{1+\omega}{1-\eta}\right)\right) + 1\right] = \\
&= \frac{1}{\varepsilon} + 2 + \frac{1}{1-\eta}\left((\eta+\omega)\ln\left(\frac{\eta+\omega}{1-\eta}\right) - (1+\omega)\left(\frac{1+\omega}{1-\eta}\right) - (1-\eta)\ln(1-\eta)\right) = \\
&= \frac{1}{\varepsilon} + 2 + \frac{1}{1-\eta}\left((\eta+\omega)\ln(\eta+\omega) - (1+\omega)\ln(1+\omega)\right) = \\
&= \frac{1}{\varepsilon} + 2 - \ln(\eta+\omega) + \frac{1+\omega}{1-\eta}\ln\left(\frac{\eta+\omega}{1+\omega}\right).
\end{aligned} \tag{G.94}$$

Using

$$\int_0^1 x\ln(x+\tilde{\omega})dx = \frac{1}{2}(1-\tilde{\omega}^2)\ln(1+\tilde{\omega}) + \frac{1}{2}\tilde{\omega}^2\ln\tilde{\omega} + \frac{1}{2}\tilde{\omega} - \frac{1}{4} \tag{G.95}$$

in the second expression involving $\sigma_p(\eta,\omega)$ one obtains

$$\begin{aligned}
\sigma_p(\eta,\omega) &= \frac{2}{\varepsilon}\int_0^1 x^{1-\varepsilon}\left((1-\eta)(x+\tilde{\omega})\right)^{-\varepsilon}dx = \\
&= 2(1-\eta)^{-\varepsilon}\left[\frac{1}{\varepsilon}\int_0^1 x^{1-\varepsilon}dx - \int_0^1 x\ln(x+\tilde{\omega})dx\right] = \\
&= 2(1-\eta)^{-\varepsilon}\left[\frac{1}{\varepsilon(2-\varepsilon)} - \frac{1}{2}(1-\tilde{\omega}^2)\ln(1+\tilde{\omega}) - \frac{1}{2}\tilde{\omega}^2\ln\tilde{\omega} - \frac{1}{2}\tilde{\omega} + \frac{1}{4}\right] = \\
&= (1-\eta)^{-\varepsilon}\left[\frac{1}{\varepsilon} + \frac{1}{2} - (1-\tilde{\omega}^2)\ln(1+\tilde{\omega}) - \tilde{\omega}^2\ln\tilde{\omega} - \tilde{\omega} + \frac{1}{2}\right] = \\
&= (1-\eta)^{-\varepsilon}\left[\frac{1}{\varepsilon} + 1 - \tilde{\omega} - \ln(1+\tilde{\omega}) - \tilde{\omega}^2\ln\left(\frac{\tilde{\omega}}{1+\tilde{\omega}}\right)\right] = \\
&= (1-\eta)^{-\varepsilon}\left[\frac{1}{\varepsilon} + 1 - \frac{\eta+\omega}{1-\eta} - \ln\left(\frac{1+\omega}{1-\eta}\right) - \frac{(\eta+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right)\right] = \\
&= \frac{1}{\varepsilon} + 2 - \frac{1+\omega}{1-\eta} - \ln(1+\omega) - \frac{(\eta+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right).
\end{aligned} \tag{G.96}$$

In addition one needs

$$2\sigma(\eta,\omega) - \sigma_p(\eta,\omega) =$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon} + 2 + \frac{1+\omega}{1-\eta} + \ln(1+\omega) - 2\ln(\eta+\omega) + \\
&\qquad\qquad +2\frac{1+\omega}{1-\eta}\ln\left(\frac{\eta+\omega}{1+\omega}\right) + \frac{(\eta+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right) = \\
&= \frac{1}{\varepsilon} + 2 + \frac{1+\omega}{1-\eta} - \ln(1+\omega) - 2\ln\left(\frac{\eta+\omega}{1+\omega}\right) + \\
&\qquad\qquad +2\frac{1+\omega}{1-\eta}\ln\left(\frac{\eta+\omega}{1+\omega}\right) + \frac{(\eta+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right) = \\
&= \frac{1}{\varepsilon} + 2 + \frac{1+\omega}{1-\eta} - \ln(1+\omega) + 2\frac{\eta+\omega}{1-\eta}\ln\left(\frac{\eta+\omega}{1+\omega}\right) + \frac{(\eta+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right) = \\
&= \frac{1}{\varepsilon} + 2 + \frac{1+\omega}{1-\eta} - \ln(\eta+\omega) + \ln\left(\frac{\eta+\omega}{1+\omega}\right) + \\
&\qquad\qquad +2\frac{\eta+\omega}{1-\eta}\ln\left(\frac{\eta+\omega}{1+\omega}\right) + \frac{(\eta+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right) = \\
&= \frac{1}{\varepsilon} + 2 + \frac{1+\omega}{1-\eta} - \ln(\eta+\omega) + \frac{(1+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right). \qquad \text{(G.97)}
\end{aligned}
$$

Appendix H

The soft part of the quark self energy

This appendix collects calculations for the two-loop correction of the quark self energy where one of the gluons is considered as soft. These calculations are not contained in the main part because they could be replaced by a more elegant way of calculation, namely the exact cut of the corresponding quark line and the calculation via the residue theorem (see Chapter 5.3). Nevertheless, it is worthwile to look at these previous calculations because they already show the main features of the soft part calculations.

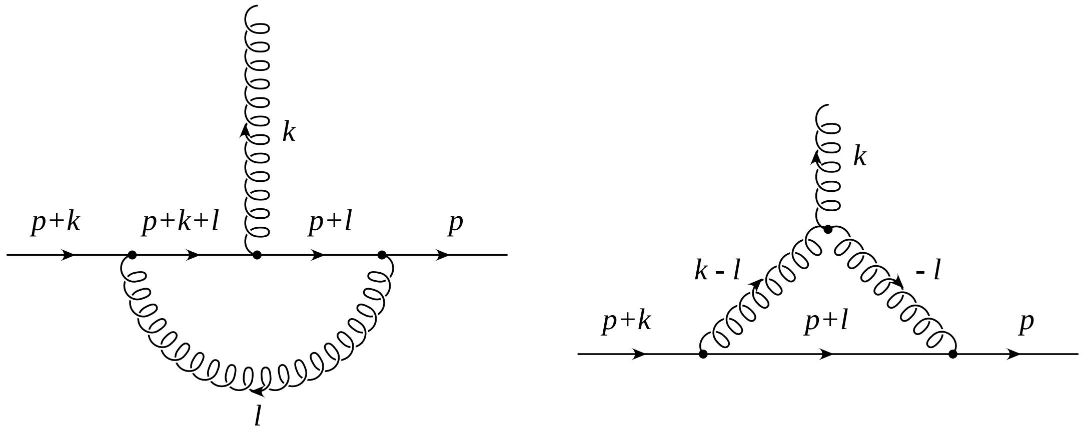


Figure H.1: the two diagrams contribution to the effective vertex given by the abelian diagram (v1, left side) and the non-abelian diagram (v2, right side)

H.1 The effective vertex corrections

Because of the assumption that one of the gluons is soft and the emitting quark remains on-shell, the calculations can be simplified to a calculation of an effective vertex correction which afterwards can be combined with the remaining (common) elements of the diagrams. This is done in the following subsections for the two diagrams in Fig. H.1.

H.1.1 The abelian effective vertex correction

Using the conventional QCD Feynman rules, the integral for the left diagram (v1) of Fig. H.1 is given by

$$\begin{aligned}
-ig_s\Lambda^\beta_{1b} &= \int \frac{d^Dl}{(2\pi)^D}(-ig_s\gamma_\alpha T_a)\frac{i}{\not p+\not l-m}(-ig_s\gamma^\beta T_b)\frac{i}{\not p+\not k+\not l-m}(-ig_s\gamma^\alpha T_a)\frac{-i}{l^2} = \\
&= -g_s^3 T_aT_bT_a\int\frac{d^Dl}{(2\pi)^D}\frac{\gamma_\alpha(\not p+\not l+m)\gamma^\beta(\not p+\not k+\not l+m)\gamma^\alpha}{((p+l)^2-m^2)((p+k+l)^2-m^2)l^2} = \\
&= -g_s^3 T_aT_bT_a\Big[\gamma_\alpha(\not p+m)\gamma^\beta(\not p+\not k+m)\gamma^\alpha I^{v1}+\gamma_\alpha(\not p+m)\gamma^\beta\gamma^\nu\gamma^\alpha I^{v1}_\nu + \\
&\qquad +\gamma_\alpha\gamma^\mu\gamma^\beta(\not p+\not k+m)\gamma^\alpha I^{v1}_\mu+\gamma_\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\alpha I^{v1}_{\mu\nu}\Big].
\end{aligned}\tag{H.1}$$

One therefore has to calculate scalar, vector, and tensor integrals with the same denominator. The combination of this effective vertex with the quark propagator (only the numerator is taken for the moment) and second vertex reads

$$\begin{aligned}
\Lambda^\beta_{1b}(\not p+\not k+m)\gamma_\beta T_b &= -ig_s^2C_1\Big[\gamma_\alpha(\not p+m)\gamma^\beta(\not p+\not k+m)\gamma^\alpha(\not p+\not k+m)\gamma_\beta I^{v1} + \\
&\quad +\gamma_\alpha(\not p+m)\gamma^\beta\gamma^\nu\gamma^\alpha(\not p+\not k+m)\gamma_\beta I^{v1}_\nu + \\
&\quad +\gamma_\alpha\gamma^\mu\gamma^\beta(\not p+\not k+m)\gamma^\alpha(\not p+\not k+m)\gamma_\beta I^{v1}_\mu + \\
&\quad +\gamma_\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\alpha(\not p+\not k+m)\gamma_\beta I^{v1}_{\mu\nu}\Big]
\end{aligned}\tag{H.2}$$

where $C_1 = T_aT_bT_aT_b = C_F(C_F - C_A/2)$ is the colour factor. The vector and scalar integrals are expressed in terms of scalar integrals,

$$\begin{aligned}
I^{v1}_\mu &= I^{v1}_k k_\mu + I^{v1}_p p_\mu, \\
I^{v1}_{\mu\nu} &= I^{v1}_g g_{\mu\nu} + I^{v1}_{kk}k_\mu k_\nu + I^{v1}_{kp}(k_\mu p_\nu + p_\mu k_\nu) + I^{v1}_{pp}p_\mu p_\nu.
\end{aligned}\tag{H.3}$$

Computer packages in MATHEMATICA have been developed to deal with Dirac structures between spinors. Because of the fact that the quark is close to being on-shell, the numerator $\Lambda^\beta_{1b}(\not p+\not k+m)\gamma_\beta T_b$ is taken placed between spinors of momentum p, namely $\bar u(p)\Lambda^\beta_{1b}(\not p+\not k+m)\gamma_\beta T_b u(p)$. One can use the Dirac equations $(\not p - m)u(p) = 0$ and $\bar u(p)(\not p - m) = 0$ to simplify the expressions. This is done in the package `effvera1.m`. The result in terms of scalar integrals is given by

$$\begin{aligned}
&\Lambda^\beta_{1b}(\not p+\not k+m)\gamma_\beta T_b = \\
&= -ig_s^2C_1\Big\{2m(D-2)^2I^{v1}_g + 4mk^2(I^{v1}+I^{v1}_k+I^{v1}_{kk}-2I^{v1}_{kp}) \\
&\quad +8m(kp)(2I^{v1}+I^{v1}_p-I^{v1}_{pp}) + 4m^3(2I^{v1}+2I^{v1}_p-I^{v1}_{pp}) + \\
&\quad -\Big((D-2)^3I^{v1}_g + 8k^2(I^{v1}_k+I^{v1}_{kk}) + 8(kp)(I^{v1}+2I^{v1}_k+I^{v1}_p+I^{v1}_{kk}+I^{v1}_{kp}) + \\
&\qquad +4m^2(2I^{v1}-2I^{v1}_k+5I^{v1}_p+2I^{v1}_{kp})\Big)\not k\Big\}.
\end{aligned}\tag{H.4}$$

There is an ambiguity in the representation, given by

$$\bar u(p)(m\not k)u(p) = \frac{1}{2}\bar u(p)(\not p\not k+\not k\not p)u(p) = \bar u(p)(kp)u(p). \tag{H.5}$$

Therefore, one can effectively replace $\not{k}$ by kp/m. In doing so one obtains

$$\begin{aligned}\frac{1}{m}\Lambda^{\beta}_{1b}(\not{p}+\not{k}+m)\gamma_\beta T_b &= -ig_s^2 C_1\Big\{2(D-2)^2 I_g^{v1} + 4m^2(2I^{v1}+2I_p^{v1}-I_{pp}^{v1}) + \\ &+\left(\frac{1}{2}(D-2)^3 I_g^{v1} - 2m^2(2I^{v1}+2I_k^{v1}-3I_p^{v1}-2I_{kp}^{v1}-2I_{pp}^{v1})\right)\eta + O\left(\eta^2\right)\Big\}\end{aligned} \tag{H.6}$$

where $\eta = -2kp/m^2$ and terms of order k^2/m^2 are omitted. The scalar integrals are calculated in Appendix G.2.1. They can be expanded in terms of η (where $p^2 = m^2$),

$$I^{v1} = \frac{-i}{(4\pi)^2 m^2}\left\{1-\ln\eta+\left(\frac{1}{4}-\frac{1}{2}\ln\eta\right)\eta+O(\eta^2)\right\}, \tag{H.7}$$

$$I_k^{v1} = \frac{i}{(4\pi)^2 m^2}\left\{\frac{1}{2}+\left(\frac{2}{9}+\frac{1}{3}\ln\eta\right)\eta+O(\eta^2)\right\}, \tag{H.8}$$

$$I_p^{v1} = \frac{i}{(4\pi)^2 m^2}\left\{1+\left(\frac{1}{4}+\frac{1}{2}\ln\eta\right)\eta+O(\eta^2)\right\}, \tag{H.9}$$

$$I_{kk}^{v1} = \frac{-i}{(4\pi)^2 m^2}\left\{\frac{1}{6}-\frac{1}{8}\eta+O(\eta^2)\right\}, \tag{H.10}$$

$$I_{kp}^{v1} = \frac{-i}{(4\pi)^2 m^2}\left\{\frac{1}{4}-\frac{1}{6}\eta+O(\eta^2)\right\}, \tag{H.11}$$

$$I_{pp}^{v1} = \frac{-i}{(4\pi)^2 m^2}\left\{\frac{1}{2}-\frac{1}{4}\eta+O(\eta^2)\right\}, \tag{H.12}$$

$$I_g^{v1} = \frac{i\Gamma(1+\varepsilon)}{4(4\pi)^{2-\varepsilon}\varepsilon}(m^2)^{-\varepsilon} + \frac{i}{(4\pi)^2}\left\{\frac{1}{4}-\frac{1}{8}\eta+O(\eta^2)\right\}. \tag{H.13}$$

Only the last integral contains a singularity which is extracted according to

$$I_{gs}^{v1} = \frac{i\Gamma(1+\varepsilon)}{4(4\pi)^{2-\varepsilon}\varepsilon}(m^2)^{-\varepsilon} =: \frac{i\Gamma(1+\varepsilon)\mu^{-2\varepsilon}}{4(4\pi)^{2-\varepsilon}}c_s, \qquad c_s = \frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+O(\varepsilon). \tag{H.14}$$

Inserting these explicit expressions, one obtains

$$\begin{aligned}\frac{1}{m}\Lambda^{\beta}_{1b}(\not{p}+\not{k}+m)\gamma_\beta T_b &= \frac{g_s^2 C_1 \mu^{-2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Big\{2\left(\frac{1}{4}(D-2)^2 c_s + 2 + 4\ln\eta\right) + \\ &+\left(\frac{1}{8}(D-2)^3 c_s + 4 + 4\ln\eta\right)\eta + O(\eta^2)\Big\}.\end{aligned} \tag{H.15}$$

The contribution from the singular part c_s will be later on absorbed by the renormalization factor. The final result for the effective abelian vertex correction reads

$$\begin{aligned}\frac{1}{m}\Lambda^{\beta}_{1b}(\not{p}+\not{k}+m)\gamma_\beta T_b &= \frac{\alpha_s C_F}{4\pi}\left(C_F-\frac{1}{2}C_A\right)\Big\{2\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+4\ln\eta\right) + \\ &+\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+1+4\ln\eta\right)\eta + O(\eta^2)\Big\}.\end{aligned} \tag{H.16}$$

H.1.2 The non-abelian vertex correction

For the non-abelian vertex correction shown on the right side of Fig. H.1 one obtains

$$\begin{aligned}
-ig_s\Lambda_2^\beta &= \int\frac{d^Dl}{(2\pi)^D}(-ig_s\gamma_\alpha T_a)\frac{i}{\not p+\not l-m}(-ig_s\gamma_\rho T_c)\frac{-i}{l^2}\frac{-i}{(k-l)^2}\ \times\\
&\qquad\times\ g_sf_{acb}\Big((l-2k)^\alpha g^{\rho\beta}+(k+l)^\rho g^{\beta\alpha}+(k-2l)^\beta g^{\alpha\rho}\Big)\ =\\
&= ig_s^3T_aT_cf_{acb}\int\frac{d^Dl}{(2\pi)^D}\frac{\gamma_\alpha(\not p+\not l+m)\gamma_\rho}{((p+l)^2-m^2)l^2(k-l)^2}\ \times\\
&\qquad\times\ \Big((l-2k)^\alpha g^{\rho\beta}+(k+l)^\rho g^{\beta\alpha}+(k-2l)^\beta g^{\alpha\rho}\Big)\ =\\
&= ig_s^3T_aT_cf_{acb}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{((p+l)^2-m^2)l^2(k-l)^2}\Big((\not l-2\not k)(\not p+\not l+m)\gamma^\beta\ +\\
&\qquad+\gamma^\beta(\not p+\not l+m)(\not k+\not l)+(k-2l)^\beta\gamma_\alpha(\not p+\not l+m)\gamma^\alpha\Big)\ =\\
&= ig_s^3T_aT_cf_{acb}\Big[\Big(-2\not k(\not p+m)\gamma^\beta+\gamma^\beta(\not p+m)\not k+k^\beta\gamma_\alpha(\not p+m)\gamma^\alpha\Big)I^{v2}\ +\\
&\qquad+\Big(\gamma^\mu(\not p+m)\gamma^\beta+\gamma^\beta\gamma^\mu\not k-2g^{\beta\mu}\gamma_\alpha(\not p+m)\gamma^\alpha\Big)I_\mu^{v2}\ +\\
&\qquad+\Big(-2\not k\gamma^\nu\gamma^\beta+\gamma^\beta(\not p+m)\gamma^\nu+k^\beta\gamma_\alpha\gamma^\nu\gamma^\alpha\Big)I_\nu^{v2}\ +\\
&\qquad+\Big(\gamma^\mu\gamma^\nu\gamma^\beta+\gamma^\beta\gamma^\mu\gamma^\nu-2g^{\beta\mu}\gamma_\alpha\gamma^\nu\gamma^\alpha\Big)I_{\mu\nu}^{v2}\Big].
\end{aligned}\tag{H.17}$$

Again one has to calculate a set of scalar integrals according to

$$\begin{aligned}
I_\mu^{v2} &= I_k^{v2}k_\mu+I_p^{v2}p_\mu,\\
I_{\mu\nu}^{v2} &= I_g^{v2}g_{\mu\nu}+I_{kk}^{v2}k_\mu k_\nu+I_{kp}^{v2}(k_\mu p_\nu+p_\mu k_\nu)+I_{pp}^{v2}p_\mu p_\nu.
\end{aligned}\tag{H.18}$$

The results can be found in Appendix G.2.2. In terms of these scalar integrals one obtains

$$\begin{aligned}
&\Lambda_{2b}^\beta(\not p+\not k+m)\gamma_\beta T_b\ =\\
&= -ig_s^2C_2\Big\{8(D-1)mI_g^{v2}+4mk^2(I^{v2}-I_k^{v2}-2I_p^{v2}+I_{kk}^{v2}+2I_{kp}^{v2})\ +\\
&\qquad-4m(kp)(I^{v2}-2I_k^{v2}+5I_p^{v2}-4I_{kp}^{v2}-2I_{pp}^{v2})+12m^3I_{pp}^{v2}\ +\\
&\quad+\Big(-4(D-2)(D-1)I_g^{v2}+8(kp)(I^{v2}-I_k^{v2}+I_p^{v2}+I_{kk}^{v2}-I_{kp}^{v2})\ +\\
&\qquad+4m^2(I^{v2}-2I_k^{v2}+2I_p^{v2}+2I_{kp}^{v2}-2I_{pp}^{v2})\Big)\not k\Big\}
\end{aligned}\tag{H.19}$$

where $C_2=-iT_aT_cT_bf_{acb}=C_FC_A/2$ is the colour factor. In using the ambiguity and replacing $\not k\to kp/m$ (and $k^2\to 0$) one obtains

$$\begin{aligned}
\frac{1}{m}\Lambda_{2b}^\beta(\not p+\not k+m)\gamma_\beta T_b &= -ig_s^2C_2\Bigg\{8(D-1)I_g^{v2}+12m^2I_{pp}^{v2}+\\
&\qquad+\Big(2(D-2)(D-1)I_g^{v2}+6m^2(I_p^{v2}-2I_{kp}^{v2})\Big)\eta+O(\eta^2)\Bigg\}.
\end{aligned}\tag{H.20}$$

Therefore, only the integrals I_p^{v2}, I_{kp}^{v2}, I_{pp}^{v2}, and I_g^{v2} have to be calculated. The expansion of these integrals in η (for $p^2=m^2$) can be taken from Appendix G. One has

$$I_p^{v2}\ =\ \frac{-i}{(4\pi)^2m^2}\Big\{\ln\eta+\eta\ln\eta+O(\eta^2)\Big\},\tag{H.21}$$

$$I_{kp}^{v2} = \frac{-i}{(4\pi)^2 m^2}\left\{\left(\frac{1}{2}+\frac{1}{2}\ln\eta\right)+\left(\frac{1}{2}+\ln\eta\right)\eta+O(\eta^2)\right\}, \tag{H.22}$$

$$I_{pp}^{v2} = \frac{-i}{(4\pi)^2 m^2}\left\{\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2}\ln\eta\right)\eta+O(\eta^2)\right\}, \tag{H.23}$$

$$I_g^{v2} = \frac{i\Gamma(1+\varepsilon)\mu^{-2\varepsilon}}{4(4\pi)^{2-\varepsilon}}c_s+\frac{i}{(4\pi)^2}\left\{\frac{3}{4}+\frac{1}{4}\eta\ln\eta+O(\eta^2)\right\} \tag{H.24}$$

with the same definition for c_s as before. Inserting these expressions, one ends up with

$$\frac{1}{m}\Lambda_{2b}^\beta(\not p+\not k+m)\gamma_\beta T_b = \frac{g_s^2C_2}{(4\pi)^2}\left\{2\left((D-1)c_s+6\right)+\left(\frac{(D-2)(D-1)}{2}c_s+9\right)\eta+O(\eta^2)\right\}. \tag{H.25}$$

One now can use the definition of c_s and

$$D-1=3-2\varepsilon+O(\varepsilon^2),\qquad (D-2)(D-1)=6-10\varepsilon+O(\varepsilon^2) \tag{H.26}$$

to obtain

$$\frac{1}{m}\Lambda_{2b}^\beta(\not p+\not k+m)\gamma_\beta T_b =$$
$$= \frac{\alpha_sC_F}{2(4\pi)}C_A\left\{2\left(\frac{3}{\varepsilon}+3\ln\left(\frac{\mu^2}{m^2}\right)+4\right)+\left(\frac{3}{\varepsilon}+3\ln\left(\frac{\mu^2}{m^2}\right)+4\right)\eta+O(\eta^2)\right\}. \tag{H.27}$$

H.1.3 The effective vertex corrections and their renormalization

The two next-to-leading order effective vertex contributions to the self energy correction of the quark have to be added to the leading order one, given by

$$\frac{1}{m}\gamma^\beta T_b(\not p+\not k+m)\gamma_\beta T_b = \frac{C_0}{m}\left(2(\not p+\not k)-D(\not p+\not k-m)\right) = \tag{H.28}$$
$$= \frac{C_0}{m}\left(2(\not k+m)-D\not k\right)=C_0\left(2+\frac{2-D}{m}\not k\right)=C_0\left(2+\frac{D-2}{2}\eta\right)=C_0\left(2+(1-\varepsilon)\eta\right)$$

where the colour factor is $C_0 = T_bT_b = C_F$. With the bare effective vertex given by $\Gamma_b^{\beta0} = \gamma^\beta T_b+\Lambda_{1b}^\beta+\Lambda_{2b}^\beta$ one therefore obtains

$$\frac{1}{m}\Gamma^{\beta0}(\not p+\not k+m)\gamma_\beta := \frac{1}{m}\Gamma_b^{\beta0}(\not p+\not k+m)\gamma_\beta T_b = \left(2+(1-\varepsilon)\eta\right)C_F+$$
$$+\frac{\alpha_sC_F}{4\pi}\left(C_F-\frac{1}{2}C_A\right)\left\{2\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+4\ln\eta\right)+\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+1+4\ln\eta\right)\eta\right\}+$$
$$+\frac{\alpha_sC_F}{2(4\pi)}C_A\left\{2\left(\frac{3}{\varepsilon}+3\ln\left(\frac{\mu^2}{m^2}\right)+4\right)+\left(\frac{3}{\varepsilon}+3\ln\left(\frac{\mu^2}{m^2}\right)+4\right)\eta\right\}+O(\alpha_s^2,\eta^2) =$$
$$= \left(2+(1-\varepsilon)\eta\right)C_F +$$
$$+\frac{\alpha_sC_F^2}{4\pi}\left\{2\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+4\ln\eta\right)+\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+1+4\ln\eta\right)\eta\right\}+$$
$$+\frac{\alpha_sC_FC_A}{4\pi}\left\{2\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+2-2\ln\eta\right)+\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+\frac{3}{2}-2\ln\eta\right)\eta\right\}+$$
$$+O(\alpha_s^2,\eta^2) =$$

$$\begin{aligned}
&= (2+(1-\varepsilon)\eta)\,C_F\left[1+\frac{\alpha_s C_F}{4\pi}\left\{\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+4\ln\eta+\eta\right\}+\right.\\
&\qquad\left.+\frac{\alpha_s C_A}{4\pi}\left\{\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+2-2\ln\eta+\frac{\eta}{4}\right\}+O(\alpha_s^2,\eta^2)\right] =\\
&= (2+(1-\varepsilon)\eta)\,C_F Z_1 \qquad\qquad \text{(H.29)}
\end{aligned}$$

where

$$\begin{aligned}
Z_1 &= 1+\frac{\alpha_s}{4\pi}Z_{11} = 1+\frac{\alpha_s C_F}{4\pi}\left\{\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+4\ln\eta+\eta\right\}+\\
&\qquad+\frac{\alpha_s C_A}{4\pi}\left\{\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+2-2\ln\eta+\frac{\eta}{4}\right\}+O(\alpha_s^2,\eta^2). \qquad \text{(H.30)}
\end{aligned}$$

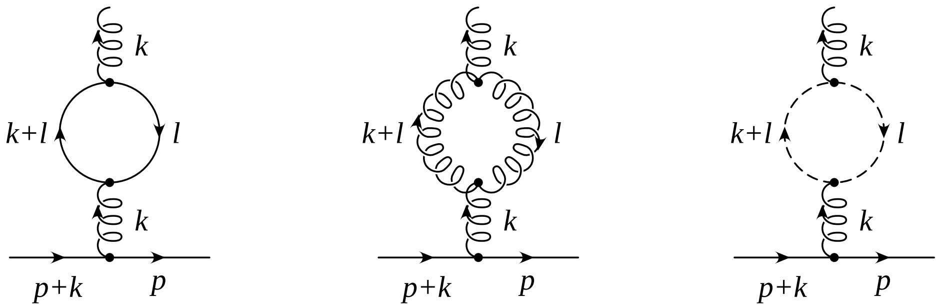


Figure H.2: the gluon self energy corrections, including quark loops (left, diagrams ($l1l$) und ($l1h$)), a gluon loop (middle, diagram ($l2$)), and a ghost loop (right, diagram ($l3$))

H.2 The gluon self energy corrections

As next one has to calculate the gluon self energy. This correction consists of the quark loop (integrals L_{1i}, gluon propagator correction in diagram (d) of Fig. 5.7), the gluon loop (L_2, diagram (e)), and the ghost loop (L_3, diagram (f)). All these integrals have Lorentz indices α and β as well as colour indices a and b at the vertices of the loop. The outer momentum is k, the loop momentum is l. The diagrams are shown in Fig. H.2.

H.2.1 The massive quark loop integral L_{1m}

The quark loop integral for a single massive quark flavour reads

$$\begin{aligned}
L_{1m} &= \int\frac{d^Dl}{(2\pi)^D}\mathrm{Tr}\left((-ig_s\gamma_\alpha T_a)\frac{i}{\not k+\not l-m}(-ig_s\gamma_\beta T_b)\frac{i}{\not l-m}\right) =\\
&= g_s^2\mathrm{Tr}(T_aT_b)\int\frac{d^Dl}{(2\pi)^D}\frac{\mathrm{Tr}(\gamma_\alpha(\not k+\not l+m)\gamma_\beta(\not l+m))}{((k+l)^2-m^2)(l^2-m^2)}. \qquad \text{(H.31)}
\end{aligned}$$

The colour factor is given by $\mathrm{Tr}(T_aT_b)=\delta_{ab}/2$ ($a,b\in\{1,\ldots,N_c^2-1\}$). By calculating the trace one obtains

$$L_{1m}=2g_s^2\delta_{ab}\left\{k_\alpha L_{1\beta}+L_{1\alpha\beta}-\left(L_{1\mu}k^\mu+L_{1\mu\nu}g^{\mu\nu}-L_1m^2\right)g_{\alpha\beta}+L_{1\alpha}k_\beta+L_{1\alpha\beta}\right\}. \quad \text{(H.32)}$$

Expressing $L_{1\alpha}$ and $L_{1\alpha\beta}$ in terms of covariants,

$$L_{1\alpha} = L_{1k}k_\alpha, \qquad L_{1\alpha\beta} = L_{1g}g_{\alpha\beta} + L_{1kk}k_\alpha k_\beta, \tag{H.33}$$

one ends up with

$$L_{1m} = -2g_s^2\delta_{ab}\left\{\left((D-2)L_{1g} + (L_{1k}+L_{1kk})k^2 - L_1 m^2\right)g_{\alpha\beta} - 2(L_{1k}+L_{1kk})k_\alpha k_\beta\right\}. \tag{H.34}$$

A fey new scalar integrals have to be calculated. This is done in Appendix G.3.1. Using

$$L_{1k} + L_{1kk} = -\frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(m^2)^{-\varepsilon}\left(\frac{1}{6\varepsilon} + \frac{k^2}{20m^2}\right) + O\left(\frac{k^4}{m^4}\right) \tag{H.35}$$

one has

$$\begin{aligned}
&(D-2)L_{1g} + (L_{1k}+L_{1kk})k^2 - L_1 m^2 \; = \\
&= \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(m^2)^{1-\varepsilon}\left\{(1-\varepsilon)\left(\left(1-\frac{k^2}{6m^2}\right)\frac{1}{\varepsilon}+1\right) - \frac{k^2}{6m^2\varepsilon} - \frac{1}{\varepsilon} - \frac{k^2}{6m^2} + O\left(\frac{k^4}{m^4}\right)\right\} = \\
&= \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(m^2)^{1-\varepsilon}\left\{\left(1-\frac{k^2}{6m^2}\right)\frac{1}{\varepsilon} + \frac{k^2}{6m^2} - \frac{k^2}{6m^2\varepsilon} - \frac{1}{\varepsilon} - \frac{k^2}{6m^2} + O\left(\frac{k^4}{m^4}\right)\right\} = \\
&= -\frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(m^2)^{1-\varepsilon}\left\{\frac{k^2}{3m^2\varepsilon} + O\left(\frac{k^4}{m^4}\right)\right\}.
\end{aligned} \tag{H.36}$$

Therefore finally, including a factor of -1 for the closed fermion loop, one obtains

$$L_{1m} = -\frac{ig_s^2\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(m^2)^{-\varepsilon}\delta_{ab}\left\{\frac{2}{3\varepsilon}k^2 g_{\alpha\beta} - \frac{2}{3\varepsilon}k_\alpha k_\beta + O\left(\frac{k^4}{m^4}\right)\right\}. \tag{H.37}$$

H.2.2 The massless quark loop integral L_{1l}

The same calculation has to be done for the massless quark loop. Here one starts with

$$\begin{aligned}
L_{1l} &= \int\frac{d^D l}{(2\pi)^D}\mathrm{Tr}\left((-ig_s\gamma_\alpha T_a)\frac{i}{\not k + \not l}(-ig_s\gamma_\beta T_b)\frac{i}{\not l}\right) = \\
&= g_s^2\mathrm{Tr}(T_a T_b)\int\frac{d^D l}{(2\pi)^D}\frac{\mathrm{Tr}(\gamma_\alpha(\not k+\not l)\gamma_\beta \not l)}{l^2(k+l)^2} = \\
&= -2g_s^2\delta_{ab}\left\{\left((D-2)L_{1g}^0 + (L_{1k}^0 + L_{1kk}^0)k^2\right)g_{\alpha\beta} - 2(L_{1k}^0+L_{1kk}^0)k_\alpha k_\beta\right\}.
\end{aligned} \tag{H.38}$$

The integrals are found in Appendix G.3.2, with

$$L_{1k}^0 + L_{1kk}^0 = -\frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\left\{\frac{1}{6\varepsilon} + \frac{5}{18}\right\}. \tag{H.39}$$

One obtains

$$\begin{aligned}
&(D-2)L_{1g}^0 + (L_{1k}^0 + L_{1kk}^0)k^2 \; = \\
&= \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{1-\varepsilon}\left\{(1-\varepsilon)\left(\frac{1}{6\varepsilon}+\frac{4}{9}\right) + \frac{1}{6\varepsilon} + \frac{5}{18}\right\} = \\
&= \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{1-\varepsilon}\left\{\frac{1}{3\varepsilon} + \frac{13}{18} - \frac{1}{6}\right\} = \frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{1-\varepsilon}\left\{\frac{1}{3\varepsilon}+\frac{5}{9}\right\}
\end{aligned} \tag{H.40}$$

and therefore

$$L_{1l} = -\frac{ig_s^2\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\delta_{ab}\left\{\left(\frac{2}{3\varepsilon}+\frac{10}{9}\right)k^2 g_{\alpha\beta} - \left(\frac{2}{3\varepsilon}+\frac{10}{9}\right)k_\alpha k_\beta\right\}. \tag{H.41}$$

H.2.3 The gluon loop integral L_2

The gluon loop integral contains two three-gluon vertices. Using the three-gluon vertex coupling structure

$$g_s f_{a_1a_2a_3}\left((k_2-k_3)_{\mu_1}g_{\mu_2\mu_3}+(k_3-k_1)_{\mu_2}g_{\mu_3\mu_1}+(k_1-k_2)_{\mu_3}g_{\mu_1\mu_2}\right) \tag{H.42}$$

at each of the three-gluon vertices (k_i are outgoing moments), one obtains

$$\begin{aligned}
L_2 &= \int\frac{d^Dl}{(2\pi)^D}\left(\frac{-i}{l^2}\right)\left(\frac{-i}{(k+l)^2}\right)\ \times\\
&\quad\times\ g_sf_{acd}\left((k+2l)_\alpha g_{\gamma\delta}+(-2k-l)_\gamma g_{\delta\alpha}+(k-l)_\delta g_{\alpha\gamma}\right)\ \times\\
&\quad\times\ g_sf_{bcd}\left((-k-2l)_\beta g^{\gamma\delta}+(2k+l)^\gamma g^\delta{}_\beta+(-k+l)^\delta g_\beta{}^\gamma\right)\ =\\
&= g_s^2f_{acd}f_{bcd}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{l^2(k+l)^2}\Big[\left(4k^2+l^2+(k+l)^2\right)g_{\alpha\beta}\ +\\
&\qquad +(D-6)k_\alpha k_\beta+(2D-3)(k_\alpha l_\beta+l_\alpha k_\beta)+2(2D-3)l_\alpha l_\beta\Big].
\end{aligned} \tag{H.43}$$

The colour factor can be calculated to be $f_{acd}f_{bcd}=C_A\delta_{ab}$. Two methods for calculation can be used here and in the next subsection. The first one is again the Passarino-Veltman method using the ansatz

$$L_2=:g_s^2C_A\delta_{ab}\left(L_{2g}k^2g_{\alpha\beta}+L_{2kk}k_\alpha k_\beta\right). \tag{H.44}$$

The scalar integrals L_{2g} and L_{2kk} are given in Appendix G.3.3. In the second method one uses symmetric integration to replace

$$l_\mu l_\nu\to\frac{1}{D}l^2g_{\mu\nu}\to\frac{m^2g_{\mu\nu}}{D-2} \tag{H.45}$$

which is valid according to the introduction of Appendix G. One obtains

$$\begin{aligned}
L_2 &= g_s^2C_A\delta_{ab}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{l^2(k+l)^2}\Big[\left(4k^2+l^2+(k+l)^2\right)g_{\alpha\beta}+\\
&\qquad +(D-6)k_\alpha k_\beta+(2D-3)(k_\alpha l_\beta+l_\alpha k_\beta)+2(2D-3)l_\alpha l_\beta\Big]\ =\\
&= g_s^2C_A\delta_{ab}\int_0^1dx\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2+2xkl+xk^2)^2}\Big[\left(4k^2+l^2+(k+l)^2\right)g_{\alpha\beta}\ +\\
&\qquad +(D-6)k_\alpha k_\beta+(2D-3)(k_\alpha l_\beta+l_\alpha k_\beta)+2(2D-3)l_\alpha l_\beta\Big]\ =\\
&= g_s^2C_A\delta_{ab}\int_0^1dx\int\frac{d^Dl}{(2\pi)^D}\frac{1}{((l+xk)^2+x(1-x)k^2)^2}\Big[\left(4k^2+l^2+(k+l)^2\right)g_{\alpha\beta}\ +\\
&\qquad +(D-6)k_\alpha k_\beta+(2D-3)(k_\alpha l_\beta+l_\alpha k_\beta)+2(2D-3)l_\alpha l_\beta\Big]\ =\\
&= g_s^2C_A\delta_{ab}\int_0^1dx\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(-l^2-x(1-x)k^2)^2}\ \times\\
&\quad\times\ \Big[\left(4k^2+x^2k^2+(1-x)^2k^2-2xkl+2(1-x)kl+2l^2\right)g_{\alpha\beta}\ +\\
&\qquad +(D-6)k_\alpha k_\beta+(2D-3)(k_\alpha l_\beta+l_\alpha k_\beta-2xk_\alpha k_\beta)\ +\\
&\qquad\quad +2(2D-3)\left(l_\alpha l_\beta-xl_\alpha k_\beta-xk_\alpha l_\beta+x^2k_\alpha k_\beta\right)\Big]\ =
\end{aligned}$$

$$
\begin{aligned}
&= g_s^2 C_A \delta_{ab} \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{1}{(-l^2 - x(1-x)k^2)^2} \times \\
&\quad \times \Big[\Big(\big(4 + x^2 + (1-x)^2\big) k^2 + 2l^2 \Big) g_{\alpha\beta} + \frac{2}{D}(2D-3) l^2 g_{\alpha\beta} + \\
&\qquad + \Big(D - 6 - 2x(2D-3) + 2x^2(2D-3)\Big) k_\alpha k_\beta \Big] = \\
&= g_s^2 C_A \delta_{ab} \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{1}{(-l^2 - x(1-x)k^2)^2} \times \\
&\quad \times \Bigg[\Big(\big(4 + x^2 + (1-x)^2\big) k^2 + \frac{6}{D}(D-1) l^2 \Big) g_{\alpha\beta} + \\
&\qquad + \Big(D - 6 - 2x(2D-3) + 2x^2(2D-3)\Big) k_\alpha k_\beta \Bigg] = \\
&= \frac{i g_s^2 C_A \Gamma(\varepsilon)}{(4\pi)^{D/2}} (-k^2)^{-\varepsilon} \delta_{ab} \int_0^1 x^{-\varepsilon}(1-x)^{-\varepsilon} dx \times \\
&\quad \times \Bigg[\Bigg(4 + x^2 + (1-x)^2 - \frac{6(D-1)}{D-2} x(1-x) \Bigg) k^2 g_{\alpha\beta} + \\
&\qquad + \Big(D - 6 - 2x(2D-3) + 2x^2(2D-3)\Big) k_\alpha k_\beta \Bigg] \qquad \text{(H.46)}
\end{aligned}
$$

The two integrals in this expression are given by

$$
\begin{aligned}
&\int_0^1 x^{-\varepsilon}(1-x)^{-\varepsilon} \big(4 + x^2 + (1-x)^2\big) dx = \\
&\qquad = 4\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} + 2\frac{\Gamma(1-\varepsilon)\Gamma(3-\varepsilon)}{\Gamma(4-2\varepsilon)} = \frac{14}{3} + \frac{85}{5}\varepsilon + O(\varepsilon^2), \\
&-\frac{6(D-1)}{D-2} \int_0^2 x^{-\varepsilon}(1-x)^{-\varepsilon} x(1-x) dx = \\
&\qquad = -\frac{6(D-1)}{D-2} \frac{\Gamma(2-\varepsilon)^2}{\Gamma(4-2\varepsilon)} = -\frac{3}{2} - 3\varepsilon + O(\varepsilon^2). \qquad \text{(H.47)}
\end{aligned}
$$

The sum of both is the coefficient of $k^2 g_{\alpha\beta}$ (up to an overall factor) which reads

$$
\frac{19}{6} + \frac{58}{9}\varepsilon + O(\varepsilon^2). \tag{H.48}
$$

This result is agrees with L_{2g} calculated in Appendix G.3.3.

The same agreement holds for the coefficient of $k_\alpha k_\beta$,

$$
\begin{aligned}
&\int_0^1 x^{-\varepsilon}(1-x)^{-\varepsilon} \Big(D - 6 - 2x(2D-3) + 2x^2(2D-3)\Big) dx = \\
&= \Gamma(1-\varepsilon) \left((D-6)\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} - 2(2D-3)\frac{\Gamma(2-\varepsilon)}{\Gamma(3-2\varepsilon)} + 2(2D-3)\frac{\Gamma(3-\varepsilon)}{\Gamma(4-2\varepsilon)} \right) = \\
&= -\frac{11}{3} - \frac{67}{9}\varepsilon + O(\varepsilon^2). \qquad \text{(H.49)}
\end{aligned}
$$

The final result, including a factor of $1/2$ for the closed gluon line, is therefore given by

$$
L_2 = \frac{i g_s^2 C_A \Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}} (-k^2)^{-\varepsilon} \delta_{ab} \left\{ \left(\frac{19}{6\varepsilon} + \frac{58}{9} \right) k^2 g_{\alpha\beta} - \left(\frac{11}{3\varepsilon} + \frac{67}{9} \right) k_\alpha k_\beta \right\}. \tag{H.50}
$$

H.2.4 The ghost loop integral L_3

The ghost loop involves vertex factors

$$g_s f_{ab_1b_2} k_\mu \tag{H.51}$$

where the indices a and μ refer to the gluon line, b_1, b_2 to the ghost lines and k is the outgoing ghost momentum (supposing that one has selected a loop direction). One starts with

$$L_3 = \int \frac{d^D l}{(2\pi)^D} \frac{-g_s^2}{l^2(k+l)^2} f_{acd} f_{bdc} l_\alpha (k+l)_\beta = g_s^2 C_A \delta_{ab} \int \frac{d^D l}{(2\pi)^D} \frac{l_\alpha (k+l)_\beta}{l^2(k+l)^2}. \tag{H.52}$$

The first method is again the Passarino-Veltman method by using the ansatz

$$L_3 =: g_s^2 C_A \delta_{ab} \left(L_{3g} k^2 g_{\alpha\beta} + L_{3kk} k_\alpha k_\beta \right). \tag{H.53}$$

The scalar integrals L_{3g} and L_{3kk} are listed in Appendix G.3.4. With the symmetric integration method one obtains

$$\begin{aligned}
L_3 &= g_s^2 C_A \delta_{ab} \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{l_\alpha (k+l)_\beta}{(l^2 + 2xkl + xk^2)^2} = \\
&= g_s^2 C_A \delta_{ab} \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{(l - xk)_\alpha (l + (1-x)k)_\beta}{(-l^2 - x(1-x)k^2)^2} = \\
&= g_s^2 C_A \delta_{ab} \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{-x(1-x) k_\alpha k_\beta + l^2 g_{\alpha\beta}/D}{(-l^2 - x(1-x)k^2)^2} = \\
&= g_s^2 C_A \delta_{ab} \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{-x(1-x) k_\alpha k_\beta - x(1-x) k^2 g_{\alpha\beta}/(D-2)}{(-l^2 - x(1-x)k^2)^2} = \\
&= -\frac{i g_s^2 \Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} (-k^2)^{-\varepsilon} \left(2k_\alpha k_\beta + \frac{k^2 g_{\alpha\beta}}{1-\varepsilon} \right) \int_0^1 (x(1-x))^{1-\varepsilon} dx = \\
&= -\frac{i g_s^2 \Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} (-k^2)^{-\varepsilon} \left(2k_\alpha k_\beta + \frac{k^2 g_{\alpha\beta}}{1-\varepsilon} \right) \frac{\Gamma(2-\varepsilon)^2}{\Gamma(4-2\varepsilon)}
\end{aligned} \tag{H.54}$$

Together with a factor of -1 for a closed ghost loop one obtains

$$L_3 = \frac{i g_s^2 C_A \Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}} (-k^2)^{-\varepsilon} \delta_{ab} \left\{ \left(\frac{1}{6\varepsilon} + \frac{4}{9} \right) k^2 g_{\alpha\beta} + \left(\frac{1}{3\varepsilon} + \frac{5}{9} \right) k_\alpha k_\beta \right\} \tag{H.55}$$

which agrees again with the previous calculation.

H.2.5 The summation of L_2, L_3, L_{1l} and L_{1m}

For the sum of the gluon loop and ghost loop term one obtains

$$\begin{aligned}
L_2 + L_3 &= \frac{i g_s^2 C_A \Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}} (-k^2)^{-\varepsilon} \delta_{ab} \left\{ \left(\frac{20}{6\varepsilon} + \frac{62}{9} \right) k^2 g_{\alpha\beta} - \left(\frac{10}{3\varepsilon} + \frac{62}{9} \right) k_\alpha k_\beta \right\} = \\
&= \frac{i g_s^2 C_A \Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}} (-k^2)^{-\varepsilon} \delta_{ab} \left\{ \left(\frac{5}{3\varepsilon} + \frac{31}{9} \right) k^2 g_{\alpha\beta} - \left(\frac{5}{3\varepsilon} + \frac{31}{9} \right) k_\alpha k_\beta \right\}.
\end{aligned} \tag{H.56}$$

This result transverse as it must be. If N_l (number of light quark flavours) times the result for the massless quarks is added, one obtains the bare gluon self energy correction

$$\begin{aligned} L^0_{\alpha\beta} &= \frac{ig_s^2\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\delta_{ab}\left\{\frac{1}{\varepsilon}\left(\frac{5}{3}C_A-\frac{2}{3}N_l\right)+\frac{31}{9}C_A-\frac{10}{9}N_l\right\}\left\{k^2g_{\alpha\beta}-k_\alpha k_\beta\right\} = \\ &= \delta_{ab}L^0\left\{k^2g_{\alpha\beta}-k_\alpha k_\beta\right\} \qquad \text{where} \qquad (H.57) \\ L^0 &= \frac{ig_s^2\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(-k^2)^{-\varepsilon}\left\{\frac{1}{\varepsilon}\left(\frac{5}{3}C_A-\frac{4}{3}N_lT_F\right)+\frac{31}{9}C_A-\frac{20}{9}N_lT_F\right\} = \\ &= \frac{i\alpha_s}{4\pi}\left(\frac{\mu^2}{-k^2}\right)^\varepsilon\left\{\frac{1}{\varepsilon}\left(\frac{5}{3}C_A-\frac{4}{3}N_lT_F\right)+\frac{31}{9}C_A-\frac{20}{9}N_lT_F\right\}. \qquad (H.58) \end{aligned}$$

For the bare gluon propagator $G_{\mu\nu}$ one obtains

$$\begin{aligned} G_{\mu\nu} &= \frac{-ig_{\mu\nu}}{k^2}+-ig_{\mu\lambda}k^2L^{0\lambda\rho}\frac{-ig_{\rho\nu}}{k^2} = \\ &= \frac{-ig_{\mu\nu}}{k^2}-\frac{L^0}{k^4}\left(k^2g_{\mu\lambda}g^{\lambda\rho}g_{\rho\nu}-g_{\mu\lambda}k^\lambda k^\rho g_{\rho\nu}\right) = \\ &= \frac{-ig_{\mu\nu}}{k^2}-\frac{L^0}{k^4}\left(k^2g_{\mu\nu}-k_\mu k_\nu\right) \stackrel{!}{=} \frac{-iZ_3^l}{k^2}\left(g_{\mu\nu}-(1-Z_\chi\chi)\frac{k_\mu k_\nu}{k^2}\right). \qquad (H.59) \end{aligned}$$

The comparison results in

$$\begin{aligned} Z_3^l &= 1+\frac{\alpha_s}{4\pi}Z_{31}^l = 1-iL^0 = \\ &= 1+\frac{\alpha_s}{4\pi}\left\{\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{-k^2}\right)\right)\left(\frac{5}{3}C_A-\frac{4}{3}N_lT_F\right)+\frac{31}{9}C_A-\frac{20}{9}N_lT_F\right\} \qquad (H.60) \end{aligned}$$

and

$$Z_3^l(1-Z_\chi\chi)=-iL^0 \quad\Leftrightarrow\quad 1-iL^0-Z_\chi\chi=-iL^0 \quad\Rightarrow\quad Z_\chi\chi=1. \qquad (H.61)$$

The transverse contribution vanishes therefore, as is usual for Lorentz gauge.

The massive loop has to be considered separately. The same considerations as the one just done for the massless loop lead to

$$G^h_{\mu\nu}=\frac{-iZ_3^h}{k^2}g_{\mu\nu}, \qquad Z_3^h=1+\frac{\alpha_s}{4\pi}Z_{31}^h=1-\frac{\alpha_s}{4\pi}\left(\frac{1}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)\right)\frac{4}{3}N_hT_F \qquad (H.62)$$

where N_h is the number of heavy quark flavours.

H.3 The quark self energy correction

For the line between the two vertices one has to calculate the self energy correction for the quark. The momentum of this line is $p+k$ according to previous conventions (see Fig. 8.3). Nevertheless, one first starts with the momentum p and later on extends this to the actual situation. The self energy is given by

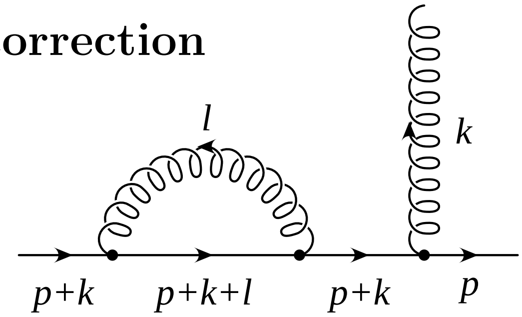


Figure 8.3: self energy correction

$$-i\Sigma(\not{p}) = \int\frac{d^Dl}{(2\pi)^D}(-ig_s\gamma_\alpha T_a)\frac{i}{\not{p}+\not{l}-m}(-ig_s\gamma^\alpha T_a)\frac{-i}{l^2} =$$

$$\begin{aligned}
&= -g_s^2 T_a^2 \int \frac{d^D l}{(2\pi)^D} \frac{\gamma_\alpha(\not{p}+\not{l}+m)\gamma^\alpha}{((p+l)^2-m^2)l^2} = -g_s^2 C_F \int \frac{d^D l}{(2\pi)^D} \frac{(2-D)(\not{p}+\not{l})+Dm}{((p+l)^2-m^2)l^2} = \\
&= g_s^2 C_F \left[(D-2)\gamma^\alpha \left(p_\alpha S(p)+S_\alpha(p)\right) - DmS(p)\right] = \\
&= g_s^2 C_F \left[(D-2)\not{p} \left(S(p)+S_p(p)\right) - DmS(p)\right] \qquad \text{(H.63)}
\end{aligned}$$

where

$$(T_a)_i^j (T_a)_j^k = \frac{1}{2}\left(\delta_i^k \delta_j^j - \frac{1}{N_C}\delta_i^j \delta_j^k\right) = \frac{N_C^2-1}{2N_C}\delta_i^k = C_F \delta_i^k \qquad \text{(H.64)}$$

as well as $S_\alpha(p) = p_\alpha S_p(p)$ is used. Replacing p by $p+k$, one obtains

$$\begin{aligned}
\Sigma(\not{p}+\not{k}) &= \frac{-g_s^2 C_F}{(4\pi)^{2-\varepsilon}}\Gamma(1+\varepsilon)(p^2)^{-\varepsilon} \times \\
&\times \Big[(1-\varepsilon)\left(2\sigma(\eta,\omega)-\sigma_p(\eta,\omega)\right)(\not{p}+\not{k}) - 2(2-\varepsilon)\sigma(\eta,\omega)m\Big] \qquad \text{(H.65)}
\end{aligned}$$

where $\omega = m^2/p^2 - 1$ parametrizes the deviation of the quark from the mass shell, η is the same as before ($\eta = -2kp/m^2$), and the integrals $\sigma(\eta,\omega)$ are found in Appendix G.4. By inserting the results one obtains

$$\begin{aligned}
\Sigma(\not{p}+\not{k}) &= \frac{-g_s^2 C_F}{(4\pi)^{2-\varepsilon}}\Gamma(1+\varepsilon)(p^2)^{-\varepsilon} \times \\
&\times \left[(1-\varepsilon)\left(\frac{1}{\varepsilon}+2+\frac{1+\omega}{1-\eta}-\ln(\eta+\omega)+\frac{(1+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right)\right)(\not{p}+\not{k}) + \right. \\
&\left. -2(2-\varepsilon)\left(\frac{1}{\varepsilon}+2-\ln(\eta+\omega)+\frac{1+\omega}{1-\eta}\ln\left(\frac{\eta+\omega}{1+\omega}\right)\right)m\right] = \\
&= -\frac{\alpha_s C_F}{4\pi}\left(\frac{p^2}{\mu^2}\right)^{-\varepsilon}\left[\left(\frac{1}{\varepsilon}+1+\frac{1+\omega}{1-\eta}-\ln(\eta+\omega)+\frac{(1+\omega)^2}{(1-\eta)^2}\ln\left(\frac{\eta+\omega}{1+\omega}\right)\right)(\not{p}+\not{k}) + \right. \\
&\left. -\left(\frac{4}{\varepsilon}+6-4\ln(\eta+\omega)+4\frac{1+\omega}{1-\eta}\ln\left(\frac{\eta+\omega}{1+\omega}\right)\right)m\right]. \qquad \text{(H.66)}
\end{aligned}$$

This expression is exact both in its dependence on η and ω. However, in order to calculate the renormalization factors for the *on-shell scheme*, one returns to the special case $k=0$ (i.e. $\eta=0$) and calculates the self energy

$$\begin{aligned}
\Sigma(\not{p}) &= -\frac{\alpha_s C_F}{4\pi}\left(\frac{\mu^2}{p^2}\right)^{\varepsilon}\left[\left(\frac{1}{\varepsilon}+2+\omega+(2+\omega)\omega\ln\omega-(1+\omega)^2\ln(1+\omega)\right)\not{p} + \right. \\
&\left. -\left(\frac{4}{\varepsilon}+6+4\omega\ln\omega-4(1+\omega)\ln(1+\omega)\right)m\right] \qquad \text{(H.67)}
\end{aligned}$$

and its derivative at $\not{p}=m$ (i.e. $\omega=0$) in order to combine them in

$$\Sigma(\not{p}) = \Sigma(m) + (\not{p}-m)\left.\frac{\partial\Sigma(\not{p})}{\partial\not{p}}\right|_{\not{p}=m}. \qquad \text{(H.68)}$$

With

$$\Sigma(\not{p}) = \not{p}\Sigma_p(\not{p}) + m\Sigma_m(\not{p}) \qquad \text{(H.69)}$$

one obtains

$$\frac{\partial}{\partial\not p}\Sigma(\not p)\Big|_{\not p=m} = \Sigma_p(m) + m\frac{\partial}{\partial\not p}\left(\Sigma_p(\not p)+\Sigma_m(\not p)\right)\Big|_{\not p=m}. \tag{H.70}$$

The application to the overall factor results in

$$\frac{\partial}{\partial\not p}\left(\frac{\mu^2}{p^2}\right)^\varepsilon = -\frac{2\varepsilon\not p}{p^2}\left(\frac{\mu^2}{p^2}\right)^\varepsilon. \tag{H.71}$$

However, up to the overall factor the remaining expression is a function of $\omega = m^2/p^2 - 1$ rather than of $\not p$. Writing

$$\Sigma_p(\not p) = -\frac{\alpha_s C_F}{4\pi}\left(\frac{\mu^2}{p^2}\right)^\varepsilon \tilde\Sigma_p(\omega), \qquad \Sigma_m(\not p) = -\frac{\alpha_s C_F}{4\pi}\left(\frac{\mu^2}{p^2}\right)^\varepsilon \tilde\Sigma_m(\omega), \tag{H.72}$$

and rewriting the derivative as

$$\frac{\partial}{\partial\not p} = \frac{\partial p^2}{\partial\not p}\frac{\partial}{\partial p^2} = \frac{\partial\not p^2}{\partial\not p}\frac{\partial}{\partial p^2} = 2\not p\frac{\partial}{\partial p^2} = -\frac{2\not p m^2}{(p^2)^2}\frac{\partial}{\partial\omega}, \tag{H.73}$$

the whole derivative reads

$$\begin{aligned}\frac{\partial}{\partial\not p}\Sigma(\not p)\Big|_{\not p=m} &= -\frac{\alpha_s C_F}{4\pi}\left(\frac{\mu^2}{m^2}\right)^\varepsilon \times \qquad \text{(H.74)}\\ &\times \lim_{\omega\to 0}\left[\tilde\Sigma_p(\omega) - 2\left\{\varepsilon\left(\tilde\Sigma_p(\omega)+\tilde\Sigma_m(\omega)\right) + \frac{\partial}{\partial\omega}\left(\tilde\Sigma_p(\omega)+\tilde\Sigma_m(\omega)\right)\right\}\right].\end{aligned}$$

Starting with

$$\tilde\Sigma_p(\omega) = \frac{1}{\varepsilon} + 2 + \omega + (2+\omega)\omega\ln\omega - (1+\omega)^2\ln(1+\omega), \tag{H.75}$$

$$\tilde\Sigma_m(\omega) = -\frac{4}{\varepsilon} - 6 - 4\omega\ln\omega + 4(1+\omega)\ln(1+\omega) \tag{H.76}$$

one obtains

$$\tilde\Sigma_p(0) = \frac{1}{\varepsilon} + 2, \qquad \varepsilon\left(\tilde\Sigma_p(\omega)+\tilde\Sigma_m(\omega)\right) = -3 + O(\varepsilon) \tag{H.77}$$

and

$$\begin{aligned}\frac{\partial}{\partial\omega}\tilde\Sigma_p(\omega) &= 1 + 2(1+\omega)\ln\omega + (2+\omega) - 2(1+\omega)\ln(1+\omega) - (1+\omega) = \\ &\to 2 + 2\ln\omega - 2\omega + 2\omega\ln\omega + O(\omega^2),\\ \frac{\partial}{\partial\omega}\tilde\Sigma_m(\omega) &= -4\ln\omega - 4 + 4\ln(1+\omega) + 4 \ \to\ -4\ln\omega + 4\omega + O(\omega^2) \qquad \text{(H.78)}\end{aligned}$$

(the arrows indicate the limit $\omega \to 0$), therefore finally

$$\begin{aligned}\Sigma(m) &= -\frac{\alpha_s C_F}{4\pi}\left(\frac{\mu^2}{m^2}\right)^\varepsilon\left[-\frac{3}{\varepsilon} - 4\right]m,\\ \frac{\partial}{\partial\not p}\Sigma(\not p)\Big|_{\not p=m} &= -\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 4 + 4\ln\omega - 4\omega - 4\omega\ln\omega + O(\omega^2)\right]. \qquad \text{(H.79)}\end{aligned}$$

In order to renormalize in the on-shell scheme one has to compare

$$\Delta^0 = \frac{i}{\not{p} - m_0 - \Sigma^0}, \qquad \Delta = \frac{i}{\not{p} - m - \Sigma}, \qquad \Delta^0 = Z_2\Delta, \quad m_0 = Z_m m \tag{H.80}$$

where the quantities with index "0" are bare quantities while the ones without index are renormalized quantities. In representing the bare self energy correction as well as the renormalized one in the same manner as above,

$$\Sigma^0(\not{p}) = \not{p}\Sigma_p^0(\not{p}) + m_0\Sigma_m^0(\not{p}), \qquad \Sigma(\not{p}) = \not{p}\Sigma_p(\not{p}) + m\Sigma_m(\not{p}). \tag{H.81}$$

One can solve for the finite self energy contributions and obtains

$$\Sigma_p(\not{p}) = 1 - Z_2\left(1 - \Sigma_p^0(\not{p})\right), \qquad \Sigma_m(\not{p}) = 1 - Z_2 Z_m\left(1 - \Sigma_m^0(\not{p})\right). \tag{H.82}$$

The condition that there should be no finite self energy correction becomes manifest in the requirement that the zeroth and first order term in the expansion

$$\Sigma(\not{p}) = \Sigma(m) + (\not{p} - m)\frac{\partial\Sigma}{\partial\not{p}}\bigg|_{\not{p}=m} + O\left((\not{p} - m)^2\right) \tag{H.83}$$

have to vanish. With Eq. (H.83) one has two equations for the renormalization factors Z_2 and Z_m which can be solved to give

$$Z_m = \frac{1 - \Sigma_p^0(m)}{1 + \Sigma_m^0(m)}, \qquad Z_2^{-1} = 1 - \Sigma_p^0(m) - m\frac{\partial}{\partial\not{p}}\left(Z_m\Sigma_m^0(\not{p}) + \Sigma_p^0(\not{p})\right)\bigg|_{\not{p}=m}. \tag{H.84}$$

In the present case one only needs the first order terms,

$$Z_m = 1 + \frac{\alpha_s}{4\pi}Z_{m1} = 1 - \Sigma_m^{01}(m) - \Sigma_p^{01}(m), \tag{H.85}$$

$$\begin{aligned} Z_2 &= 1 + \frac{\alpha_s}{4\pi}Z_{21} = 1 + \Sigma_p^{01}(m) + m\frac{\partial}{\partial\not{p}}\left(\Sigma_p^{01}(\not{p}) + \Sigma_m^{01}(\not{p})\right)\bigg|_{\not{p}=m} = \\ &= 1 + \frac{\partial}{\partial\not{p}}\left(\not{p}\Sigma_p^{01}(\not{p}) + \Sigma_m^{01}(\not{p})\right)\bigg|_{\not{p}=m} = 1 + \frac{\partial}{\partial\not{p}}\Sigma^{01}(\not{p})\bigg|_{\not{p}=m} = \\ &= 1 - \frac{\alpha_s C_F}{4\pi}\left\{\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 4(1-\omega)(1+\ln\omega) + O(\omega^2)\right\}. \end{aligned} \tag{H.86}$$

Remembering that one actually has to take the quark self energy factor Z_{21} not at momentum p but at $p+k$ which relates ω to η, one replaces

$$\begin{aligned} \omega &= \frac{m^2 - p^2}{p^2} \to \frac{m^2 - (p+k)^2}{(p+k)^2} = \frac{m^2 - p^2 - 2pk - k^2}{p^2 + 2pk + k^2} = \\ &= \frac{(\omega+\eta)p^2 - k^2}{(1-\eta)p^2 + k^2} = \frac{\omega+\eta}{1-\eta}\left(1 - \left(\frac{1}{\omega+\eta} + \frac{1}{1-\eta}\right)\frac{k^2}{p^2} + O\left(\frac{k^4}{p^4}\right)\right) = \\ &= \frac{\omega+\eta}{1-\eta} - \frac{1+\omega}{(1-\eta)^2}\frac{k^2}{p^2} + O\left(\frac{k^4}{p^4}\right). \end{aligned} \tag{H.87}$$

Taking the outer states exactly on-shell (i.e. $\omega = 0$ and $k^2 = 0$), the translation reads

$$\omega \to \frac{\eta}{1-\eta} + O(\eta^2). \tag{H.88}$$

Inserting this, one obtains

$$(1-\omega)(1+\ln\omega) \to 1 + \ln\eta - \eta\ln\eta + O(\eta^2) \tag{H.89}$$

and therefore

$$Z_{21} = -C_F\left\{\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 4 + 4\ln\eta - 4\eta\ln\eta + O(\eta^2)\right\}. \tag{H.90}$$

H.4 The soft part of the quark self energy

The final step in this appendix is to combine all the different results in one integral

$$-i\mathcal{S} = \int^{\mu_f} \frac{d^4k}{(2\pi)^4} \left(-ig_s\Gamma_a^\alpha(p,p+k)\right) Q(p+k) \left(-ig_s\Gamma_b^\beta(p+k,p)\right) D_{\alpha\beta}^{ab}(k) \tag{H.91}$$

(for the notation with μ_f see Chapter 5.3) where

$$\begin{aligned}
\Gamma_a^\alpha(p,p+k) &= -ig_s\left(\gamma^\alpha T_a + \Lambda_a^\alpha(p,p+k)\right),\\
Q(p) &= \frac{iZ_2}{\not{p}-m}, \qquad Z_2 = 1 + \frac{\alpha_s}{4\pi}Z_{21},\\
D_{\alpha\beta}^{ab}(k) &= \frac{-i(Z_3^l + Z_3^h)}{k^2}\delta_{ab}g_{\alpha\beta}, \qquad Z_3^{(l,h)} = 1 + \frac{\alpha_s}{4\pi}Z_{31}^{(l,h)},
\end{aligned} \tag{H.92}$$

the arguments of the vertex function are the outgoing momentum (first argument) and the incoming momentum (second argument). Because the integral $\mathcal{S}$ should be calculated up to second order and the integration over the (soft) momentum k is made explicit, one needs the integrand only up to first order. Therefore, the integral reads

$$\begin{aligned}
-i\mathcal{S} &= \int^{\mu_f} \frac{d^4k}{(2\pi)^4}(-ig_s\gamma^\alpha T_a)\frac{i}{\not{p}+\not{k}-m}(-ig_s\gamma^\beta T_b)\frac{-i}{k^2}\delta_{ab}g_{\alpha\beta} +\\
&\quad + \int^{\mu_f} \frac{d^4k}{(2\pi)^4}\left(-ig_s\Lambda_a^\alpha(p,p+k)\right)\frac{i}{\not{p}+\not{k}-m}(-ig_s\gamma^\beta T_b)\frac{-i}{k^2}\delta_{ab}g_{\alpha\beta} +\\
&\quad + \int^{\mu_f} \frac{d^4k}{(2\pi)^4}(-ig_s\gamma^\alpha T_a)\frac{iZ_{21}}{\not{p}+\not{k}-m}(-ig_s\gamma^\beta T_b)\frac{-i}{k^2}\delta_{ab}g_{\alpha\beta} +\\
&\quad + \int^{\mu_f} \frac{d^4k}{(2\pi)^4}(-ig_s\gamma^\alpha T_a)\frac{i}{\not{p}+\not{k}-m}\left(-ig_s\Lambda_b^\beta(p+k,p)\right)\frac{-i}{k^2}\delta_{ab}g_{\alpha\beta} +\\
&\quad + \int^{\mu_f} \frac{d^4k}{(2\pi)^4}(-ig_s\gamma^\alpha T_a)\frac{i}{\not{p}+\not{k}-m}(-ig_s\gamma^\beta T_b)\frac{-i(Z_{31}^l + Z_{31}^h)}{k^2}\delta_{ab}g_{\alpha\beta} =\\
&= -g_s^2C_F\int^{\mu_f}\frac{d^4k}{(2\pi)^4}\frac{\gamma^\alpha(\not{p}+\not{k}+m)\gamma_\alpha}{(p+k)^2-m^2} - g_s^2\int^{\mu_f}\frac{d^4k}{(2\pi)^4}\frac{\Lambda_b^\beta(p,p+k)(\not{p}+\not{k}+m)\gamma_\beta T_b}{(p+k)^2-m^2} +\\
&\quad -g_s^2C_FZ_{21}\int^{\mu_f}\frac{d^4k}{(2\pi)^4}\frac{\gamma^\alpha(\not{p}+\not{k}+m)\gamma_\alpha}{(p+k)^2-m^2} - g_s^2\int^{\mu_f}\frac{d^4k}{(2\pi)^4}\frac{\gamma_\beta T_b(\not{p}+\not{k}+m)\Lambda_b^\beta(p+k,p)}{(p+k)^2-m^2} +\\
&\quad -g_s^2C_FZ_{31}^l\int^{\mu_f}\frac{d^4k}{(2\pi)^4}\frac{\gamma^\alpha(\not{p}+\not{k}+m)\gamma_\alpha}{(p+k)^2-m^2} - g_s^2C_FZ_{31}^h\int^{\mu_f}\frac{d^4k}{(2\pi)^4}\frac{\gamma^\alpha(\not{p}+\not{k}+m)\gamma_\alpha}{(p+k)^2-m^2} =\\
&=: -i\mathcal{S}_0 - i\mathcal{S}_1 - i\mathcal{S}_2 - i\mathcal{S}_3 - i\mathcal{S}_4 - i\mathcal{S}_5.
\end{aligned} \tag{H.93}$$

Here $\mathcal{S}_1 = \mathcal{S}_3$, while $\mathcal{S}_2$, $\mathcal{S}_4$ and $\mathcal{S}_5$ are proportional to $\mathcal{S}_0$. If $\mathcal{S}$ is assumed to stand between spinors of momentum p and mass m, one can use the previous results and obtains

$$-i\mathcal{S} = -g_s^2 C_F \int^{\mu_f} \frac{d^4k}{(2\pi)^4} \frac{2+(1-\varepsilon)\eta}{(p+k)^2 - m^2} \left(1 + \frac{\alpha_s}{4\pi}\left(2Z_{11} + Z_{21} + Z_{31}^l + Z_{31}^h\right)\right) \qquad \text{(H.94)}$$

where

$$\begin{aligned}
Z_{11} &= C_F\left\{\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 4\ln\eta + \eta\right\} + C_A\left\{\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 2 - 2\ln\eta + \frac{\eta}{4}\right\} + O(\eta^2), \\
Z_{21} &= -C_F\left\{\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 4 + 4\ln\eta - 4\eta\ln\eta\right\} + O(\eta^2), \\
Z_{31}^l &= \left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-k^2}\right)\right)\left(\frac{5}{3}C_A - \frac{4}{3}N_l T_F\right) + \frac{31}{9}C_A - \frac{20}{9}N_l T_F, \\
Z_{31}^h &= -\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right)\frac{4}{3}N_h T_F. \qquad \text{(H.95)}
\end{aligned}$$

Therefore, the sum $2Z_{11} + Z_{21} + Z_{31}^l + Z_{31}^h$ gives

$$\begin{aligned}
2Z_{11} + Z_{21} + Z_{31}^l + Z_{31}^h &= -\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right)\frac{4}{3}N_h T_F + C_F\left\{\frac{2}{\varepsilon} + 2\ln\left(\frac{\mu^2}{m^2}\right) + 8\ln\eta\right\} + \\
&\quad + C_A\left\{\frac{2}{\varepsilon} + 2\ln\left(\frac{\mu^2}{m^2}\right) + 4 - 4\ln\eta\right\} - C_F\left\{\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 4 + 4\ln\eta - 4\eta\ln\eta\right\} + \\
&\quad + \left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-k^2}\right)\right)\left(\frac{5}{3}C_A - \frac{4}{3}N_l T_F\right) + \frac{31}{9}C_A - \frac{20}{9}N_l T_F + O(\eta^2) = \\
&= \frac{1}{\varepsilon}\left(C_F + \frac{11}{3}C_A - \frac{4}{3}(N_h + N_l)T_F\right) + \ln\left(\frac{\mu^2}{m^2}\right)\frac{4}{3}N_h T_F + \\
&\quad + C_F\left\{\ln\left(\frac{\mu^2}{m^2}\right) - 4 + 4\ln\eta + 4\eta\ln\eta\right\} + C_A\left\{2\ln\left(\frac{\mu^2}{m^2}\right) + 4 - 4\ln\eta\right\} + \\
&\quad + \ln\left(\frac{\mu^2}{-k^2}\right)\left(\frac{5}{3}C_A - \frac{4}{3}N_l T_F\right) + \frac{31}{9}C_A - \frac{20}{9}N_l T_F + O(\eta^2). \qquad \text{(H.96)}
\end{aligned}$$

Appendix I

The one-loop residue method

As preparation for the two-loop calculations, in this appendix the soft part of the quark self energy (as used in Section 5.3) is calculated to one-loop order for three different gauges, namely the Feynman gauge, the Coulomb gauge, and the general covariant gauge which includes both the Feynman gauge and the Landau gauge.

I.1 One-dimensional integrals and residues

As a preparation for things to come, one-dimensional integrals over the time component of the inner moment k are calculated. These integrals are of the form

$$I^{(r)}(n_1,n_2;\kappa)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(k_0)^r dk_0}{((p-k)^2-m^2)^{n_1}(k^2)^{n_2}} \tag{I.1}$$

where $p=(m;0,0,0)$ is the outer momentum. The numerator has (multiple) poles in the complex k_0-plane on the real axis which are shifted by using an additional $-i\epsilon$ for each of the propagators. In the following the short form $\kappa=|\vec{k}\,|=\sqrt{k_0^2-k^2}$ is used. Using the ansatz $k_0=k_a+ik_b\epsilon$ one can extract the position of the poles,

$$\begin{aligned}
0 &= (p-k)^2-m^2-i\epsilon \;=\; m^2-2pk+k^2-m^2-i\epsilon \;=\; k_0^2-\kappa^2-2k_0m-i\epsilon \;= \\
&= k_a^2+2ik_ak_b\epsilon-\kappa^2-2mk_a-2imk_b\epsilon-i\epsilon \qquad\Rightarrow \qquad\qquad (\mathrm{I.2})\\
&\quad k_a^2-\kappa^2-2mk_a \;=\; 0,\quad 2(k_a-m)k_b \;=\; 1 \quad\Rightarrow\quad k_0 \;=\; m\pm\left(\sqrt{\kappa^2+m^2}+i\epsilon\right),\\
0 &= k^2-i\epsilon \;=\; k_0^2-\kappa^2-i\epsilon \;=\; k_a^2+2ik_ak_b\epsilon-\kappa^2-i\epsilon \quad\Rightarrow\\
&\quad k_a^2 \;=\; \kappa^2,\quad 2k_ak_b \;=\; 1 \quad\Rightarrow\quad k_0 \;=\; \pm(\kappa+i\epsilon). \qquad\qquad (\mathrm{I.3})
\end{aligned}$$

For $r<2(n_1+n_2)$, the integral over the infinite interval can be replaced by an integral over the contour of a half disk where the integral over the infinite circle vanishes. It is matter of convenience in which of the complex half planes the interval on the real axis to this contour is closed. By closing on the upper half plane one obtains

$$\begin{aligned}
I^{(r)}(n_1,n_2;\kappa) &= \mathrm{Res}\left[\frac{(k_0)^r}{(k_0^2-\kappa^2-2k_0m)^{n_1}(k_0^2-\kappa^2)^{n_2}}\,;\,k_0=m+\sqrt{\kappa^2+m^2}\right]+ \qquad (\mathrm{I.4})\\
&\quad +\mathrm{Res}\left[\frac{(k_0)^r}{(k_0^2-\kappa^2-2k_0m)^{n_1}(k_0^2-\kappa^2)^{n_2}}\,;\,k_0=\kappa\right]-I_0^{(r)}(n_1,n_2;\kappa)
\end{aligned}$$

where the parameter ϵ can be skipped because of the choice of the contour. The term $I_0^{(r)}(n_1, n_2; \kappa)$ is the contribution from the circle with infinite radius, in this case in the upper half plane. It will vanish for $r < 2(n_1 + n_2) - 1$ and will give a finite contribution for $r = 2(n_1 + n_2) - 1$. In this case one replaces k_0 by $Re^{i\phi}$ where $\phi \in [0, \pi]$ and therefore obtains ($R \to \infty$)

$$I_0^{(2(n_1+n_2)-1)}(n_1, n_2; \kappa) = \frac{1}{2\pi i} \int_0^{\pi} \frac{(Re^{i\phi})^{2(n_1+n_2)-1} iRe^{i\varphi} d\phi}{(R^2 e^{2i\phi})^{n_1+n_2}} = \frac{1}{2}. \tag{I.5}$$

I.1.1 Calculation of the residues

With the notation

$$R_1^{(r)}(n_1, n_2; \kappa) := \mathrm{Res}\left[\frac{(k_0)^r}{(k_0^2 - \kappa^2 - 2k_0 m)^{n_1}(k_0^2 - \kappa^2)^{n_2}}\,;\, k_0 = m + \sqrt{\kappa^2 + m^2}\right] \tag{I.6}$$

one calculates the residues which will be needed in the following. These are

$$\begin{aligned}
R_1(1, 0; \kappa) &= \mathrm{Res}\left[\frac{1}{k_0^2 - \kappa^2 - 2k_0 m}\,;\, k_0 = m + \sqrt{\kappa^2 + m^2}\right] = \\
&= \left.\frac{1}{k_0 - m + \sqrt{\kappa^2 + m^2}}\right|_{k_0 = m + \sqrt{\kappa^2 + m^2}} = \frac{1}{2\sqrt{\kappa^2 + m^2}},
\end{aligned} \tag{I.7}$$

$$\begin{aligned}
R_1'(1, 0; \kappa) &= \mathrm{Res}\left[\frac{k_0}{k_0^2 - \kappa^2 - 2k_0 m}\,;\, k_0 = m + \sqrt{\kappa^2 + m^2}\right] = \\
&= \left.\frac{k_0}{k_0 - m + \sqrt{\kappa^2 + m^2}}\right|_{k_0 = m + \sqrt{\kappa^2 + m^2}} = \frac{m + \sqrt{\kappa^2 + m^2}}{2\sqrt{\kappa^2 + m^2}},
\end{aligned} \tag{I.8}$$

$$\begin{aligned}
R_1(1, 1; \kappa) &= \mathrm{Res}\left[\frac{1}{(k_0^2 - \kappa^2 - 2k_0 m)(k_0^2 - \kappa^2)}\,;\, k_0 = m + \sqrt{\kappa^2 + m^2}\right] = \\
&= \left.\frac{1}{(k_0 - m + \sqrt{\kappa^2 + m^2})(k_0^2 - \kappa^2)}\right|_{k_0 = m + \sqrt{\kappa^2 + m^2}} = \\
&= \frac{1}{(2\sqrt{\kappa^2 + m^2})(m^2 + 2m\sqrt{\kappa^2 + m^2} + m^2 + \kappa^2 - \kappa^2)} = \\
&= \frac{1}{4m\sqrt{\kappa^2 + m^2}(m + \sqrt{\kappa^2 + m^2})},
\end{aligned} \tag{I.9}$$

$$\begin{aligned}
R_1'(1, 1; \kappa) &= \mathrm{Res}\left[\frac{k_0}{(k_0^2 - \kappa^2 - 2k_0 m)(k_0^2 - \kappa^2)}\,;\, k_0 = m + \sqrt{\kappa^2 + m^2}\right] = \\
&= \left.\frac{k_0}{(k_0 - m + \sqrt{\kappa^2 + m^2})(k_0^2 - \kappa^2)}\right|_{k_0 = m + \sqrt{\kappa^2 + m^2}} = \\
&= \frac{m + \sqrt{\kappa^2 + m^2}}{(2\sqrt{\kappa^2 + m^2})(m^2 + 2m\sqrt{\kappa^2 + m^2} + m^2 + \kappa^2 - \kappa^2)} = \\
&= \frac{m + \sqrt{\kappa^2 + m^2}}{4m\sqrt{\kappa^2 + m^2}(m + \sqrt{\kappa^2 + m^2})} = \frac{1}{4m\sqrt{\kappa^2 + m^2}},
\end{aligned} \tag{I.10}$$

$$R_1''(1, 1; \kappa) = \mathrm{Res}\left[\frac{k_0^2}{(k_0^2 - \kappa^2 - 2k_0 m)(k_0^2 - \kappa^2)}\,;\, k_0 = m + \sqrt{\kappa^2 + m^2}\right] =$$

$$
\begin{aligned}
&= \left.\frac{k_0^2}{(k_0 - m + \sqrt{\kappa^2+m^2})(k_0^2-\kappa^2)}\right|_{k_0=m+\sqrt{\kappa^2+m^2}} = \\
&= \frac{(m+\sqrt{\kappa^2+m^2})^2}{(2\sqrt{\kappa^2+m^2})(m^2+2m\sqrt{\kappa^2+m^2}+m^2+\kappa^2-\kappa^2)} = \\
&= \frac{m+\sqrt{\kappa^2+m^2}}{4m\sqrt{\kappa^2+m^2}}, \qquad \text{(I.11)}
\end{aligned}
$$

$$
\begin{aligned}
R_1''(1,2;\kappa) &= \mathrm{Res}\left[\frac{k_0^2}{(k_0^2-\kappa^2-2k_0m)(k_0^2-\kappa^2)^2}\,;\, k_0 = m+\sqrt{\kappa^2+m^2}\right] = \\
&= \left.\frac{k_0^2}{(k_0 - m + \sqrt{\kappa^2+m^2})(k_0^2-\kappa^2)^2}\right|_{k_0=m+\sqrt{\kappa^2+m^2}} = \\
&= \frac{(m+\sqrt{\kappa^2+m^2})^2}{(2\sqrt{\kappa^2+m^2})(2m^2+2m\sqrt{\kappa^2+m^2})^2} = \frac{1}{8m^2\sqrt{\kappa^2+m^2}}. \qquad \text{(I.12)}
\end{aligned}
$$

With the definition

$$
R_2(n_1,n_2;r) := \mathrm{Res}\left[\frac{(k_0)^r}{(k_0^2-\kappa^2-2k_0m)^{n_1}(k_0^2-\kappa^2)^{n_2}}\,;\, k_0=\kappa\right] \qquad \text{(I.13)}
$$

and using similar techniques, one obtains

$$
R_2(1,0;\kappa) = 0, \qquad \text{(I.14)}
$$

$$
R_2'(1,0;\kappa) = 0, \qquad \text{(I.15)}
$$

$$
\begin{aligned}
R_2(1,1;\kappa) &= \mathrm{Res}\left[\frac{1}{(k_0^2-\kappa^2-2k_0m)(k_0^2-\kappa^2)}\,;\, k_0=\kappa\right] = \\
&= \left.\frac{1}{(k_0^2-\kappa^2-2k_0m)(k_0+\kappa)}\right|_{k_0=\kappa} = \frac{1}{(-2m\kappa)(2\kappa)} = \frac{-1}{4m\kappa^2}, \qquad \text{(I.16)}
\end{aligned}
$$

$$
\begin{aligned}
R_2'(1,1;\kappa) &= \mathrm{Res}\left[\frac{k_0}{(k_0^2-\kappa^2-2k_0m)(k_0^2-\kappa^2)}\,;\, k_0=\kappa\right] = \\
&= \left.\frac{k_0}{(k_0^2-\kappa^2-2k_0m)(k_0+\kappa)}\right|_{k_0=\kappa} = \frac{\kappa}{(-2m\kappa)(2\kappa)} = \frac{-1}{4m\kappa}, \qquad \text{(I.17)}
\end{aligned}
$$

$$
\begin{aligned}
R_2''(1,1;\kappa) &= \mathrm{Res}\left[\frac{k_0^2}{(k_0^2-\kappa^2-2k_0m)(k_0^2-\kappa^2)}\,;\, k_0=\kappa\right] = \\
&= \left.\frac{k_0^2}{(k_0^2-\kappa^2-2k_0m)(k_0+\kappa)}\right|_{k_0=\kappa} = \frac{\kappa^2}{(-2m\kappa)(2\kappa)} = \frac{-1}{4m}, \qquad \text{(I.18)}
\end{aligned}
$$

$$
\begin{aligned}
R_2''(1,2;\kappa) &= \mathrm{Res}\left[\frac{k_0^2}{(k_0^2-\kappa^2-2k_0m)(k_0^2-\kappa^2)^2}\,;\, k_0=\kappa\right] = \\
&= \frac{d}{dk_0}\left.\left(\frac{k_0^2}{(k_0^2-\kappa^2-2k_0m)(k_0+\kappa)^2}\right)\right|_{k_0=\kappa} = \\
&= \left[\frac{2k_0}{(k_0^2-\kappa^2-2k_0m)(k_0+\kappa)^2} - \frac{2k_0^2(k_0-m)}{(k_0^2-\kappa^2-2k_0m)^2(k_0+\kappa)^2} + \right. \\
&\qquad \left. - \frac{2k_0^2}{(k_0^2-\kappa^2-2k_0m)(k_0+\kappa)^3}\right]_{k_0=\kappa} =
\end{aligned}
$$

$$\begin{aligned} &= \frac{2\kappa}{(-2m\kappa)(2\kappa)^2} - \frac{2\kappa^2(\kappa-m)}{(-2m\kappa)^2(2\kappa)^2} - \frac{2\kappa^2}{(-2m\kappa)(2\kappa)^3} = \\ &= \frac{-1}{4m\kappa^2} - \frac{\kappa-m}{8m^2\kappa^2} + \frac{1}{8m\kappa^2} = -\frac{\kappa-m}{8m^2\kappa^2} - \frac{1}{8m\kappa^2} = \frac{-1}{8m^2\kappa}. \end{aligned} \tag{I.19}$$

These are all the residues that are needed in the following.

I.1.2 Calculation of the one-dimensional integrals

The residues can be combined for the one-dimensional integrals $I^{(r)}(n_1,n_2;\kappa)$, according to

$$I^{(r)}(n_1,n_2;\kappa) = R_1^{(r)}(n_1,n_2;\kappa) + R_2^{(r)}(n_1,n_2;\kappa) - I_0^{(r)}(n_1,n_2;\kappa). \tag{I.20}$$

One obtains

$$I(1,0;\kappa) = \frac{1}{2\sqrt{\kappa^2+m^2}}, \tag{I.21}$$

$$I'(1,0;\kappa) = \frac{m+\sqrt{\kappa^2+m^2}}{2\sqrt{\kappa^2+m^2}} - \frac{1}{2} = \frac{m}{2\sqrt{\kappa^2+m^2}}, \tag{I.22}$$

$$I(1,1;\kappa) = \frac{1}{4m\sqrt{\kappa^2+m^2}(m+\sqrt{\kappa^2+m^2})} - \frac{1}{4m\kappa^2}, \tag{I.23}$$

$$I'(1,1;\kappa) = \frac{1}{4m\sqrt{\kappa^2+m^2}} - \frac{1}{4m\kappa}, \tag{I.24}$$

$$I''(1,1;\kappa) = \frac{m+\sqrt{\kappa^2+m^2}}{4m\sqrt{\kappa^2+m^2}} - \frac{1}{4m} = \frac{1}{4\sqrt{\kappa^2+m^2}}, \tag{I.25}$$

$$I''(1,2;\kappa) = \frac{1}{8m^2\sqrt{\kappa^2+m^2}} - \frac{1}{8m^2\kappa}. \tag{I.26}$$

I.2 Two different methods to proceed

The next step towards the calculation of the self energy consists in reducing the four-dimensional integral to a one-dimensional integral. First, the integral measure can be rewritten as

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} &= \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_0^{\infty} \kappa^2 d\kappa \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = \\ &\to \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_0^{\infty} \kappa^2 d\kappa \int_0^{\pi} \sin\theta\, d\theta \to \frac{2}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_0^{\infty} \kappa^2 d\kappa \end{aligned} \tag{I.27}$$

where the last two transformations are valid for isotropy with respect to the azimuthal angle φ which is given in all cases considered here and the polar angle θ does *not* generalize to the two-loop case. In the one-loop case, however, one can use the last expression as replacement for the measure. Now one actually does not consider the full range in κ but only the range from $\kappa = 0$ to $\kappa = \mu$ which is the cut. There are two methods to proceed. The methods are equivalent for the one-loop case while only the last one works for the higher-loop cases.

I.2.1 The explicit cut method

The first method consists in writing

$$
\begin{aligned}
I_\mu^{(r)}(n_1,n_2) &= -i\int^\mu \frac{d^4k}{(2\pi)^4}\frac{(pk)^r}{((p-k)^2-m^2)^{n_1}(k^2)^{n_2}} = \\
&= -i\int^\mu \frac{d^4k}{(2\pi)^4}\frac{(k_0m)^r}{(k_0^2-\kappa^2-2k_0m-i\epsilon)^{n_1}(k_0^2-\kappa^2-i\epsilon)^{n_2}}
\end{aligned}
\tag{I.28}
$$

where the upper limit μ indicates symbolically the cut (the factorization scale of Sec. 5.3). The integrand is isotropic, so that one can replace the measure in the sense of Eq. (I.27) to obtain

$$
\begin{aligned}
I_\mu^{(r)}(n_1,n_2) &= \frac{1}{2\pi^2}\int_0^\mu \kappa^2 d\kappa\int_{-\infty}^\infty \frac{dk_0}{2\pi i}\frac{(k_0m)^r}{(k_0^2-\kappa^2-2k_0m-i\epsilon)^{n_1}(k_0^2-\kappa^2-i\epsilon)^{n_2}} = \\
&= \frac{m^r}{2\pi^2}\int_0^\mu \kappa^2 I^{(r)}(n_1,n_2;\kappa)d\kappa.
\end{aligned}
\tag{I.29}
$$

For the integration one uses the substitution

$$
\kappa = m\sinh\zeta = \frac{1}{2}m(e^\zeta-e^{-\zeta}) = \frac{m(t^2-1)}{2t}. \tag{I.30}
$$

The integration limits are transformed to $t(\kappa=0)=1$ and

$$
t(\kappa=\mu)=\sqrt{a^2+1}+a=:\tau,\qquad \tau^{-1}=\sqrt{a^2+1}-a,\qquad a:=\frac{\mu}{m}. \tag{I.31}
$$

One can then make use of

$$
d\kappa=\frac{m(t^2+1)}{2t^2}dt,\qquad \sqrt{\kappa^2+m^2}=\frac{m(t^2+1)}{2t}\quad\Rightarrow\quad \frac{d\kappa}{\sqrt{\kappa^2+m^2}}=\frac{dt}{t} \tag{I.32}
$$

and

$$
m+\sqrt{\kappa^2+m^2}=\frac{m(t+1)^2}{2t}. \tag{I.33}
$$

Finally one has

$$
\tau-\frac{1}{\tau}=2a,\qquad \tau^2-\frac{1}{\tau^2}=4a\sqrt{a^2+1}. \tag{I.34}
$$

Using this, one obtains

$$
\begin{aligned}
I_\mu(1,0) &= \frac{1}{2\pi^2}\int_0^\mu \frac{\kappa^2 d\kappa}{2\sqrt{\kappa^2+m^2}} = \frac{1}{2\pi^2}\int_1^\tau \frac{m^2(t^2-1)^2dt}{8t^3} = \\
&= \frac{m^2}{16\pi^2}\int_1^\tau\left(t-\frac{2}{t}+\frac{1}{t^3}\right)dt = \frac{m^2}{16\pi^2}\left[\frac{1}{2}t^2-2\ln t-\frac{1}{2t^2}\right]_1^\tau = \\
&= \frac{m^2}{16\pi^2}\left(\frac{1}{2}\tau^2-\frac{1}{2\tau^2}-2\ln\tau\right) = \frac{m^2}{8\pi^2}\left(a\sqrt{a^2+1}-\ln\left(\sqrt{a^2+1}+a\right)\right),
\end{aligned}
\tag{I.35}
$$

$$
I'_\mu(1,0) = \frac{m}{2\pi^2}\int_0^\mu \frac{\kappa^2 d\kappa}{2\sqrt{\kappa^2+m^2}} = mI_\mu(1,0) = \frac{m^3}{8\pi^2}\left(a\sqrt{a^2+1}-\ln\left(\sqrt{a^2+1}+a\right)\right), \tag{I.36}
$$

$$\begin{aligned}
I_\mu(1,1) &= \frac{1}{2\pi^2}\int_0^\mu \frac{\kappa^2 d\kappa}{4m\sqrt{\kappa^2+m^2}(m+\sqrt{\kappa^2+m^2})} - \frac{1}{2\pi^2}\int_0^\mu \frac{\kappa^2 d\kappa}{4m\kappa^2} = \\
&= \frac{1}{8\pi^2 m}\int_1^\tau \frac{m^2(t^2-1)^2 2t\,dt}{4t^3 m(t+1)^2} - \frac{\mu}{8\pi^2 m} = \\
&= \frac{1}{16\pi^2}\int_1^\tau \frac{(t-1)^2 dt}{t^2} - \frac{a}{8\pi^2} = \frac{1}{16\pi^2}\int_1^\tau \left(1-\frac{2}{t}+\frac{1}{t^2}\right) dt - \frac{a}{8\pi^2} = \\
&= \frac{1}{16\pi^2}\left[t - 2\ln t - \frac{1}{t}\right]_1^\tau - \frac{a}{8\pi^2} = \frac{1}{16\pi^2}\left(\tau - \frac{1}{\tau} - 2\ln\tau\right) - \frac{a}{8\pi^2} = \\
&= \frac{1}{8\pi^2}\left(a - \ln\left(\sqrt{a^2+1}+a\right) - a\right) = \frac{-1}{8\pi^2}\ln\left(\sqrt{a^2+1}+a\right), \qquad \text{(I.37)} \\
I'_\mu(1,1) &= \frac{m}{2\pi^2}\int_0^\mu \frac{\kappa^2 d\kappa}{4m\sqrt{\kappa^2+m^2}} - \frac{m}{2\pi^2}\int_0^\mu \frac{\kappa^2 d\kappa}{4m\kappa} = \\
&= \frac{1}{2} I_\mu(1,0) - \frac{1}{8\pi^2}\left[\frac{1}{2}\kappa^2\right]_0^\mu = \\
&= \frac{m^2}{16\pi^2}\left(a\sqrt{a^2+1} - \ln\left(\sqrt{a^2+1}+a\right)\right) - \frac{\mu^2}{16\pi^2} = \\
&= \frac{m^2}{16\pi^2}\left(a\sqrt{a^2+1} - a^2 - \ln\left(\sqrt{a^2+1}+a\right)\right), \qquad \text{(I.38)} \\
I''_\mu(1,1) &= \frac{m^2}{2\pi^2}\int_0^\mu \frac{\kappa^2 d\kappa}{4\sqrt{\kappa^2+m^2}} = \frac{m^2}{2} I_\mu(1,0) = \\
&= \frac{m^4}{16\pi^2}\left(a\sqrt{a^2+1} - \ln\left(\sqrt{a^2+1}+a\right)\right), \qquad \text{(I.39)} \\
I''_\mu(1,2) &= \frac{1}{4} I_\mu(1,0) - \frac{m^2}{2\pi^2}\int_0^\mu \frac{\kappa^2 d\kappa}{8m^2\kappa} = \\
&= \frac{m^2}{32\pi^2}\left(a\sqrt{a^2+1} - \ln\left(\sqrt{a^2+1}+a\right)\right) - \frac{1}{16\pi^2}\left[\frac{1}{2}\kappa^2\right]_0^\mu = \\
&= \frac{m^2}{32\pi^2}\left(a\sqrt{a^2+1} - a^2 - \ln\left(\sqrt{a^2+1}+a\right)\right). \qquad \text{(I.40)}
\end{aligned}$$

I.2.2 The implicit cut method

In the second method one represents the cut by a step function inside the integral, i.e. one writes

$$\begin{aligned}
I_\mu^{(r)}(n_1,n_2) &= -i\int \frac{d^4k}{(2\pi)^4}\frac{(pk)^r\theta(\mu^2-\kappa^2)}{((p-k)^2-m^2)^{n_1}(k^2)^{n_2}} = \\
&= -i\int \frac{d^4k}{(2\pi)^4}\frac{(k_0 m)^r\theta(\mu^2-\kappa^2)}{(k_0^2-\kappa^2-2k_0 m - i\epsilon)^{n_1}(k_0^2-\kappa^2-i\epsilon)^{n_2}}. \qquad \text{(I.41)}
\end{aligned}$$

If one now differentiates the integral with respect to μ^2, one can use

$$\int f(x)\delta(x^2-a^2)dx = \int \frac{f(\sqrt{x'})}{2\sqrt{x'}}\delta(x'-a^2)dx' = \frac{1}{2a}f(a) \qquad \text{(I.42)}$$

to obtain

$$\frac{d}{d\mu^2} I_\mu^{(r)}(n_1,n_2) = -i\int \frac{d^4k}{(2\pi)^4}\frac{(k_0 m)^r\delta(\mu^2-\kappa^2)}{(k_0^2-\kappa^2-2k_0 m - i\epsilon)^{n_1}(k_0^2-\kappa^2-i\epsilon)^{n_2}} =$$

$$
\begin{aligned}
&= \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi i}\int_0^\infty \frac{(k_0m)^r\delta(\mu^2-\kappa^2)\kappa^2d\kappa}{(k_0^2-\kappa^2-2k_0m-i\epsilon)^{n_1}(k_0^2-\kappa^2-i\epsilon)^{n_2}} = \\
&= \frac{2\mu^2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi i}\frac{(2\mu)^{-1}(k_0m)^r}{(k_0^2-\kappa^2-2k_0m-i\epsilon)^{n_1}(k_0^2-\kappa^2-i\epsilon)^{n_2}}\bigg|_{\kappa=\mu} = \\
&= \frac{m^r\mu}{(2\pi)^2}I^{(r)}(n_1,n_2;\kappa)\bigg|_{\kappa=\mu}. \qquad \text{(I.43)}
\end{aligned}
$$

Note that $\kappa=\mu$ is not yet inserted. This is postponed because one can also rewrite this equation as a differential equation in terms of κ, "reserving" μ for the later upper limit of the (postponed) integration. In terms of κ the differential equation thus reads

$$\frac{d}{d\kappa^2}I_\kappa(n_1,n_2)=\frac{m^r\kappa}{(2\pi)^2}I^{(r)}(n_1,n_2;\kappa). \qquad \text{(I.44)}$$

It is quite obvious that the two methods give the same result. However, as mentioned before, only the latter method can be used for the multi-loop case.

I.3 The quark self energy up to one-loop order

Arriving at the main topic, namely the determination of the soft part of the quark self energy, this calculation will be done for the three different gauges mentioned earlier.

I.3.1 The quark self energy in Feynman gauge

To start with, the self energy to one-loop order in Feynman gauge reads

$$
\begin{aligned}
-i\Sigma(\not{p}) &= \int^\mu \frac{d^4k}{(2\pi)^4}(-ig_s\gamma_\mu T_a)\frac{i}{\not{p}-\not{k}-m}(-ig_s\gamma^\mu T_a)\frac{-i}{k^2} = \\
&= -g_s^2C_F\int^\mu\frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma^\mu}{((p-k)^2-m^2)k^2} = -g_s^2C_F\int^\mu\frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(m\not{v}-\not{k}+m)\gamma^\mu}{((mv-k)^2-m^2)k^2} = \\
&= -g_s^2C_F\int^\mu\frac{d^4k}{(2\pi)^4}\frac{-2m\not{v}+2\not{k}+4m}{((mv-k)^2-m^2)k^2} = -g_s^2C_F\int^\mu\frac{d^4k}{(2\pi)^4}\frac{2m(2-\not{v})+2\not{k}}{(k^2-2mkv)k^2} = \\
&= -2g_s^2C_F\left\{(2-\not{v})\int^\mu\frac{d^4k}{(2\pi)^4}\frac{m}{(k^2-2mkv)k^2}+\not{v}\int^\mu\frac{d^4k}{(2\pi)^4}\frac{kv}{(k^2-2mkv)k^2}\right\} \qquad \text{(I.45)}
\end{aligned}
$$

so that finally

$$
\begin{aligned}
\Sigma(\not{p}) &= 2g_s^2C_F\left\{(2-\not{v})mI_\mu(1,1)+\not{v}I'_\mu(1,1)\right\} = \\
&= \frac{g_s^2C_F}{8\pi^2}m\left\{-2(2-\not{v})\ln\left(\sqrt{a^2+1}+a\right)+\not{v}\left(a\sqrt{a^2+1}-a^2-\ln\left(\sqrt{a^2+1}+a\right)\right)\right\} = \\
&= -\frac{\alpha_sC_F}{2\pi}m\left\{4\ln\left(\sqrt{a^2+1}+a\right)+\not{v}\left(a^2-a\sqrt{a^2+1}-\ln\left(\sqrt{a^2+1}+a\right)\right)\right\}. \qquad \text{(I.46)}
\end{aligned}
$$

For $a\ll 1$ one obtains

$$
\begin{aligned}
\Sigma(\not{p}) &= -\frac{\alpha_sC_F}{2\pi}m\left\{4a-\frac{2a^3}{3}-\not{v}\left(2a-a^2+\frac{a^3}{3}\right)+O(a^4)\right\} = \qquad \text{(I.47)} \\
&\to -\frac{\alpha_sC_F}{2\pi}m\left\{2a+a^2-a^3+O(a^4)\right\} = -\frac{\alpha_sC_F}{\pi}\mu\left\{1+\frac{\mu}{2m}-\frac{\mu^2}{2m^2}+O\left(\frac{\mu^3}{m^3}\right)\right\}
\end{aligned}
$$

where the arrow indicates the application of the equation of motion, i.e. $\not{v}\to 1$.

I.3.2 The quark self energy in Coulomb gauge

The Feynman rule for the gluon propagator in Coulomb gauge is taken from Ref. [291],

$$G^{ab}_{\mu\nu}(k)=\frac{-i\delta^{ab}}{k^2}\left[g_{\mu\nu}-\left(\frac{k_\mu k_\nu-(kv)(k_\mu v_\nu+v_\mu k_\nu)}{-\vec{k}^{\,2}}\right)\right]. \tag{I.48}$$

Because of

$$\begin{aligned}-i\Sigma^C(\not{p}) &= \int^\mu\frac{d^4k}{(2\pi)^4}(-ig_s\gamma^\mu T_a)\frac{i}{\not{p}-\not{k}-m}(-ig_s\gamma^\nu T_a)\times\\ &\quad\times\frac{-i}{k^2}\left(g_{\mu\nu}-\frac{k_\mu k_\nu-(kv)(k_\mu v_\nu+v_\mu k_\nu)}{-\vec{k}^{\,2}}\right) =\\ &= -g_s^2C_F\int^\mu\frac{d^4k}{(2\pi)^4}\frac{\Gamma^\mu(\not{p}-\not{k}+m)\gamma^\nu}{((p-k)^2-m^2)k^2}\left(g_{\mu\nu}-\frac{k_\mu k_\nu-(kv)(k_\mu v_\nu+v_\mu k_\nu)}{-\vec{k}^{\,2}}\right) =\\ &=: -i\Sigma^F(\not{p})-i\Sigma^{\Delta C}(\not{p}).\end{aligned} \tag{I.49}$$

one only has to calculate the difference $\Sigma^{\Delta C}$ to the contribution in Feynman gauge Σ^F calculated just before. One obtains

$$\begin{aligned}\Sigma^{\Delta C}(\not{p}) &= -ig_s^2C_F\int^\mu\frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma^\nu}{((p-k)^2-m^2)k^2\vec{k}^{\,2}}\left(k_\mu k_\nu-(kv)(k_\mu v_\nu+v_\mu k_\nu)\right) = \\ &= -ig_s^2C_F\int^\mu\frac{d^4k}{(2\pi)^4}\frac{\not{k}(\not{p}-\not{k}+m)\not{k}-(kv)(\not{k}(\not{p}-\not{k}+m)\not{v}+\not{v}(\not{p}-\not{k}+m)\not{k})}{((p-k)^2-m^2)k^2\vec{k}^{\,2}}.\end{aligned} \tag{I.50}$$

Inserting the on-shell momentum $p=mv$, the numerator of the integrand is given by

$$\begin{aligned}&\not{k}(m\not{v}-\not{k}+m)\not{k}-(kv)\left(\not{k}(m\not{v}-\not{k}+m)\not{v}+\not{v}(m\not{v}-\not{k}+m)\not{k}\right) =\\ &= \ldots = 2k_0k^2\not{v}-mk^2\not{v}-k^2\not{k}-2mk_0^2+mk^2 =\\ &\to 2k_0k^2\not{v}-mk^2\not{v}-k^2k_0\not{v}-2mk_0^2+mk^2 = mk^2(1-\not{v})+k_0k^2\not{v}-2mk_0^2\end{aligned} \tag{I.51}$$

where the arrow indicates that the expansion of the integral into covariants is taken into account in advance (note that the only outer vector is v). This leads to

$$\begin{aligned}\Sigma^{\Delta C}(\not{p}) &= -ig_s^2C_F\int^\mu\frac{d^4k}{(2\pi)^4}\frac{mk^2(1-\not{v})+k_0k^2\not{v}-2mk_0^2}{(k^2-2mk_0)k^2\vec{k}^{\,2}} =\\ &=: \frac{g_s^2C_F}{2\pi^2}\int_0^\mu\left(m(1-\not{v})I(1,0;\kappa)+\not{v}I'(1,0;\kappa)-2mI''(1,1;\kappa)\right)d\kappa\end{aligned} \tag{I.52}$$

where the $\vec{k}^{\,2}$ in the denominator cancels the κ^2 in the integration measure. The integral parts are not given by the integrals $I_\mu(n_1,n_2)$, instead one obtains

$$\begin{aligned}\int_0^\mu I(1,0;\kappa)d\kappa &= \int_0^\mu\frac{d\kappa}{2\sqrt{\kappa^2+m^2}} = \int_1^\tau\frac{dt}{2t} =\\ &= \frac{1}{2}\ln\tau = \frac{1}{2}\ln\left(\sqrt{a^2+1}+a\right),\end{aligned} \tag{I.53}$$

$$\int_0^\mu I'(1,0;\kappa)d\kappa = m\int_0^\mu I(1,0;\kappa)d\kappa = \frac{m}{2}\ln\left(\sqrt{a^2+1}+a\right), \tag{I.54}$$

$$\int_0^\mu I''(1,1;\kappa)d\kappa = \frac{1}{2}\int_0^\mu I(1,0;\kappa)d\kappa = \frac{1}{4}\ln\left(\sqrt{a^2+1}+a\right). \tag{I.55}$$

It is easy to see that $\Sigma^{\Delta C}(\not{p})$ vanishes since

$$\Sigma^{\Delta C}(\not{p}) = \frac{g_s^2 C_F m}{8\pi^2}\left((1-\not{v}) + \not{v} - 1\right)\ln\left(\sqrt{a^2+1}+a\right) = 0. \tag{I.56}$$

It is obvious that the result for the Coulomb gauge is the same as for Feynman gauge.

I.3.3 The quark self energy in general covariant gauge

The Feynman rule for the gluon propagator in the general covariant gauge (a generalization of the Landau gauge which is obtained for $\alpha_g = 0$) is given by

$$G^{ab}_{\mu\nu}(k) = \frac{-i\delta_{ab}}{k^2}\left(g_{\mu\nu} - (1-\alpha_g)\frac{k_\mu k_\nu}{k^2}\right). \tag{I.57}$$

Therefore, one obtains

$$\begin{aligned}
-i\Sigma^L(\not{p}) &= \int^\mu \frac{d^4k}{(2\pi)^4}(-ig_s\gamma^\mu T_a)\frac{i}{\not{p}-\not{k}-m}(-ig_s\gamma^\nu T_a)\frac{-i}{k^2}\left(g_{\mu\nu} - (1-\alpha_g)\frac{k_\mu k_\nu}{k^2}\right) = \\
&= -g_s^2 C_F \int^\mu \frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma^\nu}{((p-k)^2-m^2)k^2}\left(g_{\mu\nu} - (1-\alpha_g)\frac{k_\mu k_\nu}{k^2}\right) = \\
&=: -i\Sigma^F(\not{p}) - i(1-\alpha_g)\Sigma^{\Delta L}(\not{p}).
\end{aligned} \tag{I.58}$$

The difference term $-i(1-\alpha_g)\Sigma^{\Delta L}$ can be calculated to give

$$-i(1-\alpha_g)\Sigma^{\Delta L}(\not{p}) = (1-\alpha_g)g_s^2 C_F \int^\mu \frac{d^4k}{(2\pi)^4}\frac{\not{k}(\not{p}-\not{k}+m)\not{k}}{((p-k)^2-m^2)(k^2)^2}. \tag{I.59}$$

On the mass shell the numerator gives

$$\begin{aligned}
\not{k}(m(1+\not{v})-\not{k})\not{k} &= mk^2 + 2mk_0\not{k} - mk^2\not{v} - k^2\not{k} = \\
&\to mk^2 + 2mk_0^2\not{v} - mk^2\not{v} - k_0k^2\not{v} = \\
&= m(1-\not{v})k^2 - k_0k^2\not{v} + 2mk_0^2\not{v}
\end{aligned} \tag{I.60}$$

(the arrow indicates again the expansion in invariants) and thus

$$\begin{aligned}
\Sigma^{\Delta L}(\not{p}) &= \frac{g_s^2 C_F}{2\pi^2}\int_0^\mu \left(-(1-\not{v})mI(1,1;\kappa) + \not{v}I'(1,1;\kappa) - 2m\not{v}I''(1,2;\kappa)\right)\kappa^2 d\kappa = \\
&= g_s^2 C_F\left\{-(1-\not{v})mI_\mu(1,1) + \frac{1}{m}\not{v}I'_\mu(1,1) - \frac{2}{m}\not{v}I''_\mu(1,2)\right\} = \\
&= \frac{g_s^2 C_F}{16\pi^2}m\Bigg\{(2(1-\not{v})\ln\left(\sqrt{a^2+1}+a\right) + \\
&\qquad +\not{v}\left(a\sqrt{a^2+1} - a^2 - \ln\left(\sqrt{a^2+1}+a\right)\right) + \\
&\qquad -\not{v}\left(a\sqrt{a^2+1} - a^2 - \ln\left(\sqrt{a^2+1}+a\right)\right)\Bigg\} = \\
&= \frac{\alpha_s C_F}{2\pi}m(1-\not{v})\ln\left(\sqrt{a^2+1}+a\right).
\end{aligned} \tag{I.61}$$

$\Sigma^{\Delta L}(\not{p})$ vanishes because of the equation of motion $\not{v} - 1 = 0$.

Appendix J

Non-abelian soft self energy

In this appendix the non-abelian diagrams which contribute to the two-loop correction of the quark self energy diagram in Section 5.3 are analyzed in Coulomb gauge (Fig. 5.12).

J.1 Contributions of diagrams

Only the pure Dirac structure of the integrand for the diagrams is considered in this step. The notation (cba) denotes the vertex structure along the direction of the arrow of the quark line (the entry 0 represents the coupling to a Coulomb gluon, i, j, k the coupling to a transverse gluon). The expressions are taken between on-shell states which are represented by the projectors $(1+\gamma^0)/2$ on both sides of the structure. The first step is to simplify the expression. The gluons attached to the quark line are internal gluons. Therefore, one gluon propagator is contracted. In Coulomb gauge this propagator reads (cf. Eq. (5.98))

$$G_{00}^{ab}(k) = \frac{i\delta^{ab}}{\vec{k}^{\,2}}, \qquad G_{ij}^{ab}(k) = \frac{-i\delta^{ab}}{k^2\vec{k}^{\,2}}\left(\vec{k}^{\,2}g_{ij} + k_ik_j\right), \qquad i,j = 1,2,3. \tag{J.1}$$

If one adds these propagators in the second step, the γ^i and γ^j are modified. If the index c is a space index, γ^c is replaced by $(\vec{k}_1^2\gamma^c + k_1^c\hat{k}_1)$ where $\hat{k} = \gamma^i k_i = -\vec{\gamma}\cdot\vec{k}$ and the denominator is multiplied by $-k_1^2$. If index b is a space index, γ^b is replaced by $((\vec{k}_1 - \vec{k}_2)^2\gamma^b + (k_1 - k_2)^b(\hat{k}_1 - \hat{k}_2))$ and the denominator is multiplied by $-(k_1 - k_2)^2$. Finally, if a is a space index, γ^a is replaced by $(\vec{k}_2^2\gamma^a + k_2^a\hat{k}_2)$ and the denominator is multiplied by $-k_2^2$. In the third step one considers the coupling of these internal gluons in the *three-gluon vertex*

$$g_s f_{abc}\left((k_2 - 2k_1)_a g_{bc} + (k_1 + k_2)_b g_{ca} + (k_1 - 2k_2)_c g_{ab}\right) \tag{J.2}$$

For simplicity the same labels have been used here for the Lorentz and for the colour indices. The factor except for $g_s f_{abc}$ is called the *three-gluon factor* in the following. For the simplification of the expression one can use

$$\gamma^i\gamma_i = 3, \quad \hat{k}\hat{k} = -\vec{k}^{\,2}, \quad \gamma^i\hat{k}\gamma_i = -\hat{k}, \quad \hat{k}_a\hat{k}_b\hat{k}_a = \vec{k}_a^2\hat{k}_b - 2(\vec{k}_a\vec{k}_b)\hat{k}_a. \tag{J.3}$$

Note, finally, that the calculation is done in the quark rest frame. One can therefore use $\not{p} = m\gamma^0$.

J.1.1 The diagrams $(i00)$ und $(00i)$

For the diagram $(i00)$ (Fig. 5.12(a)) one obtains

$$\begin{aligned}
&\frac{1+\gamma^0}{2}\gamma^0(\not{p}+\not{k}_2+m)\gamma^0(\not{p}+\not{k}_1+m)\gamma^i\frac{1+\gamma^0}{2} \;=\\
&= \frac{1+\gamma^0}{2}\left(m(1+\gamma^0)+k_{20}\gamma^0+\hat{k}_2\right)\gamma^0\left(m(1+\gamma^0)+k_{10}\gamma^0+\hat{k}_1\right)\frac{1-\gamma^0}{2}\gamma^i \;=\\
&= \left(2m\frac{1+\gamma^0}{2}+k_{20}\frac{1+\gamma^0}{2}+\hat{k}_2\frac{1-\gamma^0}{2}\right)\gamma^0\left(-k_{10}\frac{1-\gamma^0}{2}+\frac{1+\gamma^0}{2}\hat{k}_1\right)\gamma^i \;=\\
&= \left((2m+k_{20})\frac{1+\gamma^0}{2}+\hat{k}_2\frac{1-\gamma^0}{2}\right)\left(k_{10}\frac{1-\gamma^0}{2}+\frac{1+\gamma^0}{2}\hat{k}_1\right)\gamma^i \;=\\
&= \left((2m+k_{20})\frac{1+\gamma^0}{2}\hat{k}_1+\hat{k}_2\frac{1-\gamma^0}{2}k_{10}\right)\gamma^i \;=\; \left((2m+k_{20})\hat{k}_1+\hat{k}_2k_{10}\right)\gamma^i\frac{1+\gamma^0}{2}.
\end{aligned} \tag{J.4}$$

The attachment of the transverse gluon propagator leads to

$$\left\{(2m+k_{20})\hat{k}_1+\hat{k}_2k_{10}\right\}\left(\vec{k}_1^2\gamma^i+k_1^i\hat{k}_1\right). \tag{J.5}$$

In addition, one obtains a factor $-k_2^2$ for the denominator (the projector have been omitted again). The contribution of the three-gluon factor is $(k_1-2k_2)_i g_{00}$. For the diagram $(00i)$ (Fig. 5.12(c)) one obtains

$$\begin{aligned}
&\frac{1+\gamma^0}{2}\gamma^i(\not{p}+\not{k}_2+m)\gamma^0(\not{p}+\not{k}_1+m)\gamma^0\frac{1+\gamma^0}{2} \;=\\
&= \gamma^i\frac{1-\gamma^0}{2}\left(m(1+\gamma^0)+k_{20}\gamma^0+\hat{k}_2\right)\gamma^0\left(m(1+\gamma^0)+k_{10}\gamma^0+\hat{k}_1\right)\frac{1+\gamma^0}{2} \;=\\
&= \gamma^i\left(-k_{20}\frac{1-\gamma^0}{2}+\hat{k}_2\frac{1+\gamma^0}{2}\right)\gamma^0\left(2m\frac{1+\gamma^0}{2}+k_{10}\frac{1+\gamma^0}{2}+\frac{1-\gamma^0}{2}\hat{k}_1\right) \;=\\
&= \gamma^i\left(k_{20}\frac{1-\gamma^0}{2}+\hat{k}_2\frac{1+\gamma^0}{2}\right)\left((2m+k_{10})\frac{1+\gamma^0}{2}+\frac{1-\gamma^0}{2}\hat{k}_1\right) \;=\\
&= \gamma^i\left(\hat{k}_2\frac{1+\gamma^0}{2}(2m+k_{10})+k_{20}\frac{1-\gamma^0}{2}\hat{k}_1\right) \;=\; \gamma^i\left(\hat{k}_2(2m+k_{10})+k_{20}\hat{k}_1\right)\frac{1+\gamma^0}{2}.
\end{aligned} \tag{J.6}$$

The attachment of the transverse gluon leads to

$$\left(\vec{k}_2^2\gamma^i+k_2^i\hat{k}_2\right)\left\{\hat{k}_2(2m+k_{10})+k_{20}\hat{k}_1\right\} \tag{J.7}$$

and a factor $-k_1^2$ for the denominator. The three-gluon factor is given by $(k_2-2k_1)_i g_{00}$.

Note that the replacement $k_1 \leftrightarrow k_2$ transforms the denominator and the three-gluon factor of diagram $(i00)$ into the denominator and the three-gluon factor of diagram $(00i)$. If one therefore sums up both contributions in the form of the first one, the Dirac structure reduces to

$$\begin{aligned}
&(2m+k_{20})\left(\vec{k}_1^2(\hat{k}_1\gamma^i+\gamma^i\hat{k}_1)-2\vec{k}_1^2k_1^i\right)+\left(k_1^2(\hat{k}_2\gamma^i+\gamma^i\hat{k}_2)+k_1^i(\hat{k}_2\hat{k}_1+\hat{k}_1\hat{k}_2)\right)k_{10} \;=\\
&= (2m+k_{20})\left(2\vec{k}_1^2k_1^i-2\vec{k}_1^2k_1^i\right)+\left(2\vec{k}_1^2k_2^i-2(\vec{k}_1\vec{k}_2)k_1^i\right)k_{10} \;=\; 2\left(\vec{k}_1^2k_2^i-(\vec{k}_1\vec{k}_2)k_1^i\right)k_{10}.
\end{aligned} \tag{J.8}$$

The part proportional to m vanishes. Therefore, this contribution is suppressed by one power of $1/m$.

J.1.2 The diagram $(0i0)$

For the symmetric diagram $(0i0)$ (Fig. 5.12(b)) one obtains

$$\begin{aligned}
&\frac{1+\gamma^0}{2}\gamma^0(\not{p}+\not{k}_2+m)\gamma^i(\not{p}+\not{k}_1+m)\gamma^0\frac{1+\gamma^0}{2} = \\
&= \frac{1+\gamma^0}{2}\left(m(1+\gamma^0)+k_{20}\gamma^0+\hat{k}_2\right)\gamma^i\left(m(1+\gamma^0)+k_{10}\gamma^0+\hat{k}_1\right)\frac{1+\gamma^0}{2} = \\
&= \left(2m\frac{1+\gamma^0}{2}+k_{20}\frac{1+\gamma^0}{2}+\hat{k}_2\frac{1-\gamma^0}{2}\right)\gamma^i\left(2m\frac{1+\gamma^0}{2}+k_{10}\frac{1+\gamma^0}{2}+\frac{1-\gamma^0}{2}\hat{k}_1\right) = \\
&= \left((2m+k_{20})\gamma^i\frac{1-\gamma^0}{2}+\hat{k}_2\gamma^i\frac{1+\gamma^0}{2}\right)\left((2m+k_{10})\frac{1+\gamma^0}{2}+\frac{1-\gamma^0}{2}\hat{k}_1\right) = \\
&= (2m+k_{20})\gamma^i\frac{1-\gamma^0}{2}\hat{k}_1+\hat{k}_2\gamma^i\frac{1+\gamma^0}{2}(2m+k_{10}) = \\
&= \left((2m+k_{20})\gamma^i\hat{k}_1+\hat{k}_2\gamma^i(2m+k_{10})\right)\frac{1+\gamma^0}{2}. \qquad (J.9)
\end{aligned}$$

The attachment of the transverse gluon propagator leads to

$$\begin{aligned}
&(2m+k_{20})\left((\vec{k}_1-\vec{k}_2)^2\gamma^i+(k_1-k_2)^i(\hat{k}_1-\hat{k}_2)\right)\hat{k}_1+ \\
&+\hat{k}_2\left((\vec{k}_1-\vec{k}_2)^2\gamma^i+(k_1-k_2)^i(\hat{k}_1-\hat{k}_2)\right)(2m+k_{10}) \qquad (J.10)
\end{aligned}$$

and a factor $-(k_1-k_2)^2$ for the denominator. The three-gluon factor reads $(k_1+k_2)_i g_{00}$. In this case, the remaining factors are symmetric under the exchange $k_1 \leftrightarrow k_2$. Therefore, one can symmetrize the Dirac structure and obtains

$$\begin{aligned}
&(2m+k_{20})\left((\vec{k}_1-\vec{k}_2)^2k_1^i-(k_1-k_2)^i(\vec{k}_1^2-\vec{k}_1\vec{k}_2)\right)+ \\
&+\left((\vec{k}_1-\vec{k}_2)^2k_2^i-(k_1-k_2)^i(\vec{k}_1\vec{k}_2-\vec{k}_2^2)\right)(2m+k_{10}). \qquad (J.11)
\end{aligned}$$

The contraction with the three-gluon factor leads to the very simple result

$$-2(4m+k_{10}+k_{20})\left(\vec{k}_1^2\vec{k}_2^2-(\vec{k}_1\vec{k}_2)^2\right). \qquad (J.12)$$

J.1.3 The diagrams $(0ji)$ and $(ij0)$

For the diagram $(0ji)$ (Fig. 5.12(d)) the Dirac structure is given by

$$\begin{aligned}
&\frac{1+\gamma^0}{2}\gamma^i(\not{p}+\not{k}_2+m)\gamma^j(\not{p}+\not{k}_1+m)\gamma^0\frac{1+\gamma^0}{2} = \\
&= \gamma^i\frac{1-\gamma^0}{2}\left(m(1+\gamma^0)+k_{20}\gamma^0+\hat{k}_2\right)\gamma^j\left(m(1+\gamma^0)+k_{10}\gamma^0+\hat{k}_1\right)\frac{1+\gamma^0}{2} = \\
&= \gamma^i\left(-k_{20}\frac{1-\gamma^0}{2}+\hat{k}_2\frac{1+\gamma^0}{2}\right)\gamma^j\left(2m\frac{1+\gamma^0}{2}+k_{10}\frac{1+\gamma^0}{2}+\frac{1-\gamma^0}{2}\hat{k}_1\right) = \\
&= \gamma^i\left(-k_{20}\gamma^j\frac{1+\gamma^0}{2}+\hat{k}_2\gamma^j\frac{1-\gamma^0}{2}\right)\left((2m+k_{10})\frac{1+\gamma^0}{2}+\frac{1-\gamma^0}{2}\hat{k}_1\right) = \\
&= \gamma^i\left(-k_{20}\gamma^j\frac{1+\gamma^0}{2}(2m+k_{10})+\hat{k}_2\gamma^j\frac{1-\gamma^0}{2}\hat{k}_1\right) = \\
&= \gamma^i\left(-k_{20}\gamma^j(2m+k_{10})+\hat{k}_2\gamma^j\hat{k}_1\right)\frac{1+\gamma^0}{2}. \qquad (J.13)
\end{aligned}$$

The attachment of transverse gluons results in

$$\begin{aligned}&-k_{20}(\vec{k}_2^2\gamma^i + k_2^i\hat{k}_2)\left((\vec{k}_1-\vec{k}_2)^2\gamma^j + (k_1-k_2)^j(\hat{k}_1-\hat{k}_2)\right)(2m+k_{10}) + \\ &+(\vec{k}_2^2\gamma^i + k_2^i\hat{k}_2)\hat{k}_2\left((\vec{k}_1-\vec{k}_2)^2\gamma^j + (k_1-k_2)^j(\hat{k}_1-\hat{k}_2)\right)\hat{k}_1\end{aligned} \tag{J.14}$$

and a factor $k_2^2(k_1-k_2)^2$ for the denominator. The three-gluon factor reads $(k_1-2k_2)_0 g_{ij}$. Contracting with the three-gluon factor, the expression can be further simplified,

$$\begin{aligned}&-(k_{10}-2k_{20})k_{20}(\vec{k}_2^2\gamma^i + k_2^i\hat{k}_2)\left((\vec{k}_1-\vec{k}_2)^2\gamma_i + (k_1-k_2)_i(\hat{k}_1-\hat{k}_2)\right)(2m+k_{10}) + \\ &\quad +(k_{10}-2k_{20})(\vec{k}_2^2\gamma^i + k_2^i\hat{k}_2)\hat{k}_2\left((\vec{k}_1-\vec{k}_2)^2\gamma_i + (k_1-k_2)_i(\hat{k}_1-\hat{k}_2)\right)\hat{k}_1 = \\ &= -(k_{10}-2k_{20})k_{20}\left(\vec{k}_2^2(\vec{k}_1-\vec{k}_2)^2 - (\vec{k}_1\vec{k}_2-\vec{k}_2^2)\hat{k}_2(\hat{k}_1-\hat{k}_2)\right)(2m+k_{10}) + \\ &\quad -(k_{10}-2k_{20})\left(\vec{k}_2^2(\vec{k}_1-\vec{k}_2)^2\hat{k}_2 + \vec{k}_2^2(\vec{k}_1\vec{k}_2-\vec{k}_2^2)(\hat{k}_1-\hat{k}_2)\right)\hat{k}_1.\end{aligned} \tag{J.15}$$

For the diagram $(ij0)$ (Fig. 5.12(f)) one obtains

$$\begin{aligned}&\frac{1+\gamma^0}{2}\gamma^0(\not{p}+\not{k}_2+m)\gamma^j(\not{p}+\not{k}_1+m)\gamma^i\frac{1+\gamma^0}{2} = \\ &= \frac{1+\gamma^0}{2}\left(m(1+\gamma^0)+k_{20}\gamma^0+\hat{k}_2\right)\gamma^j\left(m(1+\gamma^0)+k_{10}\gamma^0+\hat{k}_1\right)\frac{1-\gamma^0}{2}\gamma^i = \\ &= \left(2m\frac{1+\gamma^0}{2}+k_{20}\frac{1+\gamma^0}{2}+\hat{k}_2\frac{1-\gamma^0}{2}\right)\gamma^j\left(-k_{10}\frac{1-\gamma^0}{2}+\frac{1+\gamma^0}{2}\hat{k}_1\right)\gamma^i = \\ &= \left((2m+k_{20})\frac{1+\gamma^0}{2}+\hat{k}_2\frac{1-\gamma^0}{2}\right)\left(-k_{10}\frac{1+\gamma^0}{2}\gamma^j+\frac{1-\gamma^0}{2}\gamma^j\hat{k}_1\right)\gamma^i = \\ &= \left(-(2m+k_{20})\frac{1+\gamma^0}{2}k_{10}\gamma^j+\hat{k}_2\frac{1-\gamma^0}{2}\gamma^j\hat{k}_1\right)\gamma^i = \\ &= \left(-(2m+k_{20})k_{10}\gamma^j+\hat{k}_2\gamma^j\hat{k}_1\right)\gamma^i\frac{1+\gamma^0}{2}.\end{aligned} \tag{J.16}$$

Attaching the transverse gluons leads to

$$\begin{aligned}&-(2m+k_{20})\left((\vec{k}_1-\vec{k}_2)^2\gamma^j + (k_1-k_2)^j(\hat{k}_1-\hat{k}_2)\right)k_{10}(\vec{k}_1^2\gamma^i + k_1^i\hat{k}_1) + \\ &+\hat{k}_2\left((\vec{k}_1-\vec{k}_2)^2\gamma^j + (k_1-k_2)^j(\hat{k}_1-\hat{k}_2)\right)\hat{k}_2(\vec{k}_1^2\gamma^i + k_1^i\hat{k}_1)\end{aligned} \tag{J.17}$$

and a factor $k_1^2(k_1-k_2)^2$ to the denominator. The contraction with the three-gluon factor $(k_2-2k_1)_0 g_{ij}$ finally leads to

$$\begin{aligned}&-(2m+k_{20})(k_{20}-2k_{10})\left((\vec{k}_1-\vec{k}_2)^2\gamma_i + (k_1-k_2)_i(\hat{k}_1-\hat{k}_2)\right)k_{10}(\vec{k}_1^2\gamma^i + k_1^i\hat{k}_1) + \\ &\quad +(k_{20}-2k_{10})\hat{k}_2\left((\vec{k}_1-\vec{k}_2)^2\gamma_i + (k_1-k_2)_i(\hat{k}_1-\hat{k}_2)\right)\hat{k}_2(\vec{k}_1^2\gamma^i + k_1^i\hat{k}_1) = \\ &= -(2m+k_{20})(k_{20}-2k_{10})\left(\vec{k}_1^2(\vec{k}_1-\vec{k}_2)^2 - (\vec{k}_1^2-\vec{k}_1\vec{k}_2)(\hat{k}_1-\hat{k}_2)\hat{k}_1\right)k_{10} + \\ &\quad -(k_{20}-2k_{10})\hat{k}_2\left(\vec{k}_1^2(\vec{k}_1-\vec{k}_2)^2\hat{k}_1 + \vec{k}_1^2(\vec{k}_1^2-\vec{k}_1\vec{k}_2)(\hat{k}_1-\hat{k}_2)\right).\end{aligned} \tag{J.18}$$

The result of the symmetrization of both results in $k_1 \leftrightarrow k_2$ (in the first form) reads

$$\begin{aligned}&-k_{20}(k_{10}-2k_{20})\left(\vec{k}_2^2(\vec{k}_1-\vec{k}_2)^2 + (\vec{k}_1\vec{k}_2-\vec{k}_2^2)^2\right)(2m+k_{10}) + \\ &+\vec{k}_2^2\left((\vec{k}_1-\vec{k}_2)^2(\vec{k}_1\vec{k}_2) + (\vec{k}_1\vec{k}_2-\vec{k}_2^2)(\vec{k}_1^2-\vec{k}_1\vec{k}_2)\right).\end{aligned} \tag{J.19}$$

J.1.4 The diagram $(i0j)$

Diagrams with transverse gluons at both ends will have contributions proportional to m. Even though for this reason these diagrams can be omitted in the calculation to leading order in m, the results will be shown here as well. The first example is the diagram $(i0j)$ (Fig. 5.12(e)),

$$\begin{aligned}
&\frac{1+\gamma^0}{2}\gamma^j(\not{p}+\not{k}_2+m)\gamma^0(\not{p}+\not{k}_1+m)\gamma^i\frac{1+\gamma^0}{2} \;=\\
&= \gamma^j\frac{1-\gamma}{2}\left(m(1+\gamma^0)+k_{20}\gamma^0+\hat{k}_2\right)\gamma^0\left(m(1+\gamma^0)+k_{10}\gamma^0+\hat{k}_1\right)\frac{1-\gamma^0}{2}\gamma^i \;=\\
&= \gamma^j\left(-k_{20}\frac{1-\gamma^0}{2}+\hat{k}_2\frac{1+\gamma^0}{2}\right)\gamma^0\left(-k_{10}\frac{1-\gamma^0}{2}+\frac{1+\gamma^0}{2}\hat{k}_1\right)\gamma^i \;=\\
&= \gamma^j\left(-k_{20}\frac{1-\gamma^0}{2}+\hat{k}_2\frac{1+\gamma^0}{2}\right)\left(k_{10}\frac{1-\gamma^0}{2}+\frac{1+\gamma^0}{2}\hat{k}_1\right)\gamma^i \;=\\
&= \gamma^j\left(\hat{k}_2\frac{1+\gamma^0}{2}\hat{k}_1-k_{20}\frac{1-\gamma^0}{2}k_{10}\right)\gamma^i \;=\; \gamma^j\left(\hat{k}_2\hat{k}_1-k_{20}k_{10}\right)\gamma^i\frac{1+\gamma^0}{2}.
\end{aligned} \tag{J.20}$$

J.1.5 The diagram (ijk)

The second example is given by the diagram (ijk) (Fig. 5.12(g)),

$$\begin{aligned}
&\frac{1+\gamma^0}{2}\gamma^k(\not{p}+\not{k}_2+m)\gamma^j(\not{p}+\not{k}_1+m)\gamma^i\frac{1+\gamma^0}{2} \;=\\
&= \gamma^k\frac{1-\gamma^0}{2}\left(m(1+\gamma^0)+k_{20}\gamma^0+\hat{k}_2\right)\gamma^j\left(m(1+\gamma^0)+k_{10}\gamma^0+\hat{k}_1\right)\frac{1-\gamma^0}{2}\gamma^i \;=\\
&= \gamma^k\left(-k_{20}\frac{1-\gamma^0}{2}+\hat{k}_2\frac{1+\gamma^0}{2}\right)\gamma^j\left(-k_{10}\frac{1-\gamma^0}{2}+\frac{1+\gamma^0}{2}\hat{k}_1\right)\gamma^i \;=\\
&= \gamma^k\left(-k_{20}\gamma^j\frac{1+\gamma^0}{2}+\hat{k}_2\gamma^j\frac{1-\gamma^0}{2}\right)\left(-k_{10}\frac{1-\gamma^0}{2}+\frac{1+\gamma^0}{2}\hat{k}_1\right)\gamma^i \;=\\
&= \gamma^k\left(-k_{20}\gamma^j\frac{1+\gamma^0}{2}\hat{k}_1-\hat{k}_2\gamma^j\frac{1-\gamma^0}{2}k_{10}\right)\gamma^i \;=\; -\gamma^k\left(k_{20}\gamma^j\hat{k}_1+\hat{k}_2\gamma^j k_{10}\right)\gamma^i\frac{1+\gamma^0}{2}.
\end{aligned} \tag{J.21}$$

J.2 The main contribution in Coulomb gauge

The diagram $(0i0)$ is the main contribution to the non-abelian two-loop correction. The colour structure which was omitted up to now is given by $T_aT_bT_cf_{abc} = iC_FC_A/2$ with $C_F = (N_c^2-1)/2N_c$ and $C_A = N_c$ (N_c is the number of colours). Starting with

$$\begin{aligned}
-i\Sigma^{(b)}(\not{p}) \;=\; &\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{\vec{k}_1^2}\frac{i}{\vec{k}_2^2}\frac{i}{(k_1-k_2)^2}\left(\delta_{ij}-\frac{(k_1-k_2)_i(k_1-k_2)_j}{(\vec{k}_1-\vec{k}_2)^2}\right)\times\\
&\times g_sf_{abc}\left((k_2-2k_1)^0g^{j0}+(k_1+k_2)^jg^{00}+(k_1-2k_2)^0g^{0j}\right)\times\\
&\times\frac{1+\gamma^0}{2}(-ig_s\gamma^0T_a)\frac{i}{\not{p}+\not{k}_2-m}(-ig_s\gamma^iT_b)\frac{i}{\not{p}+\not{k}_1-m}(-ig_s\gamma^0T_c)\frac{1+\gamma^0}{2}.
\end{aligned} \tag{J.22}$$

One finally obtains

$$\Sigma^{(b)}(\not{p}) = g_s^4 C_F C_A \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \times \tag{J.23}$$

$$\times \frac{(4m + k_{10} + k_{20}) \left(\vec{k}_1^2 \vec{k}_2^2 - (\vec{k}_1 \vec{k}_2)^2 \right)}{\vec{k}_1^2 \vec{k}_2^2 (\vec{k}_1 - \vec{k}_2)^2 (k_1 - k_2)^2 ((p + k_1)^2 - m^2)((p + k_2)^2 - m^2)}. \tag{J.24}$$

J.2.1 The soft part of the self energy correction

In order to extract the soft part one makes the replacements

$$\frac{1}{(p + k_1)^2 - m^2} \to -i\pi\delta((p + k_1)^2 - m^2), \qquad \frac{1}{(p + k_2)^2 - m^2} \to -i\pi\delta((p + k_2)^2 - m^2) \tag{J.25}$$

and sums up the various contributions. This procedure will be detailed here in the case where the line with momentum k_1 is cut, i.e. for the first replacement. For the time being the part which is relevant will be considered separately,

$$I_{\text{soft1}}^{(b)} = -i\pi \int_{-\infty}^{\infty} \frac{dk_{10}}{2\pi} \frac{dk_{20}}{2\pi} \frac{(4m + k_{10} + k_{20})\delta((p + k_1)^2 - m^2)}{(k_1 - k_2)^2((p + k_2)^2 - m^2)}. \tag{J.26}$$

With respect to k_{10}, the Dirac delta function has the two singular points

$$k_{10} = -m \pm \sqrt{m^2 + \vec{k}_1^2} =: -m \pm \kappa_1' =: k_{1\pm}. \tag{J.27}$$

With

$$\delta((p + k_1)^2 - m^2) = \frac{1}{2\kappa_1'} \left(\delta(k_{10} - k_{1+}) + \delta(k_{10} - k_{1-}) \right) \tag{J.28}$$

one obtains

$$I_{\text{soft1}}^{(b)} = \frac{1}{8\pi i \kappa_1'} \left(I_{1+}^{(b)} + I_{1-}^{(b)} \right), \quad I_{1\pm}^{(b)} := \int_{-\infty}^{\infty} \frac{(4m + k_{1\pm} + k_{20})dk_{20}}{((k_{1\pm} - k_{20})^2 - (\vec{k}_1 - \vec{k}_2)^2)(k_{20}^2 + 2mk_{20} - \vec{k}_2^2)}. \tag{J.29}$$

The poles of $I_{1+}^{(b)}$ are given by

$$k_{20} = -m \pm (\kappa_2' - i\epsilon), \qquad k_{20} = -m + \kappa_1' \pm (\kappa_3 - i\epsilon) \tag{J.30}$$

and the poles of $I_{1-}^{(b)}$ by

$$k_{20} = -m \pm (\kappa_2' - i\epsilon), \qquad k_{20} = -m - \kappa_1' \pm (\kappa_3 - i\epsilon), \tag{J.31}$$

where the abbreviations

$$\kappa_i := |\vec{k}_i|, \qquad \kappa_i' := \sqrt{m^2 + \vec{k}_i^2}, \qquad \kappa_3 := |\vec{k}_1 - \vec{k}_2| \tag{J.32}$$

have been used. The contour will now be closed in the lower complex half plane and the residue theorem will be used. The residues which have to be considered are given by

$$\text{Res} \left[\frac{4m + k_{1\pm} + k_{20}}{((k_{1\pm} - k_{20})^2 - (\vec{k}_1 - \vec{k}_2)^2)(k_{20}^2 + 2mk_{20} - \vec{k}_2^2)} ; \, k_{20} = -m + \kappa_2' \right] =$$

$$
\begin{aligned}
&= \frac{4m - m \pm \kappa_1' - m + \kappa_2'}{((-m \pm \kappa_1' + m - \kappa_2')^2 - \kappa_3^2)2\kappa_2'} = \frac{2m \pm \kappa_1' + \kappa_2'}{2\kappa_2'((\kappa_1' \mp \kappa_2')^2 - \kappa_3^2)} = \\
&= \frac{2m \pm \kappa_1' + \kappa_2'}{4\kappa_2'(m^2 + \vec{k}_1\vec{k}_2 \mp \kappa_1'\kappa_2')}, \\
&\mathrm{Res}\left[\frac{4m + k_{1\pm} + k_{20}}{((k_{1\pm} - k_{20})^2 - (\vec{k}_1 - \vec{k}_2)^2)(k_{20}^2 + 2mk_{20} - \vec{k}_2^2)} \,;\, k_{20} = -m \pm \kappa_1' + \kappa_3\right] = \\
&= \frac{4m - m \pm \kappa_1' - m \pm \kappa_1' + \kappa_3}{2\kappa_3\left(m^2 \mp 2m\kappa_1' + {\kappa_1'}^2 - 2m\kappa_3 \pm 2\kappa_1'\kappa_3 + \kappa_3^2 - 2m^2 \pm 2m\kappa_1' + 2m\kappa_3 - \vec{k}_2^2\right)} = \\
&= \frac{2m \pm 2\kappa_1' + \kappa_3}{4\kappa_3(\vec{k}_1^2 - \vec{k}_1\vec{k}_2 \pm \kappa_1'\kappa_3)}.
\end{aligned}
\tag{J.33}
$$

One can use

$$
\begin{aligned}
&\frac{2m + \kappa_1' + \kappa_2'}{4\kappa_2'(m^2 + \vec{k}_1\vec{k}_2 - \kappa_1'\kappa_2')} + \frac{2m - \kappa_1' + \kappa_2'}{4\kappa_2'(m^2 + \vec{k}_1\vec{k}_2 + \kappa_1'\kappa_2')} = \\
&= \frac{2((2m + \kappa_2')(m^2 + \vec{k}_1\vec{k}_2) + {\kappa_1'}^2\kappa_2')}{4\kappa_2'((m^2 + \vec{k}_1\vec{k}_2)^2 - (m^2 + \vec{k}_1^2)(m^2 + \vec{k}_2^2))} = \\
&= \frac{(2m + \kappa_2')(m^2 + \vec{k}_1\vec{k}_2) + (m^2 + \vec{k}_1^2)\kappa_2'}{2\kappa_2'((\vec{k}_1\vec{k}_2)^2 - \vec{k}_1^2\vec{k}_2^2 - m^2(\vec{k}_1 - \vec{k}_2)^2)}
\end{aligned}
\tag{J.34}
$$

and

$$
\begin{aligned}
&\frac{2m + 2\kappa_1' + \kappa_3}{4\kappa_3(\vec{k}_1^2 - \vec{k}_1\vec{k}_2 + \kappa_1'\kappa_3)} + \frac{2m - 2\kappa_1' + \kappa_3}{4\kappa_3(\vec{k}_1^2 - \vec{k}_1\vec{k}_2 - \kappa_1'\kappa_3)} = \\
&= \frac{2((2m + \kappa_3)(\vec{k}_1^2 - \vec{k}_1\vec{k}_2) - 2{\kappa_1'}^2\kappa_3)}{4\kappa_3((\vec{k}_1^2 - \vec{k}_1\vec{k}_2)^2 - (m^2 + \vec{k}_1^2)(\vec{k}_1 - \vec{k}_2)^2)} = \\
&= \frac{(2m + \kappa_3)(\vec{k}_1^2 - \vec{k}_1\vec{k}_2) - 2(m^2 + \vec{k}_1^2)\kappa_3}{2\kappa_3((\vec{k}_1\vec{k}_2)^2 - \vec{k}_1^2\vec{k}_2^2 - m^2(\vec{k}_1 - \vec{k}_2)^2)}.
\end{aligned}
\tag{J.35}
$$

In combining all the residues one ends up with

$$
\begin{aligned}
I^{(b)}_{\mathrm{soft1}} &= \frac{-1}{8\kappa_1'\kappa_2'\kappa_3((\vec{k}_1\vec{k}_2)^2 - \vec{k}_1^2\vec{k}_2^2 - m^2(\vec{k}_1 - \vec{k}_2)^2)} \times \\
&\quad \times \Big[(2m + \kappa_2')(m^2 + \vec{k}_1\vec{k}_2)\kappa_3 + (m^2 + \vec{k}_1^2)\kappa_2'\kappa_3 + \\
&\qquad + (2m + \kappa_3)(\vec{k}_1^2 - \vec{k}_1\vec{k}_2)\kappa_2' - 2(m^2 + \vec{k}_1^2)\kappa_2'\kappa_3\Big] = \\
&= \frac{(2m + \kappa_2')(m^2 + \vec{k}_1\vec{k}_2)\kappa_3 + (2m + \kappa_3)(\vec{k}_1^2 - \vec{k}_1\vec{k}_2)\kappa_2' - (m^2 + \vec{k}_1^2)\kappa_2'\kappa_3}{8\kappa_1'\kappa_2'\kappa_3(m^2(\vec{k}_1 - \vec{k}_2)^2 + \vec{k}_1^2\vec{k}_2^2 - (\vec{k}_1\vec{k}_2)^2)} = \\
&= \frac{2m(m^2 + \vec{k}_1\vec{k}_2)\kappa_3 + 2m(\vec{k}_1^2 - \vec{k}_1\vec{k}_2)\kappa_2'}{8\kappa_1'\kappa_2'\kappa_3(m^2(\vec{k}_1 - \vec{k}_2)^2 + \vec{k}_1^2\vec{k}_2^2 - (\vec{k}_1\vec{k}_2)^2)} = \\
&= \frac{m((m^2 + \vec{k}_1\vec{k}_2)\kappa_3 + (\vec{k}_1^2 - \vec{k}_1\vec{k}_2)\kappa_2')}{4\kappa_1'\kappa_2'\kappa_3(m^2(\vec{k}_1 - \vec{k}_2)^2 + \vec{k}_1^2\vec{k}_2^2 - (\vec{k}_1\vec{k}_2)^2)}.
\end{aligned}
\tag{J.36}
$$

J.2.2 The single-soft region

Because of $\kappa_1 = |\vec{k}_1| < \mu$, for $\mu \ll m$ one can use the approximate expression

$$I^{(b)}_{\rm soft1} \to \frac{1}{4m(\vec{k}_1 - \vec{k}_2)^2}. \tag{J.37}$$

One obtains

$$\begin{aligned}
\Sigma^{(b)}_{\rm soft1} &= \frac{g_s^4 C_F C_A}{4m} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{\vec{k}_1^2 \vec{k}_2^2 - (\vec{k}_1\vec{k}_2)^2}{\vec{k}_1^2\vec{k}_2^2((\vec{k}_1 - \vec{k}_2)^2)^2} = \\
&= \frac{g_s^4 C_F C_A}{2(2\pi)^4 m} \int_0^\mu \kappa_1^2 d\kappa_1 \int_0^\infty \kappa_2^2 d\kappa_2 \int_{-1}^{+1} \frac{(\kappa_1^2\kappa_2^2 - \kappa_1^2\kappa_2^2 t^2)dt}{\kappa_1^2\kappa_2^2(\kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 t)^2} = \\
&= \frac{g_s^4 C_F C_A}{8(2\pi)^4 m} \int_0^\mu d\kappa_1 \int_0^\infty d\kappa_2 \int_{-1}^{+1} \frac{(1-t^2)dt}{(t_0 - t)^2}
\end{aligned} \tag{J.38}$$

with $t = \cos\theta(\vec{k}_1, \vec{k}_2)$ and $t_0 = (\kappa_1^2 + \kappa_2^2)/2\kappa_1\kappa_2$. For the last integral one obtains

$$\begin{aligned}
\int_{-1}^{+1} \frac{(1-t)^2 dt}{(t_0-t)^2} &= \int_{t_0-1}^{t_0+1} \left(1 - (t_0 - t')^2\right) \frac{dt'}{t'^2} = \int_{t_0-1}^{t_0+1} \left(-\frac{t_0^2 - 1}{t'^2} + \frac{2t_0}{t'} - 1 \right) dt' = \\
&= (t_0^2 - 1)\left(\frac{1}{t_0+1} - \frac{1}{t_0 - 1} \right) + 2t_0 \ln\left(\frac{t_0+1}{t_0-1} \right) - (t_0 + 1 - t_0 + 1) = \\
&= (t_0^2 - 1)\frac{t_0 - 1 - t_0 - 1}{t_0^2 - 1} + 2t_0 \ln\left(\frac{t_0+1}{t_0-1} \right) - 2 = \\
&= -4 + 2t_0 \ln\left(\frac{t_0+1}{t_0-1} \right) = -4 + 2\frac{\kappa_1^2 + \kappa_2^2}{\kappa_1\kappa_2} \ln\left| \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \right|
\end{aligned} \tag{J.39}$$

and therefore

$$\begin{aligned}
&\int_0^\infty \left(-4 + 2\frac{\kappa_1^2 + \kappa_2^2}{\kappa_1\kappa_2} \ln\left| \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \right| \right) d\kappa_2 = \\
&= \int_0^{\kappa_1} \left(-4 + 2\frac{\kappa_1^2 + \kappa_2^2}{\kappa_1\kappa_2} \ln\left(\frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \right) \right) d\kappa_2 + \\
&\qquad + \int_{\kappa_1}^\infty \left(-4 + 2\frac{\kappa_1^2 + \kappa_2^2}{\kappa_1\kappa_2} \ln\left(\frac{\kappa_1 + \kappa_2}{\kappa_2 - \kappa_1} \right) \right) d\kappa_2 = \\
&= \kappa_1 \int_0^1 \left(-4 + 2\frac{1+x^2}{x} \ln\left(\frac{1+x}{1-x} \right) \right) dx + \\
&\qquad + \kappa_1 \int_0^1 \left(-4 + 2\frac{1+x^2}{x} \ln\left(\frac{1+x}{1-x} \right) \right) \frac{dx}{x^2} = \\
&= \kappa_1 \int_0^1 \left(-4 + 2\frac{1+x^2}{x} \ln\left(\frac{1+x}{1-x} \right) \right) \frac{1+x^2}{x^2} dx = \pi^2 \kappa_1
\end{aligned} \tag{J.40}$$

where $\kappa_2 = \kappa_1 x$ and $\kappa_2 = \kappa_1/x$ are used, resp., for the two parts. Because of the high degree of divergence for $x = 0$ one might wonder why this integral exists. But if one expands the integrand in x near $x = 0$ one actually obtains a term of order $O(x^0)$ which can be integrated. Therefore, the final result reads

$$\Sigma^{(b)}_{\rm soft1} = \frac{(4\pi\alpha_s)^2 C_F C_A}{8(2\pi)^4 m} \pi^2 \int_0^\mu \kappa_1 d\kappa_1 = \frac{\alpha_s^2 C_F C_A}{16m} \mu^2. \tag{J.41}$$

J.3 The non-abelian diagram in Feynman gauge

Because it is not obvious from the previous considerations that only the diagram in Fig. 5.12(b) contributes, the calculation is done also in Feynman gauge where there is only one diagram. The contribution is given by

$$\begin{aligned}
-i\Sigma(\not p) &= \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\left(\frac{-i}{k_1^2}\right)\left(\frac{-i}{k_2^2}\right)\left(\frac{-i}{(k_1-k_2)^2}\right)\times\\
&\qquad \times(-ig_s\gamma^\alpha T_a)\frac{i}{\not p+\not k_2-m}(-ig_s\gamma^\beta T_b)\frac{i}{\not p+\not k_1-m}(-ig_s\gamma^\rho T_c)\times\\
&\qquad\quad \times g_sf_{abc}\left((k_2-2k_1)_\alpha g_{\beta\rho}+(k_1+k_2)_\beta g_{\rho\alpha}+(k_1-2k_2)_\rho g_{\alpha\beta}\right) =\\
&= g_s^4f_{abc}T_aT_bT_c\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{\gamma^\alpha(\not p+\not k_2+m)\gamma^\beta(\not p+\not k_1+m)\gamma^\rho}{k_1^2k_2^2(k_1-k_2)^2((p+k_1)^2-m^2)((p+k_2)^2-m^2)}\times\\
&\qquad \times\left((k_2-2k_1)_\alpha g_{\beta\rho}+(k_1+k_2)_\beta g_{\rho\alpha}+(k_1-2k_2)_\rho g_{\alpha\beta}\right) =\\
&= \frac{i}{2}g_s^4C_FC_A\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{1}{k_1^2k_2^2(k_1-k_2)^2((p+k_1)^2-m^2)((p+k_2)^2-m^2)}\times\\
&\qquad \times\Big[(\not k_2-2\not k_1)(\not p+\not k_2+m)\gamma_\alpha(\not p+\not k_1+m)\gamma^\alpha+\\
&\qquad\quad +\gamma_\alpha(\not p+\not k_2+m)(\not k_1+\not k_2)(\not p+\not k_1+m)\gamma^\alpha+\\
&\qquad\qquad +\gamma_\alpha(\not p+\not k_2+m)\gamma^\alpha(\not p+\not k_1+m)(\not k_1-2\not k_2)\Big].
\end{aligned}\tag{J.42}$$

For the evaluation of the Dirac structure sandwiched between the two on-shell spinors with momentum p and mass m an evaluation package has been used which was of help also for the calculations in Appendix H. The result (symmetrized with respect to k_1 and k_2 and with the effective replacement $k_i\to k_ip/m=k_{i0}$) gives rise to the following numerator N and denominator D of the integrand,

$$\begin{aligned}
N &= 4\Big[2(k_1k_2)k_{10}+2mk_{10}^2+2(k_1k_2)k_{20}+2mk_{20}^2-2mk_{10}k_{20}+\\
&\qquad -2k_1^2k_{20}-2k_2^2k_{10}-m(k_1k_2)+mk_1^2+mk_2^2\Big] =\\
&= 4\Big[2k_{10}^2k_{20}-2(\vec k_1\vec k_2)k_{10}+2mk_{10}^2+2k_{10}k_{20}^2-2(\vec k_1\vec k_2)k_{20}+2mk_{20}^2+\\
&\qquad -2mk_{10}k_{20}-2k_{10}^2k_{20}+2\vec k_1^2k_{20}-2k_{10}k_{20}^2+2\vec k_2^2k_{10}+\\
&\qquad\quad -mk_{10}k_{20}+m(\vec k_1\vec k_2)+mk_{10}^2-m\vec k_1^2+mk_{20}^2-m\vec k_2^2\Big] =\\
&= 4\Big[3m(k_{10}^2-k_{10}k_{20}+k_{20}^2)+(m-2k_{10}-2k_{20})\vec k_1\vec k_2-(m-2k_{20})\vec k_1^2-(m-2k_{10})\vec k_2^2\Big],
\end{aligned}\tag{J.43}$$

$$D = (k_{10}^2-\kappa_1^2-i\epsilon)(k_{20}^2-\kappa_2^2-i\epsilon)((k_{10}-k_{20})^2-\kappa_3^2-i\epsilon)(k_{20}^2+2mk_{20}-\kappa_2^2-i\epsilon). \tag{J.44}$$

The poles of the denominator with respect to k_{20} are found to be

$$\begin{aligned}
k_{20} &= \pm(\kappa_2+i\epsilon),\\
k_{20} &= k_{10}\pm(\kappa_3+i\epsilon),\\
k_{20} &= -m\pm(\kappa_2'+i\epsilon).
\end{aligned}\tag{J.45}$$

Now the residue theorem can be used to close the integration over the time component of k_2 in the upper complex half plane. The corresponding denominator factors of the residues are called D_{20}, D_{30} and D'_{20} for these three zeros, respectively. They read

$$\begin{aligned}
D_{20} &= (k_{10}^2-\kappa_1^2)(k_{20}+\kappa_2)\left((k_{10}-k_{20})^2-\kappa_3^2\right)\left((m+k_{20})^2-\kappa_2^2-m^2\right)\Big|_{k_{20}=\kappa_2} = \\
&= 2\kappa_2(k_{10}^2-\kappa_1^2)\left((k_{10}-\kappa_2)^2-\kappa_3^2\right)\left((m+\kappa_2)^2-\kappa_2^2-m^2\right) = \qquad (J.46)\\
&= 2\kappa_2(k_{10}^2-\kappa_1^2)\left((k_{10}-\kappa_2)^2-\kappa_3^2\right)2m\kappa_2 = 4m\kappa_2^2(k_{10}^2-\kappa_1^2)\left((k_{10}-\kappa_2)^2-\kappa_3^2\right),\\
D_{30} &= (k_{10}^2-\kappa_1^2)(k_{20}^2-\kappa_2^2)(k_{20}-k_{10}+\kappa_3)\left((m+k_{20})^2-\kappa_2^2-m^2\right)\Big|_{k_{20}=k_{10}+\kappa_3} = \\
&= 2\kappa_3(k_{10}^2-\kappa_1^2)\left((k_{10}+\kappa_3)^2-\kappa_2^2\right)\left((m+k_{10}+\kappa_3)^2-\kappa_2^2-m^2\right), \qquad (J.47)\\
D'_{20} &= (k_{10}^2-\kappa_1^2)(k_{20}^2-\kappa_2^2)\left((k_{20}-k_{10})^2-\kappa_3\right)(k_{20}+m+\kappa'_2)\Big|_{k_{20}=-m+\kappa'_2} = \\
&= (k_{10}^2-\kappa_1^2)\left((-m+\kappa'_2)^2-\kappa_2^2\right)\left((-m+\kappa'_2-k_{10})^2-\kappa_3^2\right)2\kappa'_2 = \\
&= 2\kappa'_2(k_{10}^2-\kappa_1^2)\left(m^2-2m\kappa'_2+{\kappa'_2}^2-\kappa_2^2\right)\left((-m+\kappa'_2-k_{10})^2-\kappa_3^2\right) = \\
&= 4m\kappa'_2(k_{10}^2-\kappa_1^2)(m-\kappa'_2)\left((-m+\kappa'_2-k_{10})^2-\kappa_3^2\right). \qquad (J.48)
\end{aligned}$$

If one replaces the propagator factor $((p+k_1)^2-m^2)^{-1}$ by $-i\pi\delta((p+k_1)^2-m^2)$, one obtains two parts according to the two solutions of the delta function with respect to k_{10},

$$k_{1\pm} = -m \pm \sqrt{m^2+\kappa_1^2} = -m \pm \kappa'_1. \qquad (J.49)$$

They were previously called the *scattering* and the *annihilation part*. The denominator factors obtained by using these solutions are called $D_{2\pm}$, $D_{3\pm}$, and $D'_{2\pm}$. They read

$$\begin{aligned}
D_{2\pm} &= 4m\kappa_2^2\left((-m\pm\kappa'_1)^2-\kappa_1^2\right)\left((-m\pm\kappa'_1-\kappa_2)^2-\kappa_3^2\right) = \\
&= 4m\kappa_2^2(m^2\mp 2m\kappa'_1+{\kappa'_1}^2-\kappa_1^2)\left(m^2\mp m\kappa'_1+2m\kappa_2+{\kappa'_1}^2\mp 2\kappa'_1\kappa_2+\kappa_2^2-\kappa_3^2\right) = \\
&= 8m^2\kappa_2^2(m\mp\kappa'_1)\left(m^2+2m\kappa_2+m^2+\vec k_1^2+\vec k_2^2-\vec k_1^2+2\vec k_1\vec k_2-\vec k_2^2\mp 2(m+\kappa_2)\kappa'_1\right) = \\
&= 16m^2\kappa_2^2(m\mp\kappa'_1)\left(m^2+m\kappa_2+\vec k_1\vec k_2\mp(m+\kappa_2)\kappa'_1\right) = \\
&= 16m^2\kappa_2^2(m\mp\kappa'_1)\left(\vec k_1\vec k_2+(m\mp\kappa'_1)(m+\kappa_2)\right) =: 16m^2\kappa_2^2A_{2\pm}, \qquad (J.50)\\
D_{3\pm} &= 2\kappa_3\left((-m\pm\kappa'_1)^2-\kappa_1^2\right)\left((-m\pm\kappa'_1+\kappa_3)^2-\kappa_2^2\right)\left((\pm\kappa'_1+\kappa_3)^2-\kappa_2^2-m^2\right) = \\
&= 4m\kappa_3(m\mp\kappa'_1)\left(m^2\mp 2m\kappa'_1-2m\kappa_3+{\kappa'_1}^2\pm 2\kappa'_1\kappa_3+\kappa_3^2-\kappa_2^2\right)\times\\
&\qquad\times\left({\kappa'_1}^2\pm 2\kappa'_1\kappa_3+\kappa_3^2-\kappa_2^2-m^2\right) = \\
&= 4m\kappa_3(m\mp\kappa'_1)\left(m^2\mp 2m\kappa'_1-2m\kappa_3+\vec k_1^2+m^2\pm 2\kappa'_1\kappa_3+\vec k_1^2-2\vec k_1\vec k_2+\vec k_2^2-\vec k_2^2\right)\times\\
&\qquad\times\left(\vec k_1^2+m^2\pm 2\kappa'_1\kappa_3+\vec k_1^2-2\vec k_1\vec k_2+\vec k_2^2-\vec k_2^2-m^2\right) = \\
&= 16m\kappa_3(m\mp\kappa'_1)\left(m^2\mp m\kappa'_1-m\kappa_3\pm\kappa'_1\kappa_3+\vec k_1^2-\vec k_1\vec k_2\right)\left(\vec k_1^2-\vec k_1\vec k_2\pm\kappa'_1\kappa_3\right) = \\
&= 16m\kappa_3(m\mp\kappa'_1)\left(\vec k_1(\vec k_1-\vec k_2)+(m\mp\kappa'_1)(m-\kappa_3)\right)\left(\vec k_1(\vec k_1-\vec k_2)\pm\kappa'_1\kappa_3\right) = \\
&=: 16m\kappa_3A_{3\pm}, \qquad (J.51)
\end{aligned}$$

$$\begin{aligned}
D'_{2\pm} &= 4m\kappa'_2\left((-m\pm\kappa'_1)^2-\kappa_1^2\right)(m-\kappa'_2)\left((-m+\kappa'_2+m\mp\kappa'_1)^2-\kappa_3^2\right) = \\
&= 8m^2\kappa'_2(m\mp\kappa'_1)(m-\kappa'_2)\left({\kappa'_2}^2\mp 2\kappa'_1\kappa'_2+{\kappa'_1}^2-\kappa_3^2\right) = \\
&= 8m^2\kappa'_2(m\mp\kappa'_1)(m-\kappa'_2)\left(2m^2+2\vec{k}_1\vec{k}_2\mp 2\kappa'_1\kappa'_2\right) = \\
&= 16m^2\kappa'_2(m\mp\kappa'_1)(m-\kappa'_2)\left(\vec{k}_1\vec{k}_2+m^2\mp\kappa'_1\kappa'_2\right) =: 16m^2\kappa'_2(m-\kappa'_2)A'_{2\pm}. \qquad (J.52)
\end{aligned}$$

The corresponding numerators are calculated in the same manner. One obtains

$$\begin{aligned}
N_{2\pm} &= 4\Big[3m\left(\vec{k}_1(\vec{k}_1+\vec{k}_2)+(2m+\kappa_2)(m\mp\kappa'_1)\right) + \\
&\qquad +2\kappa_2\vec{k}_1(\vec{k}_1-\vec{k}_2)\mp 2\kappa'_1(\vec{k}_1-\vec{k}_2)\vec{k}_2\Big], \qquad (J.53)\\
N_{3\pm} &= 4\Big[3m\left(\vec{k}_1^2+(2m-\kappa_3)(m\mp\kappa'_1)\right)-m\vec{k}_1\vec{k}_2 + \\
&\qquad +2\kappa_3\vec{k}_1(\vec{k}_1-\vec{k}_2)\pm 2\kappa'_1(\vec{k}_1-\vec{k}_2)^2\Big], \qquad (J.54)\\
N'_{2\pm} &= 4\Big[3m\left(2(\vec{k}_1\vec{k}_2+m^2)+(m-\kappa'_2)(m\mp\kappa'_1)\right)-m\vec{k}_1\vec{k}_2 + \\
&\qquad +2\kappa'_2\vec{k}_1(\vec{k}_1-\vec{k}_2)\mp 2\kappa'_1(\vec{k}_1-\vec{k}_2)\vec{k}_2\Big]. \qquad (J.55)
\end{aligned}$$

One now combines the scattering and the annihilation part for each of the three contributions according to the rule

$$\frac{N_{i+}}{D_{i+}}+\frac{N_{i-}}{D_{i-}}=\frac{N_{i+}A_{i-}+N_{i-}A_{i+}}{D_i}=\frac{N_i}{D_i} \qquad (J.56)$$

where $D_i = D_{i+}A_{i-} = D_{i-}A_{i+}$. The necessary manipulations were done by a MATHEMATICA package which was designed to calculate the needed residues (i.e. localizing the poles in the upper complex half plane and calculating the residues).

J.3.1 The limit $\mu \ll m$

The rather lengthy result of the above calculations cannot be shown here. But it can be simplified drastically in the special case $\mu \ll m$. The substitutions which are used are the same as the ones applied to the Coulomb gauge case, i.e. taking the variable t for $\cos\theta$ with $t_0 = (\kappa_1^2+\kappa_2^2)/2\kappa_1\kappa_2$, using $\kappa_2 = \kappa_1 x$ (where x can be replaced by $1/x$ for a part of the integration) and finally chosing $\kappa_1 = my$. The integration measure changes accordingly to

$$\begin{aligned}
\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} &= \frac{2}{(2\pi)^4}\int_0^\mu\kappa_1^2d\kappa_1\int_0^\infty\kappa_2^2d\kappa_2\int_{-1}^{+1}dt = \qquad (J.57)\\
= \frac{2}{(2\pi)^4}\int_0^\mu\kappa_1^5d\kappa_1\int_0^\infty x^2dx\int_{-1}^{+1}dt &= \frac{2m^6}{(2\pi)^4}\int_0^{\mu/m}y^5dy\int_0^\infty x^2dx\int_{-1}^{+1}dt.
\end{aligned}$$

Because the y range extends up to $\mu/m \ll 1$, one can expand in y. The integrand is then given by

$$\frac{1+x^2-2x(t_0-t)}{4m^5x^3y^4(t_0-t)}. \qquad (J.58)$$

Including all overall factors, including the factor $-i\pi/2\pi = -i/2$ from the soft part and the factor $2\pi i/2\pi = i$ from the residues, one obtains

$$
\begin{aligned}
\Sigma_{\rm soft1} &= \frac{m}{4(2\pi)^4}\int_0^{\mu/m} y\,dy\int_0^\infty dx\int_{-1}^{+1}\left(-2+\frac{1+x^2}{x(t_0-t)}\right)dt \;= \\
&= \frac{mg_s^4C_FC_A}{4(2\pi)^4}\int_0^{\mu/m} y\,dy\int_0^\infty dx\int_{t_0-1}^{t_0+1}\left(-2+\frac{1+x^2}{xt'}\right)dt' \;= \\
&= \frac{mg_s^4C_FC_A}{4(2\pi)^4}\int_0^{\mu/m} y\,dy\int_0^\infty dx\left(-4+\frac{1+x^2}{x}\ln\left(\frac{t_0+1}{t_0-1}\right)\right) \;= \\
&= \frac{mg_s^4C_FC_A}{4(2\pi)^4}\int_0^{\mu/m} y\,dy\int_0^\infty \left(-4+2\frac{1+x^2}{x}\ln\left|\frac{1+x}{1-x}\right|\right)dx \;= \\
&= \frac{mg_s^4C_FC_A}{4(2\pi)^4}\int_0^{\mu/m} y\,dy\Bigg[\int_0^1 \left(-4+2\frac{1+x^2}{x}\ln\left(\frac{1+x}{1-x}\right)\right)dx \;+ \\
&\quad (x'=1/x) \qquad\qquad + \int_0^1\left(-4+2\frac{1+x'^2}{x'}\ln\left(\frac{1+x'}{1-x'}\right)\frac{dx'}{x'^2}\right)\Bigg] \;= \\
&= \frac{mg_s^4C_FC_A}{4(2\pi)^4}\int_0^{\mu/m} y\,dy\int_0^1 \left(-4+2\frac{1+x^2}{x}\ln\left(\frac{1+x}{1-x}\right)\right)\frac{1+x^2}{x^2}dx \;= \\
&= \frac{mg_s^4C_FC_A\pi^2}{4(2\pi)^4}\int_0^{\mu/m} y\,dy \;=\; \frac{m\alpha_s^2C_FC_A}{8}\frac{\mu^2}{2m^2} \;=\; \frac{\alpha_s^2C_FC_A}{16m}\mu^2. \qquad (J.59)
\end{aligned}
$$

This result coincides with the one presented in Eq. (J.41).

Appendix K

The trigluon diagram calculations

This appendix contains calculations for the quark loop diagram with one external photon and three external gluon lines. This diagram is needed in Section 7.5 to find the low-energy contribution to the spectral density.

K.1 The vector contribution to the effective action

As it is shown in the main text, the contribution to the effective action for four external bosons is given by $\Gamma_4[\mathcal{B}]$ where

$$i\frac{\partial}{\partial m}\Gamma_4[\mathcal{B}] = \mathrm{Tr}\,(S_0\gamma^\mu\mathcal{B}_\mu S_0\gamma^\nu\mathcal{B}_\nu S_0\gamma^\rho\mathcal{B}_\rho S_0\gamma^\sigma\mathcal{B}_\sigma S_0) \tag{K.1}$$

with $\mathcal{B}_\mu = eA_\mu + g_s B^a_\mu t_a$ as elements of the algebra of the gauge group $SU(N_c)\otimes U(1)$. Using $\mathcal{B}_\mu(x) = x^\alpha G_{\alpha\mu}/2$, the right hand side can be rewritten as

$$i\frac{\partial}{\partial m}\Gamma_4[\mathcal{B}] = \frac{1}{16}t(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma)\mathrm{Tr}\,(\mathcal{G}_{\alpha\mu}\mathcal{G}_{\beta\nu}\mathcal{G}_{\gamma\rho}\mathcal{G}_{\delta\sigma})\,. \tag{K.2}$$

In momentum space ($x_\mu \to -i\partial/\partial p^\mu$), the coefficient $t(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma)$ is given by

$$t(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma) = \mathrm{Tr}\,\Big(S(p)\gamma^\mu\partial^\alpha S(p)\gamma^\nu\partial\beta S(p)\gamma^\rho\partial^\gamma S(p)\gamma^\sigma\partial^\delta S(p)\Big)\,. \tag{K.3}$$

The derivative rule

$$-i\partial^\mu S(p) = -i\frac{\partial}{\partial p_\mu}S(p) = S(p)\gamma^\mu S(p) \tag{K.4}$$

can be used to calculate the trace as a long chain of elements $S(p)$ and γ^μ. This trace can be considered independently from the (colour) trace of the field strength tensors, if the antisymmetric structure is transferred to it. This can be done by using

$$\mathcal{G}_{\lambda_1\lambda_2} = f_{\lambda_1\lambda_2}(\alpha,\mu)\mathcal{G}^{\alpha\mu}, \qquad f_{\lambda_1\lambda_2}(\alpha,\mu) = \frac{1}{2}\left(g_{\lambda_1\alpha}g_{\lambda_2\mu} - g_{\lambda_2\alpha}g_{\lambda_1\mu}\right). \tag{K.5}$$

The trace is calculated and the integral over p is taken according to

$$\int\frac{d^4p}{(2\pi)^4}\frac{1}{(-p^2+m^2)^a} = \frac{i}{(4\pi)^2}\frac{\Gamma(a-2)}{\Gamma(a)}(m^2)^{2-a} = \frac{i}{(4\pi)^2}\frac{(m^2)^{2-a}}{(a-1)(a-2)} \quad \text{for } a>2. \tag{K.6}$$

Because there is no outer moment, tensor integrals with odd rank vanish. For tensor integrals with even rank one can use the rules (cf. the beginning of Appendix G, the dimension D is kept explicit for a moment)

$$p^{\lambda_1}p^{\lambda_2} \;\to\; \frac{m^2}{D}g^{\lambda_1\lambda_2}, \tag{K.7}$$

$$p^{\lambda_2}p^{\lambda_2}p^{\lambda_3}p^{\lambda_4} \;\to\; \frac{m^4}{3D(D+2)}\Big(g^{\lambda_1\lambda_2}g^{\lambda_3\lambda_4}+\text{perm.}\Big) \tag{K.8}$$

where the permutations are all pairings of an even number of indices. The common normalization can be calculated by contracting the starting pairing with the permuted ones in a kind of a "domino game", each closed circle resulting in a factor D. The results one obtains (also for the lower order terms) using the package `triadi.add` are given by

$$t(\alpha,\mu;\beta,\nu) \;=\; -\frac{8i}{3m(4\pi)^2}\mathrm{Tr}(f(\alpha,\mu)f(\beta,\nu)), \tag{K.9}$$

$$t(\alpha,\mu;\beta,\nu;\gamma,\rho) \;=\; -\frac{8i}{3m^3(4\pi)^2}\mathrm{Tr}(f(\alpha,\mu)f(\beta,\nu)f(\gamma,\rho)), \tag{K.10}$$

$$\begin{aligned} t(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma) \;=\; & \frac{4\times 4i}{45m^5(4\pi)^2}\Big[-7\mathrm{Tr}(f(\alpha,\mu)f(\beta,\nu)f(\gamma,\rho)f(\delta,\sigma))+\\ & +36\mathrm{Tr}(f(\beta,\nu)f(\gamma,\rho)f(\alpha,\mu)f(\delta,\sigma))+27\mathrm{Tr}(f(\gamma,\rho)f(\alpha,\mu)f(\beta,\nu)f(\delta,\sigma))\;+\\ & -2\mathrm{Tr}(f(\alpha,\mu)f(\beta,\nu))\mathrm{Tr}(f(\gamma,\rho)f(\delta,\sigma))-13\mathrm{Tr}(f(\alpha,\mu)f(\gamma,\rho))\mathrm{Tr}(f(\beta,\nu)f(\delta,\sigma))\;+\\ & -5\mathrm{Tr}(f(\beta,\nu)f(\gamma,\rho))\mathrm{Tr}(f(\alpha,\mu)f(\delta,\sigma))\Big]. \end{aligned} \tag{K.11}$$

where the traces are understood as traces with respect to the Lorentz indices. The additional factor 4 in the last expression results from the four (cyclic symmetric) possibilities to obtain the coefficient to eg_s^3. A remark on the procedure to obtain this result is of order here. For the last expression there are two main structures available, denoted by $\mathrm{Tr}(f(\ldots)f(\ldots)f(\ldots)f(\ldots))$ and $\mathrm{Tr}(f(\ldots)f(\ldots))\mathrm{Tr}(f(\ldots)f(\ldots))$. The decision on which of these structures corresponds to a specific term in the result can be made by looking at the path that emerges from concatenating the index pairs (α,μ), (β,ν), (γ,ρ), and (δ,σ) and the index pairs occurring in the specific term (as arguments of the $f(\ldots)$), again like a domino game. If this path is a single (cyclic) one, the term belongs to a structure of the first kind, in the other case (when there are two cyclic paths) it belongs to a structure of the second kind (however, up to a factor 4). This path can be visualized by writing the indices α, β, γ, and δ in a column, next to a column built by μ, ν, ρ, and σ. Joining the index pairs (horizontal lines) as well as the pairs combined by the $f(\ldots)$ factors (diagonal lines; horizontal lines are excluded because of the antisymmetry), one obtains the path(s). The order within the trace can be found by rearranging the lines to the canonical one (with a zig-zag line down and a diagonal up again, no matter what direction down one takes).

One now can collect the relevant contributions. The coefficients multiplying g_s^2, g_s^3, and eg_s^3 are found to be

$$t_2=\mathrm{Tr}(T_aT_b)t_2^{ab},\quad t_3=\mathrm{Tr}(T_aT_bT_c)t_3^{abc},\quad t_{3V}=\mathrm{Tr}(T_aT_bT_c)t_{3V}^{abc} \tag{K.12}$$

where

$$\begin{aligned} t_2^{ab} &= -\frac{8i}{3m(4\pi)^2}\mathrm{Tr}(G^aG^b), && \text{(K.13)}\\ t_3^{abc} &= -\frac{8i}{3m^3(4\pi)^2}\mathrm{Tr}(G^aG^bG^c), && \text{(K.14)}\\ t_{3V}^{abc} &= \frac{16i}{45m^5(4\pi)^2}\Big[-7\mathrm{Tr}(FG^aG^bG^c)+36\mathrm{Tr}(FG^bG^cG^a)+27\mathrm{Tr}(FG^cG^aG^b)\ + \\ &\quad -5\mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c)-13\mathrm{Tr}(FG^b)\mathrm{Tr}(G^cG^a)-2\mathrm{Tr}(FG^c)\mathrm{Tr}(G^aG^b)\Big] && \text{(K.15)} \end{aligned}$$

(traces again only for the Lorentz indices). For the traces of the Gell–Mann-Matrices one obtains

$$2\mathrm{Tr}(T_aT_b)=\delta_{ab},\qquad 4\mathrm{Tr}(T_aT_bT_c)=2\mathrm{Tr}(T_a[T_b,T_c])+2\mathrm{Tr}(T_a\{T_b,T_c\})=if_{abc}+d_{abc} \tag{K.16}$$

such that (with $f_{abc}\mathrm{Tr}(FG^aG^bG^c)=0$, $f_{abc}\mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c)=0$)

$$\begin{aligned} t_2 &= -\frac{4i}{3m(4\pi)^2}\delta_{ab}\mathrm{Tr}(G^aG^b), && \text{(K.17)}\\ t_3 &= -\frac{2i}{3m^3(4\pi)^2}(if_{abc}+d_{abc})\mathrm{Tr}(G^aG^bG^c), && \text{(K.18)}\\ t_{3V} &= \frac{16i}{45m^5(4\pi)^2}d_{abc}\Big[14\mathrm{Tr}(FG^aG^bG^c)-5\mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c)\Big]. && \text{(K.19)} \end{aligned}$$

Finally, one integrates over m which is trivial. Therefore, the contribution of the one-photon three-gluon diagram to the effective action is given by

$$\Gamma_{3V}=\frac{eg_s^3d_{abc}}{180m^4(4\pi)^2}\Big[14\mathrm{Tr}(FG^aG^bG^c)-5\mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c)\Big]. \tag{K.20}$$

K.1.1 Calculation of the vector current

For the calculation of the electromagnetic current one uses

$$eJ^\mu=-\frac{\delta F_{\mu'\nu'}}{\delta A_\mu}\frac{\delta\Gamma[A]}{\delta F_{\mu'\nu'}}=-(\partial_{\mu'}\delta^\mu_{\nu'}-\partial_{\nu'}\delta^\mu_{\mu'})\frac{\delta\Gamma[A]}{\delta F_{\mu'\nu'}}=-2\partial_\nu\frac{\delta\Gamma[A]}{\delta F_{\nu\mu}}. \tag{K.21}$$

and obtains

$$J^\mu=\partial_\nu\mathcal{O}^{\mu\nu} \tag{K.22}$$

where $\mathcal{O}$ is an antisymmetric operator,

$$\mathcal{O}^{\mu\nu}=-\frac{2}{e}\frac{\delta\Gamma[A]}{\delta F_{\nu\mu}}=\frac{-g_s^3d_{abc}}{90m^4(4\pi)^2}\left[14(G^aG^bG^c)^{\mu\nu}-5(G^a)^{\mu\nu}\mathrm{Tr}(G^bG^c)\right] \tag{K.23}$$

which can also be expressed by the operator expressions

$$\mathcal{O}_1^{\mu\nu}=-d_{abc}(G^a)^{\mu\nu}\mathrm{Tr}(G^bG^c),\qquad \mathcal{O}_2^{\mu\nu}=d_{abc}(G^aG^bG^c)^{\mu\nu}. \tag{K.24}$$

This structure automatically guarantees current conservation $\partial_\mu J^\mu=0$.

K.1.2 The correlator function

As a first step towards the calculation of the vacuum polarization function one has to calculate the vacuum expectation value of the time ordered product of two currents at locations x and x',

$$\langle TJ_\mu(x)J_{\mu'}(x')\rangle = \partial^x_\nu\partial^{x'}_{\nu'}\langle T\mathcal{O}_{\mu\nu}(x)\mathcal{O}_{\mu'\nu'}(x')\rangle \tag{K.25}$$

(for simplicity all Lorentz indices will be written as lower indices for the time being) where

$$\begin{aligned}\langle T\mathcal{O}_{\mu\nu}(x)\mathcal{O}_{\mu'\nu'}(x')\rangle &= \frac{g_s^6}{8100m^8(4\pi)^4}\Big[25\langle\mathcal{O}_{1\mu\nu}(x)\mathcal{O}_{1\mu'\nu'}(x')\rangle + 70\langle\mathcal{O}_{1\mu\nu}(x)\mathcal{O}_{2\mu'\nu'}(x')\rangle\ + \\ &\quad +70\langle\mathcal{O}_{2\mu\nu}(x)\mathcal{O}_{1\mu'\nu'}(x')\rangle + 196\langle\mathcal{O}_{2\mu\nu}(x)\mathcal{O}_{2\mu'\nu'}(x')\rangle\Big].\end{aligned} \tag{K.26}$$

Each of the operators can then be written as a product of three field strength tensor components. Taking, for example, the last term, one can use Wick's theorem and obtains

$$\begin{aligned}&\langle\mathcal{O}_{2\mu\nu}(x)\mathcal{O}_{2\mu'\nu'}(x')\rangle\ = \\ &\quad = d_{abc}d_{a'b'c'}\langle G^a_{\mu\alpha}(x)G^b_{\alpha\beta}(x)G^c_{\beta\nu}(x)G^{a'}_{\mu'\alpha'}(x')G^{b'}_{\alpha'\beta'}(x')G^{c'}_{\beta'\nu'}(x')\rangle\ = \\ &\quad = d_{abc}d_{a'b'c'}\Big[\langle G^a_{\mu\alpha}(x)G^{a'}_{\mu'\alpha'}(x')\rangle\langle G^b_{\alpha\beta}(x)G^{b'}_{\alpha'\beta'}(x')\rangle\langle G^c_{\beta\nu}(x)G^{c'}_{\beta'\nu'}(x')\rangle + 5\text{ perm.}\Big]\end{aligned} \tag{K.27}$$

where "5 perm." stands for the 5 permutations which result from the five other pair groupings of $\{a,b,c\}$ and $\{a',b',c'\}$. Each of the factors of this sum can now be divided up again into four terms,

$$\begin{aligned}\langle G^a_{\mu\nu}(x)G^{a'}_{\mu'\nu'}(x')\rangle &= \langle(\partial^x_\mu B^a_\nu(x) - \partial^x_\nu B^a_\mu(x))(\partial^{x'}_{\mu'}B^{a'}_{\nu'}(x') - \partial^{x'}_{\nu'}B^{a'}_{\mu'}(x'))\rangle\ = \\ &= \langle\partial^x_\mu B^a_\nu(x)\partial^{x'}_{\mu'}B^{a'}_{\nu'}(x')\rangle - \langle\partial^x_\mu B^a_\nu(x)\partial^{x'}_{\nu'}B^{a'}_{\mu'}(x')\rangle\ + \\ &\quad -\langle\partial^x_\nu B^a_\mu(x)\partial^{x'}_{\mu'}B^{a'}_{\nu'}(x')\rangle + \langle\partial^x_\nu B^a_\mu(x)\partial^{x'}_{\nu'}B^{a'}_{\mu'}(x')\rangle\ = \\ &= \partial^x_\mu\partial^{x'}_{\mu'}\langle B^a_\nu(x)B^{a'}_{\nu'}(x')\rangle - \partial^x_\mu\partial^{x'}_{\nu'}\langle B^a_\nu(x)B^{a'}_{\mu'}(x')\rangle\ + \\ &\quad -\partial^x_\nu\partial^{x'}_{\mu'}\langle B^a_\mu(x)B^{a'}_{\nu'}(x')\rangle + \partial^x_\nu\partial^{x'}_{\nu'}\langle B^a_\mu(x)B^{a'}_{\mu'}(x')\rangle\end{aligned} \tag{K.28}$$

K.1.3 Correlator function in terms of the scalar propagator

Next one has

$$\langle B^a_\mu(x)B^{a'}_{\mu'}(x')\rangle = -i\delta^{aa'}g_{\mu\mu'}D(x-x') \tag{K.29}$$

where $D(x-x')$ is the propagator of a scalar particle. Therefore, one obtains

$$\begin{aligned}\langle G^a_{\mu\nu}(x)G^{a'}_{\mu'\nu'}(x')\rangle &= -i\delta^{aa'}\Big[g_{\nu\nu'}\partial^x_\mu\partial^{x'}_{\mu'}D(x-x') - g_{\nu\mu'}\partial^x_\mu\partial^{x'}_{\nu'}D(x-x')\ + \\ &\quad -g_{\mu\nu'}\partial^x_\nu\partial^{x'}_{\mu'}D(x-x') + g_{\mu\mu'}\partial^x_\nu\partial^{x'}_{\nu'}D(x-x')\Big].\end{aligned} \tag{K.30}$$

One can use $\partial^{x'}_{\mu'}D(x-x') = -\partial^x_{\mu'}D(x-x')$ and can go to the point $x'=0$,

$$\begin{aligned}\langle G^a_{\mu\nu}(x)G^{a'}_{\mu'\nu'}(0)\rangle &= i\delta^{aa'}\Big[g_{\nu\nu'}\partial^x_\mu\partial^x_{\mu'}D(x) - g_{\nu\mu'}\partial^x_\mu\partial^x_{\nu'}D(x)\ + \\ &\quad -g_{\mu\nu'}\partial^x_\nu\partial^x_{\mu'}D(x) + g_{\mu\mu'}\partial^x_\nu\partial^x_{\nu'}D(x)\Big].\end{aligned} \tag{K.31}$$

Using finally [107, 108, 112]

$$D(x)=\frac{i\Gamma(\lambda)}{4\pi^{\lambda+1}(-x^2)^{\lambda}}=\frac{-i\Gamma(1)}{4\pi^2x^2}=\frac{-i}{4\pi^2x^2} \tag{K.32}$$

(note that in contrast to Refs. [107, 108, 112] the Minkowskian metric is used), the innermost building block for the vacuum expectation value is given by

$$\langle G^a_{\mu\nu}(x)G^{a'}_{\mu'\nu'}(0)\rangle=\frac{\delta^{aa'}}{4\pi^2}\Big[g_{\nu\nu'}\partial_\mu\partial_{\mu'}-g_{\nu\mu'}\partial_\mu\partial_{\nu'}-g_{\mu\nu'}\partial_\nu\partial_{\mu'}+g_{\mu\mu'}\partial_\nu\partial_{\nu'}\Big]\frac{1}{x^2}. \tag{K.33}$$

All this as well as the two derivatives with respect to the indices ν and ν' (also converted from x' to x) are implemented in the package `wicki.add`. The result of this automatic calculation is given by

$$\langle TJ_\mu(x)J_{\mu'}(0)\rangle=\frac{g_s^6d_{abc}d_{abc}}{8100m^8(4\pi)^4}417792\frac{11g_{\mu\mu'}x^2-14x_\mu x_{\mu'}}{(4\pi^2)^3x^{16}}. \tag{K.34}$$

This has to be compared with

$$\Big(\partial_\mu\partial_{\mu'}-g_{\mu\mu'}\partial^2\Big)\frac{1}{x^{12}}=\frac{-12}{x^{16}}(11g_{\mu\nu'}x^2-14x_\mu x_{\mu'}), \tag{K.35}$$

Therefore, one obtains

$$\langle TJ_\mu(x)J_{\mu'}(0)\rangle=\frac{-34d_{abc}d_{abc}}{2025\pi^4m^8}\left(\frac{\alpha_s}{\pi}\right)^3\Big(\partial_\mu\partial_{\mu'}-g_{\mu\mu'}\partial^2\Big)\frac{1}{x^{12}}. \tag{K.36}$$

K.1.4 The spectral density – conventional approach

There are two ways to obtain the spectral density. The first (conventional) one consists of first calculating the vacuum polarization function in momentum space, i.e. to calculate the integral

$$12\pi^2 i\int\langle TJ_\mu(x)J_\nu(0)\rangle e^{iqx}d^4x. \tag{K.37}$$

Here one uses the integration-by-parts technique,

$$\begin{aligned}&\int\Big[(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)\frac{1}{x^{12}}\Big]e^{iqx}d^4x\ =\\ &=\ \int\frac{1}{x^{12}}\Big[(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)e^{iqx}\Big]d^4x\ =\ -(q_\mu q_\nu-g_{\mu\nu}q^2)\int\frac{1}{x^{12}}e^{iqx}d^4x,\end{aligned} \tag{K.38}$$

followed by (cf. Refs. [61, 292], transformed here to Minkowskian space)

$$\int\frac{e^{2ikx}d^Dx}{(-x^2)^{\lambda+1-\alpha}}=\frac{-i\pi^{\lambda+1}}{(-k^2)^{2\alpha}}\frac{\Gamma(\alpha)}{\Gamma(\lambda+1-\alpha)},\qquad D=2(\lambda+1),\quad\lambda=1-\varepsilon \tag{K.39}$$

which for $\lambda+1-\alpha=6$ and $k=q/2$ results in

$$\begin{aligned}\int\frac{e^{iqx}}{x^{12}}d^Dx\ &=\ \frac{-i\pi^{\lambda+1}\Gamma(\alpha)}{(-k^2)^\alpha\Gamma(\lambda+1-\alpha)}\ =\ \frac{-i\pi^{\lambda+1}\Gamma(\lambda-5)}{(-k^2)^{\lambda-5}\Gamma(6)}\ =\\ &=\ \frac{-i\pi^{2-\varepsilon}\Gamma(-4-\varepsilon)}{(-k^2)^{-4-\varepsilon}\Gamma(6)}\ =\ -i\pi^{2-\varepsilon}\left(\frac{\mu}{2}\right)^{2\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(6)}(-k^2)^4\left(\frac{-4k^2}{\mu^2}\right)^{\varepsilon}\frac{\Gamma(-4-\varepsilon)}{\Gamma(1-\varepsilon)}\ =\\ &=\ -i\pi^{2-\varepsilon}\left(\frac{\mu}{2}\right)^{2\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(6)}\left(\frac{q^2}{4}\right)^4\left(\frac{-1}{24\varepsilon}+\frac{25}{288}-\frac{1}{24}\ln\left(\frac{-q^2}{\mu^2}\right)+O(\varepsilon)\right)\end{aligned} \tag{K.40}$$

Near the origin in momentum space, i.e. for small values of q^2, only the logarithmic part is relevant. Then one can set $\varepsilon = 0$. With $24 = \Gamma(5)$ one obtains

$$\int \frac{e^{iqx}}{x^{12}} d^4x \to \frac{-i\pi^2}{\Gamma(6)\Gamma(5)} \left(\frac{q^2}{4}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right). \tag{K.41}$$

Therefore, the vacuum polarization function in momentum space is given by

$$12\pi^2 i \int \langle T J_\mu(x) J_\nu(0)\rangle e^{iqx} d^4x = C_g (q_\mu q_\nu - g_{\mu\nu} q^2) \left(\frac{q^2}{4m^2}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right) \tag{K.42}$$

and

$$\Pi(-q^2) = C_g \left(\frac{q^2}{4m^2}\right)^4 \ln\left(\frac{\mu^2}{-q^2}\right), \qquad C_g = \frac{17 d_{abc} d_{abc}}{243000} \left(\frac{\alpha_s}{\pi}\right)^3. \tag{K.43}$$

Taking the discontinuity of this expression, divided by $2\pi i$, one obtains

$$\rho(s) = \frac{1}{2\pi i} \left(\Pi(s+i0) - \Pi(s-i0)\right) = C_g \left(\frac{s}{4m^2}\right)^4. \tag{K.44}$$

K.1.5 The spectral density – advanced approach

For the propagator with mass m in four-dimensional Minkowskian space,

$$D(x^2, m^2) = \frac{im\sqrt{-x^2} K_1(m\sqrt{-x^2})}{4\pi^2(-x^2)} \tag{K.45}$$

($K_1(z)$ is the McDonald function of order 1), one obtains

$$\begin{aligned}
\int_0^\infty D(x^2, s) s^4 ds &= \int_0^\infty \frac{i\sqrt{-sx^2} K_1(\sqrt{-sx^2})}{4\pi(-x^2)} s^4 ds = \qquad (t = \sqrt{-sx^2}) \\
&= \int_0^\infty \frac{itK_1(t)}{4\pi^2(-x^2)} \left(\frac{-t^2}{x^2}\right)^4 \frac{-2t\,dt}{x^2} = \\
&= \frac{i}{2\pi^2} \left(\frac{1}{x^2}\right)^6 \int_0^\infty t^{10} K_1(t) dt = \frac{2^8 \Gamma(6)\Gamma(5)}{\pi^2} \frac{i}{x^{12}}
\end{aligned} \tag{K.46}$$

(cf. Ref. [108], Eq. (18)). Using this, one can immediately read off the spectral density.

K.2 Results for the tensor current

In a similar way the tensor current

$$J^{\mu\nu} = g_T \bar\psi \sigma^{\mu\nu} \psi, \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{K.47}$$

can be considered. In contrast to the vector current, in this case there is no problem with the action of the potential on the vacuum state. Therefore, one has

$$i\Gamma_T[\mathcal{B}] = g_T \mathrm{Tr}\left(\sigma^{\alpha\mu} F_{\alpha\mu} S_0 \gamma^\nu \mathcal{B}_\nu S_0 \gamma^\rho \mathcal{B}_\rho S_0 \gamma^\sigma \mathcal{B}_\sigma S_0\right) \tag{K.48}$$

where one of the field strength tensors is already given explicitly, including the "charge" g_T. Because of the asymmetry of this expression, it is not derived from the effective action. Using the aforementioned gauge, one now substitutes for the three potentials in terms of the field strength tensors. This gives rise to a factor 1/8 and one obtains

$$i\Gamma_T[\mathcal{B}] = \frac{g_T}{8} t'(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma)\mathrm{Tr}(F_{\alpha\mu}\mathcal{G}_{\beta\nu}\mathcal{G}_{\gamma\rho}\mathcal{G}_{\delta\sigma}). \tag{K.49}$$

where (note that $x^\mu \to -i\partial_p^\mu$)

$$t'(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma) = (-i)^3 \mathrm{Tr}\left(S(p)\sigma^{\alpha\mu}S(p)\gamma^\nu\partial^\beta S(p)\gamma^\rho\partial^\gamma S(p)\gamma^\sigma\partial^\delta S(p)\right). \tag{K.50}$$

Following the same steps as before, one obtains

$$\begin{aligned} t(\alpha,\mu;\beta,\nu;\gamma,\rho;\delta,\sigma) \; &= \; \frac{4ig_T}{3m^3(4\pi)^2}\Big[4\mathrm{Tr}(f(\alpha,\mu)f(\beta,\nu)f(\gamma,\rho)f(\delta,\sigma)) + \\ & -6\mathrm{Tr}(f(\alpha,\mu)f(\gamma,\rho)f(\delta,\sigma)f(\beta,\nu)) - 6\mathrm{Tr}(f(\alpha,\mu)f(\delta,\sigma)f(\beta,\nu)f(\gamma,\rho)) \; + \\ & +\mathrm{Tr}(f(\alpha,\mu)f(\beta,\nu))\mathrm{Tr}(f(\gamma,\rho)f(\delta,\sigma)) + 2\mathrm{Tr}(f(\alpha,\mu)f(\gamma,\rho))\mathrm{Tr}(f(\delta,\sigma)f(\beta,\nu)) \; + \\ & +\mathrm{Tr}(f(\alpha,\mu)f(\delta,\sigma))\mathrm{Tr}(f(\beta,\nu)f(\gamma,\rho))\Big]. \end{aligned} \tag{K.51}$$

In this case there is no additional factor because there is only one way to extract a further factor g_s^3. Multiplying this with the trace of the field strength tensors, one obtains

$$\begin{aligned} t_{3W}^{abc} \; &= \; \frac{4ig_T g_s^3}{3m^3(4\pi)^2}\Big[4\mathrm{Tr}(FG^aG^bG^c) - 6\mathrm{Tr}(FG^bG^cG^a) - 6\mathrm{Tr}(FG^cG^aG^b) \; + \\ & +\mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c) + 2\mathrm{Tr}(FG^b)\mathrm{Tr}(G^cG^a) + \mathrm{Tr}(FG^c)\mathrm{Tr}(G^aG^b)\Big]. \end{aligned} \tag{K.52}$$

As a next step one obtains

$$\begin{aligned} t_{3W} \; &= \; \mathrm{Tr}(T_aT_bT_c)t_{3W}^{abc} \; = \; \frac{1}{4}(if_{abc} + d_{abc})t_{3W}^{abc} \; = \\ &= \; \frac{ig_T g_s^3 d_{abc}}{3m^3(4\pi)^2}\Big[4\mathrm{Tr}(FG^aG^bG^c) - 6\mathrm{Tr}(FG^bG^cG^a) - 6\mathrm{Tr}(FG^cG^aG^b) \; + \\ & \quad +\mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c) + 2\mathrm{Tr}(FG^b)\mathrm{Tr}(G^cG^a) + \mathrm{Tr}(FG^c)\mathrm{Tr}(G^aG^b)\Big] \; = \\ &= \; \frac{-4ig_T g_s^3 d_{abc}}{3m^3(4\pi)^2}\Big[2\mathrm{Tr}(FG^aG^bG^c) - \mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c)\Big] \end{aligned} \tag{K.53}$$

and therefore (the factor 1/8 included)

$$\Gamma_{3W} = \frac{-g_T g_s^3 d_{abc}}{6m^3(4\pi)^2}\Big[2\mathrm{Tr}(FG^aG^bG^c) - \mathrm{Tr}(FG^a)\mathrm{Tr}(G^bG^c)\Big]. \tag{K.54}$$

The current is obtained by variation with respect to F,

$$g_T J^{\mu\nu} = -\frac{\delta\Gamma_{3W}}{\delta F_{\mu\nu}} = \frac{-g_T g_s^3 d_{abc}}{6m^3(4\pi)^2}\Big[2(G^aG^bG^c)^{\mu\nu} - (G^a)^{\mu\nu}\mathrm{Tr}(G^bG^c)\Big]. \tag{K.55}$$

K.2.1 The direct correlator

Calculating the time-ordered product of the two currents in configuration space, one obtains

$$\begin{aligned}\langle TJ_{\mu\nu}(x)J_{\mu'\nu'}(0)\rangle &= \frac{g_s^6 d_{abc}d_{abc}}{36m^6(4\pi)^4}\frac{1024}{(4\pi^2)^3x^{14}}\Big[x^2(g_{\mu\nu'}g_{\nu\mu'}-g_{\mu\mu'}g_{\nu\nu'})+\\ &\quad +2(x_\mu x_{\mu'}g_{\nu\nu'}-x_\mu x_{\nu'}g_{\nu\mu'}-x_\nu x_{\mu'}g_{\mu\nu'}+x_\nu x_{\nu'}g_{\mu\mu'})\Big] =\\ &= \frac{g_s^6 d_{abc}d_{abc}}{9\pi^4(4\pi^2)^3m^6x^{14}}\Big[-x^2(g_{\mu\mu'}g_{\nu\nu'}-g_{\mu\nu'}g_{\nu\mu'})+\\ &\quad +2(x_\mu x_{\mu'}g_{\nu\nu'}-x_\mu x_{\nu'}g_{\nu\mu'}-x_\nu x_{\mu'}g_{\mu\nu'}+x_\nu x_{\nu'}g_{\mu\mu'})\Big].\end{aligned} \tag{K.56}$$

In order to integrate this expression over d^4x, one uses the fact that

$$\int\frac{1}{x^{12}}e^{iqx}d^Dx = -i\pi^{2-\varepsilon}\frac{\Gamma(-4-\varepsilon)}{\Gamma(6)}\left(-\frac{q^2}{4}\right)^{4+\varepsilon}. \tag{K.57}$$

On the other hand, the tensor integral is given by the two-fold application of the q-differentiation to Eq. (K.57),

$$\int\frac{1}{x^{14}}e^{iqx}d^Dx = +i\pi^{2-\varepsilon}\frac{\Gamma(-5-\varepsilon)}{\Gamma(7)}\left(-\frac{q^2}{4}\right)^{5+\varepsilon}. \tag{K.58}$$

For this one calculates

$$\partial_q^\nu\left(-\frac{q^2}{4}\right)^{5+\varepsilon} = (5+\varepsilon)\left(-\frac{q^2}{4}\right)^{4+\varepsilon}\left(-\frac{1}{2}q^\nu\right) = -\frac{1}{2}(5+\varepsilon)\left(-\frac{q^2}{4}\right)^{4+\varepsilon}q^\nu, \tag{K.59}$$

$$\partial_q^\mu\partial_q^\nu\left(-\frac{q^2}{4}\right)^{5+\varepsilon} = \frac{1}{4}(5+\varepsilon)(4+\varepsilon)\left(-\frac{q^2}{4}\right)^{3+\varepsilon}q^\mu q^\nu-\frac{1}{2}(5+\varepsilon)\left(-\frac{q^2}{4}\right)^{4+\varepsilon}g^{\mu\nu} \tag{K.60}$$

and thus obtains

$$\begin{aligned}\int\frac{x^\mu x^\nu}{x^{14}}e^{iqx}d^Dx &= -\partial_q^\mu\partial_q^\nu\int\frac{1}{x^{14}}e^{iqx}d^Dx =\\ &= -i\pi^{2-\varepsilon}\left(\frac{\mu}{2}\right)^\varepsilon\frac{\Gamma(1-\varepsilon)}{\Gamma(7)}\left(-\frac{q^2}{4}\right)^3\Bigg[\frac{1}{4}(5+\varepsilon)(4+\varepsilon)\frac{\Gamma(-5-\varepsilon)}{\Gamma(1-\varepsilon)}\left(-\frac{q^2}{\mu^2}\right)^\varepsilon q^\mu q^\nu+\\ &\quad +\frac{1}{8}(5+\varepsilon)\frac{\Gamma(-5-\varepsilon)}{\Gamma(1-\varepsilon)}\left(-\frac{q^2}{\mu^2}\right)^\varepsilon q^2g^{\mu\nu}\Bigg] =\\ &= -i\pi^{2-\varepsilon}\left(\frac{\mu}{2}\right)^\varepsilon\frac{\Gamma(1-\varepsilon)}{\Gamma(7)}\left(-\frac{q^2}{4}\right)^3\Bigg[\left(\frac{1}{24\varepsilon}-\frac{11}{144}+\frac{1}{24}\ln\left(\frac{-q^2}{\mu^2}\right)\right)q^\mu q^\nu+\\ &\quad +\frac{1}{8}\left(\frac{1}{24\varepsilon}-\frac{25}{288}+\frac{1}{24}\ln\left(\frac{-q^2}{\mu^2}\right)\right)q^2g^{\mu\nu}+O(\varepsilon)\Bigg].\end{aligned} \tag{K.61}$$

For small values of q^2 one can again skip the singular parts, therefore

$$\int\frac{1}{x^{12}}e^{iqx}d^4x \to \frac{-i\pi^2}{\Gamma(6)\Gamma(5)}\left(\frac{q^2}{4}\right)^4\ln\left(\frac{\mu^2}{-q^2}\right), \tag{K.62}$$

$$\int\frac{x^\mu x^\nu}{x^{14}}e^{iqx}d^4x \to \frac{-i\pi^2}{6\Gamma(6)\Gamma(5)}\left(\frac{q^2}{4}\right)^3\left(q^\mu q^\nu+\frac{1}{8}q^2g^{\mu\nu}\right)\ln\left(\frac{\mu^2}{-q^2}\right). \tag{K.63}$$

Therefore, one obtains

$$\begin{aligned}
&\int \frac{1}{x^{14}}\Big[x^2\left(g_{\mu\nu'}g_{\nu\mu'}-g_{\mu\mu'}g_{\nu\nu'}\right)+\\
&\qquad +2\left(x_\mu x_{\mu'}g_{\nu\nu'}-x_\mu x_{\nu'}g_{\nu\mu'}-x_\nu x_{\mu'}g_{\mu\nu'}+x_\nu x_{\nu'}g_{\mu\mu'}\right)\Big] =\\
&= \frac{-i\pi^2}{\Gamma(6)\Gamma(5)}\left(\frac{q^2}{4}\right)^3\ln\left(\frac{\mu^2}{-q^2}\right)\Bigg[\frac{q^2}{4}\left(g_{\mu\nu'}g_{\nu\mu'}-g_{\mu\mu'}g_{\nu\nu'}\right)+\\
&\qquad +\frac{1}{3}(q_\mu q_{\mu'}+\frac{1}{8}q^2g_{\mu\mu'})g_{\nu\nu'}-\frac{1}{3}(q_\mu q_{\nu'}+\frac{1}{8}q^2g_{\mu\nu'})g_{\nu\mu'}+\\
&\qquad -\frac{1}{3}(q_\nu q_{\mu'}+\frac{1}{8}q^2g_{\nu\mu'})g_{\mu\nu'}+\frac{1}{3}(q_\nu q_{\nu'}+\frac{1}{8}q^2g_{\nu\nu'})g_{\mu\mu'}\Bigg] =\\
&= \frac{-i\pi^2}{\Gamma(6)\Gamma(5)}\left(\frac{q^2}{4}\right)^3\ln\left(\frac{\mu^2}{-q^2}\right)\Bigg[\frac{q^2}{24}\left(6(g_{\mu\nu'}g_{\nu\mu'}-g_{\mu\mu'}g_{\nu\nu'})+2g_{\mu\mu'}g_{\nu\nu'}-2g_{\mu\nu'}g_{\nu\mu'}\right)+\\
&\qquad\qquad +\frac{1}{3}\left(q_\mu q_{\mu'}g_{\nu\nu'}-q_\mu q_{\nu'}g_{\nu\mu'}-q_\nu q_{\mu'}g_{\mu\nu'}+q_\nu q_{\nu'}g_{\mu\mu'}\right)\Bigg] =\\
&= \frac{-i\pi^2}{6\Gamma(6)\Gamma(5)}\left(\frac{q^2}{4}\right)^3\ln\left(\frac{\mu^2}{-q^2}\right)\Bigg[-q^2(g_{\mu\mu'}g_{\nu\nu'}-g_{\mu\nu'}g_{\nu\mu'})+\\
&\qquad\qquad +2\left(q_\mu q_{\mu'}g_{\nu\nu'}-q_\mu q_{\nu'}g_{\nu\mu'}-q_\nu q_{\mu'}g_{\mu\nu'}+q_\nu q_{\nu'}g_{\mu\mu'}\right)\Bigg]. \qquad \text{(K.64)}
\end{aligned}$$

The correlator function is given by

$$\begin{aligned}
&12\pi^2 i\int\langle TJ_{\mu\nu}(x)J_{\mu'\nu'}(0)\rangle e^{iqx}d^4x \;=\\
&= 12\pi^2 i\,\frac{g_s^6 d_{abc}d_{abc}}{9\pi^4(4\pi^2)^3m^6}\,\frac{-i\pi^2}{6\Gamma(6)\Gamma(5)}\left(\frac{q^2}{4}\right)^3\ln\left(\frac{\mu^2}{-q^2}\right)\Bigg[-q^2(g_{\mu\mu'}g_{\nu\nu'}-g_{\mu\nu'}g_{\nu\mu'})+\\
&\qquad\qquad +2\left(q_\mu q_{\mu'}g_{\nu\nu'}-q_\mu q_{\nu'}g_{\nu\mu'}-q_\nu q_{\mu'}g_{\mu\nu'}+q_\nu q_{\nu'}g_{\mu\mu'}\right)\Bigg] =\\
&= \frac{d_{abc}d_{abc}}{3240}\left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{q^2}{4m^2}\right)^3\ln\left(\frac{\mu^2}{-q^2}\right)\Bigg[-\frac{q^2}{4}(g_{\mu\nu'}g_{\nu\mu'}-g_{\mu\mu'}g_{\nu\nu'})+\\
&\qquad\qquad +\frac{1}{2}\left(q_\mu q_{\mu'}g_{\nu\nu'}-q_\mu q_{\nu'}g_{\nu\mu'}-q_\nu q_{\mu'}g_{\mu\nu'}+q_\nu q_{\nu'}g_{\mu\mu'}\right)\Bigg]. \qquad \text{(K.65)}
\end{aligned}$$

K.2.2 The mixed correlator

One can also calculate the mixed correlator of vector and tensor current,

$$12\pi^2 i\int\langle TJ_\mu(x)J_{\mu'\nu'}(0)\rangle e^{iqx}d^4x. \qquad \text{(K.66)}$$

Starting with

$$\mathcal{O}^{\mu\nu}=\frac{-g_s^3}{90m^4(4\pi)^2}[5\mathcal{O}_1^{\mu\nu}+14\mathcal{O}_2^{\mu\nu}],\qquad \tilde{\mathcal{O}}^{\mu\nu}=\frac{-g_s^3}{6m^3(4\pi)^2}[\mathcal{O}_1^{\mu\nu}+2\mathcal{O}_2^{\mu\nu}] \qquad \text{(K.67)}$$

one obtains

$$\begin{aligned}\langle TJ_\mu(x)J_{\mu'\nu'}(0)\rangle &= \partial^\nu\langle T\mathcal{O}_{\mu\nu}(x)\tilde{\mathcal{O}}_{\mu'\nu'}\rangle = \\ &= \frac{g_s^6\partial^\nu}{540m^7(4\pi)^4}\Big[5\langle T\mathcal{O}_{1\mu\nu}(x)\mathcal{O}_{1\mu'\nu'}\rangle + 10\langle T\mathcal{O}_{1\mu\nu}(x)\mathcal{O}_{2\mu'\nu'}\rangle + \\ &\qquad +14\langle T\mathcal{O}_{2\mu\nu}(x)\mathcal{O}_{1\mu'\nu'}\rangle + 28\langle T\mathcal{O}_{2\mu\nu}(x)\mathcal{O}_{2\mu'\nu'}\rangle\Big] = \\ &= \frac{-g_s^6 d_{abc}d_{abc}}{540m^7(4\pi)^4}49152\frac{g_{\mu\mu'}x_{\nu'}-g_{\mu\nu'}x_{\mu'}}{(4\pi^2)^3x^{14}} = \frac{-16g_s^6d_{abc}d_{abc}}{45\pi^4(4\pi^2)^3m^7}\,\frac{g_{\mu\mu'}x_{\nu'}-g_{\mu\nu'}x_{\mu'}}{x^{14}}. \qquad \text{(K.68)}\end{aligned}$$

For the Fourier integral in D dimensions one obtains (using Eq. (K.59))

$$\begin{aligned}\int\frac{x^\mu}{x^{14}}e^{iqx}d^Dx &= -i\partial_q^\mu\int\frac{1}{x^{14}}e^{iqx}d^Dx = \\ &= i\pi^{2-\varepsilon}\left(\frac{\mu}{2}\right)^\varepsilon\frac{\Gamma(1-\varepsilon)}{\Gamma(7)}\left(-\frac{q^2}{4}\right)^4\left[\frac{i}{2}(5+\varepsilon)\frac{\Gamma(-5-\varepsilon)}{(1-\varepsilon)}\left(\frac{-q^2}{\mu^2}\right)^\varepsilon q^\mu\right] = \\ &= -\pi^{2-\varepsilon}\left(\frac{\mu}{2}\right)^\varepsilon\frac{\Gamma(1-\varepsilon)}{6\Gamma(6)}\left(\frac{q^2}{4}\right)^4\left(\frac{1}{48\varepsilon}-\frac{25}{576}+\frac{1}{48}\ln\left(\frac{-q^2}{\mu^2}\right)+O(\varepsilon)\right) \qquad \text{(K.69)}\end{aligned}$$

and therefore (for $|q^2|\to 0$)

$$\int\frac{x^\mu}{x^{14}}e^{iqx}d^4x \to \frac{\pi^2}{12\Gamma(6)\Gamma(5)}\left(\frac{q^2}{4}\right)^4\ln\left(\frac{\mu^2}{-q^2}\right)q^\mu. \qquad \text{(K.70)}$$

One therefore ends up with

$$\begin{aligned}&12\pi^2i\int\langle TJ_\mu(x)J_{\mu'\nu'}(0)\rangle e^{iqx}d^4x = \\ &= \frac{-16g_s^6d_{abc}d_{abc}12\pi^2im}{45\pi^4(4\pi^2)^3}\,\frac{\pi^2}{12\Gamma(6)\Gamma(5)}\left(\frac{q^2}{4m^2}\right)^4\ln\left(\frac{\mu^2}{-q^2}\right)(g_{\mu\mu'}q_{\nu'}-g_{\mu\nu'}q_{\mu'}) = \\ &= \frac{-id_{abc}d_{abc}}{8100}\left(\frac{\alpha_s}{\pi}\right)^3\frac{q^2}{4m}\left(\frac{q^2}{4m^2}\right)^3\ln\left(\frac{\mu^2}{-q^2}\right)(g_{\mu\mu'}q_{\nu'}-g_{\mu\nu'}q_{\mu'}). \qquad \text{(K.71)}\end{aligned}$$

K.3 A short note about the Gordon decomposition

Using $\sigma^{\mu\nu}=i/2[\gamma^\mu,\gamma^\nu]$ and $g^{\mu\nu}=1/2\{\gamma^\mu,\gamma^\nu\}$, one obtains

$$\begin{aligned}\bar\psi m\gamma^\mu\psi &= \bar\psi\gamma^\mu(i\not{D})\psi = i\bar\psi\gamma^\mu\gamma^\nu D_\nu\psi = \\ &= i\bar\psi(g^{\mu\nu}-i\sigma^{\mu\nu})D_\nu\psi = i\bar\psi D^\mu\psi+\bar\psi\sigma^{\mu\nu}D_\nu\psi. \qquad \text{(K.72)}\end{aligned}$$

Using an arrow to indicate the action of the operator D_ν, one obtains by complex conjugation

$$\begin{aligned}m\bar\psi\gamma^\mu\psi &= i\bar\psi\overrightarrow{D}^\mu\psi+\bar\psi\sigma^{\mu\nu}\overrightarrow{D}_\nu\psi \\ m\bar\psi\gamma^\mu\psi &= -i\bar\psi\overleftarrow{D}^\mu\psi+\bar\psi\sigma^{\mu\nu}\overleftarrow{D}_\nu\psi\end{aligned}$$

Because of

$$\partial_\nu(\bar\psi\sigma^{\mu\nu}\psi)=\bar\psi\sigma^{\mu\nu}\overrightarrow{D}_\nu\psi+\bar\psi\sigma^{\mu\nu}\overleftarrow{D}_\nu\psi, \qquad \text{(K.73)}$$

the sum of both equations leads to

$$2m\bar\psi\gamma^\mu\psi=\bar\psi i\overleftrightarrow{D}^\mu\psi+\partial_\nu(\bar\psi\sigma^{\mu\nu}\psi), \qquad \overleftrightarrow{D}_\mu=\overrightarrow{D}_\mu-\overleftarrow{D}_\mu, \qquad D_\mu=\partial_\mu-iB_\mu. \qquad \text{(K.74)}$$

K.4 The regularized moments

In order to avoid the problems with higher moments at the origin, the regularized moments

$$\mathcal{M}_n(\Delta) = \frac{1}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=-\Delta} = \int \frac{\rho(s)ds}{(s+\Delta)^{n+1}} \tag{K.75}$$

were proposed in Ref. [274]. Here the leading order contribution starting at $s = 4m^2$ as well as the newly calculated three-gluon contribution starting at $s = 0$ are calculated. For the leading order contribution with

$$\rho_0(s) = \sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right) = \frac{1}{2}v(3-v^2) \tag{K.76}$$

where

$$v = \sqrt{1-\frac{4m^2}{s}} \quad \Rightarrow \quad s = \frac{4m^2}{1-v^2}, \quad ds = \frac{8m^2 v\,dv}{(1-v^2)^2} \tag{K.77}$$

one obtains

$$\begin{aligned}
\mathcal{M}_n^0(\Delta) &= \int_{4m^2}^{\infty} \frac{\rho_0(s)ds}{(s+\Delta)^{n+1}} = \int_{4m^2}^{\infty} \frac{ds}{(s+\Delta)^{n+1}}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right) = \\
&= \int_0^1 \frac{8m^2 v}{(1-v^2)^2}\left(\frac{4m^2}{1-v^2}+\Delta\right)^{-n-1} \frac{v}{2}(3+v^2)dv = \\
&= \frac{1}{(4m^2)^n}\int_0^1 \frac{v^2(3-v^2)}{(1-v^2)^2}\left(\frac{1}{1-v^2}+\frac{\Delta}{4m^2}\right)^{-n-1} dv = \\
&= 4m^2\int_0^1 \frac{v^2(3-v^2)(1-v^2)^{n-1}}{(4m^2+\Delta(1-v^2))^{n+1}}dv = \qquad \left(t=v^2,\ v=\sqrt{t},\ dv=\frac{1}{2\sqrt{t}}dt\right) \\
&= 2m^2\int_0^1 \frac{\sqrt{t}(3-t)(1-t)^{n-1}}{(4m^2+\Delta(1-t))^{n+1}}dt = \qquad \left(z=\frac{\Delta}{4m^2+\Delta}\right) \\
&= \frac{2m^2}{(\Delta+4m^2)^{n+1}}\left[3\int_0^1 t^{1/2}(1-t)^{n-1}(1-zt)^{-n-1}dt\ + \right. \\
&\qquad\qquad \left. -\int_0^1 t^{3/2}(1-t)^{n-1}(1-zt)^{-n-1}dt\right] = \\
&= \frac{2m^2}{(\Delta+4m^2)^{n+1}}\left[\frac{3\Gamma(3/2)\Gamma(n)}{\Gamma(n+3/2)}\ {}_2F_1(3/2,n+1;n+3/2;z)\ + \right. \\
&\qquad\qquad \left. -\frac{\Gamma(5/2)\Gamma(n)}{\Gamma(n+5/2)}\ {}_2F_1(5/2,n+1;n+5/2;z)\right] = \\
&= \frac{4m^2}{(4m^2+\Delta)^{n+1}}\frac{\Gamma(5/2)\Gamma(n)}{2\Gamma(n+5/2)} \times \\
&\qquad \times\left[\frac{3(n+3/2)}{3/2}\ {}_2F_1(3/2,n+1;n+3/2;z) - {}_2F_1(5/2,n+1;n+5/2;z)\right] = \\
&= \frac{4m^2}{(4m^2+\Delta)^{n+1}}\frac{\Gamma(5/2)\Gamma(n)}{2\Gamma(n+5/2)}\left[(2n+3)\ {}_2F_1\left(3/2,n+1;n+3/2;\frac{\Delta}{4m^2+\Delta}\right)\ + \right. \\
&\qquad \left. -{}_2F_1\left(5/2,n+1;n+5/2;\frac{\Delta}{4m^2+\Delta}\right)\right].
\end{aligned} \tag{K.78}$$

	$\Delta = 1\,GeV^2$		$\Delta = 2\,GeV^2$	
	$\mathcal{M}_n^{(a)}/\mathcal{M}_n^{(c)}$	$\mathcal{M}_n^{(b)}/\mathcal{M}_n^{(c)}$	$\mathcal{M}_n^{(a)}/\mathcal{M}_n^{(c)}$	$\mathcal{M}_n^{(b)}/\mathcal{M}_n^{(c)}$
$n = 5$	0.000	0.854	0.000	0.841
$n = 6$	0.002	1.151	0.001	1.133
$n = 7$	0.085	1.507	0.011	1.481
$n = 8$	4.173	1.932	0.280	1.897
$n = 9$	239.3	2.439	8.089	2.392

Table K.1: The ratios of the moments from different squared energy regions for the bottom quark ($m_b = 4.8\,GeV$) with $E = 1\,GeV$ and two different values for Δ.

For $\Delta \to 0$ the expression in square brackets reduces to $2(n+1)$, so that

$$\mathcal{M}_n^0(\Delta = 0) = \frac{(n+1)}{(4m^2)^n}\frac{\Gamma(5/2)\Gamma(n)}{\Gamma(n+5/2)}. \tag{K.79}$$

For the new part

$$\rho_g(s) = C_g\left(\frac{s}{4m^2}\right)^4, \qquad C_g = \frac{17 d_{abc}d_{abc}}{243000}\left(\frac{\alpha_s}{\pi}\right)^3 \tag{K.80}$$

one obtains

$$\begin{aligned}
\mathcal{M}_n^g(\Delta) &= \frac{1}{n!}\left(\frac{d}{dq^2}\right)^n \Pi_g(q^2)\Big|_{q^2=-\Delta} = \int_0^\infty \frac{\rho_g(s)ds}{(s+\Delta)^{n+1}} = \\
&= C_g\int_0^\infty \frac{ds}{(s+\Delta)^{n+1}}\left(\frac{s}{4m^2}\right)^4 = \qquad (s = \Delta x) \\
&= \frac{C_g}{\Delta^n}\left(\frac{\Delta}{4m^2}\right)^4\int_0^\infty \frac{x^4dx}{(1+x)^{n+1}} = \qquad (x' = 1+x) \\
&= \frac{C_g}{\Delta^n}\left(\frac{\Delta}{4m^2}\right)^4\int_1^\infty \frac{(x'-1)^4dx'}{(x')^{n+1}} = \qquad \left(x'' = \frac{1}{x'},\ dx' = -\frac{1}{x''^2}dx''\right) \\
&= \frac{C_g}{\Delta^n}\left(\frac{\Delta}{4m^2}\right)^4\int_0^1 \frac{(1/x''-1)^4dx''}{x''^2(1/x'')^{n+1}} = \\
&= \frac{C_g}{\Delta^n}\left(\frac{\Delta}{4m^2}\right)^4\int_0^1 (x'')^{n-5}(1-x'')^4dx'' = \frac{C_g}{\Delta^n}\left(\frac{\Delta}{4m^2}\right)^4\frac{\Gamma(n-4)\Gamma(5)}{\Gamma(n+1)}.
\end{aligned} \tag{K.81}$$

For a numerical comparison of the contribution below threshold with the "regular" contribution one compares the moments

$$\mathcal{M}_n^{(a)} = \int_0^{4m^2}\frac{\rho_g(s)ds}{(s+\Delta)^{n+1}}, \quad \mathcal{M}_n^{(b)} = \int_{4m^2}^{(2m+E)^2}\frac{\rho_0(s)ds}{(s+\Delta)^{n+1}}, \quad \mathcal{M}_n^{(c)} = \int_{(2m+E)^2}^\infty \frac{\rho_0(s)ds}{(s+\Delta)^{n+1}} \tag{K.82}$$

where the range between $s = 4m^2$ and $s = (2m+E)^2$ is the resonance region. As typical values one takes $E = 1\,GeV$, $\Delta = 1\,GeV^2$ or $\Delta = 2\,GeV^2$ and for the quark masses $m_c = 1.3\,GeV$ and $m_b = 4.8\,GeV$. In Table K.1 the ratios $\mathcal{M}_n^{(a)}/\mathcal{M}_n^{(c)}$ and $\mathcal{M}_n^{(b)}/\mathcal{M}_n^{(c)}$ for different values of n for the bottom quark are shown. For the charm quark one can neglect the contributions of the region below threshold for the moments ($n < 7$).

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Appendix L

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