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Solving differential and integral equations by the Haar wavelet method; revisited

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ABSTRACT

Solving an n-th order differential equation by the Haar wavelet method usually the highest derivative $y^{(n)}$ is expanded into the series of Haar functions. It is shown in the present paper that if we develop into Haar series the derivative $y^{(n+1)}$ then the results for the same number of grid points are considerably more exact. The same approach is applicable also for integro-differential equations. Four numerical examples are presented.

Keywords: Haar wavelets, ordinary differential equations, boundary value problems, nonlinear equations.

Mathematics Subject Classification: 34A30, 34B05, 65T60.

1 Introduction

Among all the wavelet families mathematically the simpliest are the Haar wavelets – they are made up of pairs of piecewise constant functions. Due to simplicity the Haar wavelet method has been an effective tool for solving several problems of differential and integral calculus. For the sake of conciseness these papers are not considered here; the necessary information can be found from other papers (e.g. (Chen and Hsiao, 1997), (Lepik 2003), (Lepik 2006), (Lepik 2007), (Lepik 2008b)).

The Haar wavelets have also an essential shortcoming – they are not continuous and therefore cannot be differentiated in the points of discontinuity. This difficulty was overridden by Chen and Hsiao (1997), who recommended to expand into the Haar series not the function itself, but its highest derivative appearing in the equation. The other derivatives (and the function itself) are obtained though integrations. As much as we know this approach was up to now applied only for solving first and second order ordinary differential equations. Solution of higher order differential equations was discussed by Lepik (2008).

Let us consider an n-th order differential or integro-differential equation

$$F(x, y, y', ..., y^{(n)}) = 0\,, \quad x \in [A, B]\,. \tag{1.1}$$

According to Chen and Hsiao we shall develop into Haar series the derivative $y^{(n)}$:

$$y^{(n)}(x) = \sum_{i=1}^{2M} a_i h_i(x) \,. \tag{1.2}$$

Here $h_i(x)$ are the Haar wavelet functions; a_i the wavelet coefficients which must be calculated in the course of the solution.

For (1.2) the derivative $y^{(n)}$ is a piecewise constant function. From here the idea to develop into the Haar series the function arises:

$$y^{(n+\mu)}(x) = \sum_{i=1}^{2M} a_i h_i(x) \,, \quad \mu = 1, 2, \ldots \,. \tag{1.3}$$

Now the derivative $y^{(n)}$ is continuous and it can be expected that in the case of (1.3) the exactness of the solution is higher as for (1.2). The aim of the paper is to test this assumption. This property is illustrated and tested in Sections (3)–(5) with exact solution known four different examples. Local and global error estimates are calculated. Computer simulations were carried out with the aid of the MATLAB programs.

In this paper linear differential or integro-differential equations are dealt with, but the recommended method remains applicable also for nonlinear problems (consult (Lepik 2006), (Lepik 2008a).

2 Haar wavelet method

Consider the interval $x \in [A, B]$, where A and B are given constants. This interval is partiated into $2M$ subintervals of equal length, where $M = 2^J$ (J is the maximal level of resolution). The length of each subinterval is $\Delta x = (B - A)/(2M)$. Next two parameters are introduced: the dilatation parameter $j = 0, 1, ..., J$ and the translation parameter $k = 0, 1, ..., m - 1$, where $m = 2^j$. The wavelet number i is identified as $i = m + k + 1$.

The i-th Haar wavelet is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } \ x \in [\xi_1(i), \xi_2(i)] \,, \\ -1 & \text{for } \ x \in [\xi_2(i), \xi_3(i)] \,, \\ 0 & \text{elsewhere} \,, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} \xi_1(i) &= A + 2k\mu\Delta x \,, & \xi_2(i) &= A + (2k+1)\mu\Delta x \,, \\ \xi_3(i) &= A + 2(k+1)\mu\Delta x \,, & \mu &= M/m \,. \end{aligned} \tag{2.2}$$

The case $i = 1$ corresponds to the scaling function, where $h_1(x) = 1$ for $x \in [A, B]$.

By integrating (2.1) α-times we obtain (Lepik 2008a).

$$p_{\alpha,i}(x) = \begin{cases} 0 \\ \qquad\qquad \text{for } \ x < \xi_1(i)\,, \\ \dfrac{1}{\alpha!}[x-\xi_1(i)]^\alpha \\ \qquad\qquad \text{for } \ x \in [\xi_1(i), \xi_2(i)]\,, \\ \dfrac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2[x-\xi_2(i)]^\alpha\right\} \\ \qquad\qquad \text{for } \ x \in [\xi_2(i), \xi_3(i)]\,, \\ \dfrac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2[x-\xi_2(i)]^\alpha + [x-\xi_3(i)]^\alpha]\right\} \\ \qquad\qquad \text{for } \ x > \xi_3(i)\,. \end{cases} \tag{2.3}$$

These formulas hold for $i > 1$. If $i = 1$ we have $\xi_1 = A\,,\ \xi_2 = \xi_3 = B$ and

$$p_{\alpha,1}(x) = \frac{1}{\alpha!}(x-A)^\alpha\,. \tag{2.4}$$

In the present paper the solutions are based on the collocation method. The grid points are

$$\tilde{x}_l = A + l\Delta x,\ \ l = 0, 1, 2, ..., 2M\,;$$

the collocation points

$$x_l = 0.5(\tilde{x}_{l-1} + \tilde{x}_l),\ \ l = 1, 2, ..., 2M\,.$$

3 Initial value problem for n-th order ODE

Consider the equation

$$\sum_{\nu=0}^{n} A_\nu(x) y^{(\nu)}(x) = f(x)\,, \quad x \in [A, B] \tag{3.1}$$

with the initial conditions

$$y^{(\nu)}(A) = y_0^{(\nu)}, \quad \nu = 0, 1, ..., n-1\,. \tag{3.2}$$

Let us seek the Haar wavelet solution in the form (1.3). Here $A_\nu(x),\ f(x)$ are prescribed functions, $y_0^{(\nu)}$ given constants. By integrating this equation ν times we find keeping in view (2.3)–(2.4), that

$$y^{(n+\mu-\nu)}(x) = \sum_{i=1}^{2M} a_i p_{\nu,i}(x) + Z_\nu(x)\,, \tag{3.3}$$

where

$$Z_\nu(x) = \sum_{\sigma=0}^{\nu-1} \frac{1}{\sigma!} y_0^{(n+\mu+\sigma-\nu)}(x-A)^\sigma\,, \quad \nu = 1, 2, ..., n+\mu\,. \tag{3.4}$$

If $\nu = n + \mu$ we get the function $y(x)$ sought for.

As to the parameter μ we consider here only the values $\mu = 0$ and $\mu = 1$. The case $\mu = 0$ corresponds to the conventional solution (1.2). As to the values $\mu = 2, 3, ...$ then the calculations

showed that they have generally only a small effect for improving the solution and therefore the cases $\mu > 1$ are not considered in the following analysis.
If $\mu = 1$ we need a complementary initial condition for $y^{(n)}(A)$; it can be calculated from the equation (3.1).
Next (3.3)–(3.4) are discretized by replacing $x \to x_l$. Substituting these results into (3.1) we get a system of linear equations for the wavelet coefficients a_i.
Example 1: Solve

$$y^{(6)} + y = \sin\frac{3}{2}x \sin\frac{x}{2}\,, \quad x \in [0,\pi] \tag{3.5}$$

for the initial conditions $y_0 = 1,\ y_0' = -1,\ y_0'' = 0.3,\ y_0''' = 0,\ y_0^{IV} = 1,\ y_0^{V} = 1.$
In the case $\mu = 0$ this equation was solved by (Lepik 2008a),. Now let us consider the case $\mu = 1$. Calculating from (3.5) the complementary initial condition $y^{(6)}(0)$ we find $y^{(6)}(0) = -1.$
Now we shall take

$$y^{(7)}(x) = \sum_{i=1}^{2M} a_i h_i(x)\,.$$

By integrating it 7 times and replacing the results into (3.5) we obtain

$$\sum_{i=1}^{2M} a_i[p_{1,i}(x_l) + p_{7,i}(x_l)] = 1 + Z(l) + F(l)\,, \tag{3.6}$$

where

$$\begin{aligned} Z(l) &= \sum_{\sigma=0}^{6} \frac{1}{\sigma!} y_0^{(\sigma)} x_l^{\sigma}\,, \\ F(l) &= \sin\frac{3}{2}x_l \sin\frac{x_l}{2}\,. \end{aligned}$$

From (3.6) the wavelet coefficients are calculated. The function to be sought is

$$y(x_l) = \sum_{i=1}^{2M} a_i p_{7,i}(x_l) + Z(l)\,. \tag{3.7}$$

The error estimates are defined as in (Lepik 2008a):

$$\begin{aligned} \delta &= \max_l \left| \frac{y(x_l)}{y_{ex}(x_l)} - 1 \right| \ \text{local estimate}\,, \\ \sigma &= \parallel y - y_{ex} \parallel /(2M) \ \text{global estimate}\,. \end{aligned} \tag{3.8}$$

Computer simulation gave the following results.
It follows from this Table that the errors estimates for $\mu = 1$ are about 10 times lesser as for $\mu = 0$.

4 Boundary value problems

Consider again (3.1). Now we assume that only $n_1 < n$ initial conditions are prescribed at $x = A$; the remaining $n - n_1$ conditions are specified in some internal points $x < B$ or at the boundary $x = B$. Values of the missing initial conditions can be calculated from (10)–(11). The subsequent course of solution proceeds as indicated in Section 3.

Example 2: Solve

$$x^2y''' + 6xy'' + 6y' = 6\,, \quad x \in [2,4] \tag{4.1}$$

for $y(2) = y(4) = 0\,,\; y'(3) = 0.$

First consider the conventional case $\mu = 0$, where the solution is sought in the form (1.2). By integrating this equation we find

$$\begin{aligned} y'(x) &= \sum_{i=1}^{2M} a_i p_{2,i}(x) + (x-2)y_0'' + y_0' \\ y(x) &= \sum_{i=1}^{2M} a_i p_{3,i}(x) + \frac{1}{2}(x-2)^2 y_0'' + (x-2)y_0' + y_0\,. \end{aligned} \tag{4.2}$$

Satisfying the boundary conditions $y(4) = y'(3) = 0$ we get the system

$$\begin{aligned} 0 &= \sum_{i=1}^{2M} a_i P_2(i,3) + y_0'' + y_0' \\ 0 &= \sum_{i=1}^{2M} a_i P_3(i,4) + 2y_0'' + 2y_0'\,. \end{aligned} \tag{4.3}$$

Here the notation

$$P_\alpha(i, x_*) = p_{\alpha,i}(x = x_*) \tag{4.4}$$

was introduced.

From the system (4.3) we should evaluate the missing initial values $y_0',\; y_0''$, but in the present case it is not possible. From here the conclusion, that for $\mu = 0$ the Haar wavelet method is unable to solve the set up boundary value problem, can be made.

Let us pass to the case $\mu = 1$. Now we assume

$$y^{IV} = \sum_{i=1}^{2M} a_i h_i(x)\,. \tag{4.5}$$

Making use of (3.3)–(3.4) we find

$$y^{(4-\nu)}(x) = \sum_{i=1}^{2M} a_i p_{\nu,i}(x) + \sum_{\sigma=0}^{\nu-1} \frac{1}{\sigma!} y_0^{(4+\sigma-\nu)}(x-2)^\sigma\,, \quad \nu = 1,2,3,4\,. \tag{4.6}$$

Instead of (4.3) we get now the system

$$\begin{aligned} 0 &= \sum_{i=1}^{2M} a_i P_3(i,3) + \frac{1}{2}y_0''' + y_0'' + y_0' \\ 0 &= \sum_{i=1}^{2M} a_i P_4(i,4) + \frac{4}{3}y_0''' + 2y_0'' + 2y_0'\,. \end{aligned} \tag{4.7}$$

Satisfying (4.1) at the boundary $x = 2$ we obtain

$$4y_0''' + 12y_0'' + 6y_0' = 6\,. \tag{4.8}$$

From (22)–(23) the missing initial values are calculated; by doing this we find

$$\begin{aligned} y_0' &= \sum_{i=1}^{2M} a_i \left[P_4(i,4) - 4P_3(i,3)\right] - 2 \\ y_0'' &= \frac{1}{2}\sum_{i=1}^{2M} a_i P_4(i,4) + 2 \\ y_0''' &= -3\sum_{i=1}^{2M} a_i \left[P_4(i,4) - 2P_3(i,3)\right] . \end{aligned} \tag{4.9}$$

Next the results (4.6), (4.9) are substituted into (4.1). Satisfying this equation in the collocation points x_l we get for a_i the system

$$\sum_{i=1}^{2M} a_i S(i,l) = -12(x_l - 2)\,, \quad (l = 1, 2, ..., 2M)\,, \tag{4.10}$$

where

$$\begin{aligned} S(i,l) = p_{1,i}(l)x_l^2 + 6p_{2,i}(l)x_l + 6p_{3,i}(l) + \\ +12(5x_l^2 - 12x_l + 4)P_3(i,3) - 6(5x_l^2 - 13x_l + 6)P_4(i,4)\,. \end{aligned} \tag{4.11}$$

The wanted function $y(x)$ is calculated from (4.6) by taking $\nu = 4$.
This boundary value problem has an exact solution

$$y_{ex} = x - \frac{15}{2} + \frac{17}{x} - \frac{12}{x^2}\,. \tag{4.12}$$

The error estimates (3.8) are presented in Table 2.
It follows from this Table that the exactness of the obtained solutions is rather good: already in the case of $2M = 8$ calculation points the curves of the exact and wavelet solutions visually coincide.

5 Fredholm integro-differential equation

Consider the Fredholm integro-differential equation of the second kind

$$\begin{aligned} Cy'(x) + Dy(x) = \int_A^B K(t,x)\left[y(t) + Ey'(t)\right]dt + f(x)\,, \\ x, t \in [A, B]\,, \quad y(A) = y_0\,, \end{aligned} \tag{5.1}$$

where C, D, E, A, B, y_0 are given constants and K, f are given functions.
The wavelet solution of this equation is sought in the form

$$y^{\mu+1}x = \sum_{i=1}^{2M} a_i h_i(x)\,. \tag{5.2}$$

Integrating (5.2) we obtain

$$
\begin{aligned}
y^{(\mu)}(x) &= \sum_{i=1}^{2M} a_i p_(1,i)(x) + y_0^{(\mu)} \\
y^{(\mu-1)}(x) &= \sum_{i=1}^{2M} a_i p_(2,i)(x) + y_0^{(\mu)}(x-A) + y_0^{(\mu-1)} .
\end{aligned}
\tag{5.3}
$$

The initial value y_0' is calculated from (5.1) by assuming $x = A$.
The results (5.2)–(5.3) are replaced into (5.1), which will be satisfied in the collocation points x_l. In this way we again get a system of linear equations for calculating the wavelet coefficients a_i. The most labour-consuming operation here to calculate the integrals

$$
\int_A^B K(t, x_l) p_{\alpha,i}(t) dt . \tag{5.4}
$$

Details of the solution will be cleared up with the aid of the subsequent Examples.
Example 3: Solve

$$
y'(x) + y(x) = \int_0^1 (x+t) y(t) dt + f(x) , \quad y(0) = 1 , \tag{5.5}
$$

where

$$
f(x) = x^4 + 4x^3 - \frac{6}{5}x + \frac{1}{3} .
$$

The solution starts with calculating $\xi_1(i), \xi_2(i), \xi_3(i)$ from (2.2). Let us denote $\Delta\xi(i) = \xi_2(i) - \xi_1(i)$. Next making use of (2.3)–(2.4) the following integrals are evaluated (for conciseness sake the index i at $\xi_1, \xi_2, \xi_3, \Delta\xi$ is omitted):

$$
\begin{aligned}
Q_{\mu+1}(i) &= \int_0^1 p_{\mu+1,i}(t) dt = \\
&= \frac{1}{(\mu+2)!} \left[(1-\xi_1)^{\mu+2} - 2(1-\xi_2)^{\mu+2} + (1-\xi_3)^{\mu+2} \right] \\
R_{\mu+1}(i) &= \int_0^1 t p_{\mu+1,i}(t) dt = \\
&= \frac{1}{(\mu+1)!(\mu+3)} \left[(1-\xi_1)^{\mu+3} - 2(1-\xi_2)^{\mu+3} + (1-\xi_3)^{\mu+3} \right] + \\
&+ \frac{1}{(\mu+2)!} \left[\xi_1(1-\xi_1)^{\mu+2} - 2\xi_2(1-\xi_2)^{\mu+2} + \xi_3(1-\xi_3)^{\mu+2} \right] .
\end{aligned}
\tag{5.6}
$$

At first consider the case $\mu = 1$. According to (5.3) we have

$$
\begin{aligned}
y'(x_l) &= \sum_{i=1}^{2M} a_i p_{1,i}(x_l) + y'(0) \\
y(x_l) &= \sum_{i=1}^{2M} a_i p_{2,i}(x_l) + y_0' x_l + 1
\end{aligned}
\tag{5.7}
$$

$$\int_0^1 y(t)dt = \sum_{i=1}^{2M} a_i Q_2(i) + \frac{1}{2}y_0' + 1$$
$$\int_0^1 ty(t)dt = \sum_{i=1}^{2M} a_i R_2(i) + \frac{1}{3}y_0' + \frac{1}{2}\,. \tag{5.8}$$

The initial condition is calculated from (5.5) assuming $x = 0$; doing this we find

$$y_0' = \frac{3}{2}\left[\sum_{i=1}^{2M} a_i R_2(i) + f(0) - 0.5\right]\,. \tag{5.9}$$

Replacing (5.7)–(5.9) into (5.5) we get the system

$$\sum_{i=1}^{2M} a_i \left[p_{2,i}(x_l) + p_{1,i}(x_l) - x_l Q_2(i) + \frac{3}{4}x_l R_2(i)\right] = \\ = \frac{11}{8}x_l + f(x_l) - \left(\frac{3}{4}x_l + 1\right) f(0)\,, \quad (l = 1, 2, ..., 2M)\,. \tag{5.10}$$

Similar calculations were carried out also for $\mu = 0$; instead of (5.10) we get now the system

$$\sum_{i=1}^{2M} a_i \left[p_{1,i}(x_l) + h_i(x_l) - x_l Q_1(i) - R_1(i)\right] = x_l - 0.5 + f(x)\,. \tag{5.11}$$

The function $y(x)$ is evaluated from

$$y(x) = \sum_{i=1}^{2M} a_i p_{2,i}(x) + y_0 + xy_0'\,, \quad \text{for} \quad \mu = 1$$
$$y(x) = \sum_{i=1}^{2M} a_i p_{1,i}(x) + y_0\,, \quad \text{for} \quad \mu = 0\,. \tag{5.12}$$

The exact solution of the problem is

$$y(x) = x^4 + 1\,. \tag{5.13}$$

Error estimates for $\mu = 0$ and $\mu = 1$ are presented in Table 3.

Again we can state that the results for $\mu = 1$ are considerably more exact as for $\mu = 0$ (e.g. for 64 calculation points the error estimates decrease more than 60 times).

Example 4:

Consider once more (5.5), but the external force $f(x)$ is now taken in the form

$$f(x) = x^2 + \frac{2}{3}x + \frac{1}{4}\,. \tag{5.14}$$

Making again use of (5.9)–(5.12) we find that for $\mu = 0$ the error estimates are $\delta = 6.5E{-}3,\ \sigma = 1.2E - 3$ for $2M = 32$ and $\delta = 6.5E - 3,\ \sigma = 8.2E - 4$ for $2M = 64$.

Carrying out the calculations for $\mu = 1$ we find that $a_1 = 2,\ a_i = 0$ for $i > 1$ and at each level of resolution J the wavelet solution is exact in the collocation points. This circumstance is not unexpected since due to (5.9), (5.12) we have $y_0' = 0,\ y(x) = x^2 + 1$, which coincides with the exact solution of the problem (5.5), (5.14).

6 Conclusion

A new approach for solving differential and integral equations by the aid of Haar wavelet method is proposed. It is demonstrated that it allows to obtain considerably more exact results as the convenient method.
A benefit of the proposed method is its universality: it can be applied to for solving a wide class of differential and integral equations. By it is expedient to put together subprograms for calculating the functions $p_{\alpha,i}(x)$; these subprograms can be used for solving different problems without changes.

Table 1: Error estimates for the problem (3.5).

	δ		σ	
$2M$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$
8	3.3E-3	2.5E-4	4.1E-4	3.2E-5
16	8.3E-4	6.4E-5	5.2E-5	4.0E-6
32	2.7E-4	1.6E-5	6.5E-6	5.1E-7

Table 2: Error estimates for the problem (4.1).

$2M$	δ	σ
8	1.3E-3	4.7E-5
16	1.7E-4	4.2E-6
32	7.8E-5	1.2E-6
64	2.2E-5	2.3E-7

Table 3: Error estimates for the problem (5.5).

	δ		σ	
$2M$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = 1$
16	4.3E-3	7.7E-4	1.4E-3	2.7E-4
32	3.5E-3	1.9E-4	8.03E-4	4.7E-5
64	3.3E-3	4.9E-5	5.3E-4	8.3E-6

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References

Chen, C. F. and Hsiao, C. H., 1997, *Haar wavelet method for solving lumped and distributed-parameter systems*. IEE Proc, number 144 in Control Theory Appl., 87–94.

Lepik, Ü., 2003, *Numerical solution of differential equations using Haar wavelets*. Math. Comp. in Simulation, (68); 127–143.

Lepik, Ü., 2006, *Haar wavelet method for non-linear integro-differential equations*. Appl. Math. Comput., (176); 324–333.

Lepik, Ü., 2007, *Application of the Haar wavelet transform to solving integral and differential equations*. Proc. Estonian Acad. Sci. Phys. Math., (56); 28–46.

Lepik, Ü., 2008a, *Solving higher order differential equations by the Haar wavelet method*. Appl. Math. Comput., (in print).

Lepik, Ü., 2008b, *Solving integral- and differential equations by the aid of nonuniform Haar wavelets*. Appl. Math. Comput., (198); 326–332.