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Haar wavelet method for solving higher order differential equations

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ABSTRACT

Up to now the Haar wavelets are applied mostly for solving first and second order differential equations. The aim of the present paper is to demonstrate that this method is valuable also in the case of higher order equations. Initial and boundary value problems are discussed (also in the case of nonlinear equations). To show the effeciency of the method tree problems are solved. Computer simulations show that the necessary exactness of the results is achieved already for a small number of grid points.

Keywords: Haar wavelets, ordinary differential equations, boundary value problems, nonlinear equations.

Mathematics Subject Classification: 34A34, 34B15, 65T60.

1 Introduction

The Haar wavelets are made up of pairs of piecewise constant functions and are mathematically the simpliest orthonormal wavelets with a compact support. Due to the mathematical simplicity the Haar wavelet method has turned out to be an effective tool for solving many problems, among these are also differential and integral equations.

In the present paper the Haar wavelet method for solution of ordinary differential equations (ODE) is discussed. Here we meet with the fact that the Haar wavelets are not continuous and cannot be differentiated in the points of discontinuity. By this reason it is not possible to apply the Haar wavelets directly for solving ODE. There are at least two possibilities of ending this impasse. First, the piecewice constant Haar functions can be regularized with interpolation splines. This greatly complicates the solution process and the main advantage of the Haar wavelets – their simplicity – gets lost.

Another possibility was proposed by Chen and Hsiao [1997a,1997b]. They recommended to expand into the Haar series not the function itself, but its highest derivative appearing in the differential equation. The other derivatives (and the function) are obtained through integrations.

All these ingredients are then incorporated into the whole system, discretized by Galerkin or collocation method. This approach was applied in papers [Hsiao and Wang, 1999, 2000, 2001], [Lepik 2003].

In these papers first or second order ODE-s are discussed, but the benefits of the Haar method become apparent especially in the case of higher order ODE-s. The aim of the present paper is to demonstrate how to integrate such equations.

The paper is organized as follows. In Section 2 the Haar wavelet method is reported. The traditional Haar wavelets are defined for the argument $x \in [0,1]$, but in Section 2 a new variant of the Haar wavelets, which is applicable for an arbitrary region of the variable x, is presented. In this Section the integrals of the Haar wavelet are also calculated.

In Section 3 the Haar wavelet solution for linear ODE-s is presented. In Section 4 the treatment of boundary value problems for linear ODE-s is discussed. Solution of initial and boundary problems for nonlinear ODE-s is presented in Section 5.

2 Haar wavelets and their integrals

Let us consider the interval $x \in [A,B]$, where A and B are given constants. We shall define the quantity $M = 2^J$, where J is the maximal level of resolution. The interval $[A,B]$ is partiated into $2M$ subintervals of equal length; the length of each subinterval is $\Delta x = (B-A)/(2M)$. Next two parameters are introduced: the dilatation parameter $j = 0,1,...,J$ and the translation parameter $k = 0,1,...,m-1$ (here the notation $m = 2^J$ is introduced). The wavelet number i is identified as $i = m+k+1$.

The i-th Haar wavelet is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1(i), \xi_2(i)]\,, \\ -1 & \text{for } x \in [\xi_2(i), \xi_3(i)]\,, \\ 0 & \text{elsewhere}\,, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} \xi_1(i) &= A + 2k\mu\Delta x\,, \qquad & \xi_2(i) &= A + (2k+1)\mu\Delta x\,, \\ \xi_3(i) &= A + 2(k+1)\mu\Delta x\,, \qquad & \mu &= M/m\,. \end{aligned} \tag{2.2}$$

The case $i = 1$ corresponds to the scaling function: $h_1(x) = 1$ for $x \in [A,B]$ and $h_1(x) = 0$ elsewhere.

If we want to solve a n-th order ODE we need the integrals

$$\begin{gathered} p_{\nu,i}(x) = \underbrace{\int_A^x \int_A^x \dots \int_A^x}_{\nu-\text{times}} h_i(t)dt^\nu = \frac{1}{(\nu-1)!} \int_A^x (x-t)^{\nu-1} h_i(t)dt \\ \nu = 1,2,...,n\,, \quad i = 1,2,...,2M\,. \end{gathered} \tag{2.3}$$

The case $\nu = 0$ corresponds to function $h_i(t)$.

Taking account of (2.1) these integrals can be calculated analytically; by doing it we obtain

$$p_{\alpha,i}(x) = \begin{cases} 0 \\ \qquad\qquad \text{for } \; x < \xi_1(i)\,, \\ \dfrac{1}{\alpha!}[x-\xi_1(i)]^\alpha \\ \qquad\qquad \text{for } \; x \in [\xi_1(i), \xi_2(i)]\,, \\ \dfrac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2[x-\xi_2(i)]^\alpha\right\} \\ \qquad\qquad \text{for } \; x \in [\xi_2(i), \xi_3(i)]\,, \\ \dfrac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2[x-\xi_2(i)]^\alpha + [x-\xi_3(i)]^\alpha]\right\} \\ \qquad\qquad \text{for } \; x > \xi_3(i)\,. \end{cases} \tag{2.4}$$

These formulas hold for $i > 1$. In the case $i = 1$ we have $\xi_1 = A\,,\ \xi_2 = \xi_3 = B$ and

$$p_{\alpha,1}(x) = \frac{1}{\alpha!}(x-A)^\alpha\,. \tag{2.5}$$

In the present paper the collocation method for solving the ODEs is applied. The collocation points are $x_l = 0.5[\tilde{x}_{l-1} + \tilde{x}_l],\ l = 1, 2, ..., 2M$: the symbol $\tilde{x}_l$ denotes the l-th grid point $\tilde{x}_l = A + l\Delta x,\ l = 1, 2, ..., 2M$. Next (2.1), (2.4) are discretized by replacing $x \to x_l$.

It is convenient to introduce the Haar matrices $H(i,l) = h_i(x_l),\ P_\nu(i,l) = p_{\nu,i}(x_l)$. In the following Sections computer simulations are carried out with the aid of the *Matlab* programs; for which the matrix representation is effective.

3 Haar wavelet solution for linear ODEs

Consider the n-th order linear differential equation

$$\sum_{\nu=0}^{n} A_\nu(x) y^{(\nu)}(x) = f(x)\,, \quad x \in [\alpha, \beta] \tag{3.1}$$

with the initial conditions

$$y^{(\nu)}(A) = y_0^{(\nu)}, \quad \nu = 0, 1, ..., n-1\,. \tag{3.2}$$

Here $A_\nu(x),\ f(x)$ are prescribed functions, $y_0^{(\nu)}$ given constants.

The Haar wavelet solution is sought in the form

$$y^{(n)}(x) = \sum_{i=1}^{2M} a_i h_i(x)\,, \tag{3.3}$$

where a_i are the wavelet coefficients.

By integrating (3.3) $n - \nu$ times we obtain

$$y^{(\nu)}(x) = \sum_{i=1}^{2M} a_i p_{n-\nu,i}(x) + Z_\nu(x)\,, \tag{3.4}$$

where

$$Z_\nu(x) = \sum_{\sigma=0}^{n-\nu-1} \frac{1}{\sigma!}(x-A)^\sigma y_0^{(\nu+\sigma)} . \quad (3.5)$$

We shall satisfy (3.1), (3.3)–(3.5) in the collocation points $x(l)$. If we substitute $x \to x(l)$ and replace (3.3)–(3.5) into equation (3.1) we get a system of linear equations for calculating the wavelet coefficients a_i . After solving this system the wanted function $y = y(x)$ is calculated from (3.4) by assuming $\nu = 0$.

It is essential to estimate the exactness of the achieved results. For this purpose we introduce the following to error estimates.

(i) If the exact solution $y = y_{ex}(x)$ is known we define the error estimates as

$$\delta_{ex} = \max_l \left| \frac{y(x_l)}{y_{ex}(x_l)} - 1 \right| \quad \text{(local estimate)}$$

and

$$\sigma_{ex} = \| y - y_{ex} \| / (2M) \quad \text{(global estimate)} .$$

(ii) If the exact solution is unknown we define the error vector

$$\varepsilon(l) = \sum_{\nu=0}^{n} A_\nu(x_l) y^{(\nu)}(x_l) - f(x_l) . \quad (3.6)$$

The smaller $\| \varepsilon(l) \| / (2M)$ the more exact is the solution ($\| \varepsilon \| = 0$ in the case of the exact solution).

If $\| \varepsilon(l) \| / (2M)$ is not small enough, we have to go higher level of resolution J and repeat the calculations.

Example 1: Solve the equation

$$y^{(6)} + y = \sin \frac{3}{2} x \sin \frac{1}{2} x , \quad x \in [0, \pi] \quad (3.7)$$

for the initial conditions $y_0 = 1,\ y_0' = -1,\ y_0'' = 0.3,\ y_0''' = 0,\ y_0^{IV} = 0,\ y_0^{V} = 1.$

According to (3.3)–(3.5) we find

$$\begin{aligned} y^{(6)}(x_l) &= \sum_{i=1}^{2M} a_i h_i(x_l) \\ y(x_l) &= \sum_{i=1}^{2M} a_i p_{6,i}(x_l) + \sum_{\sigma=0}^{6} \frac{1}{\sigma!} y_0^{(\sigma)} x_l^\sigma . \end{aligned} \quad (3.8)$$

Replacing these results into (3.7) we get

$$\sum_{i=1}^{2M} a_i [h_i(x_l) + p_{6,i}(x_l)] + Z(l) = F(l) , \quad (3.9)$$

where $F(l) = \sin \frac{3}{2} x_l \sin \frac{1}{2} x_l$.

The matrix form of (3.9) is

$$a(H + P_6) = F - Z . \quad (3.10)$$

For the "exact solution" the Runge-Kutta solution, realized by the MATLAB program *ode45* was taken. Computer simulation gave the following error estimates

$$\begin{array}{lll} (i) & \delta_{ex} = 3.3E-3\,, \quad \sigma_{ex} = 4.1E-4\,, & \text{for} \quad J=3 \\ (ii) & \delta_{ex} = 8.3E-4\,, \quad \sigma_{ex} = 5.2E-5\,, & \text{for} \quad J=4 \\ (iii) & \delta_{ex} = 2.7E-4\,, \quad \sigma_{ex} = 6.5E-6\,, & \text{for} \quad J=5\,. \end{array}$$

The wavelet solution $y = y(x)$ is plotted in Fig. 1a; already in the case $J = 3$ (16 collocation points) it visually coincides with the Runge-Kutta solution. High accuracy of the wavelet results for a small number of calculation points is caused from the fact that by increasing the wavelet number i the coefficients a_i rapidly decrease (Fig 1b).

4 Treatment of the boundary value problems

Let us consider once more (3.1). We assume that for $\mu < m$ are given the initial conditions (specified at $x = A$), but the remaining $n - \mu$ conditions are prescribed in some internal points $x_1 < B$ or at the boundary $x = B$.
Consider one of the boundary conditions $y^{(\nu)}(x_*) = y_*^{(\nu)}$, where $y_*^{(\nu)}$ is a given number.
It follows from (3.4)–(3.5)

$$\begin{aligned} &\sum_{i=1}^{2M} a_i p_{n-\nu,i}(x_*) + Z_\nu(x_*) = y_*^{(\nu)} \\ &Z_\nu(x_*) = \sum_{\sigma=0}^{n-\nu-1} \frac{1}{\sigma!}(x_* - A)^\sigma y_0^{(\nu+\sigma)}\,. \end{aligned} \tag{4.1}$$

The functions $p_{n-\nu,i}(x_*)$ are calculated from (2.4) replacing there $x \to x_*$. Such analysis is carried out for each boundary condition. By doing this we get a system of $n-\nu$ linear equations from which the missing $n - \nu$ initial values can be calculated (of course these formulas contain also the wavelet coefficients a_i).
The following course of solution is the same as demonstrated in Section 3. Details are explained by the help of the following Example.
Example 2: Solve the equation

$$xy^{IV} + 5y''' = 24\,, \quad x \in [1, 3] \tag{4.2}$$

for $y(1) = 0\,,\ y''(1) = 200\,,\ y(3) = 0\,,\ y'''(3) = 0.$
Making use of (3.3)–(3.5) we find

$$\begin{aligned} y^{IV}(x) &= \sum_{i=1}^{2M} a_i h_i(x) \\ y'''(x) &= \sum_{i=1}^{2M} a_i p_{1,i}(x) + y'''(1) \\ y(x) &= \sum_{i=1}^{2M} a_i p_{4,i}(x) + \frac{1}{6}y'''(1)(x-1)^3 + \frac{1}{2}y''(1)(x-1)^2 + \\ &\quad y'(1)(x-1) + y(1)\,. \end{aligned} \tag{4.3}$$

Taking in view the initial and boundary conditions we find from (4.3):

$$\begin{aligned} y(3) &= \sum_{i=1}^{2M} a_i p_{4,i}(3) + 2y'(1) + 400 + \frac{4}{3}y'''(1) = 0 \\ y'''(3) &= \sum_{i=1}^{2M} a_i p_{1,i}(3) + y'''(1) = 0\,. \end{aligned} \tag{4.4}$$

From this system we calculate the values of the missing initial conditions:

$$\begin{aligned} y'''(1) &= -\sum_{i=1}^{2M} a_i p_{1,i}(3) \\ y'(1) &= \sum_{i=1}^{2M} a_i \left[\frac{2}{3}p_{1,i}(3) - \frac{1}{2}p_{4,i}(3)\right] - 200\,. \end{aligned} \tag{4.5}$$

It follows from (3.4) that

$$y'''(x) = \sum_{i=1}^{2M} a_i p_{1,i}(x) + y'''(1)\,. \tag{4.6}$$

Substituting all these results into (4.2) and discretizising the result $x \to x_l$ we get a system of linear equations for calculating the wavelet coefficients a_i:

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l)x_l + 5p_{1,i}(x_l) - 5p_{1,i}(3)\right] = 24\,, \quad l = 1, 2, ..., 2M\,. \tag{4.7}$$

After solving (4.7) we can evaluate the function $y = y(x)$ from (4.3).
Exact solution of (4.2) is

$$y_{ex}(x) = \frac{1}{5}\left(243/x^2 - 1026 + 1020x - 241x^2 + 4x^3\right)\,. \tag{4.8}$$

Computer simulation was carried out for some values of J. The error estimates were

$$\begin{aligned} \delta_4 &= 0.088\,, & \sigma_4 &= 2.7E-3 \\ \delta_5 &= 0.021\,, & \sigma_5 &= 3.3E-4 \\ \delta_6 &= 0.0054\,, & \sigma_6 &= 4.2E-5\,. \end{aligned}$$

The solution for $J = 5$ is plotted in Fig. 2a. In the case $J = 6$ our solution and the exact solution visually coincide.
The wavelet coefficients for $J = 5$ are plotted in Fig. 2b. It follows from this Figure that the values a_i are significantly bigger as for Example 1; this is the reason why the rate of convergence is here somewhat smaller as for Example 1.

5 Nonlinear equations

The n-th order nonlinear ODE has the form

$$F(x, y, y', y'', ..., y^{(n)}) = 0\,, \quad x \in [A, B] \tag{5.1}$$

where F is a nonlinear function.

We shall satisfy (5.1) in the collocation points $x_l\,,\ l = 1, ..., 2M$. Next y and its derivatives are expanded into Haar wavelet series and the results are placed into (5.1). By doing this we get a system of nonlinear equations

$$\Phi_l(x_l, a_1, a_2, ..., a_{2M}) = 0\,, \quad l = 1, 2, ..., 2M \tag{5.2}$$

from which the wavelet coefficients a_i must be calculated. Solving (5.2), especially if the number of equations is great, is as a rule a quite complicated problem.
For solving (5.2) we make use of the Newton's method. One possible course of solution was proposed by [Lepik 2006]. According to this technique it is assumed that (5.2) is already solved at some level of resolution J, the wavelet coefficients at this level we denote as $a_i^{(0)}\,,\ i = 1, 2, ..., 2M$. Our aim is to put together the solution for the next level $J + 1$. We shall estimate the wavelet coefficients at the new level as

$$\tilde{a}_i = \begin{cases} a_i^0 & \text{for } i = 1, 2, ..., 2M\,, \\ 0 & \text{for } i = 2M+1, 2M+2, ..., 4M\,. \end{cases} \tag{5.3}$$

These estimates are corrected by the Newton's method, which leads us to the system

$$\sum_{i=1}^{4M} S(i, l)\Delta a_i = -\Phi_l\,, \quad (l = 1, 2, ..., 4M)\,, \tag{5.4}$$

where

$$S(i, l) = \partial\Phi_l/\partial a_i\,.$$

The matrix form of (5.4) is

$$S\Delta a = -\Phi\,, \tag{5.5}$$

which has the solution

$$\Delta a = -\Phi/S\,. \tag{5.6}$$

The corrected values of the wavelet coefficients are

$$a = \tilde{a} + \lambda\Delta a\,. \tag{5.7}$$

The coefficient λ is selected to guarantee decrease in $\parallel \Phi \parallel$ (for an exact solution $\parallel \Phi \parallel = 0$).
This Newton step is repeated until the minimal value at this level is reached. After that we can pass to the next level of resolution.
The assumption (5.3) is motivated by the fact that higher coefficients of the sequence a_i are usually small.
The question how to start this procedure remains. We recommend to begin with a small number of collocation points (2 or 4 points), for which the solution of (5.2) is more simple.
We have checked this method for solving several examples: in all cases the speed of convergency was quite good. Details are clarified with the aid of the following example.
Example 3: Solve the boundary value problem

$$\begin{aligned} &2y'y''' - 3y''^2 = 0\,, \quad x \in [0, 2] \\ &y(0) = -1\,,\ y'(0.5) = 4/3\,,\ y(2) = 1\,. \end{aligned} \tag{5.8}$$

In view of (3.4)–(3.5) we have

$$\begin{aligned} y'''(x) &= \sum_{i=1}^{2M} a_i h_i(x) \\ y''(x) &= \sum_{i=1}^{2M} a_i p_{1,i}(x) + y''(0) \\ y'(x) &= \sum_{i=1}^{2M} a_i p_{2,i}(x) + y''(0)x + y'(0) \\ y(x) &= \sum_{i=1}^{2M} a_i p_{3,i}(x) + \frac{1}{2}y''(0)x^2 + y'(0)x + y(0)\,. \end{aligned} \tag{5.9}$$

Satisfying the boundary conditions $y'(0.5) = 4/3\,,\ y(2) = 1$ we obtain

$$\begin{aligned} \frac{4}{3} &= \sum_{i=1}^{2M} a_i p_{2,i}(0.5) + \frac{1}{2}y''(0)x + y'(0) \\ 1 &= \sum_{i=1}^{2M} a_i p_{3,i}(2) + 2y''(0) + 2y'(0) - 1\,. \end{aligned} \tag{5.10}$$

Evaluating from this system $y'(0)\,,\ y''(0)$, we get

$$\begin{aligned} y'(0) &= \frac{5}{3} + \frac{1}{2}\sum_{i=1}^{2M} a_i q_{1,i} \\ y''(0) &= -\frac{2}{3} - \sum_{i=1}^{2M} a_i q_{2,i}\,, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} q_{1,i} &= p_{3,i}(2) - 4p_{1,i}(0.5) \\ q_{2,i} &= p_{3,i}(2) - 2p_{2,i}(0.5)\,. \end{aligned} \tag{5.12}$$

Equation (5.2) obtains the form

$$\Phi_l = 2y'(x_l)y''(x_l) - 3\left[y''(x_l)\right]^2\,, \tag{5.13}$$

where the quantities $y'(x_l),\ y''(x_l),\ y'''(x_l),$ must be replaced according to formulas (5.9), (5.11), (5.12).

The elements of the matrix S are

$$S(i,l) = 2\left[y'(x_l)\frac{\partial y'''(x_l)}{\partial a_i} + y'''(x_l)\frac{\partial y'(x_l)}{\partial a_i} - 3y''(x_l)\frac{\partial y''(x_l)}{\partial a_i}\right]\,. \tag{5.14}$$

In view of (5.9) we have

$$\begin{aligned} \frac{\partial y'''(x_l)}{\partial a_i} &= h_i(x_l)\,, \\ \frac{\partial y''(x_l)}{\partial a_i} &= p_{1,i}(x_l) - q_{2,i}\,, \\ \frac{\partial y'(x_l)}{\partial a_i} &= p_{2,i}(x_l) - q_{2,i}(x_l) + \frac{1}{2}q_{1,i}\,. \end{aligned} \tag{5.15}$$

Next some results of the computer simulation are presented. For beginning two collocation points $x_1 = 0.5,\ x_2 = 1.5$ are taken. Computations showed that $\| \ \Phi \ \|$ is minimal for $a_1 = 0.70,\ a_2 = 0.6$.
At the level $J = 1$ (four collocation points) the estimate (5.3) is $\tilde{a}_i = (0.7,\, 0.6,\, 0,\, 0)$. Correcting it with the Newton's method we find that $\| \ \Phi \ \| = \min$ for $\lambda = 0.14$ and $a = (0.98,\, 0.83,\, 0.40,\, 0.02)$; the error estimates were $\delta_1 = 2.61E - 2,\ \sigma_1 = 6.5E - 3$. This process is continued until the necessary exactness is achieved.
The error estimates for the subsequent levels of resolution were

$$\begin{aligned} \delta_2 &= 1.5E-2\,, \quad \sigma_2 = 1.8E-3 \\ \delta_3 &= 2.1E-3\,, \quad \sigma_3 = 1.3E-4 \\ \delta_4 &= 1.8E-3\,, \quad \sigma_4 = 5.6E-5\,. \end{aligned}$$

The results of the computations for $J = 4$ (32 collocation points) are plotted in Fig. 3.

6 Conclusion

It was shown that the Haar wavelet method is effective for solving higher order ODE; already a small number of grid points guarantees the necessary exactness of the obtained results. The treatment of boundary value problems is also very simple. By our mind the main benefit of the proposed method is solving nonlinear boundary value problems (in this case other methods of solution are considerably more complicated).

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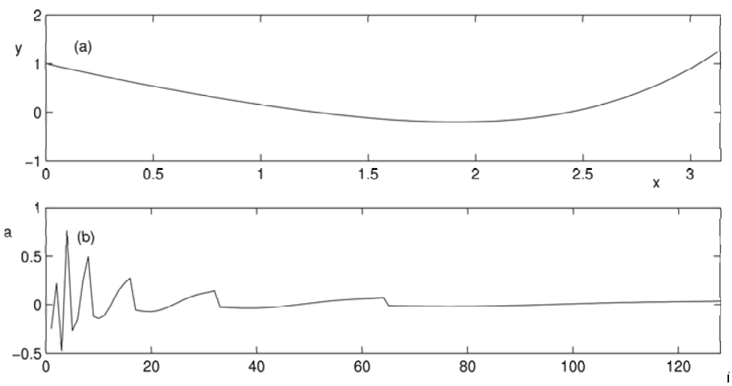

Figure 1: Solution of (3.7) for $J = 6$
(a) Function $y = y(x)$; (b) wavelet coefficients.

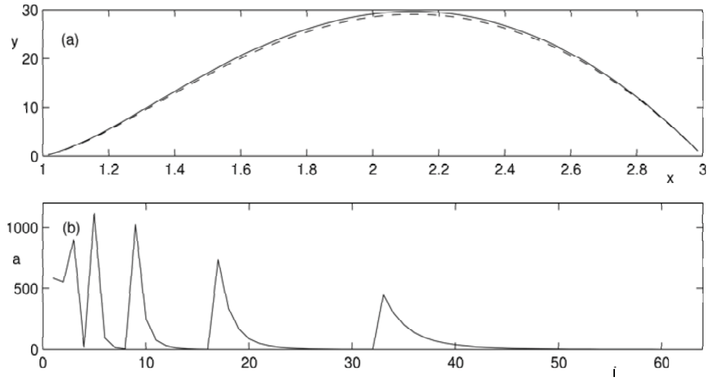

Figure 2: Solution of (4.2) for $J = 5$
(a) - Haar wavelet solution, - - exact solution; (b) wavelet coefficients.

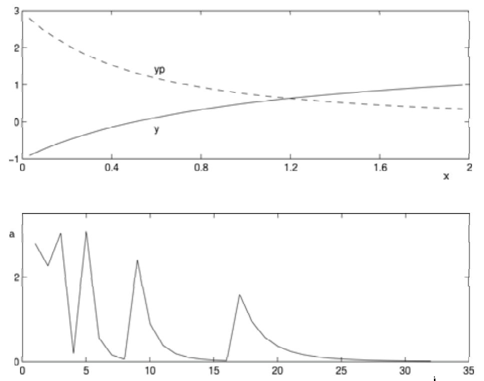

Figure 3: Solution of (5.8) for $J = 4$
(a) $y = y(x)$, $y' = y'(x)$; (b) wavelet coefficients.