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Solving integral and differential equations by the aid of non-uniform Haar wavelets

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Abstract

A modification of the Haar wavelet method, for which the stepsize of the argument is variable, is proposed. To establish the efficiency of the method three test problems, for which exact solution is known, are considered. Computer simulations show clear preference of the suggested method compared with the Haar wavelet method of a constant stepsize.

Keywords: Haar wavelet method; Variable stepsize; Integral equations; Ode-s

1. Introduction

The Haar wavelet method has turned out to be an effective tool for solving differential and integral equations [1–9]. The achieved results are mathematically very simple; the necessary exactness of the results is obtained already for a small number of grid points (for details consult the state-of-the-art paper [10]). On the other hand, this approach has also some limitations. The conventional form of the Haar wavelet approach is applicable for the range of the argument $x \in [0, 1]$; besides it is assumed that this interval is distributed into subintervals of equal length. If we want to raise the exactness of the results we must increase the number of the grid points. In the course of the solution we have to invert some matrices, but by increasing the number of calculation points these matrices become nearly singular and therefore the inverse matrices cannot be evaluated with necessary accuracy [10]. There are also many problems where the uniform Haar wavelet method is not suitable (e.g. differential equations under local excitations, boundary layer problems, weakly singular equations, problems for which the region of the variation of the argument is infinite). In [4] for solving such problems the method of segmentation was suggested, but in this case the solution becomes more complicated. One possibility to find a way out of these difficulties is to make use of the non-uniform Haar method for which the length of the subintervals is unequal. This idea was proposed in [11] and applied for analysing the function approximation problems. In the present paper a new variant of the non-uniform Haar wavelet method is presented, which is applicable for an arbitrary region of variation of the argument $x \in [a, b]$. For testing the efficiency of the method three problems for which the exact solution is known are discussed.

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2. Non-uniform Haar wavelets

Haar wavelets are characterized by two numbers: the dilatation parameter $j = 0, 1, \ldots, J$ (J is maximal level of resolution) and the translation parameter $k = 0, 1, \ldots, m-1$, where $m = 2^j$. The number of the wavelet is identified as $i = m + k + 1$. The maximal value is $i = 2M$ where $M = 2^J$. Consider the interval $x \in [a, b]$. We shall partition this interval into $2M$ subintervals; the coordinates of the grid points are denoted by $x(l), l = 0, 1, \ldots, 2M$. We shall define the ith wavelet as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ -c_i & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

Here the following notations are introduced:

$$\begin{aligned} &\xi_1(i) = x(2k\mu), \quad \xi_2(i) = x[(2k+1)\mu], \\ &\xi_3(i) = x[(2k+1)\mu], \quad \mu = M/m. \end{aligned} \tag{2}$$

The coefficient is calculated from the requirement

$$\int_a^b h_i(x)\mathrm{d}x = 0, \tag{3}$$

which gives

$$c_i = \frac{\xi_2(i) - \xi_1(i)}{\xi_3(i) - \xi_2(i)}. \tag{4}$$

In the following we need also the integrals of the wavelets (for conciseness sake the wavelet index i is in Eqs. (4)–(7) omitted)

$$p_\alpha(x) = \int_{\xi_1}^x p_{\alpha-1}(x)\mathrm{d}x, \tag{5}$$

where $p_0(x) = h(x)$. Making use of (1) all these integrals can be analytically evaluated. Doing this we find

$$p_1(x) = \begin{cases} 0 & \text{for } x < \xi_1, \\ x - \xi_1 & \text{for } x \in [\xi_1, \xi_2], \\ c(\xi_3 - x) & \text{for } x \in [\xi_2, \xi_3], \\ 0 & \text{for } x \geqslant \xi_3, \end{cases} \tag{6}$$

$$p_2(x) = \begin{cases} 0 & \text{for } x < \xi_1, \\ 0.5(x - \xi_1)^2 & \text{for } x \in [\xi_1, \xi_2], \\ K - 0.5(\xi_3 - x)^2 & \text{for } x \in [\xi_2, \xi_3], \\ K & \text{for } x \geqslant \xi_3, \end{cases} \tag{7}$$

where

$$K = 0.5(\xi_2 - \xi_1)(\xi_3 - \xi_1). \tag{8}$$

This analysis does not involve the case $i = 1$, which corresponds to the scaling function $h_1(x) = 1$ for $x \in [a, b]$ and $h_1(x) = 0$ elsewhere. Here $\xi_1(1) = a$, $\xi_2(1) = \xi_3(1) = b$ and we find

$$p_{1,1}(x) = x - a, \quad p_{2,1}(x) = 0.5(x - a)^2. \tag{9}$$

The Haar functions $h(x)$, $p_1(x)$, $p_2(x)$ for $i > 1$ are plotted in Fig. 1.

The problems discussed in this paper are solved by the collocation method. The collocation points are

$$x_c(l) = 0.5[x(l-1) + x(l)], \quad l = 1, 2, \ldots, 2M. \tag{10}$$

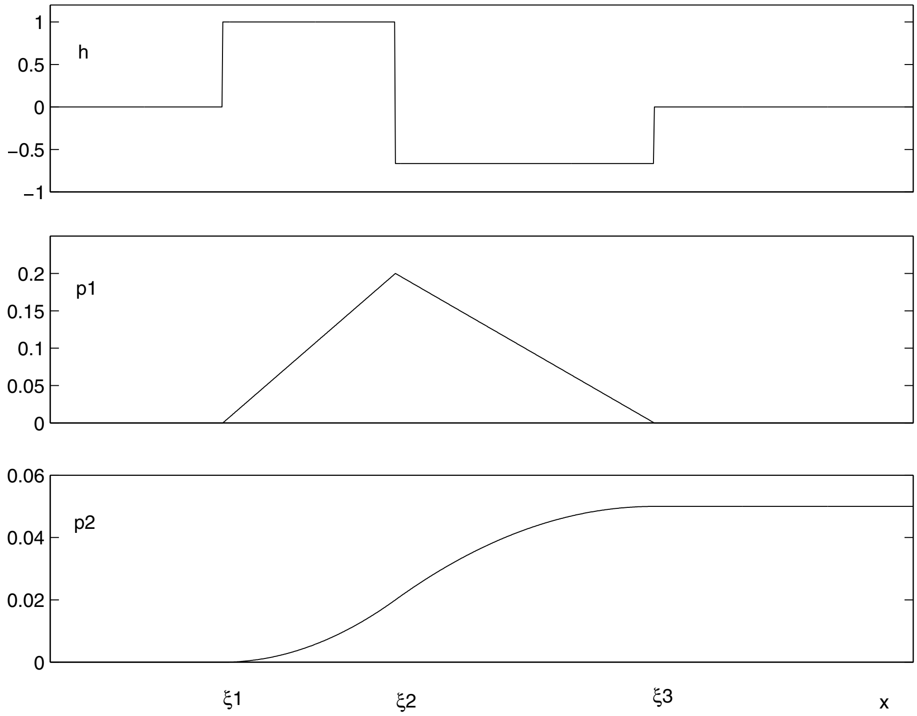

Fig. 1. Haar diagrams.

We shall apply Eqs. (1), (6) and (7) in the collocation points. It is convenient to introduce the Haar matrices

$$H(i,l) = h_i[x_c(l)], \quad P_1(i,l) = p_{1,i}[x_c(l)], \quad P_2(i,l) = p_{2,i}[x_c(l)]. \tag{11}$$

With the purpose to demonstrate the good features of the presented method three problems are solved. Since for all these problems the exact solution is known we can calculate the error of the wavelet solutions.

3. Fredholm integral equation

For the first problem we shall take the integral equation

$$y(x) - 4\int_0^\infty e^{-(x+t)} y(t)\mathrm{d}t = (x-1)e^{-x}, \tag{12}$$

which has the exact solution

$$y_{ex}(x) = xe^{-x}. \tag{13}$$

For getting numerical results the interval of integration $[0,\infty]$ is replaced by $[0,L]$, where L is large enough constant. The partition of this interval into subintervals is carried out as follows. Let us denote the length of the lth subinterval by $\Delta x_l = x_l - x_{l-1}$, $l = 1,2,\ldots,2M$. It is assumed that $\Delta x_{l+1} = q\Delta x_l$, where $q > 1$ is a given constant. By summing up all the length of the subintervals we find

$$\Delta x_1(1 + q + q^2 + \cdots + q^{2M-1}) = L, \tag{14}$$

where $L = b - a$. This formula can be put into the form

$$\Delta x_1 = L\frac{q-1}{q^{2M}-1}. \tag{15}$$

The grid points are

$$x_l = L\frac{q^l - 1}{q^{2M}-1}, \quad l = 1,2,\ldots,2M. \tag{16}$$

Table 1
Error estimates for Eq. (12)

J	$2M$	$q=1.0$		$q=1.1$		$q=1.2$	
		$\delta_{\max}$	δ_n	$\delta_{\max}$	δ_n	$\delta_{\max}$	δ_n
3	16	6.0E−2	4.4E−3	1.5E−2	1.3E−3	3.0E−3	3.4E−4
4	32	1.7E−2	8.0E−4	6.4E−4	5.1E−5	6.1E−4	7.0E−5
5	64	4.7E−3	1.4E−4	1.7E−4	6.4E−5	7.0E−4	7.3E−5

The collocation points are calculated according to (10). Wavelet solution of (12) is sought in the form

$$y(x) = \sum_{i=1}^{2M} a_i h_i(x), \tag{17}$$

where $a = (a_i)$ is a row vector of the wavelet coefficients. Next we shall evaluate the integral

$$G_i = \int_0^L \mathrm{e}^{-t} h_i(t)\mathrm{d}t. \tag{18}$$

In view of (1) we find

$$G_i = \mathrm{e}^{-\xi_1(i)} + [1 + c(i)]\mathrm{e}^{-\xi_2(i)} + c(i)\mathrm{e}^{-\xi_3(i)}. \tag{19}$$

Making use of (17)–(19) and satisfying (12) in the collocation points x_c we obtain the matrix equation

$$aS = F, \tag{20}$$

where

$$S(i,l) = H(i,l) - 4G_i \mathrm{e}^{-x_c(l)}, \quad F(l) = [x_c(l) - 1]\mathrm{e}^{-x_c(l)}. \tag{21}$$

From here the vector a is calculated and the function $y(x)$ is found from (17). For estimating the accuracy of the obtained results the following error estimates were introduced

$$\delta_{\max} = \max_l(|\Delta_l|), \quad \delta_n = \mathrm{norm}(\Delta_l), \tag{22}$$

where

$$\Delta_l = y[x_c(l)] - y_{ex}[x_c(l)]. \tag{23}$$

Computer simulations were carried out for $q = 1$ (uniform Haar solution) and $q = 1.1$, $q = 1.2$. For the parameter L the value $L = 10$ was taken. The results are presented in Table 1.

Calculations with $L = 20$ were also carried out but the results did not essentially differ from the data of Table 1. It follows from Table 1 that the results with variable stepsize are considerably more accurate as in the case of a constant stepsize.

4. Locally disturbed vibrations

Consider the differential equation

$$y'' + k^2 y = f, \quad x \in [0, 0.5], \tag{24}$$

where

$$f = \begin{cases} d\sin(\alpha x) & \text{for } x \in [x_*, x_{**}], \\ 0 & \text{elsewhere.} \end{cases} \tag{25}$$

If x is interpreted as time then (24) presents a locally disturbed sinusoidal motion. Eq. (24) can be integrated analytically but for conciseness sake we do not show these results here. Subsequent calculations were carried out for the parameters $k = \pi$, $d = 400$, $\alpha = 40\pi$, $x_* = 0.2$, $x_{**} = 0.25$ and for the initial conditions $y(0) = 1$, $y'(0) = 0$. Let us divide the interval $[0, 0.5]$ into three parts $[0, 0.2]$, $[0.2, 0.25]$, $[0.25, 0.5]$; with the length of the subintervals Δx_1, Δx_2, Δx_3 which are assumed to be constants. It is reasonable to choose the stepsizes so that $\Delta x_1 \approx \Delta x_3$, $\Delta x_1 \gg \Delta x_2$. If we take $\Delta x_1 = 0.4/M$, $\Delta x_2 = 0.05/M$, $\Delta x_3 = 0.5/M$ then to the interval $[0, 0.2]$ belong $M/2$ collocation points, to $[0.2, 0.25]$ M points and to $[0.25, 0.5]$ also $M/2$ points. The coordinates of the grid points are

$$\begin{aligned} &x(l) = l\Delta x_1 \quad \text{for } l = 1, \ldots, M/2, \\ &x(l + M/2) = x_* + l\Delta x_2 \quad \text{for } l = 1, \ldots, M, \\ &x(l + 3M/2) = x_{**} + l\Delta x_3 \quad \text{for } l = 1, \ldots, M/2. \end{aligned} \tag{26}$$

We shall satisfy (24) in the collocation points x_l. If Y is a row vector for which $Y(l) = y[x_c(l)]$ then the solution of (24) can be sought in the matrix form

$$Y'' = aH. \tag{27}$$

By integrating this equation twice we obtain

$$Y' = aP_1 + y'(0)E, \quad Y = aP_2 + y'(0)X_c + y(0)E. \tag{28}$$

Here X_c is a row vector with the components $x_c(l)$ and E is a $2M$-dimensional unit vector. Replacing (27) and (28) into (24) we get

$$a(H + k^2P_2) = F - k^2[y'(0)X_l + y(0)E], \tag{29}$$

where $F(l) = f[x_c(l)]$. Again the vector a is calculated from (29); the solution of the problem is found according to (28). Results of the computer simulation are presented in Table 2 and in Fig. 2.

It follows from Table 2 and Fig. 2 that in case of variable stepsize the accuracy of the results is very good – already in the case of 16 points (i.e. for $J = 3$) our results visually do not differ from the exact ones. For uniform Haar solution the error $\delta_{\max}$ is quite significant, it practically does not become smaller if we increase the number of collocation points to 128. As to the estimate δ_n then it is $\sim$1000 times greater as for non-uniform Haar solution. From here the conclusion, that the uniform Haar method for the solution of this problem is inappropriate, can be made.

Table 2
Error estimates for Eq. (24)

J	$2M$	Uniform Haar		Non-uniform Haar	
		$\delta_{\max}$	δ_n	$\delta_{\max}$	δ_n
3	16	1.59	2.0E−1	4.5E−3	6.9E−4
4	32	0.36	3.0E−2	1.1E−3	1.2E−4
5	64	0.37	2.2E−2	2.8E−4	2.1E−5
6	128	0.34	1.4E−2	6.9E−5	3.7E−6

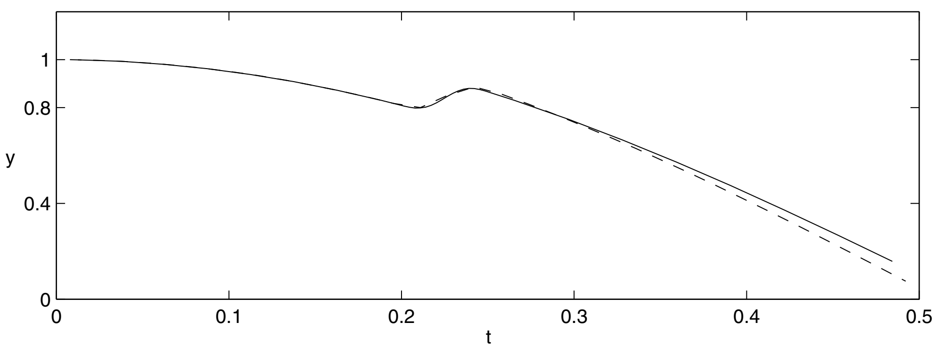

Fig. 2. Solution of Eq. (24) for $J = 5$, — exact solution, – – – uniform wavelet solution.

5. Boundary layer problem

Let us solve the differential equation

$$-\epsilon y'' + y' = 0, \quad x \in [0,1], \quad y(0) = y(1) = 0, \tag{30}$$

where $\epsilon > 0$ is a small parameter. The exact solution of (30) is

$$y_{ex} = x - \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}. \tag{31}$$

If $\epsilon \ll 1$ a boundary layer develops near the edge $x = 1$ (Fig. 3). This fact puts an extra requirement to the computational algorithm and therefore (30) is a good test problem. It was solved by Bertoluzza [12] who made use of the Deslaurier–Dubuc interpolating wavelets. The Haar wavelet method in connection with the segmentation technique was applied by Lepik [4]. Now let us put together a Haar wavelet solution with the variable stepsize. We shall assume that the lengths of the subintervals fulfil the condition $\Delta x_l = q\Delta x_{l-1}$, where $q < 1$ and $l = 1, \ldots, 2M$. In this case formulae (16) and (10) for calculating the grid and collocation points hold. The solution is sought again in the matrix form (27); also (28) remains valid. In the present case we have a boundary problem $y(1) = 0$. Satisfying this condition in view of (28) we find $y'_0 = -aP_2(x = 1)$. It follows from (7) that $P_2(x = 1) = K^{\mathrm{T}}$. Substitution this result into (28) gives

$$Y' = a(P_1 - K^{\mathrm{T}} \otimes E), \quad Y = a(P_2 + K^{\mathrm{T}} \otimes X_l), \tag{32}$$

where the symbol $\otimes$ denotes the Kronecker tensor product. In view of (27) and (32) Eq. (30) obtains the form

$$a(-\epsilon H + P_1 - K^{\mathrm{T}} \otimes E) = E, \tag{33}$$

from which the wavelet coefficients are calculated. The second equation of (32) gives the solution of the problem. Calculations are carried out for $\epsilon = 0.01$ and $\epsilon = 0.001$; for the parameter the values $q = 1$, $q = 0.9$ and

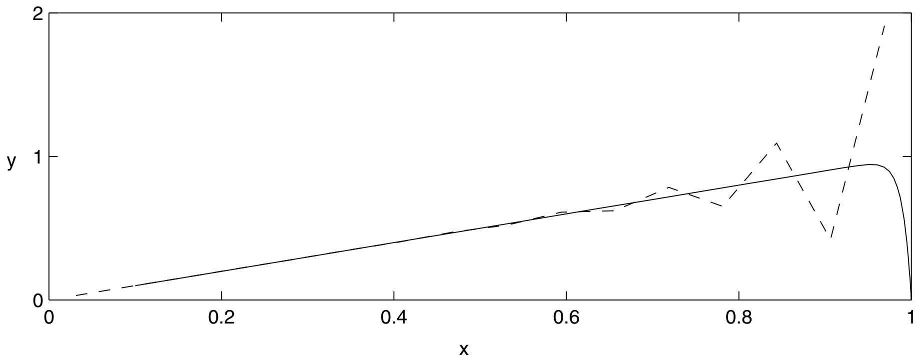

Fig. 3. Solution of Eq. (30) for $\epsilon = 0.01$; $J = 3$, — exact solution and wavelet solution for $q = 0.8$, – – – wavelet solution for $q = 1$.

Table 3
Error estimates for Eq. (30)

J	$2M$	$q = 1.0$		$q = 0.9$		$q = 0.8$	
		$\delta_{\max}$	δ_n	$\delta_{\max}$	δ_n	$\delta_{\max}$	δ_n
$\epsilon = 0.01$							
3	16	*	*	0.19	1.2E−2	1.9E−2	1.8E−3
4	32	0.30	9.2E−3	5.8E−3	3.8E−4	2.1E−3	1.4E−4
5	64	0.068	1.3E−3	4.7E−4	2.3E−5	1.9E−3	6.3E−5
$\epsilon = 0.001$							
3	16	*	*	*	*	*	*
4	32	*	*	*	*	4.8E−3	2.6E−4
5	64	*	*	1.3E−3	5.8E−5	1.8E−3	6.3E−5

The asterisk (*) indicates oscillating solutions.

$q = 0.8$ were taken. The error estimates are shown in Table 3. In some cases the solution is instable-oscillatory (Fig. 3).

From the data of Table 3 the preference of the Haar method with variable stepsize becomes again evident. To demonstrate this more clearly let us consider the solution for the case $\epsilon = 0.01$, $J = 3$ (Fig. 3). For $q = 0.8$ the wavelet solution visually coincides with the exact solution, while for $q = 1$ (uniform Haar method) the solution is oscillating. If $\epsilon = 0.001$ the solution for $q = 1$ remains instable in spite of increasing number of collocation points.

6. Conclusions

Traditional Haar wavelet method is applicable in the region $x \in [0, 1]$; it is assumed that all the subintervals Δx have the same length. In the present paper a modification of the method, which is free from these restrictions, is proposed. The Haar method with a variable stepsize is appropriate in the case of problems, where in the solution abrupt changes take place. The efficiency of the method is demonstrated in Sections 3–5 for solving three problems. Simplicity of the Haar wavelet method is retained; computing time is practically the same as for the conventional Haar wavelet method.

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