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Numerical solution of evolution equations by the Haar wavelet method

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Abstract

An efficient numerical method for solution of nonlinear evolution equations based on the Haar wavelets approach is proposed. The method is tested in the case of Burgers and sine-Gordon equations. Numerical results, obtained by computer simulation, are compared with other available solutions. These calculations demonstrate that the accuracy of the Haar wavelet solutions is quite high even in the case of a small number of grid points.
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1. Introduction

Consider the evolution problem

$$\alpha^* \frac{\partial^2 u}{\partial t_*^2} + \beta^* \frac{\partial u}{\partial t_*} = F\left(t_*, u, \frac{\partial u}{\partial x_*}, \frac{\partial^2 u}{\partial x_*^2}\right), \tag{1}$$

where $t_* \in [t_{\text{in}}, t_{\text{fin}}]$, $x_* \in [x_{\text{in}}, x_{\text{fin}}]$.

Here F is a nonlinear function, α^*, β^*, t_{in}, t_{fin}, x_{in}, x_{fin} are given constants. In this paper the following two cases are discussed in detail:

(i) Burgers equation

$$\alpha^* = 0, \quad \beta^* = 1, \quad F = v^* \frac{\partial^2 u}{\partial x_*^2} - u \frac{\partial u}{\partial x_*}. \tag{2}$$

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(ii) sine-Gordon equation

$$\alpha^* = 1, \quad \beta^* = 0, \quad F = \frac{\partial^2 u}{\partial x_*^2} - \sin u. \tag{3}$$

Numerical solution of these equations has been discussed in many papers. From conventional methods of solution we would mention here

(i) Fourier spectral method.
(ii) Galerkin and collocation methods.
(iii) Finite element and finite difference methods.

An overview of these (and other) methods can be found e.g. in [1–3].

Although all these methods have been successfully applied for many evolution equations they have also some shortcomings. In regions where singularities or sharp transitions occur the solutions may be oscillating (Gibbs phenomenon) and for accurate representation of the results adaptive numerical schemes must be used what complicates the solution.

Beginning from 1980s wavelets have been used for solution of partial differential equations (PDE). The good features of this approach are possibility to detect singularities, irregular structure and transient phenomena exhibited by the analyzed equations. Most of the wavelet algorithms can handle easily periodic boundary conditions.

The wavelet algorithms for solving PDE usually are based on the Galerkin technics [5–7] or on the collocation method [5,8]. As to other approaches then the pseudo-wavelet method was applied in [9], filter-bank methods in [10]; periodic spline wavelets of even order are used in [11].

In most papers the Daubechies wavelets are used. The main advantage of these wavelets is their regularity and smoothness. Unfortunately they do not have any explicit expression and therefore analytical differentiation or integration is not possible. This fact in particular complicates the solution of nonlinear PDE, where integrals of products of wavelets and their derivatives must be computed. This can be done by introducing the connection coefficients [12,13], but this approach can be realized only for a narrow class of nonlinearities; besides numerical evaluation of these coefficients is often unstable or inaccurate.

The complexity of the wavelet solutions has induced some pessimistic estimates. So Jameson [14] writes: "... nonlinearities etc., when treated in a wavelet subspace, are often unnecessarily complicated. There appears to be no compelling reason to work with Galerkin-style coefficients in a wavelet method".

Evidently all attempts to simplify the wavelet solutions for PDE are welcome. One possibility for this is to make use of the Haar wavelet family. Haar wavelets (which are Daubechies wavelets of order 1) consist of piecewise constant functions (see Section 2) and are therefore the simplest orthonormal wavelets with a compact support. A drawback of the Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points it is not possible to apply the Haar wavelets for solving PDE directly.

There are two possibilities for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (e.g. B-splines or Deslaurier–Dubuc interpolating wavelets). This approach has been applied by Cattani [15], but the regularization process considerably complicates the solution and the main advantage of the Haar wavelets – the simplicity gets to some extent lost.

The other way is to make use of the integral method which was proposed by Chen and Hsiao [16] in 1997. The main idea of this approach is to expand into the Haar series the highest derivative appearing in the differential equation. This approximation is integrated for getting the derivatives of lower order and the function to be calculated. This idea is used in the present paper for solving some evolution equations.

The paper is organized in the following way. For completeness sake the Haar wavelet method is presented in Section 2. The method of solution the PDE is proposed in Section 3. It is applied for solution of the Burgers equation (Section 4) and the sine-Gordon equation (Section 5). Some conclusions are drawn in Section 6.

2. Haar wavelets

The Haar wavelet family for $x \in [0,1]$ is defined as follows:

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1, \xi_2), \\ -1 & \text{for } x \in [\xi_2, \xi_3), \\ 0 & \text{elsewhere,} \end{cases} \tag{4}$$

where

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k+0.5}{m}, \quad \xi_3 = \frac{k+1}{m}. \tag{5}$$

In these formulae integer $m = 2^j$, $j = 0, 1, \ldots, J$ indicates the level of the wavelet; $k = 0, 1, \ldots, m-1$ is the translation parameter. Maximal level of resolution is J. The index i in Eq. (4) is calculated from the formula $i = m + k + 1$; in the case of minimal values $m = 1$, $k = 0$ we have $i = 2$. The maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [-1, 1), \\ 0 & \text{elsewhere.} \end{cases} \tag{6}$$

In the following analysis integrals of the wavelets

$$p_i(x) = \int_0^x h_i(x)\,\mathrm{d}x, \quad q_i(x) = \int_0^x p_i(x)\,\mathrm{d}x \tag{7}$$

must be calculated. This can be done with the aid of (4):

$$p_i(x) = \begin{cases} x - \xi_1 & \text{for } x \in [\xi_1, \xi_2), \\ \xi_3 - x & \text{for } x \in [\xi_2, \xi_3), \\ 0 & \text{elsewhere,} \end{cases} \tag{8}$$

$$q_i(x) = \begin{cases} 0 & \text{for } x \in [0, \xi_1), \\ 0.5(x - \xi_1)^2 & \text{for } x \in [\xi_1, \xi_2), \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3 - x)^2 & \text{for } x \in [\xi_2, \xi_3), \\ \frac{1}{4m^2} & \text{for } x \in [\xi_3, 1]. \end{cases} \tag{9}$$

In the present paper the wavelet-collocation method is applied. The collocation points are defined as

$$x_l = (l - 0.5)/(2M), \quad l = 1, 2, \ldots, 2M. \tag{10}$$

It is expedient to introduce the $2M \times 2M$ matrices H, P, Q with the elements $H(i,l) = h_i(x_l)$, $P(i,l) = p_i(x_l)$, $Q(i,l) = q_i(x_l)$.

3. Method of solution

For solving the PDEs the two-dimensional wavelet transform could be applied, as it was proposed e.g. by Newland [17], but more convenient seems to be the following algorithm.

Since the Haar wavelets are defined for $x \in [0, 1]$ we must first normalize Eq. (1) in regard to x. Let in the following dots and primes denote differentiation with respect to $t = t_*$ and x, respectively. Eq. (1) can now be rewritten in the form

$$\alpha\ddot{u} + \beta\dot{u} = f(t, u, u', u''), \quad x \in [0, 1], \ t \in [t_{\text{in}}, t_{\text{fin}}]. \tag{11}$$

To this equation belong the initial and boundary conditions which will be specified later on.

Next let us divide the interval $[t_{\text{in}}, t_{\text{fin}}]$ into N equal parts of length $\Delta t = (t_{\text{fin}} - t_{\text{in}})/N$ and denote $t_s = (s-1)\Delta t$, $s = 1, 2, \ldots, N$. For the subinterval $t \in [t_s, t_{s+1}]$ the solution is sought in the form

$$\text{for} \begin{cases} \alpha \neq 0: & \ddot{u}^{(\mu)}(x,t) \\ \alpha = 0: & \dot{u}^{(\mu)}(x,t) \end{cases} = \sum_{i=1}^{2M} a_s(i) h_i(x). \tag{12}$$

Here (μ) denotes the order of the highest spacial derivative in Eq. (11); a_s is a $2M$ dimensional row vector, it is regarded for a vector constant in the subinterval $t \in [t_s, t_{s+1}]$.

Eq. (12) is integrated with respect to x in the limits $[0, x]$ and with respect to t in the limits $[t_s, t]$. This process is repeated until all of the functions, u, $\dot{u}$, $\ddot{u}$, u', u'', $\dot{u}'$, $\dot{u}''$ are calculated. The boundary conditions are incorporated in these expressions as integration constants.

Eq. (11) will be satisfied in the grid points (x_l, t_s). Let us denote $U_s(l) = u(x_l, t_s)$; similar notations will be introduced for the derivatives of u. In the following the quantity $U_s(l)$ and his derivatives will be treated as $2M$-dimensional vectors. If we denote

$$a_s(:)H(:,l) = \sum_{i=1}^{2M} a_s(i)h_i(x_l), \tag{13}$$

then (11) can be rewritten in the matrix form

$$\text{for} \begin{Bmatrix} \alpha \neq 0: & \ddot{U}_s^{(\mu)}(l) \\ \alpha = 0: & \dot{U}_s^{(\mu)}(l) \end{Bmatrix} = a_s(:)H(:,l). \tag{14}$$

Further details concerning integration of Eq. (14) and solution of Eq. (11) are explained in Sections 4 and 5. All computations were carried out by the use of MATLAB programs.

4. Burgers equation

Burgers equation (1) and (2) has proved to be a touchstone for new numerical methods of solution. This is caused by two reasons. First Burgers equation is the simplest nonlinear PDE incorporating both diffusion and advection. The second reason is that for a small viscosity v^* the solution develops into a saw-tooth wave at the origin. Performance of a numerical method can be judged from its ability to resolve the large gradient region that develops in the solution.

There are several papers about numerical solution of the Burgers equation. An overview of conventional methods is presented in [1]. Different wavelet approaches have been applied in [4,6,7,9,11,19,20]. A comparison of numerical results according to these (and other) papers can be found in [19]. It should be mentioned that in the case of several solutions localized or spread oscillations appear.

There exists also an analytical solution of the Burgers equation (consult e.g. [1]), but it is quite difficult to compute.

Now let us put together the Haar wavelet solution. Following [19] the equation:

$$\frac{\partial u}{\partial t_*} + u\frac{\partial u}{\partial x_*} = v_* \frac{\partial^2 u}{\partial x_*^2}, \quad x^* \in [-1, 1],\ t \geqslant 0 \tag{15}$$

with initial and boundary conditions $u(x_*, 0) = -\sin(\pi x_*)$, $u(\pm 1, t) = 0$ is considered.

Carrying out the change of variables $x_* = 2x - 1$, $v = v^*/4$ we get the problem

$$\begin{aligned} &\dot{u} + \frac{1}{2}uu' = vu'', \quad x \in [0,1],\ t > 0, \\ &u(x,0) = \sin 2\pi x, \quad u(0,t) = u(1,t) = 0. \end{aligned} \tag{16}$$

The Haar wavelet solution is sought in the form

$$\dot{u}''(x,t) = \sum_{i=1}^{2M} a_s(i)h_i(x), \tag{17}$$

where the row vector a_s is constant in the subinterval $t \in [t_s, t_{s+1}]$.

By integrating Eq. (17) with respect to t in the limits $[t_s, t]$ and twice with respect to x in the limits $[0, x]$ we obtain

$$
\begin{aligned}
u''(x,t) &= (t-t_s)\sum_{i=1}^{2M} a_s(i)h_i(x) + u''(x,t_s),\\
u'(x,t) &= (t-t_s)\sum_{i=1}^{2M} a_s(i)p_i(x) + u'(x,t_s) - u'(0,t_s) + u'(0,t),\\
u(x,t) &= (t-t_s)\sum_{i=1}^{2M} a_s(i)q_i(x) + u(x,t_s) - u(0,t_s) + x[u'(0,t) - u'(0,t_s)] + u(0,t),\\
\dot{u}(x,t) &= \sum_{i=1}^{2M} a_s(i)q_i(x) + \dot{u}(0,t) + x\dot{u}'(0,t).
\end{aligned}
\tag{18}
$$

On the grounds of the boundary conditions we have $u(0,t) = u(0,t_s) = \dot{u}(0,t) = 0$. The conditions $u(1,t) = \dot{u}(1,t) = 0$ give

$$
\begin{aligned}
u'(0,t) - u'(0,t_s) &= -(t-t_s)\sum_{i=1}^{2M} a_s(i)q_i(1),\\
\dot{u}'(0,t) &= -\sum_{i=1}^{2M} a_s(i)q_i(1).
\end{aligned}
\tag{19}
$$

It follows from (9) that

$$
\tilde{q}(i) = q_i(1) = \begin{cases} 0.5 & \text{if } i = 1,\\ \frac{1}{4m^2} & \text{if } i > 1. \end{cases}
\tag{20}
$$

Substituting Eq. (19) into Eq. (18) and discretizising the results by assuming $x \to x_l$, $t \to t_{s+1}$ we obtain

$$
\begin{aligned}
U''_{s+1}(l) &= \Delta t a_s(:)H(:,l) + U''_s(l),\\
U'_{s+1}(l) &= \Delta t a_s(:)[P(:,l) - E(l)\tilde{q}(:)] + U'_s(l),\\
U_{s+1}(l) &= \Delta t a_s(:)[Q(:,l) - x(l)\tilde{q}(:)] + U_s(l),\\
\dot{U}_{s+1}(l) &= a_s(:)[Q(:,l) - x(l)\tilde{q}(:)].
\end{aligned}
\tag{21}
$$

In these equations $E(l)$ denotes the $2M$-dimensional unit vector.

There are several possibilities for treating the nonlinearity in Eq. (16). In the following the scheme

$$
\dot{U}^{(l)}_{s+1} = -\frac{1}{2}U^{(l)}_s U t_s(l) + vU''_s(l),
\tag{22}
$$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting Eq. (21) into Eq. (22) we obtain

$$
a_s(:)[Q(:,l) - \tilde{q}(:)x(l)] = -0.5U_s(l)U'_s(l) - vU''_s(l).
\tag{23}
$$

From Eq. (23) the wavelet coefficients a_s can be successively calculated. This process is started with

$$
\begin{aligned}
U_0(l) &= \sin[2\pi x(l)],\\
U'_0(l) &= 2\pi\cos[2\pi x(l)],\\
U''_0(l) &= -4\pi^2\sin[2\pi x(l)].
\end{aligned}
\tag{24}
$$

In the case of a small viscosity v the solution develops into a saw-tooth wave and at $x = 0.5$ a shock develops. For estimating the efficiency of the solution it is expedient to calculate the maximum value of the gradient $|\partial u/\partial x|$ at $x = 0.5$. According to the analytical solution [1,8] for $v = (400\pi)^{-1}$ the theoretical maximum is $|\partial u/\partial x| = 304.0$ and takes place at $t_{\max} = 0.51$.

Computer simulation was carried out for $v = (400\pi)^{-1}$. Result for $J = 5$ (32 collocation points), $\Delta t = 0.001$, $t = 0.35$ are plotted in Fig. 1a. It follows from this Figure that in spite of the small number of the collocation points the solution describes quite well the sawtooth effect. With increasing t oscillations near the point $x = 0.5$

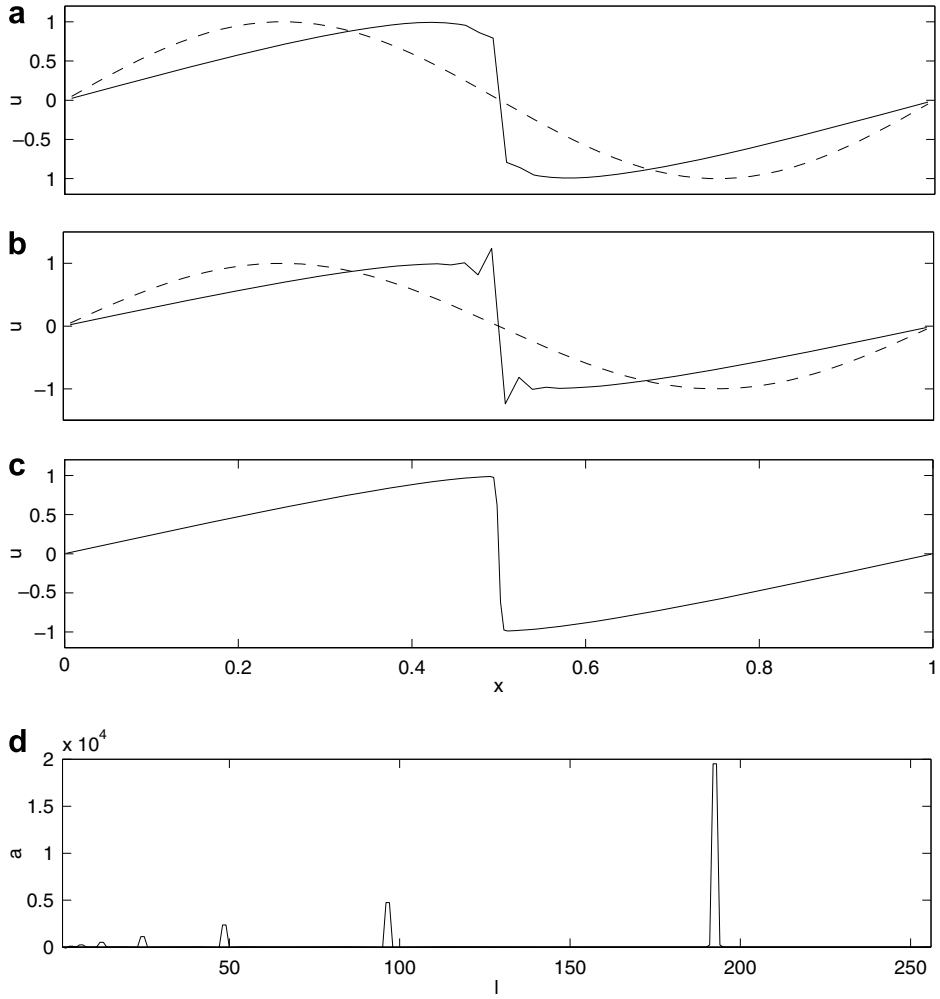

Fig. 1. Solutions of the Burgers equation for $v = (400\pi)^{-1}$, $\Delta t = 0.001$: (a) solution for $J = 5$, $t = 0.35$; (b) $J = 5$, $t = 0.37$ (dashed line denotes the solution for $t = 0$); (c) $J = 7$, $t = 0.51$; (d) wavelet coefficients for the case c (l is the number of one collocation point).

appear (Fig. 1b). For getting more exact results the value of J must be increased. Results for $J = 7$, $t = 0.51$ are plotted in Fig. 1c. In this case $\max|\partial u/\partial x| = 400.4$. That number is different from the analytical value 304.0, but this fact seems to be nonessential: if we calculate the angle $\delta = \arctan(\max|\partial u/\partial x|)$, then our solution gives $\delta = 89.86°$, while in the case of the analytic solution we have $\delta = 89.81°$.

Results of numerical solution for the Burgers equation with the aid of other methods can be found in [1,9,18]. Comparison with these algorithms shows that the Haar wavelet method is competitive and efficient. The advantages of our method are its simplicity and speed of convergence, which is caused from the sparseness of the transform matrices H, P, Q. As it follows from Fig. 1d the number of significant wavelet coefficients is quite small while in the case of other wavelet methods it can be up to 200–400 [9].

5. Sine-Gordon equation

Numerical solution of different modifications of the sine-Gordon Eqs. (1) and (3) was discussed in many papers, from which we cite here [20–26]. In most papers the conventional methods as difference methods, spectral methods or integration of the Hamiltonian systems were applied. In paper [26] the Gaussian wavelets were used.

In this paper the classical form of the sine-Gordon equation

$$\frac{\partial^2 u}{\partial x_*^2} - \frac{\partial^2 u}{\partial t_*^2} = \sin u, \quad x_* \in [x_{\text{in}}, x_{\text{fin}}], \quad t_* \geqslant t_{\text{in}} \tag{25}$$

is considered.

Changing the variables

$$x = \frac{1}{L}(x_* - x_{\text{in}}), \quad t = t_* - t_{\text{in}}, \quad L = x_{\text{fin}} - x_{\text{in}}$$

we obtain

$$\frac{1}{L^2} u'' - \ddot{u} = \sin u, \quad x \in [0,1], \ t \geqslant 0. \tag{26}$$

This equation has an analytical solitary wave solution

$$u(x,t) = 4 \arctan[\exp(z)], \quad z = \alpha(x - \beta t). \tag{27}$$

Eq. (27) is satisfied if

$$\alpha = \frac{L}{\sqrt{1 - L^2\beta^2}}. \tag{28}$$

To Eq. (26) belong the initial and boundary conditions

$$\begin{aligned} &u(0,t) = \varphi(t), \quad u'(0,t) = \psi(t), \\ &u(x,0) = f(x), \quad \dot{u}(x,0) = g(x), \end{aligned} \tag{29}$$

where φ, ψ, f, g are prescribed functions.

The Haar wavelet solution for $t \in [t_s, t_{s+1}]$ is sought in the form

$$\ddot{u}''(x,t) = \sum_{i=1}^{2M} a_s(i) h_i(x). \tag{30}$$

By integrating Eq. (26) with respect to x and t and putting the results into discrete form we obtain

$$\begin{aligned} \dot{U}''_{s+1}(l) &= \Delta t a_s(:) H(:,l) + \dot{U}''_s(l), \\ U''_{s+1}(l) &= \frac{1}{2}\Delta t^2 a_s(:) H(:,l) + \Delta t \dot{U}''_s(l) + U''_s(l), \\ \ddot{U}_{s+1}(l) &= a_s(:) Q(:,l) + x_l \ddot{\varphi}'(t_{s+1}) + \ddot{\varphi}(t_{s+1}), \\ \dot{U}_{s+1}(l) &= \Delta t a_s(:) Q(:,l) + \dot{U}_s(l) + \dot{\varphi}(t_{s+1}) - \dot{\varphi}(t_s) + x_l[\dot{\varphi}'(t_{s+1}) - \dot{\varphi}'(t_s)], \\ U_{s+1}(l) &= \frac{1}{2}\Delta t^2 a_s(:) Q(:,l) + \Delta t \dot{U}_s(l) + U_s(l) + \varphi(t_{s+1}) - \varphi(t_s) - \Delta t \dot{\varphi}(t_s) \\ &\quad + x_l[\varphi'(t_{s+1}) - \varphi'(t_s) - \Delta t \dot{\varphi}'(t_s)]. \end{aligned} \tag{31}$$

Discrete version of Eq. (26) gets the form

$$a_s(:) Q(:,l) = \frac{1}{L^2} U''_s(l) + \sin U_s(l) - x_l \ddot{\varphi}(t_{s+1}) - \ddot{\varphi}(t_{s+1}). \tag{32}$$

Eq. (32) is the matrix equation for calculating the wavelet coefficients a_s.

Now let us specify the initial and boundary conditions. If we want to get the classical solitary wave solution we must take according to Eq. (27):

$$
\begin{aligned}
f(x) &= u(x,0) = 4\arctan[\exp(\alpha x)],\\
g(x) &= u'(x,0) = \alpha V(\alpha x),\\
\varphi(t) &= u(0,t) = 4\arctan[\exp(-\alpha\beta t)],\\
\psi(t) &= \dot{u}(0,t) = -\alpha\beta V(-\alpha\beta t),
\end{aligned}
\tag{33}
$$

where $V(z) = \frac{4e^z}{1+e^{2z}}$.

For solving Eq. (32) we need in addition the functions $u''(x,0)$, $\dot{u}''(x,0)\ddot{\varphi}(t)$, $\ddot{\varphi}'(t)$, $\varphi'(t)$, $\dot{\varphi}'(t)$, which can be calculated by differentiating Eq. (33).

Computer simulation was carried out for $t_{\text{in}} = 10$, $t_{\text{fin}} = 30$, $L = 20$, $\beta = 0.025$. The computed results were compared with the exact solution Eq. (27). The accuracy of our approach was estimated by the error function

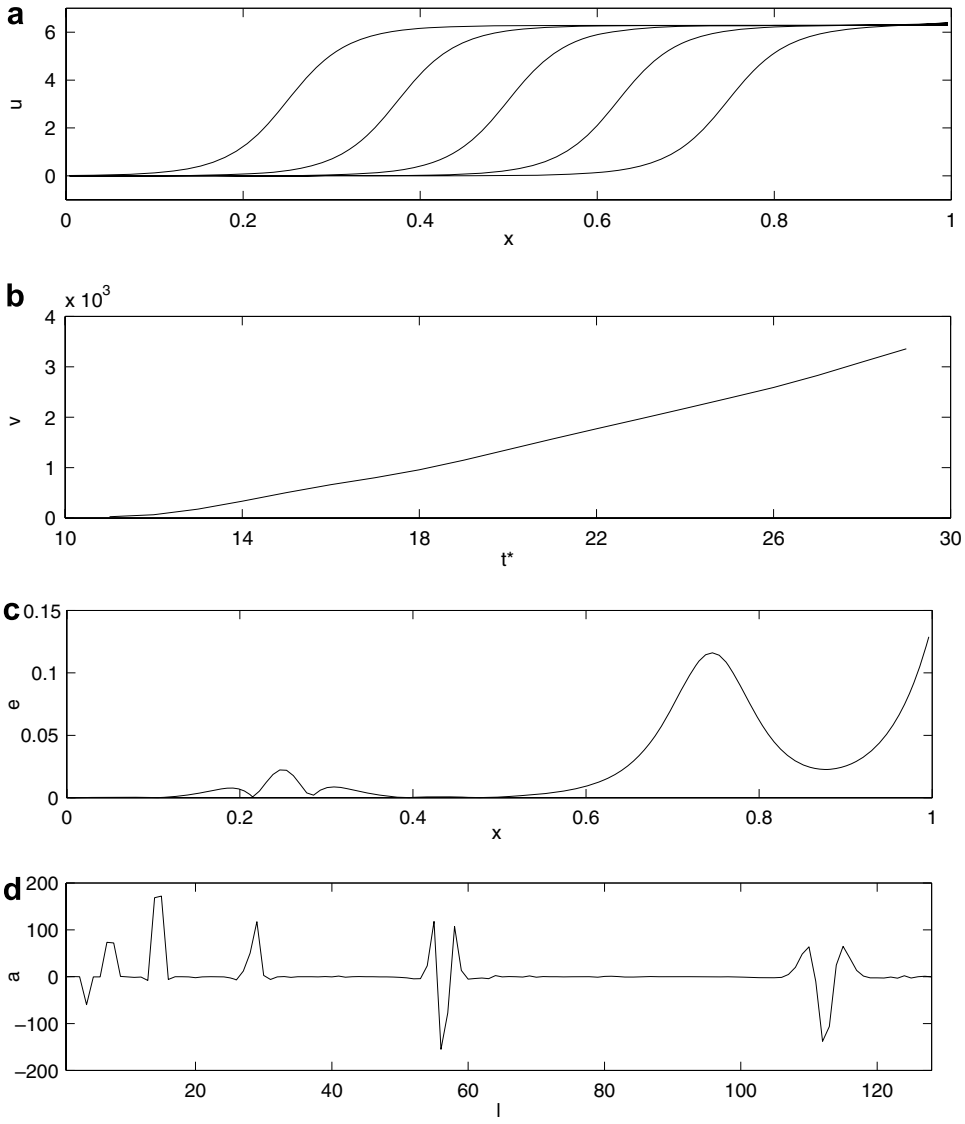

Fig. 2. Solution of the sine-Gordon equation for $J = 6$, $\Delta t = 0.001$: (a) solution for the instants $t_* = 10 + 5i$, $i = 0,1,2,3,4$; (b) error function $v(t_*)$; (c) error function $e(x) = |u(x,t_{\max}) - u_{lx}(x,t_{\max})|$; (d) wavelet coefficients at $t_* = 30$.

$$v(t) = \frac{1}{2M}\|u(x,t) - u_{\mathrm{ex}}(x,t)\| = \frac{1}{2M}\left\{\sum_{i=1}^{2M}[u(x_i,t) - u_{\mathrm{ex}}(x_i,t)]^2\right\}^{\frac{1}{2}}. \quad (34)$$

The calculations showed that the function $v(t)$ increases monotonically; therefore for the error estimate is taken $v(t_{\max})$. Some results of computation are presented in the table and in Fig. 2.

It follows from this table that already in the cases $J = 4$ or $J = 5$ we get results which visually coincide with the exact solution. For the time step was taken $\Delta t = 0.005$ or $\Delta t = 0.001$; further diminution of Δt did not give any essential effect. It can be seen from Fig. 2c that the distribution of the error v distribution along the x-axis not uniform (as it was to be awaited the biggest values of v appear in the large gradient regions). According to Fig. 2d the number of significant wavelet coefficients is again quite small.

J	M	Δt	$v(t_{\max})$
4	16	0.005	0.051
4	16	0.001	0.038
5	32	0.005	0.018
5	32	0.001	0.0090
6	64	0.005	0.0096
6	64	0.001	0.0036

6. Conclusions

A new method for solution of the PDE-s is proposed. According to this method the spacial operators are approximated by the Haar wavelet method and the time derivation operators by the finite difference method (for sake of simplicity the Euler scheme is used).

The main advantages of this method is its simplicity and small computation costs: it is due to the sparcity of the transform matrices and to the small number of significant wavelet coefficients.

The method with far less degrees of freedom and with a smaller CPU time provides better solutions than classical ones. The proposed algorithm resolves well the shock formation and solution in the large gradient regions. It is also very convenient for solving the boundary value problems, since in the solution the boundary conditions are taken care of automatically.

By our opinion the method is wholly competitive and efficient in comparison with the classical methods.

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