Applied Mathematics and Computation 214 (2009) 468–478

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Solving fractional integral equations by the Haar wavelet method

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ARTICLE INFO

Keywords:
Fractional calculus
Haar wavelets
Integral equations
Fractional vibrations

ABSTRACT

Haar wavelets for the solution of fractional integral equations are applied. Fractional Volterra and Fredholm integral equations are considered. The proposed method also is used for analysing fractional harmonic vibrations. The efficiency of the method is demonstrated by three numerical examples.

1. Introduction

Although the conception of the fractional derivatives was introduced already in the middle of the 19th century by Riemann and Liouville, the first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [1] published in 1974. After that the number of publications about the fractional calculus has rapidly increased. The reason for this is that some physical processes as anomalous diffusion, complex viscoelasticity, behaviour of mechatronic and biological systems, rheology etc. cannot be described adequately by the classical models.

At the present time we possess several excellent monographs about fractional calculus for example the book [2] by Kilbas et al., to which is also included a rather large and up-to-date Bibliography (928 items). Because of the enormous number of the papers about this topic we shall cite here only some papers which are more close to subject of this paper.

In a number of papers fractional differential equations are discussed; mostly these equations are transformed to fractional Volterra integral equations. For solution different techniques, as Fourier and Laplace transforms, power spectral density, Adomian decomposition method, path integration etc., are applied.

One-dimensional fractional harmonic oscillator is analysed in [3–6]. In [3,4] the solution is obtained in terms of Mittag–Leffler functions using Laplace transforms; several cases of the forcing function equation are considered. In [5] the fractional equation of motion is solved by the path integral method. In [6] the case, where the fractional derivatives only slightly differ from the ordinary derivatives, is analysed. Fractional Hamilton's equations are discussed in [7]. In [8] multiorder fractional differential equations are solved by using the Adomian decomposition. In several papers fractional chaotic systems are discussed. In [8] a three-dimensional fractional chaotic oscillator model is proposed. Chaotic dynamics of the fractionally damped Duffing equation is investigated in [9]. Two chaotic models for third-order chaotic nonlinear systems are analysed in [10].

It is somewhat surprising that among different solution techniques the wavelet method has not attained much attention. We found only one paper [11] in which the wavelet method is applied for solving fractional differential equations; for this purpose the Daubechies wavelet functions are used.

Among the different wavelet families mathematically most simple are the Haar wavelets. Due to the simplicity the Haar wavelets are very effective for solving differential and integral equations (see e.g. [12–15]). Therefore the idea, to apply Haar wavelet technique also for solving problems of fractional calculus, arises. This is the main aim of the present paper.

The paper is organized as follows. Sections 2 and 3 are preparative: in Section 2, basic equations of the fractional calculus are briefly reviewed, in Section 3, the Haar wavelet method is described. In Section 4, two error estimates for the results,

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2009.04.015

obtained by the Haar wavelet method, are introduced. In Section 5, the wavelet solution for fractional Volterra integral equations is presented. The one-dimensional fractional harmonic vibrations are discussed in Section 6. The solution for the fractional Fredholm integral equation is presented in Section 7.

2. About the fractional calculus

Let us briefly consider some basic formulae about the fractional calculus.

The Riemann–Liouville fractional integrals of order α are defined by

$$\left(I^{\alpha}_{A+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_A^x f(t)(x-t)^{\alpha-1}\,dt \quad (x > A,\ [\alpha] > 0) \tag{1}$$

and

$$\left(I^{\alpha}_{B-}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^B f(t)(t-x)^{\alpha-1}\,dt \quad (x < B,\ [\alpha] > 0). \tag{2}$$

Here $\Gamma(\alpha)$ is the gamma function and $[\alpha]$ the integer part of α. The integrals (1) and (2) are called left-sided and right-sided fractional integrals.

As to the fractional derivatives, then in this paper we shall use the Caputo derivatives defined as

$$D^{\alpha}u(x) = \frac{1}{\Gamma(n-\alpha)}\int_A^x u^{(n)}(t)(x-t)^{n-1-\alpha}\,dt, \tag{3}$$

where $u^{(n)}_{(x)} = d^n u/dx^n$ and $n = [\alpha] + 1$.

If $\alpha + 1 = n \in N$ then $D^n u(x)$ coincides with the ordinary derivative $u^{(n)}(x)$.

3. Haar wavelets

Usually the Haar wavelets are defined for the interval $x \in [0,1]$. In this paper the more general case $x \in [A,B]$, is considered. Let us define the quantity $M = 2^J$, where J is the maximal level of resolution. We shall divide the interval $[A,B]$ into $2M$ subintervals of equal length; each subinterval has the length $\Delta x = (B-A)/(2M)$. Next two parameters are introduced: the dilatation parameter j for which $j = 0,1,\ldots,J$ and the translation parameter $k = 0,1,\ldots,m-1$ (here the notation $m = 2^j$ is introduced). The wavelet number i is identified as $i = m + k + 1$.

The ith Haar wavelet is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1(i),\ \xi_2(i)], \\ -1 & \text{for } x \in [\xi_2(i),\ \xi_3(i)], \\ 0 & \text{elsewhere}, \end{cases} \tag{4}$$

where

$$\begin{aligned} &\xi_1(i) = A + 2k\mu\Delta x, \quad \xi_2(i) = A + (2k+1)\mu\Delta x, \\ &\xi_3(i) = A + 2(k+1)\mu\Delta x, \quad \mu = M/m. \end{aligned} \tag{5}$$

The case $i = 1$ corresponds to the scaling function, here $h_1(x) = 1$ for $x \in [A,B]$ and $h_1(x) = 0$ elsewhere.

The expansion of a given function $u(x)$ into the Haar wavelet series is

$$u(x) = \sum_{i=1}^{2M} a_i h_i(x), \quad x \in [A,B], \tag{6}$$

where a_i are the wavelet coefficients.

There are different possibilities for calculating a_i. In this paper we apply the collocation method; the collocation points are identified as

$$x_l = A + (l - 0.5)\Delta x, \quad l = 1,2,\ldots,2M. \tag{7}$$

The discrete version of (6) is

$$u(x_l) = \sum_{i=1}^{2M} a_i h_i(x_l). \tag{8}$$

It is convenient to put this result into the matrix form

$$u = aH, \tag{9}$$

where u and a are $2M$-dimensional row vectors and $H(i,l) = h_i(x_l)$ is the element of a $2M \times 2M$ matrix.

In Section 5, the integrals

$$\begin{aligned} p_i(x) &= \int_A^x h_i(x)dx, \\ q_i(x) &= \int_A^x \int_A^x h_i(x)dx\,dx = \int_A^x (x-y)h_i(y)dy \end{aligned} \tag{10}$$

are needed. In view of (4) they can be analytically evaluated. Doing this we find

$$p_i(x) = \begin{cases} 0 & \text{for } x < \xi_1(i), \\ x - \xi_1(i) & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ \xi_3(i) - x & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ 0 & \text{for } x \geqslant \xi_3(i), \end{cases} \tag{11}$$

$$q_i(x) = \begin{cases} 0 & \text{for } x < \xi_1(i), \\ 0.5[x - \xi_1(i)]^2 & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ D - 0.5[\xi_3(i) - x]^2 & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ D & \text{for } x \geqslant \xi_3(i), \end{cases} \tag{12}$$

where $D = [\xi_2(i) - \xi_1(i)]^2 = (\mu\Delta x)^2$.

It is also appropriate here to introduce the integral matrices $P(i,l) = p_i(x_l)$, $Q(i,l) = q_i(x_l)$.

Now let us consider how to solve a fractional differential or integral equation by the Haar wavelet method. Since the Haar wavelets are not continuous and non-differentiable in the points of discontinuity we shall develop into the wavelet series not the function $u = u(x)$ sought for, but according to [16] its highest derivative in the equation $u^{(\nu)} = u^{(\nu)}(x)$. The derivatives of lower order and the function $u = u(x)$ itself are calculated by integration. By replacing all these results into the equation to be solved we obtain a system of equations for calculating the wavelet coefficients a_i, $i = 1, 2, \ldots, 2M$. Details of this approach are cleared up in Sections 5–7.

4. Error estimates

It is essential to estimate the exactness of the obtained solutions, for this purpose in the following two error estimates are defined. Here we have to distinguish the two following situations.

(i) If the exact solution of the problem $u = u_{ex}(x)$ is known we shall calculate the differences $\Delta_{ex}(l) = u(x_l) - u_{ex}(x_l)$, $l = 1, 2, \ldots, 2M$ and define the error estimates as $\delta_{ex} = \max_l|\Delta_{ex}(l)|$ (local estimate) or $\sigma_{ex} = \|u - u_{ex}\|/2M$ (global estimate).

(ii) Mostly the exact solution is unknown. For this case the following procedure is recommended. First we solve our equation for some level of resolution J, the result is denoted by $u_J(x)$; then we repeat these calculations for $J+1$ getting in this way the function $u_{J+1}(x)$. Next we define the differences

$$\Delta_J(x_l) = u_J(x_l) - u_{J+1}(x_l), \tag{13}$$

where x_l, $l = 1, 2, \ldots, 2M$ are the collocation points at the level J. The error estimates we shall define as

$$\delta_J = \max_l |\Delta_J(x_l)|, \quad \sigma_J = \|\Delta_J(x_l)\|/(2M). \tag{14}$$

By increasing the parameter J the error estimates usually decrease. But computer simulation has shown that there exist also problems where the error estimates decrease up to some level $J = \tilde{J}$ and after that begin to increase. This is caused from the following fact. For calculating the wavelet coefficients a_i we must invert some matrix, which is put together from the Haar matrices. In some cases by increasing the level J this matrix may turn out to be nearly singular and it is not possible to calculate the coefficients a_i with the necessary exactness (see Example 2 in Section 6). If the exactness obtained at the level $\tilde{J}$ is insufficient we must find more exact results by some other method (e.g. in [13] for this purpose the segmentation method was proposed).

5. Fractional Volterra integral equation

The fractional Volterra integral equation has the form [2]

$$u(x) - \frac{1}{\Gamma(\alpha)} \int_0^x K(x,t)(x-t)^{\alpha-1} u(t)dt = f(x), \quad 0 \leqslant x \leqslant 1. \tag{15}$$

The kernel $K(x,t)$ and the right-side function $f(x)$ are given, $\alpha > 0$ is a real number. The value $\alpha = 1$ corresponds to the ordinary (nonfractional) Volterra equation.

According to the Haar wavelet method, the solution of (15) is sought in the form (8). Replacing (8) into (15) and satisfying this equation in the collocation points we obtain

$$\sum_{i=1}^{2M} a_i[h_i(x_l) - g_i(x_l)] = f(x_l), \quad l = 1,2,\ldots,2M. \tag{16}$$

Here the symbol $g_i(x_l)$ denotes the function

$$g_i(x_l) = \frac{1}{\Gamma(\alpha)}\int_0^{x_l} K(x_l,t)^{\alpha-1}(x_l - t)h_i(t)dt. \tag{17}$$

The matrix form of (16) is

$$a(H-G) = F, \tag{18}$$

where $G(i,l) = g_i(x_l)$, $F(l) = f(x_l)$.

The solution of (18) is

$$a = F/(H-G). \tag{19}$$

The function $u(x)$ can be calculated from (8).

The solution presented here is very simple. The most labour-consuming operation is the evaluation of the matrix G. In view of (4) the Eq. (17) can be rewritten in the following form (for conciseness sake the argument i at ξ_1, ξ_2, ξ_3 is omitted):

$$\begin{aligned}
&(i)\quad G(i,l) = 0 \quad \text{for } x_l < \xi_1,\\
&(ii)\quad G(i,l) = \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{x_l} K(x_l,t)(x_l-t)^{\alpha-1}\,dt \quad \text{for } x_l \in [\xi_1,\xi_2],\\
&(iii)\quad G(i,l) = \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{\xi_2} K(x_l,t)(x_l-t)^{\alpha-1}\,dt - \frac{1}{\Gamma(\alpha)}\int_{\xi_2}^{x} K(x_l,t)(x_l-t)^{\alpha-1}\,dt \quad \text{for } x_l \in [\xi_2,\xi_3],\\
&(iv)\quad G(i,l) = \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{\xi_2} K(x_l,t)(x_l-t)^{\alpha-1}\,dt - \frac{1}{\Gamma(\alpha)}\int_{\xi_2}^{\xi_3} K(x_l,t)(x_l-t)^{\alpha-1}\,dt \quad \text{for } x_l \geqslant \xi_3.
\end{aligned} \tag{20}$$

The integrals in (20) can be evaluated by some numerical techniques; but for some simpler forms of $K(x_l,t)$ analytical integration is possible.

Example 1. Consider the case $K(x,t) = \exp(x-t)$. Let us introduce the function

$$\varphi(x_l,\gamma_1,\gamma_2) = \int_{\gamma_1}^{\gamma_2} (x_l - t)^{\alpha-1}e^{-t}\,dt. \tag{21}$$

Now (20) can be put into the form

$$\begin{aligned}
&(i)\quad G(i,l) = 0 \quad \text{for } x_l < \xi_1,\\
&(ii)\quad G(i,l) = \frac{1}{\Gamma(\alpha)}\exp(x_l)\varphi(x_l,\xi_1,x_l) \quad \text{for } x_l \in [\xi_1,\xi_2],\\
&(iii)\quad G(i,l) = \frac{1}{\Gamma(\alpha)}\exp(x_l)[\varphi(x_l,\xi_1,\xi_2) - \varphi(x_l,\xi_2,x_l)] \quad \text{for } x_l \in [\xi_2,\xi_3],\\
&(iv)\quad G(i,l) = \frac{1}{\Gamma(\alpha)}\exp(x_l)[\varphi(x_l,\xi_1,\xi_2) - \varphi(x_l,\xi_2,\xi_3)] \quad \text{for } x_l > \xi_3.
\end{aligned} \tag{22}$$

These integrals were evaluated with the aid of the *Matlab* program *quad*.

Let us at first consider the nonfractional case $\alpha = 1$; here (15) has the exact solution $u_{ex} = exp(2x)$. Computer simulation for $J = 5$ (64 collocation points) gave the error estimates $\delta_{ex} = 8.9\text{E}-4$, $\sigma_{ex} = 5.0\text{E}-5$; for $J = 6$ (128 collocation points) we found $\delta_{ex} = 2.3\text{E}-4$, $\sigma_{ex} = 9.0\text{E}-6$.

Calculations, which were carried out for some values of α, are plotted in Fig. 1. Error estimates were computed for $\alpha = 0.5$; these results are presented in Table 1.

6. Fractional harmonic vibrations

Consider the equation

$$D^\alpha u(x) + \lambda D^\beta u(x) + \nu u(x) = f(x), \quad x \in [0,B], \tag{23}$$

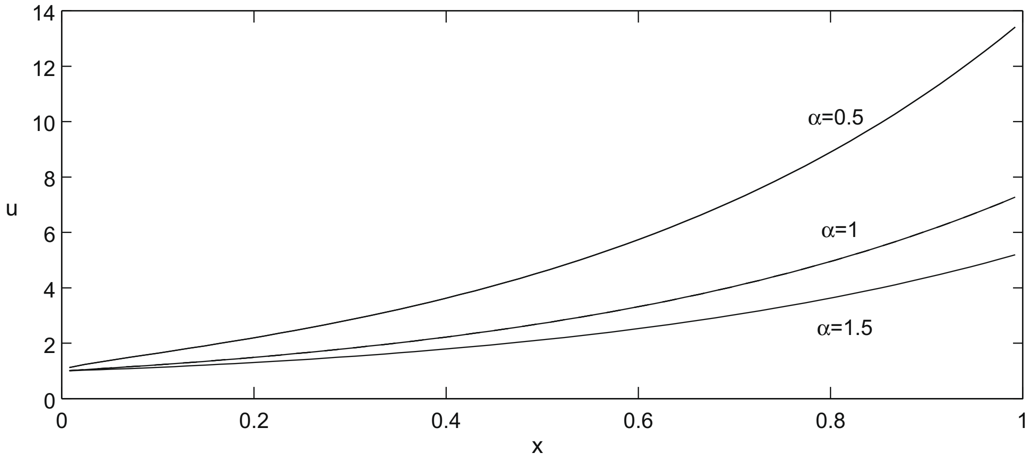

Fig. 1. Solution of the Volterra integral equation (15) for $K(x,t) = \exp(x-t)$.

Table 1
Error estimates for the Eq. (15); $\alpha = 0.5$.

J	$2M$	δ_{ex}	σ_{ex}
3	16	0.332	4.5E−2
4	32	0.186	1.7E−2
5	64	0.010	6.3E−3

where $1 < \alpha < 2,\ 0 < \beta < 1,\ \lambda,\ \nu$ are prescribed constants, $f(x)$ is the forcing term. To (23) belong initial conditions $u(0) = u_0,\ u'(0) = v_0$. If $\alpha = 2$ and $\beta = 1$ we get the usual differential equation of the harmonic oscillator.

The symbols D^α, D^β denote left-sided Caputo derivatives, which are defined by (3). Since in the present case $n_\alpha = [\alpha] + 1 = 2,\ n_\beta = [\beta] + 1 = 1$ the equation (23) gets the form

$$\frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{1-\alpha} u''(t)dt + \frac{\lambda}{\Gamma(1-\beta)} \int_0^x (x-t)^{-\beta} u'(t)dt + \nu u(x) = f(x). \tag{24}$$

This is a Volterra integral equation. Let us solve it by the Haar wavelet method.

$$\begin{aligned} u''(x) &= \sum_{i=1}^{2M} a_i h_i(x), \\ u'(x) &= \sum_{i=1}^{2M} a_i p_i(x) + u'(0), \\ u(x) &= \sum_{i=1}^{2M} a_i q_i(x) + u'(0)x + u(0), \end{aligned} \tag{25}$$

where the functions $h_i,\ p_i,\ q_i$ are calculated from (4), (11), (12). Replacing (25) into (24) and satisfying this equation in the collocation points x_l, we find

$$\sum_{i=1}^{2M} a_i G(i,l) = f(x_l) - \nu(v_0 x_l + u_0) - \frac{\lambda}{\Gamma(2-\beta)} v_0 x_l^{1-\beta}, \tag{26}$$

where

$$G(i,l) = \frac{1}{\Gamma(2-\alpha)} \int_0^{x_l} (x_l - t)^{1-\alpha} h_i(t)dt + \frac{\lambda}{\Gamma(1-\beta)} \int_0^{x_l} (x_l - t)^{-\beta} p_i(t)dt + \nu q_i(x_l). \tag{27}$$

The integrals in (27) can be evaluated with the aid of (4),(11), (12). By doing this, we get the following formulae:

$$\begin{aligned}
&G(i,l) = 0 \quad \text{for } x_l < \xi_1, \\
&G(i,l) = \frac{1}{\Gamma(3-\alpha)}(x_l-\xi_1)^{2-\alpha} + \lambda N(\beta)(x_l-\xi_1)^{2-\beta} + 0.5v(x_l-\xi_1)^2 \quad \text{for } x_l \in [\xi_1,\xi_2], \\
&G(i,l) = \frac{1}{\Gamma(3-\alpha)}\left[(x_l-\xi_1)^{2-\alpha} - 2(x_l-\xi_2)^{2-\alpha}\right] - \lambda N(\beta)\left[(x_l-\xi_1)^{2-\beta} - 2(x_l-\xi_2)^{2-\beta}\right] \\
&\qquad + 0.5v[D-(\xi_3-x_l)^2] \quad \text{for } x_l \in [\xi_2,\xi_3], \\
&G(i,l) = \frac{1}{\Gamma(3-\alpha)}\left[(x_l-\xi_1)^{2-\alpha} + (x_l-\xi_3)^{2-\alpha} - 2(x_l-\xi_2)^{2-\alpha}\right] \\
&\qquad + \lambda\Bigg\{N(\beta)(x_l-\xi_1)^{2-\beta} - \frac{1}{\Gamma(2-\beta)}(x_l-\xi_1)(x_l-\xi_2)^{1-\beta} \\
&\qquad - \frac{1}{\Gamma(2-\beta)}(\xi_3-x_l)\left[(x_l-\xi_3)^{1-\beta} - (x_l-\xi_2)^{1-\beta}\right] \\
&\qquad + \frac{1}{(2-\beta)\Gamma(1-\beta)}\left[2(x_l-\xi_2)^{2-\beta}(x_l-\xi_3)^{2-\beta}\right]\Bigg\} + 0.5vD \quad \text{for } x_l > \xi_3,
\end{aligned} \tag{28}$$

where

$$D = (\mu\Delta x)^2, \quad N(\beta) = \frac{1}{\Gamma(2-\beta)} - \frac{1}{(2-\beta)\Gamma(1-\beta)}. \tag{29}$$

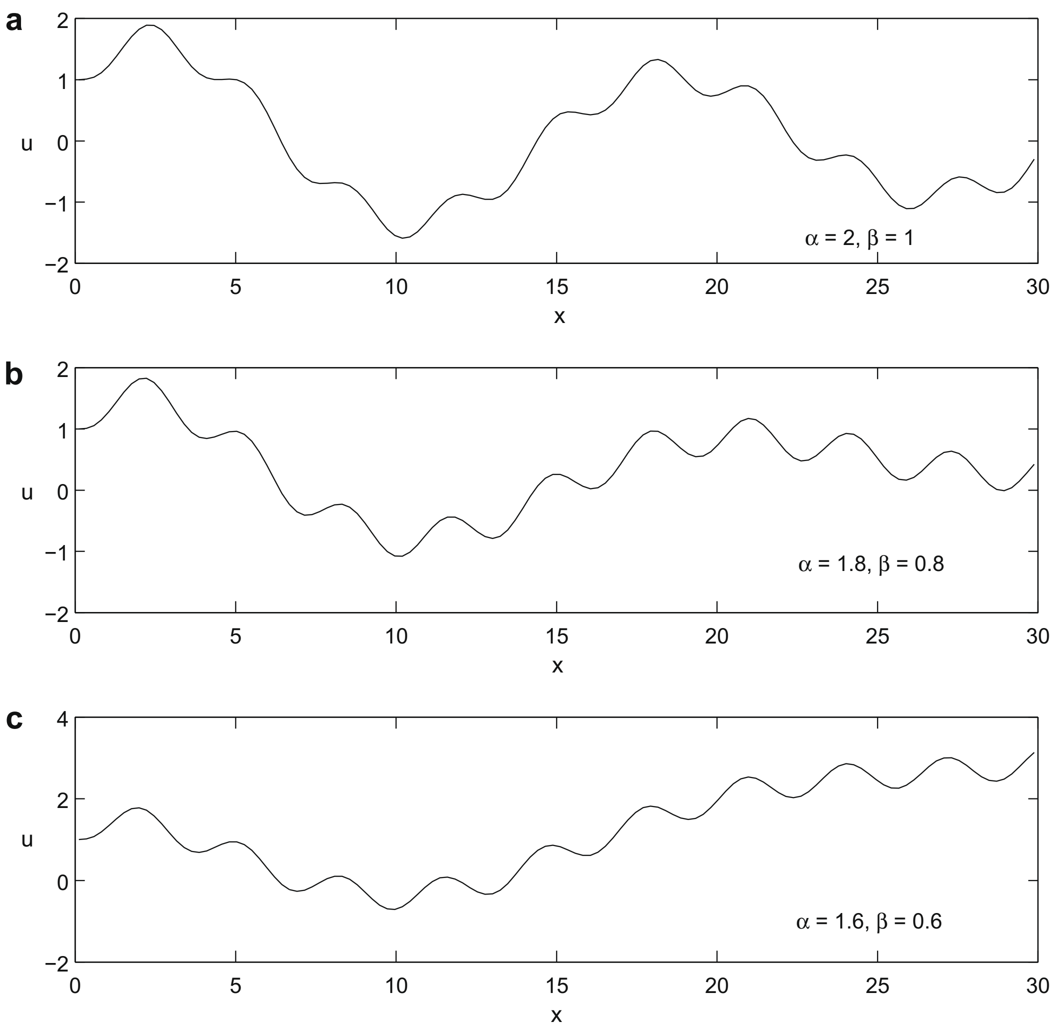

Fig. 2. Solution of (23) for different values of α and β.

In the following course of solution we shall again make use of the matrix representation. Now (26) obtains the form

$$aG = F + \Phi, \tag{30}$$

where $F(l) = f(l),\ \ \Phi(l) = -\nu(v_0 x_l + u_0) - \frac{\lambda v_0}{\Gamma(1-\beta)} x_l^{1-\beta}$. The solution of (30) is

$$a = (F + \Phi)/G. \tag{31}$$

The function u to be sought is calculated from

$$u = aQ + v_0 X + u_0 E, \tag{32}$$

where $X = \{x(l)\}$ and E is a $2M$ row vector of ones.

Example 2. Let us take for the force function $f(x) = r\sin\omega t$ and assign to parameters the values $\lambda = 0.05,\ \nu = 0.15,\ r = 1,\ \omega = 2,\ B = 30,\ u_0 = 1,\ v_0 = 0.$

In order to estimate the exactness of the presented solution at first again the case $\alpha = 2,\ \beta = 1$ is considered. The exact solution of (23) is

$$u_{ex}(x) = L\exp(-0.5\lambda x)\sin(\omega_1 x + \varepsilon) + R(\sin\omega x + \delta), \tag{33}$$

where

$$\begin{aligned} &\omega_1 = \sqrt{\nu - 0.25\lambda^2}, \quad \tan\delta = -\frac{\lambda\omega}{\nu - \omega^2}, \\ &R = \frac{r}{\sqrt{\lambda^2\omega^2 + (\nu - \omega^2)^2}}. \end{aligned} \tag{34}$$

The coefficients $L,\ \varepsilon$ are calculated from the boundary conditions $u_0 = 1,\ v_0 = 0.$

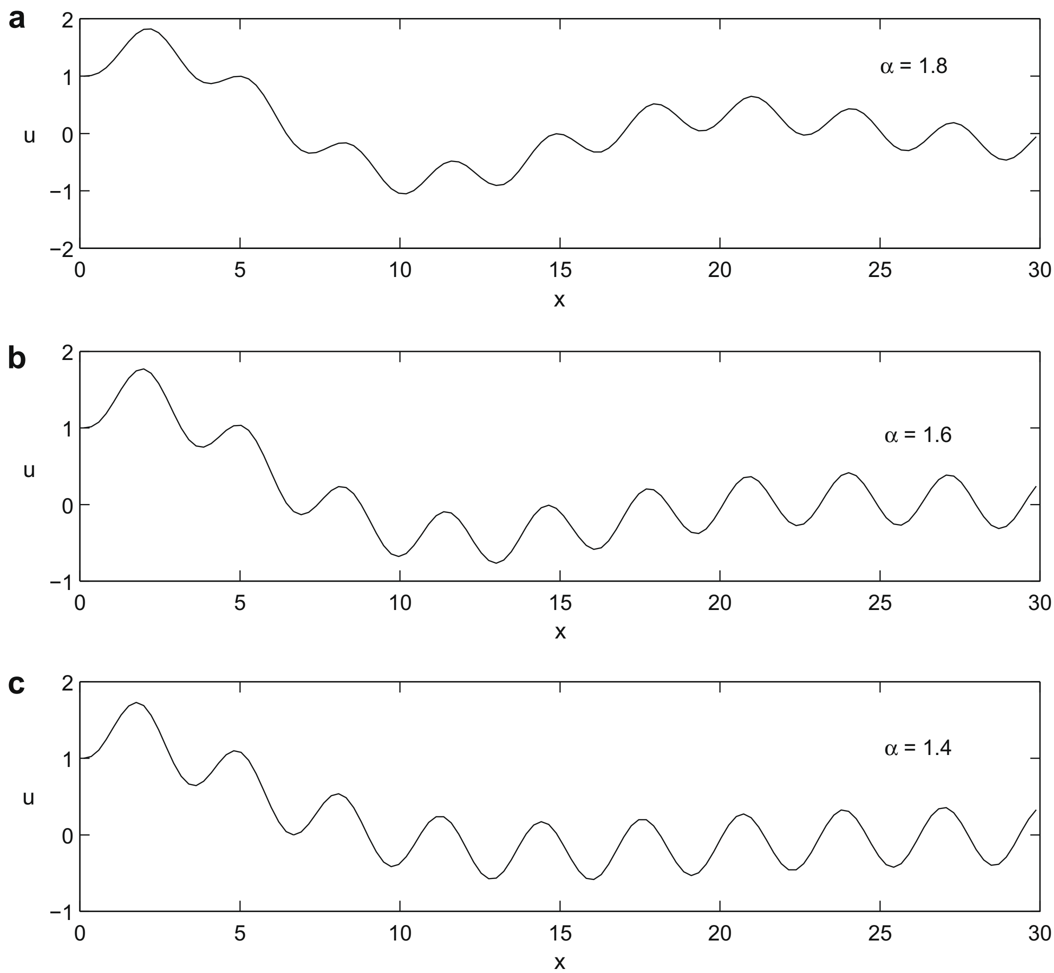

Fig. 3. Solution of (23) for $\beta = 1$.

The wavelet solution was computed for $J = 6$. The error estimates gave $\delta_{ex} = 0.005$, $\sigma_{ex} = 2.4\text{E}-4$; consequently the value $J = 6$ guarantees the necessary accuracy of the results. The function $u_{ex}(x)$ is plotted in Fig. 2a. Computer simulations were carried out for different values of α and β. The results are presented in Figs. 2–4.

Error estimates δ_J, σ_J were calculated for the case $\alpha = 1.6, \beta = 0.6$. It followed from the computations that $\delta_4 = 0.789$, $\sigma_4 = 0.068$, $\delta_5 = 0.075$, $\sigma_5 = 0.005$. At subsequent increase of the resolution level J these estimates became worse. The reason for this lies in solving (30). If we calculate the determinant $|G|$, then we obtain $|G| = 1.91E+7$ for $J = 4$, $|G| = 3.26E+6$ for $J = 5$, $|G| = 3.6\text{E}-3$ for $J = 6$, $|G| = 1.59\text{E}-36$ for $J = 7$. Consequently, we have here the unfavourable case, where the matrix $|G|$ turns out to be nearly singular (this case was discussed in Section 4).

Eq. (23) was treated in several papers from which we cite here [1,3,4,9,17]. In [3,4] the case $1 < \alpha \leqslant 2$, $\lambda = 0$ (our symbols) was analysed, in [3] it is assumed that $f \equiv 0$; in [4] different cases of the forcing function $f(x)$ were discussed. The solution was obtained in terms of Mittag–Leffler functions using Laplace transforms. In [9] Eq. (23) was solved by the Fourier transform for $\alpha = 2$, $0 < \beta < 2$, $\beta \neq 1$; the external force is impulsive. The almost free damping oscillator for which $\alpha = 2 + \varepsilon$, $\beta = 1 + 0.5\varepsilon$, $f = 0$, $|\varepsilon| \ll 1$ was discussed in [1]; particular solutions of (23) were sought in the form $u = \exp(\lambda x)$.

7. Fractional Fredholm integral equation

The usual Fredholm integral equation has the form

$$u(x) - \int_A^B K(x,t)u(t)dt = f(x), \quad x \in [A,B]. \tag{35}$$

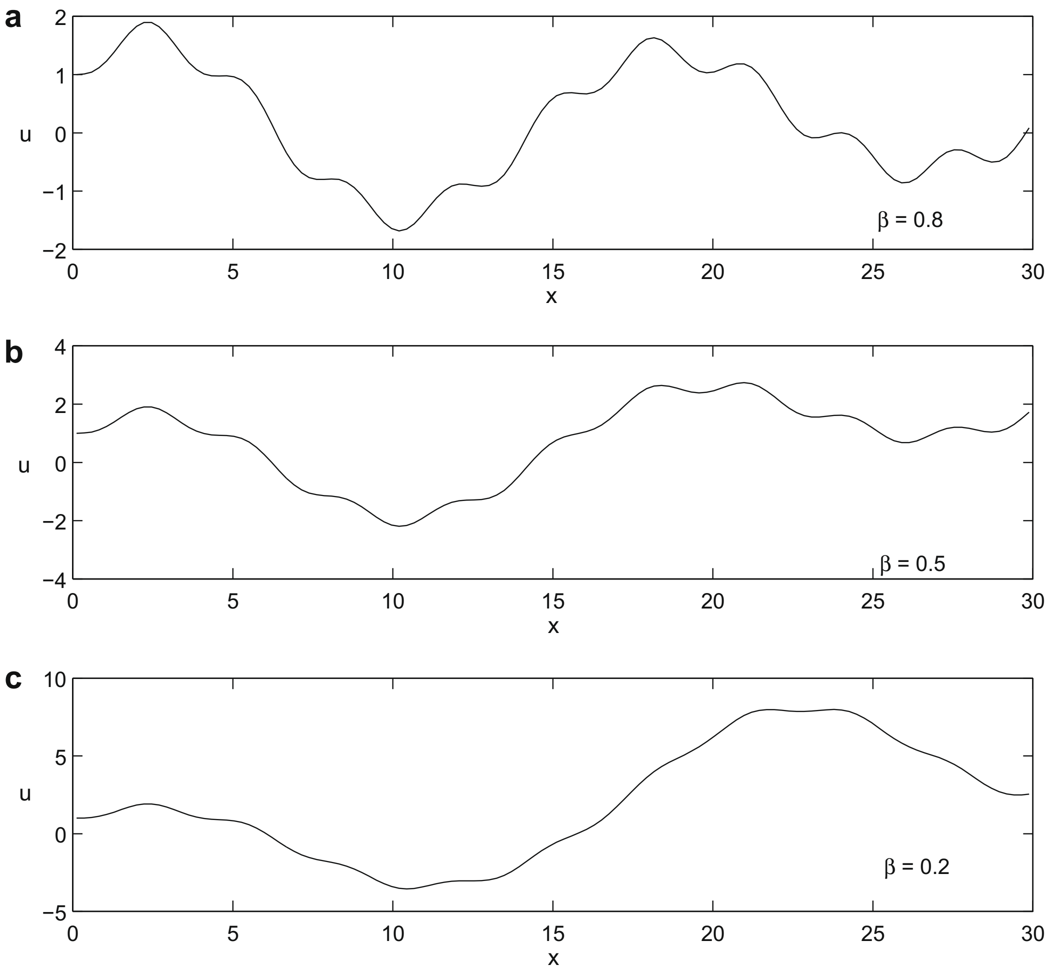

Fig. 4. Solution of (23) for $\alpha = 2$.

If we want to get its fractional analogue we must take into account the fact that in the case of non-integer α the expression $(x-t)^{\alpha}$ has sense only for $x \geqslant t$. Therefore it seems reasonable to define the fractional Fredholm integral equation in the form

$$u(x) - \frac{1}{\Gamma(\alpha)}\left[\int_A^x (x-t)^{\alpha-1}K(x,t)u(t)dt + \int_x^B (t-x)^{\alpha-1}K(x,t)u(t)dt\right] = f(x). \tag{36}$$

In (36) stand the left-sided and right-sided Riemann–Liouville integrals, respectively. As in Section 4, the solution of (36) is sought in the form (9). Let us replace $x \to x_l$ and introduce the matrix

$$G(i,l) = \frac{1}{\Gamma(\alpha)}\left[\int_A^{x_l} (x_l-t)^{\alpha-1}K(x_l,t)h_i(t)dt + \int_{x_l}^B (t-x_l)^{\alpha-1}K(x_l,t)h_i(t)dt\right]. \tag{37}$$

On the grounds of (4) this integral can be evaluated either analytically or numerically.

Now (36) can be put into the form

$$a(H-G) = F, \tag{38}$$

where $H(i,l) = h_i(x_l)$, $F(l) = f(x_l)$. The solution of (37) is $a = F/(H-G)$.

Example 3. Consider the equation

$$u(x) - \frac{1}{\Gamma(\alpha)}\left[\int_0^x (2-x-t)(x-t)^{\alpha-1}u(t)dt + \int_x^3 (2-x-t)(t-x)^{\alpha-1}u(t)dt\right] = x^2 + 15/4. \tag{39}$$

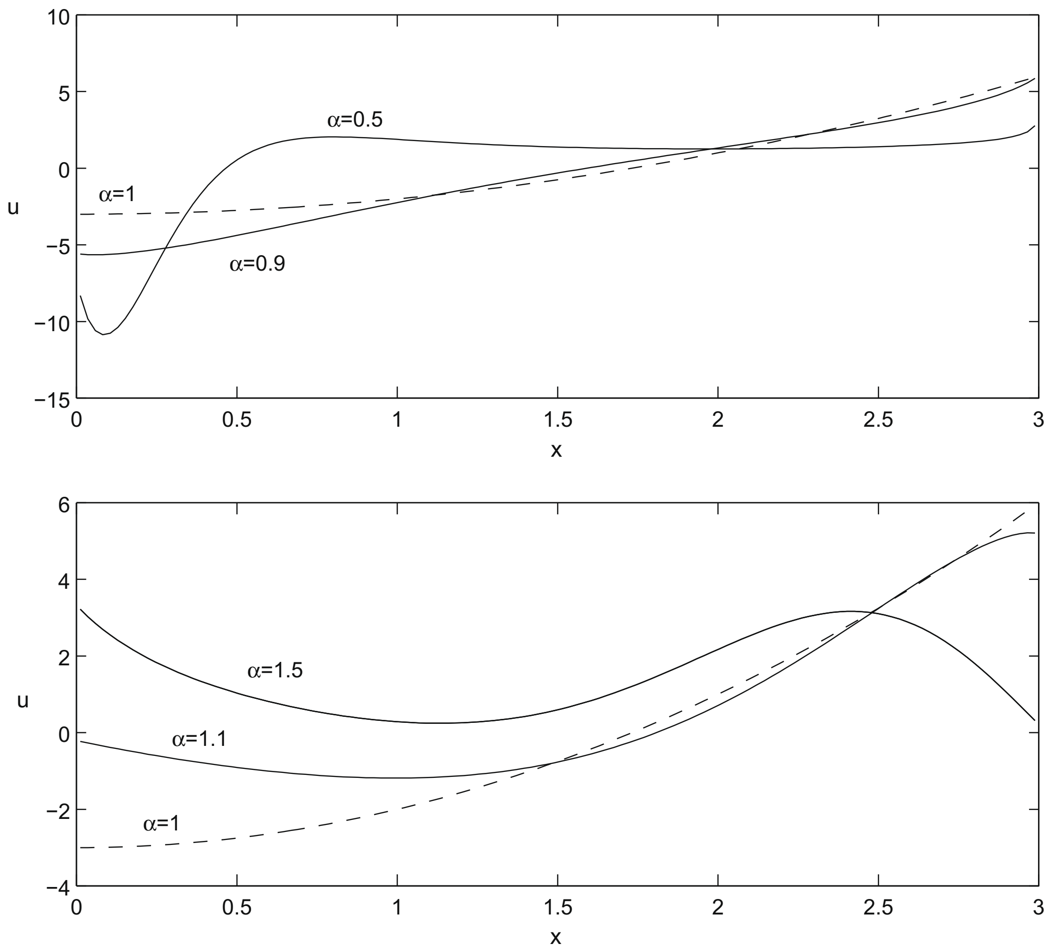

Fig. 5. Solution of the Fredholm equation (39) for different values of α.

Table 2
Error estimates for the Eq. (39); $\alpha = 1.5$.

J	$2M$	δ_{ex}	σ_{ex}
3	16	0.386	4.3E−2
4	32	0.200	1.5E−2
5	64	0.100	5.4E−3
6	128	0.050	1.9E−3
7	256	0.026	6.8E−4

Now the kernel is $K(2 - x - t)$ and evaluating the integrals in (35) we find

$$\begin{aligned}
&(i)\quad G(i,l) = \frac{2(1-x_l)}{\Gamma(\alpha+1)}\left[2(\xi_2 - x_l)^{\alpha} - (\xi_1 - x_l)^{\alpha} - (\xi_3 - x_l)^{\alpha}\right] \\
&\qquad + \frac{\alpha}{\Gamma(\alpha+2)}\left[-2(\xi_2 - x_l)^{\alpha+1} + (\xi_1 - x_l)^{\alpha+1} + (\xi_3 - x_l)^{\alpha+1}\right] \quad \text{for } x_l < \xi_1, \\
&(ii)\quad G(i,l) = \frac{2(1-x_l)}{\Gamma(\alpha+1)}\left[(x_l - \xi_1)^{\alpha} + 2(\xi_2 - x_l)^{\alpha} - (\xi_3 - x_l)^{\alpha}\right] \\
&\qquad + \frac{\alpha}{\Gamma(\alpha+2)}\left[(x_l - \xi_1)^{\alpha+1} - 2(\xi_2 - x_l)^{\alpha+1} + (\xi_3 - x_l)^{\alpha+1}\right] \quad \text{for } x_l \in [\xi_1, \xi_2], \\
&(iii)\quad G(i,l) = \frac{2(1-x_l)}{\Gamma(\alpha+1)}\left[(x_l - \xi_1)^{\alpha} - 2(x_l - \xi_2)^{\alpha} - (\xi_3 - x_l)^{\alpha}\right] \\
&\qquad + \frac{\alpha}{\Gamma(\alpha+2)}\left[(x_l - \xi_1)^{\alpha+1} - 2(x_l - \xi_2)^{\alpha+1} + (\xi_3 - x_l)^{\alpha+1}\right] \quad \text{for } x_l \in [\xi_2, \xi_3], \\
&(iv)\quad G(i,l) = \frac{2(1-x_l)}{\Gamma(\alpha+1)}\left[(x_l - \xi_1)^{\alpha} - 2(x_l - \xi_2)^{\alpha} + (x_l - \xi_3)^{\alpha}\right] \\
&\qquad + \frac{\alpha}{\Gamma(\alpha+2)}\left[(x_l - \xi_1)^{\alpha+1} - 2(x_l - \xi_2)^{\alpha+1} + (x_l - \xi_3)^{\alpha+1}\right] \quad \text{for } x_l \geqslant \xi_3.
\end{aligned} \tag{40}$$

Let us begin from the case $\alpha = 1$, which corresponds to the nonfractional equation. This equation has the exact solution

$$u_{ex}(x) = x^2 - 3, \quad x \in [0, 3]. \tag{41}$$

The computer simulation for $\alpha = 1$ gave the following error estimates

$$\begin{aligned}
&(i)\quad \delta_{ex} = 7.7\mathrm{E}-3, \quad \sigma_{ex} = 1.8\mathrm{E}-4 \quad \text{for } J = 5, \\
&(ii)\quad \delta_{ex} = 6.8\mathrm{E}-4, \quad \sigma_{ex} = 3.2\mathrm{E}-5 \quad \text{for } J = 6.
\end{aligned}$$

The results of the computations for $\alpha = 0.5,\ \alpha = 0.9,\ \alpha = 1.0,\ \alpha = 1.1,\ \alpha = 1.5$ are plotted in Fig. 5.

Error analysis was carried out for $\alpha = 1.5$, the results are given in Table 2.

Calculations show that for this case the determinant $|H - G|$ essentially differs from zero (e.g. $|H - G| = -9.7E + 18$ for $J = 4$ and $|H - G| = -5.6E + 153$, consequently, no problems for solving (38)) occur.

8. Conclusion

In the present paper three nonfractional equations are solved by the Haar wavelet method, these problems must be considered as examples for the recommended method of solution, since this approach can be easily carried over to solving some other problems, as e.g.

(i) first kind linear integral equations [12]:
(ii) integro–differential equations [12,14];
(iii) weakly singular integral equations [12].

In the present paper only linear equations are considered, but the method is applicable also for nonlinear systems; in this case the wavelet coefficients must be calculated by some numerical technique (e.g. Newton method, predictor–corrector methods) [14,18].

The main advantages of the presented method are its simplicity and small computation costs: it is due to the sparcity of the transform matrices and to the small number of significant wavelet coefficients. In our opinion the method is wholly competitive in comparison with the classical methods.

Acknowledgement

Financial support from the Estonian Science Foundation under Grant ETF-6697 is gratefully acknowledged.

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