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CAN NOISE ANNIHILATE CHAOS?

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Abstract
It is shown that adding noise to a chaotic system the motion may turn regular and is terminated in some of the fixed points. Analysis with the aid of the Lyapunov exponents affirms the fact that chaotical motion is really supressed. Computer simulations are carried out for the Duffing equation and the forced motion of the pendulum.

Key words
Dynamical systems, chaos, Gaussian noise, Lyapunov exponents.

1 Introduction

Chaotic behavior of dynamical systems under stochastic excitation has attracted great attention in the last time. From the numerous papers about this topic we cite here the followings.

The papers [Kisliakov, 1996] and [Kapitaniak and El Naschie, 1991] are devoted to the philosophy of stochastical chaos relations. It is noted [Kapitaniak and El Naschie, 1991] and [Bontempi, Casciati and Faravelli 1991] that the problem is quite complicated and it is often impossible to distinguish between these types of behavior. Nonlinear systems exhibiting chaotic, noisy chaotic and random behaviors are analysed in [Lin and Yim, 1996].

In several papers the analysis is carried out for concrete differential equations as the Duffing equation [Wei and Leng, 1997] and [Liua and Zhu, 2001], Duffing-Van der Pol equation [Feng, Xu, Rong and Wang, 2009] and [Huang, Zhu, Ni and Ko, 2002]. Some authors [Bontempi, Casciati and Faravelli 1991], [Wei and Leng, 1997], [Huang, Zhu, Ni and Ko, 2002], [Yang, Xu, Sun, 2006] have shown that by adding noise we may stabilize the system.

In most of the papers additional, multiplicative or bounded Gaussian noise is applied. In the present paper the way for introducing the noise is somewhat different. We assume that the forcing term in equations of motion has the form $f cos(\omega t)$ where f is the amplitude and t denotes time. The quantity ω can be intepreted as the angular velocity of some driver. By physical reasons ω is not strictly constant but carries some stochastic oscillations. By this reason we assume that ω is a narrow-band random variable. This approach was applied in [Lepik, 2003] and [Lepik, 2005]. In these papers a chaotic case of the Duffing equation was examined. It turned out that by adding noise to the angular velocity ω the initially chaotic motion becomes regular and is terminated in one of the focuses. This unexpected result was controlled and applied to some other dynamical systems in the papers [Lepik, Hein, 2005], [Hein, Lepik, 2007]; the aim of the present paper is to resume and develop the results of these papers.

2 Modeling stochastic vibrations

In this paper the following type of equations

$$\ddot{x} + g(t, x, \dot{x}) = f \cos \omega t \qquad 0 \leq t \leq T \tag{1}$$

is considered. Here g is a prescribed function and $\omega(t)$ the angular velocity of some driver. By physical reasons ω is not strictly constant but carries out some stochastic oscillations

$$\omega(t) = \omega_0 \left[1 + \alpha \xi(t)\right], \tag{2}$$

where $\xi(t)$ denotes the Gaussian noise with zero mean and standard deviation 1. The coefficient $0 \leq \alpha \leq 1$ characterizes the noise intensity; for $\alpha = 0$ the motion is deterministic.

Due to the inertia of the driver stochastic oscillations cannot change abruptly and some smoothness of $\omega(t)$ must take place. By this reason we propose the following model. We choose a number of time instants N_s

in which the motion is disturbed. For simplicity sake we assume that these instants are distributed uniformly over the whole interval $t \in [0, T]$ and

$$t_j = j\frac{T}{N_s},\ j = 1, 2, ..., N_s. \tag{3}$$

In view of (2) we can calculate $\omega(t)$ for the instant t_j. Making use of the cubic spline interpolation for the points t_j we find a stochastic realization for the modeled angular velocity $\tilde{\omega}(t)$. The corresponding modeled external force is

$$\tilde{f}(t) = f \cos\left[\tilde{\omega}(t), t\right]. \tag{4}$$

By integrating (1) with the aid of the Runge-Kutta method we find a stochastic realization for $x(t)$. By repeating this procedure ν times we calculate the mean $x_m(t)$ and standard deviation from the formulae

$$\left.\begin{array}{l} x_m(t) = \frac{1}{\nu}\sum_{i=1}^{\nu} x^{(i)}(t), \\ D[x(t)] = \frac{1}{\nu-1}\sum_{i=1}^{\nu}[x^{(i)}(t) - x_m(t)]^2, \\ \sigma = \sqrt{D(x(t))}. \end{array}\right\} \tag{5}$$

If N_s is a small number we have low-frequency stochastic excitations; if N_s is a relatively great number high-frequency excitations are obtained. The modeled angular velocity $\tilde{\omega}(t)$ and external force $\tilde{f}$ for $N_s = 3$ and $N_s = 20$ are plotted in Fig.1.

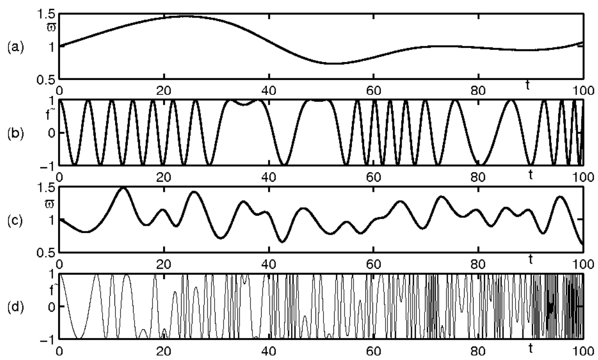

Figure 1. Modeled angular velocity $\tilde{\omega}$ and external force $\tilde{f}$: (a)-(b) for $N_s = 3$; (c)-(d) for $N_s = 20$.

3 Examples

For the first example the Duffing equation

$$\left.\begin{array}{l} \dot{x} = y, \\ \dot{y} = -py - qx + rx^3 + f\cos\omega t \end{array}\right\} \tag{6}$$

is taken. Here p, q, r, f, ω are the prescribed constants.

Computer simulation was carried out for $p = 0.25, q = -1, r = 1, s = 0.3, \omega_0 = 1, x(0) = 0, y(0) = 1$. In this case the system has two stabil fixed points $x = \pm 1, y = 0$, which are foci [Lepik, Hein, 2005]. The motion is chaotic. Computer results for the deterministic case $\alpha = 0$ are plotted in Fig. 2.

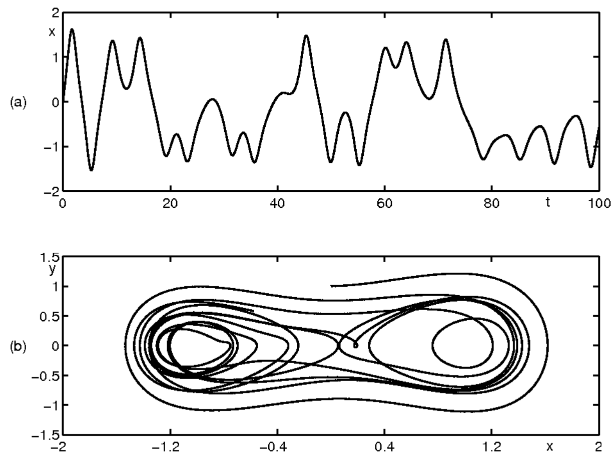

Figure 2. Duffing equation in the deterministic case: (a) time history; (b) phase diagram.

For the stochastic case the values $\alpha = 0.2$ and $N_s = 3$ or $N_s = 20$ were taken. The mean $x_m(t)$ and standard deviation $\sigma(t)$ were calculated from 20 stochastic realizations; the results are plotted in Fig. 3. It follows from this Figure that by adding noise the motion turns regular and is terminated in one of the foci $x = \pm 1, y = 0$.

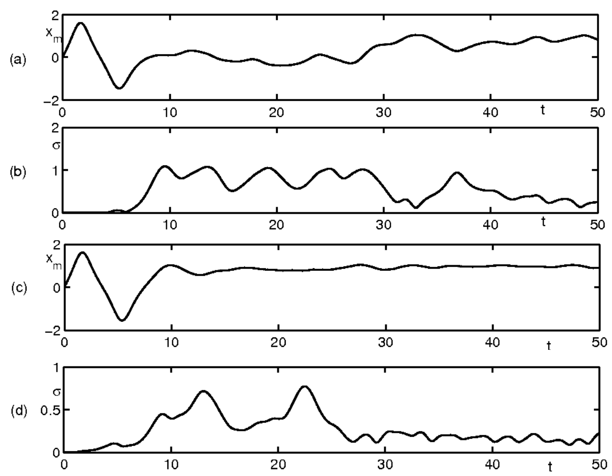

Figure 3. Duffing equation in the stochastic case: (a)-(b) for $N_s = 3$; (c)-(d) for $N_s = 20$.

Assuming $q = 0$ in (6) the Ueda equation is obtained. This equation has only one fixed point $x = y = 0$

which is a degenerated focus [Lepik, Hein, 2005]. Computer results for $p = 0.05, q = 0, r = 1, f = 7.5, \omega_0 = 1$ and for high-frequency excitations are plotted in Fig. 4. No convergence between different stochastic realizations is observed; the standard deviation σ also differs essentially from zero values.

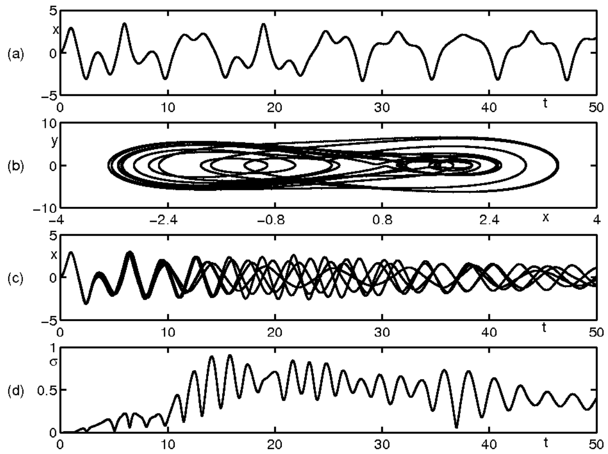

Figure 4. Ueda equation: (a)-(b) time history and phase diagram for the deterministic case; (c) stochastic realizations for x; (d) standard deviation.

For the next example let us consider the mathematical pendulum which is driven by the force $f cos\omega t$. The equations of motion can be put into the form [Lepik, Hein, 2005]:

$$\left.\begin{array}{l} \dot{x} = y, \\ \dot{y} = -sinx(1 + a\cos\omega t) - by \end{array}\right\} \qquad (7)$$

Fixed points of (7) are $x = k\pi, y = 0$ where k is an integer. It is shown in [Parker, Chua, 1989] that if k is an even number the fixed points are stable foci and saddle points if k is odd. Computations were carried out for $a = 8, b = 1, \omega_0 = 0.5\pi, x(0) = 2, y(0) = 0$; the results are plotted in Fig. 5. The deterministic motion is irregular, it consists of successive librations and rotations. The phase diagram has a rather complicated form. As to noisy motion (high frequency excitations were assumed) then it is very simple: the vibrations die away soon and the motion terminates in the focus $x = -2\pi$.

4 Establishing chaos by means of Lyapunov exponents

The computer simulations showed that after adding noise chaotic motion may turn regular. The question arises is this noisy motion chaotic or not. The answer can be given by applying the Lyapunov exponents: if the largest Lyapunov exponent is negative the motion is regular, otherwise it is chaotic.

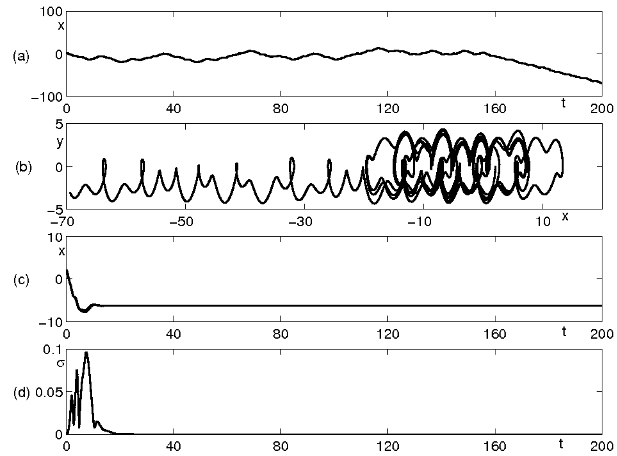

Figure 5. Driven pendulum (7): (a)-(b) deterministic case; (c)-(d) stochastic case for $\alpha = 0.2$.

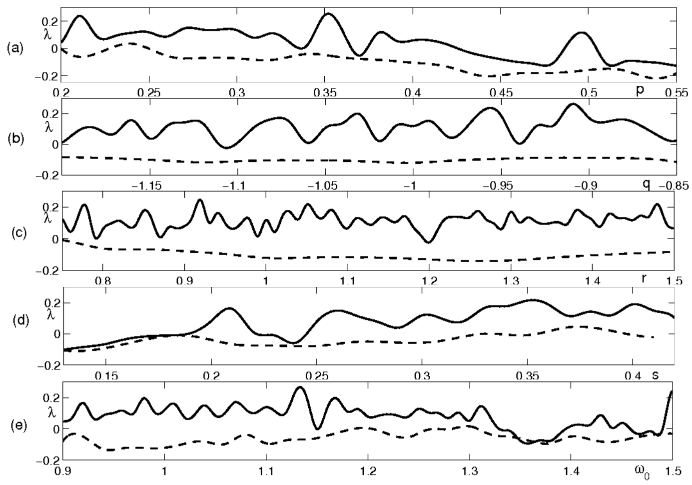

Figure 6. Lyapunov exponents versus parameters for the Duffing equation (6):— deterministic case, — — — stochastic case.

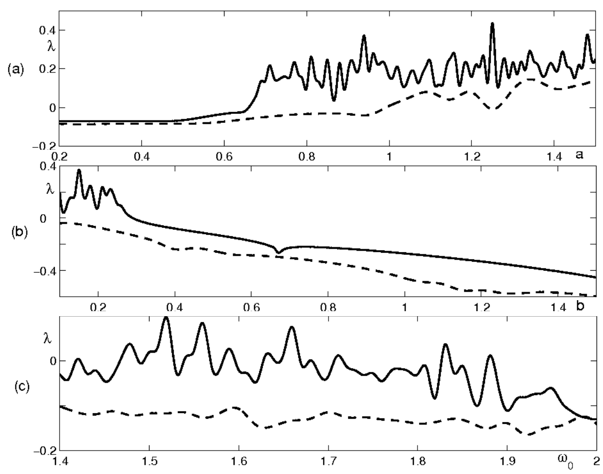

Figure 7. Lyapunov exponents versus parameters for the driven pendulum (7):— deterministic case, — — — stochastic case.

For calculating the largest Lyapunov exponent versus time we have used the Wolf's algorithm [Wolf, Swift, Swinney, Vastano, 1985]. Most interesting are the plots of Lyapunov exponents versus parameters in the equations (6) and (7). For getting such diagrams all system parameters except one are fixed. Calculations were carried out for $N_s = 20, \alpha = 0.2$; the largest Lyapunov exponent was calculated for $t = 50$ (Duffing equation) and for $t = 200$ (pendulum); a mean of 6 stochastic realizations was taken. The results are plotted in Figs 6 and 7.

It follows from these diagrams that in all cases the values of the largest Lyapunov exponents are considerably reduced. In most cases $\lambda < 0$ and the chaotic motion is annihilated (an exception is Fig 7a were the motion remains chaotic for $a > 1$).

5 Conclusion

Forced vibrations of dynamical systems for which the angular velocity is stochastic are investigated. Computer simulation shows that if the system has stabil fixed points then the chaotic motion is suppressed, becomes regular and is terminated in one of the fixed points. Similar results hold also for other dynamical equations discussed in [Lepik, Hein, 2005] and [Hein, Lepik, 2007].

Acknowledgements

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