Tüübituletus

- Church'i-stiilis λ-arvutustes sisaldavad λ-termid piisavalt tüübiannotatsioone ning tüübikontroll ja tüübituletus on lihtne.
- Curry-stiilis süsteemides on termid ilma tüübiannotatsioonideta ja tüübid on termi korrektsuse predikaadid.
- Üldjuhul on Curry-stiilis tüübisüsteemides tüübituletus (ja ka tüübikontroll) lahendamatud ülesanded.
  - Teist-järku λ-arvutus.
- Kuid lihtsamate süsteemide jaoks on tüübituletus võimalik.
  - Lihtsalt tüübitud λ-arvutus.
  - Hindley-Milner'i polümorfism.

Curry-stiilis lihtsalt tüübitud λ-arvutus

- Tüübid (sama kui $\lambda\!\!\rightarrow$ a'la Church):

$$\begin{array}{rcll} \tau & ::= & \alpha & \text{tüübimuutuja} \\ & \mid & \tau_1 \to \tau_2 & \text{funktsioonitüüp} \end{array}$$

- Termid (sama kui puhtas λ-arvutuses):

$$\begin{array}{rcll} e & ::= & x & \text{muutuja} \\ & \mid & e_1\ e_2 & \text{aplikatsioon} \\ & \mid & \lambda x.\, e & \text{abstraktsioon} \end{array}$$

- Tüüpimisreeglid:

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \qquad \frac{\Gamma, x : \sigma \vdash e : \tau}{\Gamma \vdash \lambda x.\, e : \sigma \to \tau}$$

$$\frac{\Gamma \vdash e_1 : \sigma \to \tau \quad \Gamma \vdash e_2 : \sigma}{\Gamma \vdash e_1\ e_2 : \tau}$$

Curry-stiilis lihtsalt tüübitud λ-arvutus

- Tähistused:
  - $S, S', \ldots$ tüübisubstitutsioonid
  - $\tau \succ \tau' \iff \exists S\,[\tau' = S(\tau)]$;
  - $\Gamma \succ \Gamma' \iff \exists S\,[\Gamma' \supseteq S(\Gamma)]$.
- Definitsioon: (Γ, τ) on termi e printsipiaalne paar parajasti siis, kui
  (i) $\Gamma \vdash e : \tau$;
  (ii) $\Gamma' \vdash e : \tau' \iff \Gamma \succ \Gamma' \;\wedge\; \tau \succ \tau'$.
- Kinnise termi e printsipiaalses paaris $(\emptyset, \tau)$ olevat tüüpi τ nimetatakse printsipiaalseks tüübiks.
- Teoreem: Iga tüübitava termi e jaoks leidub printsipiaalne paar (Γ, τ). See paar on unikaalne tüübimuutujate ümbernimetamise täpsuseni.

Tüübituletus $\lambda\!\rightarrow$ a'la Curry jaoks

Tüübituletuse algoritm:

- Annoteerida iga alamavaldis ja muutuja siduvesinemine unikaalse tüübimuutujaga.
- Genereeri kitsenduste süsteem kasutades järgnevaid reegleid:

$$\frac{x^\alpha \in \Gamma}{\Gamma \vdash x^\beta \Rightarrow \{\alpha = \beta\}} \qquad \frac{\Gamma, x^\alpha \vdash e^\beta \Rightarrow E}{\Gamma \vdash (\lambda x^\alpha.\, e^\beta)^\gamma \Rightarrow \{\gamma = \alpha \rightarrow \beta\} \cup E}$$

$$\frac{\Gamma \vdash e_1^\alpha \Rightarrow E_1 \qquad \Gamma \vdash e_2^\beta \Rightarrow E_2}{\Gamma \vdash (e_1^\alpha\, e_2^\beta)^\gamma \Rightarrow \{\alpha = \beta \rightarrow \gamma\} \cup E_1 \cup E_2}$$

- Lahenda kitsenduste süsteem leides selle kõige üldisema unifitseerija.
  - Kui seda ei leidu $\Longrightarrow$ term ei ole tüübitav.

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5}z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$\dfrac{\dfrac{}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\begin{array}{c}
\hline
x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \\
\hline
x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \\
\hline
x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \\
\hline
\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}
\end{array}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x^{\alpha_4}} \qquad \dfrac{}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Tüübituletus $\lambda\!\to$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x^{\alpha_4}} \qquad \dfrac{\dfrac{}{\Gamma \vdash y^{\alpha_5}} \qquad \dfrac{}{\Gamma \vdash z^{\alpha_6}}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{}{\Gamma \vdash y^{\alpha_5}} \qquad \dfrac{}{\Gamma \vdash z^{\alpha_6}}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Kitsendused:

$$E_1 \quad = \quad \{\alpha_1 = \alpha_4\}$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \quad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{}{\Gamma \vdash z^{\alpha_6}}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_4\} \\
E_2 & = & \{\alpha_2 = \alpha_5\}
\end{array}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \quad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_4\} \\
E_2 & = & \{\alpha_2 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_6\}
\end{array}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_4\} \\
E_2 & = & \{\alpha_2 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_6\} \\
E_4 & = & \{\alpha_5 = \alpha_6 \rightarrow \beta_1\} \\
 & & \quad \cup\ E_2 \cup E_3
\end{array}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \quad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_4\} \\
E_2 & = & \{\alpha_2 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_6\} \\
E_4 & = & \{\alpha_5 = \alpha_6 \rightarrow \beta_1\} \\
 & & \quad \cup\ E_2 \cup E_3
\end{array}
\qquad
E_5 \quad = \quad \{\alpha_4 = \beta_1 \rightarrow \beta_2\} \cup E_1 \cup E_4
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_4\} \\
E_2 & = & \{\alpha_2 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_6\} \\
E_4 & = & \{\alpha_5 = \alpha_6 \rightarrow \beta_1\} \\
 & & \quad \cup\ E_2 \cup E_3
\end{array}
\qquad
\begin{array}{lcl}
E_5 & = & \{\alpha_4 = \beta_1 \rightarrow \beta_2\} \cup E_1 \cup E_4 \\
E_6 & = & \{\gamma_1 = \alpha_3 \rightarrow \beta_2\} \cup E_5
\end{array}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_4\} \\
E_2 & = & \{\alpha_2 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_6\} \\
E_4 & = & \{\alpha_5 = \alpha_6 \rightarrow \beta_1\} \\
 & & \quad \cup\ E_2 \cup E_3
\end{array}
\qquad
\begin{array}{lcl}
E_5 & = & \{\alpha_4 = \beta_1 \rightarrow \beta_2\} \cup E_1 \cup E_4 \\
E_6 & = & \{\gamma_1 = \alpha_3 \rightarrow \beta_2\} \cup E_5 \\
E_7 & = & \{\gamma_2 = \alpha_2 \rightarrow \gamma_1\} \cup E_6
\end{array}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_4\} \\
E_2 & = & \{\alpha_2 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_6\} \\
E_4 & = & \{\alpha_5 = \alpha_6 \rightarrow \beta_1\} \\
 & & \quad \cup\ E_2 \cup E_3
\end{array}
\qquad
\begin{array}{lcl}
E_5 & = & \{\alpha_4 = \beta_1 \rightarrow \beta_2\} \cup E_1 \cup E_4 \\
E_6 & = & \{\gamma_1 = \alpha_3 \rightarrow \beta_2\} \cup E_5 \\
E_7 & = & \{\gamma_2 = \alpha_2 \rightarrow \gamma_1\} \cup E_6 \\
E_8 & = & \{\gamma_3 = \alpha_1 \rightarrow \gamma_2\} \cup E_7
\end{array}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Kitsendused:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \rightarrow \beta_1,\ \alpha_4 = \beta_1 \rightarrow \beta_2, \\
& \gamma_1 = \alpha_3 \rightarrow \beta_2, \\
& \gamma_2 = \alpha_2 \rightarrow \gamma_1, \\
& \gamma_3 = \alpha_1 \rightarrow \gamma_2\}
\end{aligned}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Kitsendused:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \rightarrow \beta_1,\ \alpha_4 = \beta_1 \rightarrow \beta_2, \\
& \gamma_1 = \alpha_3 \rightarrow \beta_2, \\
& \gamma_2 = \alpha_2 \rightarrow \gamma_1, \\
& \gamma_3 = \alpha_4 \rightarrow \gamma_2\}
\end{aligned}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Kitsendused:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \rightarrow \beta_1,\ \alpha_4 = \beta_1 \rightarrow \beta_2, \\
& \gamma_1 = \alpha_3 \rightarrow \beta_2, \\
& \gamma_2 = \alpha_5 \rightarrow \gamma_1, \\
& \gamma_3 = \alpha_4 \rightarrow \gamma_2\}
\end{aligned}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\begin{array}{c}
\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4} \\
\hline
x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5 \\
\hline
x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6 \\
\hline
x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7 \\
\hline
\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8
\end{array}
$$

Kitsendused:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \rightarrow \beta_1,\ \alpha_4 = \beta_1 \rightarrow \beta_2, \\
& \gamma_1 = \alpha_6 \rightarrow \beta_2, \\
& \gamma_2 = \alpha_5 \rightarrow \gamma_1, \\
& \gamma_3 = \alpha_4 \rightarrow \gamma_2\}
\end{aligned}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Kitsendused:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \rightarrow \beta_1,\ \alpha_4 = \beta_1 \rightarrow \beta_2, \\
& \gamma_1 = \alpha_6 \rightarrow \beta_2, \\
& \gamma_2 = (\alpha_6 \rightarrow \beta_1) \rightarrow \gamma_1, \\
& \gamma_3 = \alpha_4 \rightarrow \gamma_2\}
\end{aligned}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\begin{array}{c}
\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4} \\
\hline
x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5 \\
\hline
x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6 \\
\hline
x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7 \\
\hline
\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8
\end{array}
$$

Kitsendused:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \rightarrow \beta_1,\ \alpha_4 = \beta_1 \rightarrow \beta_2, \\
& \gamma_1 = \alpha_6 \rightarrow \beta_2, \\
& \gamma_2 = (\alpha_6 \rightarrow \beta_1) \rightarrow \gamma_1, \\
& \gamma_3 = (\beta_1 \rightarrow \beta_2) \rightarrow \gamma_2\}
\end{aligned}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Kitsendused:

$$
\begin{aligned}
E_8 \quad = \quad \{&\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6,\\
&\alpha_5 = \alpha_6 \rightarrow \beta_1,\ \alpha_4 = \beta_1 \rightarrow \beta_2,\\
&\gamma_1 = \alpha_6 \rightarrow \beta_2,\\
&\gamma_2 = (\alpha_6 \rightarrow \beta_1) \rightarrow \alpha_6 \rightarrow \beta_2,\\
&\gamma_3 = (\beta_1 \rightarrow \beta_2) \rightarrow \gamma_2\}
\end{aligned}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \quad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Kitsendused:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \rightarrow \beta_1,\ \alpha_4 = \beta_1 \rightarrow \beta_2, \\
& \gamma_1 = \alpha_6 \rightarrow \beta_2, \\
& \gamma_2 = (\alpha_6 \rightarrow \beta_1) \rightarrow \alpha_6 \rightarrow \beta_2, \\
& \gamma_3 = (\beta_1 \rightarrow \beta_2) \rightarrow (\alpha_6 \rightarrow \beta_1) \rightarrow \alpha_6 \rightarrow \beta_2\}
\end{aligned}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \quad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Tuletatud tüüp:

$$\gamma_3 \quad = \quad (\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$$

Võrrandite lahendamine unifitseerimisega

- Võrrandeid saab lahendada lihtsustusreeglite korduva rakendamise kaudu.
- Kaks peamist reeglit:
  - asenda võrdus kujul $\tau_1 \to \tau_2 = \tau_3 \to \tau_4$ kahe võrdusega $\tau_1 = \tau_3$ ja $\tau_2 = \tau_4$;
  - olgu võrdus kujul $\alpha = \tau$. Kui $\alpha \in \mathrm{FV}(\tau)$ siis raporteeri viga, vastasel korral asenda kõigis võrdustes α asemel τ.
- Abireeglid:
  - eemalda reeglid kujul $\alpha = \alpha$, $\mathrm{Bool} = \mathrm{Bool}$, jne.;
  - asenda $\tau = \alpha$ võrdusega $\alpha = \tau$;
  - kui leidub võrdus $\tau_1 = \tau_2$, kus peatüübikonstruktorid on erinevad (näit. $\mathrm{Bool} = \alpha_1 \to \alpha_2$), siis raprorteeri viga.

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\frac{}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}} \qquad \dfrac{}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6}}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{}{x^{\alpha_1} \vdash x^{\alpha_2}}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}} \qquad \dfrac{\dfrac{}{x^{\alpha_4} \vdash x^{\alpha_5}}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6}}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6}}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_2\} \\
E_2 & = & \{\alpha_4 = \alpha_5\}
\end{array}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}$$

Kitsendused:

$$\begin{array}{lcl} E_1 & = & \{\alpha_1 = \alpha_2\} \\ E_2 & = & \{\alpha_4 = \alpha_5\} \\ E_3 & = & \{\alpha_3 = \alpha_1 \rightarrow \alpha_2\} \cup E_1 \\ E_4 & = & \{\alpha_6 = \alpha_4 \rightarrow \alpha_5\} \cup E_2 \end{array}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_2\} \\
E_2 & = & \{\alpha_4 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_1 \rightarrow \alpha_2\} \cup E_1 \\
E_4 & = & \{\alpha_6 = \alpha_4 \rightarrow \alpha_5\} \cup E_2 \\
E_5 & = & \{\alpha_3 = \alpha_6 \rightarrow \alpha_7\} \cup E_3 \cup E_4
\end{array}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}$$

Kitsendused:

$$\begin{aligned} E_5 \quad = \quad & \{\alpha_1 = \alpha_2,\ \alpha_4 = \alpha_5, \\ & \alpha_3 = \alpha_1 \rightarrow \alpha_2, \\ & \alpha_6 = \alpha_4 \rightarrow \alpha_5, \\ & \alpha_3 = \alpha_6 \rightarrow \alpha_7\} \end{aligned}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}$$

Kitsendused:

$$\begin{aligned} E_5 \quad = \quad \{ \quad & \alpha_4 = \alpha_5, \\ & \alpha_3 = \alpha_2 \rightarrow \alpha_2, \\ & \alpha_6 = \alpha_4 \rightarrow \alpha_5, \\ & \alpha_3 = \alpha_6 \rightarrow \alpha_7 \} \end{aligned}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{
\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3}
\qquad
\dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}
}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Kitsendused:

$$
\begin{aligned}
E_5 \quad = \quad \{ & \\
& \alpha_3 = \alpha_2 \rightarrow \alpha_2, \\
& \alpha_6 = \alpha_5 \rightarrow \alpha_5, \\
& \alpha_3 = \alpha_6 \rightarrow \alpha_7 \}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{
\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3}
\qquad
\dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}
}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Kitsendused:

$$
\begin{aligned}
E_5 \quad = \quad \{ & \\
& \alpha_6 = \alpha_5 \rightarrow \alpha_5, \\
& \alpha_2 \rightarrow \alpha_2 = \alpha_6 \rightarrow \alpha_7 \}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}$$

Kitsendused:

$$\begin{aligned} E_5 \quad = \quad \{ & \\ & \alpha_2 = \alpha_6, \\ & \alpha_6 = \alpha_5 \rightarrow \alpha_5, \\ & \alpha_2 = \alpha_7 \} \end{aligned}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}$$

Kitsendused:

$$\begin{aligned} E_5 \quad = \quad \{ & \\ & \alpha_2 = \alpha_5 \rightarrow \alpha_5, \\ & \alpha_2 = \alpha_7 \} \end{aligned}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{
\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3}
\qquad
\dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}
}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Kitsendused:

$$
E_5 = \{ \alpha_7 = \alpha_5 \rightarrow \alpha_5 \}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$\frac{}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$\dfrac{\dfrac{}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4}}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{}{x^{\alpha_1} \vdash x^{\alpha_2}} \qquad \dfrac{}{x^{\alpha_1} \vdash x^{\alpha_3}}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4}}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}
$$

Tüübituletus $\lambda\!\rightarrow$ a'la Curry jaoks

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4}}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}$$

Kitsendused:

$$\begin{array}{lcl} E_1 & = & \{\alpha_1 = \alpha_2\} \\ E_2 & = & \{\alpha_1 = \alpha_3\} \end{array}$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}$$

Kitsendused:

$$\begin{array}{lcl} E_1 & = & \{\alpha_1 = \alpha_2\} \\ E_2 & = & \{\alpha_1 = \alpha_3\} \\ E_3 & = & \{\alpha_2 = \alpha_3 \rightarrow \alpha_4\} \cup E_1 \cup E_2 \end{array}$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}
$$

Kitsendused:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_2\} \\
E_2 & = & \{\alpha_1 = \alpha_3\} \\
E_3 & = & \{\alpha_2 = \alpha_3 \rightarrow \alpha_4\} \cup E_1 \cup E_2 \\
E_4 & = & \{\alpha_5 = \alpha_1 \rightarrow \alpha_4\} \cup E_3
\end{array}
$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}
$$

Kitsendused:

$$
\begin{array}{rcl}
E_4 & = & \{\alpha_1 = \alpha_2,\ \alpha_1 = \alpha_3, \\
 & & \ \alpha_2 = \alpha_3 \rightarrow \alpha_4, \\
 & & \ \alpha_5 = \alpha_1 \rightarrow \alpha_4 \\
 & & \}
\end{array}
$$

Tüübituletus $\lambda\!\rightarrow$ a'la Curry jaoks

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}$$

Kitsendused:

$$\begin{array}{rcl} E_4 & = & \{ \quad \alpha_2 = \alpha_3, \\ & & \alpha_2 = \alpha_3 \rightarrow \alpha_4, \\ & & \alpha_5 = \alpha_2 \rightarrow \alpha_4 \\ & & \} \end{array}$$

Tüübituletus $\lambda\!\!\rightarrow$ a'la Curry jaoks

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}$$

Kitsendused:

$$\begin{array}{lcl} E_4 & = & \{ \\ & & \quad \alpha_3 = \alpha_3 \rightarrow \alpha_4, \\ & & \quad \alpha_5 = \alpha_3 \rightarrow \alpha_4 \\ & & \} \end{array}$$

Tüübituletus $\lambda\!\rightarrow$ a'la Curry jaoks

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}$$

Kitsendused:

$$\begin{array}{lcl} E_4 & = & \{ \\ & & \quad \alpha_3 = \alpha_3 \rightarrow \alpha_4, \\ & & \quad \alpha_5 = \alpha_3 \rightarrow \alpha_4 \\ & & \} \end{array}$$

Error!

Tüübituletus

- Väga suure tüübiga term:

$$
\begin{aligned}
&\text{let } pair = \lambda xyz.z\ x\ y \text{ in}\\
&\text{let } x_1 = \lambda y.pair\ y\ y \text{ in}\\
&\text{let } x_2 = \lambda y.x_1(x_1\ y) \text{ in}\\
&\text{let } x_3 = \lambda y.x_2(x_2\ y) \text{ in}\\
&\text{let } x_4 = \lambda y.x_3(x_3\ y) \text{ in}\\
&\text{let } x_5 = \lambda y.x_4(x_4\ y) \text{ in}\\
&x_5(\lambda y.y)
\end{aligned}
$$

Motivatsioon

Näide:
Pilditöötlusprogramm, mis rakendab pildile (baitide massiiv) mingit filtrit.

- Puhtal FP-l on eeliseid aga peame muutmisel tegema koopiaid.
- Lahendus 1: Modelleerime programmi puhaste arvutuste ja massiivimuudatuste jadana. (sarnaselt IO-le)
  - Puhta FP eelised kaovad ära.
- Lahendus 2: Kitsendame keelt nii, et "vana" väärtust ei tohi kasutada ja kopeerimise asemel hoopis muudame väärtust.

Tehniline lahendus (2): kvantitatiivne tüübiteooria, ehk muutujatele lisatakse info, mis määral seda tohib kasutada.

Kvantitatiivne tüübiteooria

- Kolm arvu: *null*, *üks* ja *suva*.
- Arve saab kirjutada let-i muutuja ja funktsiooni tüübi parameetri ette. Näiteks `(1 x : Int) → Bool`.
- let-i muutuja arvu püüab Idris tuletada.
- Vaikimisi on arvuks *suva*.
- Arvud on ainult vahend. API looja vastutab turvalisuse eest.

Null

- Null arvuga muutujate ei tohi/saa "kasutada".
- S.t. seda saab kasutada vaid argumendis null arvuga parameetrile.
- Null arvuga parameetrid kustutatakse kompileerimisel.

Näide:

```
test0 : (0 _: Int) → Int
test0 x =
    let 0 y : Int = x + 10 in
    100
```

NB! Implitsiitsed (tüübi)parameetrid on vaikimisi arvuga null.
Näiteks `a` tüübis `a → a`.

Üks

- Üks arvuga muutujat peab täpselt üks kord kasutama.
- S.t. lisaks ühele kasutuskorrale suvaline arv null-kasutuskordi.
- Üks arvuga argumendi puhul on ka tagastusväätrus üks-arvuga.

Näide:

```
test1 : (1 _: Int) → Int
test1 x =
    let 0 y : Int = test0 x + 10 in
    let 1 z : Int = test1 x in  -- lõpmatu tsükkel
    let 0 y : Int = z in
    z
```

Suva

- Suva arvuga muutujat saab suvaline arv kordi kasutada.
- S.h. null või üks parameetri argumendina.
- Suva arvuga argumentide puhul on ka tagastusväätrus suva-arvuga.

Näide:

```
tests : Int → Int
tests x =
    let y : Int = test0 x in
    let z : Int = test1 x in
    z + z
```

Lineaarsus

Kirjanduses kasutatakse *lineaarsuse* mõistet.

Meie kontekstis:

- `x`-ga tehakse kõigepealt teisendus `f`, siis `g` ja siis `h`. Ehk `g (f x)`.
- Välistab, et väärtust `x` ja `f x` tulemust peaks korraga mälus hoidma.

Turvalisus

- Mustrisobitus on üks "kasutus", ehk kui andmestruktuuri sees pole lineaarset resurssi, siis saame lineaarsust rikkuda.

```
testb : (1 _ : Bool) → Int
testb x =
    case x of
        True  ⇒ 1
        False ⇒ 2
```

- API looja ülesanne on, et lineaarset resurssi saaks turvaliselt kasutada.
- Enamasti on arvuga üks:
  - abstraktsed tüübid või liidesed,
  - spetsiaalsed lineaarsed konteinerid.

Näide: lineaarsed massiivid

```
infix 5 #

public export
data L : Type → Type → Type where
    (#) : (1 _ : a) → b → L a b

export
interface Array arr where
  write   : (1 a : arr) → Nat → Double → arr
  read    : (1 a : arr) → Nat → L arr Double
  size    : (1 a : arr) → L arr Nat
  withArray : Nat → ((1 a : arr) → L arr c) → c

export
Array (List Double) where
    ...

export
Array LinArray where        -- LinArray "private"
    ...
```

Näide: testimiseks listid

```
export
Array (List Double) where
    write [] n d = []
    write (x :: xs) 0 d = d :: xs
    write (x :: xs) (S k) d = x :: write xs k d

    read [] n = [] # 0.0
    read (x :: xs) 0  = (x::xs) # x
    read (x :: xs) (S n) =
        let _ # r = read xs n in
        x::xs # r

    size []        = [] # 0
    size (x :: xs) = (x :: xs) # (length (x :: xs))

    withArray l f =
        let _ # r = f (replicate l 0.0) in
        r
```

Näide: kompileerimiseks LinArray-d

```
export
data LinArray = MkLinArray (IOArray Double)

private
newIOArr : List Double → IO (IOArray Double)
newIOArr xs = ...

export
Array LinArray where
    withArray n f =
        unsafePerformIO (do a ← newArray (cast n)
                            let (_ # r) = f (MkLinArray a)
                            pure r )

    size (MkLinArray a) = MkLinArray a # (cast (max a))

    write (MkLinArray a) i v =
        unsafePerformIO (do ok ← writeArray a (cast i) v
                            pure (MkLinArray a))
    read (MkLinArray a) i =
        unsafePerformIO (do r ← readArray a (cast i)
                            case r of
                                Just v  ⇒ pure (MkLinArray a # v)
                                Nothing ⇒ pure (MkLinArray a # 0))
```