Canonical variational completion of 4D Gauss-Bonnet gravity

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Outline

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Gauss-Bonnet gravity

- Action for Gauss-Bonnet gravity in D dimensions:

$$S = \int d^D x \sqrt{-g} \left[\frac{M_P^2}{2} R - \Lambda_0 + \alpha \mathcal{G} \right] + S_m \,. \tag{1}$$

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$$\begin{aligned} \mathcal{G}_{\mu\nu} &= 15 g_{\mu[\nu} R^{\rho\sigma}{}_{\rho\sigma} R^{\omega\tau}{}_{\omega\tau]} \\ &= \frac{1}{2} \mathcal{G} g_{\mu\nu} - 2 R_{\mu\lambda\rho\sigma} R_\nu{}^{\lambda\rho\sigma} + 4 R_{\mu\rho\nu\sigma} R^{\rho\sigma} + 4 R_{\mu\rho} R_\nu{}^\rho - 2 R R_{\mu\nu} \,. \end{aligned} \tag{4}$$

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↯ Gauss-Bonnet contribution vanishes identically in $D = 4$.

4D Gauss-Bonnet gravity?

- Renormalization of Gauss-Bonnet term [Glavan, Lin '20]:

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⇒ Non-vanishing contribution to solutions even in $D = 4$.

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? Try to find correction term to make them variational.

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- Consider system of partial differential equations (PDEs):

$$\mathcal{E}_A(x^\mu, y^B, y^B{}_\mu, y^B{}_{\mu\nu}) = 0\,. \tag{11}$$

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⇒ Idea of *canonical variational completion* [Voicu, Krupka '15]:
- Original system $\mathcal{E}_A$ is variational if and only if $\mathcal{E}_A = \tilde{\mathcal{E}}_A$.
- Otherwise, $H_A = \tilde{\mathcal{E}}_A - \mathcal{E}_A$ is canonical correction term.

Example: Einstein-Hilbert Lagrangian

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5. Completed field equations:
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↯ Note term $\propto t^{D/2-3}$ yielding factor $(D/2-2)^{-1} = 2(D-4)^{-1}$.

Consistency check: full field equations $E_{\mu\nu}$ with $W_{\mu\nu}$

- Calculate traces:

$$A^{\mu}{}_{\mu} = \frac{D-3}{D-2}\left(2R_{\mu\nu}R^{\mu\nu} - \frac{DR^2}{2(D-1)}\right)\,, \tag{22a}$$

$$W^{\mu}{}_{\mu} = (D-4)\left(\frac{2R_{\mu\nu}R^{\mu\nu}}{D-2} - \frac{R^2}{(D-1)(D-2)} - \frac{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}}{2}\right)\,. \tag{22b}$$

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⇒ Note that traces satisfy:

$$A^{\mu}{}_{\mu} + \frac{W^{\mu}{}_{\mu}}{D-4} = -\frac{1}{2}\mathcal{G}. \tag{23}$$

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Consistency check: full field equations $E_{\mu\nu}$ with $W_{\mu\nu}$

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↯ No variational completion for $D = 4$ Gauss-Bonnet gravity.

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Further reading

MH, C. Pfeifer and N. Voicu,
"Canonical variational completion of 4D Gauss-Bonnet gravity",
arXiv:2009.05459 [gr-qc].