Asymptotically flat spacetimes and their symmetries

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1 Asymptotic flatness

In a few previous lectures we have considered the gravitational waves emitted by what we described as an *isolated system*, i.e., a system whose matter content is confined to a compact spatial region, and we have assumed that the spacetime is *asymptotically flat*, i.e., it "approaches Minkowski space far from the source" or even "at infinity". We will now make this statement precise and give a proper definition of such spacetimes. In the following, we will denote by $\tilde{M}$ the physical spacetime, equipped with a physical metric $\tilde{g}_{\mu\nu}$. To describe the geometry "at the infinity" of $\tilde{M}$, we need to bring this infinity to some finite manifold. In the following, we define our spacetime to be asymptotically flat if the following is true:

1. There exists a smooth, injective mapping $\psi : \tilde{M} \to M$ into some manifold M, such that the image $\psi(\tilde{M}) \subset M$ of $\tilde{M}$ under ψ is diffeomorphic to $\tilde{M}$ and has a non-vanishing boundary $I = \partial(\psi(\tilde{M})) \subset M$.
2. M is equipped with a Lorentzian metric $g_{\mu\nu}$ and a real function Ω such that:
   (a) $\Omega > 0$ on the image $\psi(\tilde{M})$.
   (b) On the boundary I, one has $\Omega = 0$, $\partial_\mu\Omega \neq 0$ and $g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega = 0$.
   (c) The physical metric is given by the pullback of the conformally rescaled metric $\tilde{g} = \psi^*(\Omega^{-2}g)$.

In the following, we will use the abbreviation

$$n_\mu = \partial_\mu\Omega\,. \tag{1.1}$$

Note that the objects given in the definition above are not uniquely defined. Given a positive function $\omega : M \to \mathbb{R}^+$, one may define

$$\Omega' = \omega\Omega\,, \quad g'_{\mu\nu} = \omega^2 g_{\mu\nu}\,, \tag{1.2}$$

from which further follows

$$n'_\mu = \omega n_\mu\,, \quad n'^\mu = \omega^{-1}n^\mu\,, \tag{1.3}$$

and one easily checks that Ω' and $g'_{\mu\nu}$ will satisfy the same properties given above for Ω and $g_{\mu\nu}$.

2 Vacuum Einstein equations

In the following, we assume that the Einstein equations of general relativity hold for the physical metric $\tilde{g}_{\mu\nu}$, and that the matter source is confined to a compact region within the physical spacetime $\tilde{M}$. This means in particular that near the boundary, the rescaled unphysical metric $\Omega^{-2}g_{\mu\nu}$ satisfies the vacuum Einstein equations, i.e., it has vanishing Ricci tensor, which is given by

$$0 = \tilde{R}_{\mu\nu} = R_{\mu\nu} + 2\frac{\nabla_\mu\nabla_\nu\Omega}{\Omega} + g_{\mu\nu}\left(\frac{\Box\Omega}{\Omega} - 3\frac{\nabla_\rho\Omega\nabla^\rho\Omega}{\Omega^2}\right)\,. \tag{2.1}$$

Taking the trace with the unphysical metric $g_{\mu\nu}$ yields the relation

$$0 = R + 6\frac{\Box\Omega}{\Omega} - 12\frac{\nabla_\rho\Omega\nabla^\rho\Omega}{\Omega^2}\,, \tag{2.2}$$

which allows us to rewrite the vacuum Einstein equations as

$$0 = R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu} + 2\frac{\nabla_\mu\nabla_\nu\Omega}{\Omega} - g_{\mu\nu}\frac{\nabla_\rho\Omega\nabla^\rho\Omega}{\Omega^2}\,. \tag{2.3}$$

In the following we will multiply this equation by Ω, and it follows from the smoothness of Ω that also the resulting equation

$$0 = \Omega\left(R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu}\right) + 2\nabla_\mu\nabla_\nu\Omega - g_{\mu\nu}\frac{\nabla_\rho\Omega\nabla^\rho\Omega}{\Omega} \tag{2.4}$$

holds. Now the first term in brackets is a smooth tensor field on M, which follows from the fact that $g_{\mu\nu}$ is a smooth metric on M. Hence, its product with Ω, which vanishes at the boundary I, also vanishes at the boundary. This means that on I we have

$$0 = 2\nabla_\mu\nabla_\nu\Omega - g_{\mu\nu}\frac{\nabla_\rho\Omega\nabla^\rho\Omega}{\Omega} = (\mathcal{L}_n g)_{\mu\nu} - \frac{n_\rho n^\rho}{\Omega}g_{\mu\nu}\,, \tag{2.5}$$

using the definition of n_μ and the Lie derivative of a metric. Now the first term is a smooth tensor field on M, and so the same must also be true for the second term, and hence for the function $f = \Omega^{-1}n_\mu n^\nu$. We keep this relation in mind when we further discuss the properties of these tensor fields on the boundary I.

3 Tensor fields at the boundary

In the following, it turns out to be useful to develop a description of the asymptotic geometry which is intrinsic to the boundary. For this purpose, one considers a diffeomorphism $\zeta : \mathscr{I} \to I \subset M$, so that the manifold $\mathscr{I}$ is diffeomorphic to the boundary I. Then one constructs a number of tensor fields on $\mathscr{I}$, which will be used to model the asymptotic geometry. First, note that by construction, the vector field $n^\mu\partial_\mu$ defined by

$$n^\mu = g^{\mu\nu}n_\nu = g^{\mu\nu}\partial_\nu\Omega \tag{3.1}$$

is tangent to I. This follows from

$$n^\mu\partial_\mu\Omega = n^\mu n_\mu = g^{\mu\nu}n_\mu n_\nu = 0\,, \tag{3.2}$$

since n_μ is null on I by assumption. Hence, its restriction to I gives rise to a vector field $\mathfrak{n}^a\partial_a$ on $\mathscr{I}$, where we use Latin indices for coordinates on the three-dimensional manifold $\mathscr{I}$. Further, the pullback along ζ defines a metric on $\mathscr{I}$, given by

$$\mathfrak{g}_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b = \zeta^*(g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu)\,. \tag{3.3}$$

Note that this metric is degenerate, since by construction it satisfies $\mathfrak{n}^a\mathfrak{g}_{ab} = 0$. Hence, $\mathfrak{n}^a$ defines the non-trivial kernel of $\mathfrak{g}_{ab}$. In total, one finds that the signature of this metric is $(0,+,+)$. Further, from the relation (2.5) follows that

$$(\mathcal{L}_\mathfrak{n}\mathfrak{g})_{ab} = \mathfrak{f}\mathfrak{g}_{ab}\,, \tag{3.4}$$

and so $\mathfrak{n}^a$ is a conformal Killing vector field of $\mathfrak{g}_{ab}$, where $\mathfrak{f} = f \circ \zeta$. Finally, recalling that one can obtain an equivalent description of the same asymptotic geometry via the transformations (1.2) and (1.3), we see that two pairs $(\mathfrak{g}_{ab}, \mathfrak{n}^a)$ and $(\mathfrak{g}'_{ab}, \mathfrak{n}'^a)$ define the same asymptotic geometry if there exists a function $\omega : \mathscr{I} \to \mathbb{R}^+$ such that

$$\mathfrak{g}'_{ab} = \omega^2\mathfrak{g}_{ab}\,, \quad \mathfrak{n}'^a = \omega^{-1}\mathfrak{n}^a\,. \tag{3.5}$$

Hence, we will define an asymptotic geometry as an equivalence class of pairs $(\mathfrak{g}_{ab}, \mathfrak{n}^a)$, where $\mathfrak{g}_{ab}$ is of signature $(0,+,+)$, $\mathfrak{n}^a\mathfrak{g}_{ab} = 0$ and two pairs $(\mathfrak{g}_{ab}, \mathfrak{n}^a)$ and $(\mathfrak{g}'_{ab}, \mathfrak{n}'^a)$ are equivalent if and only if there exists a positive function ω such that (3.5) holds. This gauge freedom under conformal transformations can be taken into account in different ways. One possibility is to describe the tensor fields on $\mathscr{I}$ in terms of a tensor field

$$\Xi^{ab}{}_{cd} = \mathfrak{n}^a\mathfrak{n}^b\mathfrak{g}_{cd}\,, \tag{3.6}$$

which is clearly invariant under the transformation (3.5), and thus depends only on the aforementioned equivalence class and is uniquely defined by it. We may thus use $\Xi^{ab}{}_{cd}$ in order to describe the asymptotic geometry. However, note that not any arbitrary $(2,2)$ tensor field on $\mathscr{I}$ can be written in the form given above, but one must impose the following conditions:

1. Non-degeneracy: $\Xi \neq 0$.
2. Symmetry in upper and lower indices: $\Xi^{ab}{}_{cd} = \Xi^{(ab)}{}_{(cd)}$.
3. Exchange of $\mathfrak{n}^a$: $\Xi^{a[b}{}_{cd}\Xi^{e]f}{}_{gh} = 0$.
4. Orthogonality of $\mathfrak{n}^a$ and $\mathfrak{g}_{ab}$: $\Xi^{ab}{}_{cb} = 0$.
5. Signature of $\mathfrak{g}_{ab}$: $\mathfrak{w}_a\mathfrak{w}_b\mathfrak{v}^c\mathfrak{v}^d\Xi^{ab}{}_{cd} > 0$ for all $\mathfrak{v}^a, \mathfrak{w}_a$ satisfying $\mathfrak{w}_c\mathfrak{v}^{[a}\Xi^{b]c}{}_{de} \neq 0$.
6. Conformal Killing property: $(\mathcal{L}_{\mathfrak{v}}\Xi)^{ab}{}_{cd} = \mathfrak{f}\Xi^{ab}{}_{cd}$ for some function $\mathfrak{f}$ for all $\mathfrak{v}^a$ satisfying $\mathfrak{v}^{[a}\Xi^{b]c}{}_{de} = 0$.

Any tensor $\Xi^{ab}{}_{cd}$ uniquely defines an equivalence class of pairs $(\mathfrak{g}_{ab}, \mathfrak{n}^a)$ and vice versa. To make a convenient choice for a representative of this class, we recall the relation (3.4). Under the conformal transformation (3.5), this relation is preserved, provided that we replace $\mathfrak{f}$ by

$$\mathfrak{f}' = \omega^{-1}\mathfrak{f} + 2\omega^{-2}\mathcal{L}_{\mathfrak{n}}\omega\,. \tag{3.7}$$

Starting from an arbitrary representative, we can always find $(\mathfrak{g}_{ab}, \mathfrak{n}^a)$ such that $\mathfrak{f} = 0$ by applying a suitable conformal transformation (at least locally). In the following we will assume that we have chosen a representative for which this is the case. Note that this does not fix the representative uniquely, since one may still perform conformal transformations satisfying $\mathcal{L}_{\mathfrak{n}}\omega = 0$, i.e., such that ω is constant along the flow lines of $\mathfrak{n}^a$.

4 Projection of the boundary

We denote by B the space of all maximally extended integral curves of $\mathfrak{n}^a$ on $\mathscr{I}$. Note that this space is independent of the choice of representatives, since these integral curves are preserved under conformal transformations (up to a change of parametrization). In the following, we will assume that B can be equipped with the structure of a smooth manifold, such that the projection $\pi : \mathscr{I} \to B$, which assigns to each point in $\mathscr{I}$ the unique integral curve passing through this point, is a smooth mapping. This assumption allows us to study geometric objects on B and their relation with geometric objects on $\mathscr{I}$. In particular, we can consider the metric $\mathfrak{g}_{ab}$ on $\mathscr{I}$. Recall that this metric is degenerate and that its kernel is given by the vectors proportional to $\mathfrak{n}^a$, i.e., the π-vertical vectors, whose pushforward along π vanishes. From this property, together with our choice $(\mathcal{L}_{\mathfrak{n}}\mathfrak{g})_{ab} = 0$, follows that the Lie derivative of $\mathfrak{g}_{ab}$ with respect to *any* π-vertical vector field, i.e., any multiple of $\mathfrak{n}^a$, vanishes. These two properties guarantee that there exists a metric γ_{AB} on B such that $\mathfrak{g} = \pi^*\gamma$, which is uniquely determined by this property and the choice of $\mathfrak{g}_{ab}$. However, keep in mind that $\mathfrak{g}_{ab}$ itself is not uniquely defined by the conditions we imposed in the previous section, but only up to a conformal rescaling by a function ω which is constant along the flow lines of $\mathfrak{n}^a$. Hence, we see that we have an analogous freedom of conformal transformations of the metric γ_{AB} on B, the pullback of which then yields the transformation ω on $\mathscr{I}$.

5 The Bondi-Metzner-Sachs algebra

We now come to the question which class of diffeomorphisms $\varphi : \mathscr{I} \to \mathscr{I}$ leaves the asymptotic geometry defined above in terms of $\Xi^{ab}{}_{cd}$ invariant, i.e., satisfy the condition

$$\varphi^*\Xi = \Xi\,. \tag{5.1}$$

We aim to express this condition in terms of a representative $(\mathfrak{g}_{ab}, \mathfrak{n}^a)$ with $(\mathcal{L}_{\mathfrak{n}}\mathfrak{g})_{ab} = 0$. Denoting the pullback by a prime, we have

$$\Xi'^{ab}{}_{cd} = \mathfrak{n}'^a\mathfrak{n}'^b\mathfrak{g}'_{cd}\,, \tag{5.2}$$

and so $\Xi'^{ab}{}_{cd} = \Xi^{ab}{}_{cd}$ if and only if

$$\mathfrak{g}'_{ab} = \mathfrak{k}^2\mathfrak{g}_{ab}\,, \quad \mathfrak{n}'^a = \mathfrak{k}^{-1}\mathfrak{n}^a \tag{5.3}$$

for some function $\mathfrak{k}$ on $\mathscr{I}$. We can restrict this function further by recalling that the Lie derivative is preserved under diffeomorphisms, so that

$$0 = (\mathcal{L}_{\mathfrak{n}'}\mathfrak{g}')_{ab} = 2(\mathcal{L}_{\mathfrak{n}}\mathfrak{k})\mathfrak{g}_{ab}\,, \tag{5.4}$$

and thus $\mathfrak{k}$ is necessarily constant along the integral curves of $\mathfrak{n}^a$.
For simplicity, we will restrict ourselves now to infinitesimal transformations, generated by vector fields ξ^a on $\mathscr{I}$. Such a vector field generates an infinitesimal symmetry if and only if

$$(\mathcal{L}_\xi \Xi)^{ab}{}_{cd} = 0\,, \tag{5.5}$$

or in other words, if and only if there exists a function $\kappa : \mathscr{I} \to \mathbb{R}$ such that

$$(\mathcal{L}_\xi \mathfrak{g})_{ab} = 2\kappa \mathfrak{g}_{ab}\,, \quad (\mathcal{L}_\xi \mathfrak{n})^a = -\kappa \mathfrak{n}^a\,, \tag{5.6}$$

where $\mathcal{L}_{\mathfrak{n}}\kappa = 0$. Our aim is to find a simple parametrization for these vector fields, which will constitute the Lie algebra $\mathfrak{L}$ of infinitesimal asymptotic symmetries. We will do so in two steps, and show how the algebra can be decomposed into two parts.
Consider first a vector field ξ^a on $\mathscr{I}$ given by

$$\xi^a = \mathfrak{a}\mathfrak{n}^a \tag{5.7}$$

for some arbitrary function $\mathfrak{a}$ on $\mathscr{I}$. One then easily calculates the Lie derivatives

$$(\mathcal{L}_\xi \mathfrak{g})_{ab} = 0\,, \quad (\mathcal{L}_\xi \mathfrak{n})^a = -(\mathcal{L}_{\mathfrak{n}}\mathfrak{a})\mathfrak{n}^a\,. \tag{5.8}$$

Hence, this vector field generates an asymptotic symmetry (with $\kappa = 0$) if and only if $\mathcal{L}_{\mathfrak{n}}\mathfrak{a} = 0$, i.e., $\mathfrak{a}$ is constant along the integral curves of $\mathfrak{n}^a$. This is the case if and only if $\mathfrak{a}$ is the pullback of a function α on B, $\mathfrak{a} = \pi^*\alpha$. We will denote the set of these infinitesimal symmetries, which we call supertranslations, by $\mathfrak{S}$. Note that the commutator of two supertranslations vanishes, and that the sum of two supertranslations is again a supertranslation, so that they form an abelian subalgebra of $\mathfrak{L}$. Further, for any supertranslation $\mathfrak{a}\mathfrak{n}^a \in \mathfrak{S}$ and arbitrary symmetry $\xi^a \in \mathfrak{L}$ one has

$$[\xi, \mathfrak{a}\mathfrak{n}]^a = (\mathcal{L}_\xi \mathfrak{a} - \mathfrak{a}\kappa)\mathfrak{n}^a\,. \tag{5.9}$$

Note that this is an element of $\mathfrak{L}$, since $\mathfrak{L}$ is a Lie algebra with its Lie bracket given by the commutator of vector fields, and since it is a multiple of $\mathfrak{n}^a$, it is even an element of $\mathfrak{S}$, which means that the factor in brackets must be constant along the integral curves of $\mathfrak{n}^a$. Indeed, one has

$$\mathcal{L}_{\mathfrak{n}}(\mathcal{L}_\xi \mathfrak{a} - \mathfrak{a}\kappa) = \mathcal{L}_{\mathfrak{n}}\mathcal{L}_\xi \mathfrak{a} - \mathcal{L}_{\mathfrak{n}}(\mathfrak{a}\kappa) = \mathcal{L}_{[\mathfrak{n},\xi]}\mathfrak{a} = \mathcal{L}_{\kappa\mathfrak{n}}\mathfrak{a} = 0\,. \tag{5.10}$$

We have thus shown that $\mathfrak{S}$ is an ideal of $\mathfrak{L}$, which is parametrized by functions $\alpha : B \to \mathbb{R}$. This means that in order to determine the full algebra $\mathfrak{L}$, we can now study the quotient algebra $\mathfrak{L}/\mathfrak{S}$, which is constituted by equivalence classes of elements of $\mathfrak{L}$, whose difference lies in $\mathfrak{S}$.
To determine the quotient algebra $\mathfrak{L}/\mathfrak{S}$, consider an arbitrary vector field ξ^a on $\mathscr{I}$, and define $\mathfrak{b}_a = \mathfrak{g}_{ab}\xi^b$. Note that $\mathfrak{b}_a$ satisfies $\mathfrak{b}_a\mathfrak{n}^a = 0$ and is uniquely defined by ξ^a. Conversely, ξ^a is not fully determined by $\mathfrak{b}_a$, but only up to adding $\mathfrak{a}\mathfrak{n}^a$ with an arbitrary function $\mathfrak{a} : \mathscr{I} \to \mathbb{R}$. We now aim to show that the condition of $\xi^a + \mathfrak{a}\mathfrak{n}^a$ being a symmetry can be expressed as conditions on $\mathfrak{a}$ and $\mathfrak{b}_a$, which determine $\mathfrak{a}$ up to a function $\tilde{\mathfrak{a}}$ satisfying $\mathcal{L}_{\mathfrak{n}}\tilde{\mathfrak{a}} = 0$, i.e., up to adding an element of $\mathfrak{S}$, so that $\mathfrak{b}_a$ will be sufficient to determine a unique element of $\mathfrak{L}/\mathfrak{S}$. For this purpose, let us first use the linearity of the Lie derivative to calculate the second condition (5.6) as

$$-\kappa\mathfrak{n}^a = (\mathcal{L}_{\xi+\mathfrak{a}\mathfrak{n}^a}\mathfrak{n})^a = (\mathcal{L}_\xi \mathfrak{n})^a - (\mathcal{L}_{\mathfrak{n}}\mathfrak{a})\mathfrak{n}^a\,, \tag{5.11}$$

with some κ satisfying $\mathcal{L}_{\mathfrak{n}}\kappa = 0$. The second term is proportional to $\mathfrak{n}^a$, and by a suitable choice of $\mathfrak{a}$, it can be *any* vector field proportional to $\mathfrak{n}^a$. Hence, if also the first term is proportional to $\mathfrak{n}^a$, it can be canceled by the second term, where $\mathfrak{a}$ is determined by integrating $\mathcal{L}_{\mathfrak{n}}\mathfrak{a}$ along the flow lines of $\mathfrak{n}^a$, up to the aforementioned function $\tilde{\mathfrak{a}}$. We thus only need to demand that $(\mathcal{L}_\xi\mathfrak{n})^a$ is proportional to $\mathfrak{n}^a$, which can equivalently be expressed as

$$0 = \mathfrak{g}_{ab}(\mathcal{L}_\xi \mathfrak{n})^b = -\mathfrak{g}_{ab}(\mathcal{L}_{\mathfrak{n}}\xi)^b = -(\mathcal{L}_{\mathfrak{n}}\mathfrak{b})_a\,. \tag{5.12}$$

Together with the property $\mathfrak{b}_a \mathfrak{n}^a = 0$ this means that $\mathfrak{b}_a$ must be the pullback of some β_A on B, $\mathfrak{b} = \pi^* \beta$. We now aim to find a condition on this β_A that will be equivalent to the first condition (5.6) and determine κ in the equation above. Recall that on B we have a (non-degenerate) metric γ_{AB}, which allows us to raise one index to get a vector field $\beta^A = \gamma^{AB}\beta_B$. This vector field generates a flow $\phi : \mathbb{R} \times B \to B$ on B, which gives rise to the Lie derivative

$$(\mathcal{L}_\beta \gamma)_{AB} = \lim_{t\to 0} \frac{(\phi_t^* \gamma)_{AB} - \gamma_{AB}}{t} \,. \tag{5.13}$$

This is obviously a symmetric, covariant tensor field on B, and so we can take its pullback to $\mathscr{I}$. To understand the result of this operation, let us compare it with the flow $\varphi : \mathbb{R} \times \mathscr{I} \to \mathscr{I}$ on $\mathscr{I}$. Note that by construction we have $\beta = \pi_* \circ \xi$, and so

$$\phi_t \circ \pi = \pi \circ \varphi_t \,. \tag{5.14}$$

But using the fact that the metrics on B and $\mathscr{I}$ are related by $\mathfrak{g} = \pi^*\gamma$, we have

$$(\pi^* \mathcal{L}_\beta \gamma)_{ab} = \lim_{t\to 0} \frac{(\pi^* \phi_t^* \gamma)_{ab} - (\pi^*\gamma)_{ab}}{t} = \lim_{t\to 0} \frac{(\varphi_t^* \pi^* \gamma)_{ab} - (\pi^*\gamma)_{ab}}{t} = \lim_{t\to 0} \frac{(\varphi_t^* \mathfrak{g})_{ab} - \mathfrak{g}_{ab}}{t} = (\mathcal{L}_\xi \mathfrak{g})_{ab} \,. \tag{5.15}$$

Using the fact that the Lie derivative of $\mathfrak{g}_{ab}$ with respect to any multiple of $\mathfrak{n}^a$ vanishes, we thus find that ξ^a, and thus $\xi^a + \mathfrak{a}\mathfrak{n}^a$, satisfies the first condition with $\kappa = \pi^* k$ if and only if

$$(\mathcal{L}_\beta \gamma)_{AB} = 2k\gamma_{AB} \,, \tag{5.16}$$

i.e., if β^A generates a conformal symmetry of γ_{AB}. Hence, there is a one-to-one correspondence between such conformal symmetries and elements of $\mathfrak{L}/\mathfrak{S}$.
In summary, we have thus found a simple description of the algebra $\mathfrak{L}$ of infinitesimal asymptotic symmetries. This algebra is known as the Bondi-Metzner-Sachs algebra, or BMS algebra.

6 Examples

It is helpful to consider a few examples which satisfy the asymptotic flatness condition and derive the corresponding formulas. Here we look at Schwarzschild and Minkowski spacetime.

6.1 Schwarzschild spacetime

In the usual spherical coordinates $(x^\mu) = (t, r, \vartheta, \varphi)$, the Schwarzschild metric $\tilde{g}_{\mu\nu}$ on the physical spacetime $\tilde{M}$ is given by

$$\tilde{g}_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\left(1 - \frac{R}{r}\right) \mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{R}{r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r + r^2 \left(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \,\mathrm{d}\varphi \otimes \mathrm{d}\varphi\right) \,, \tag{6.1}$$

where $R = 2Gm$ is the Schwarzschild radius. In the following we will only consider the exterior spacetime $r > R$. On this region we can define new coordinates

$$u = t - r - R\ln\left(\frac{r}{R} - 1\right) \,, \quad \rho = \frac{1}{r} \,. \tag{6.2}$$

In these new coordinates, the exterior Schwarzschild spacetime is given by $0 < \rho < R^{-1}$. The physical metric on this region reads

$$\tilde{g}_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \frac{1}{\rho^2} \left[\mathrm{d}u \otimes \mathrm{d}\rho + \mathrm{d}\rho \otimes \mathrm{d}u - \rho^2(1 - R\rho)\mathrm{d}u \otimes \mathrm{d}u + \left(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \,\mathrm{d}\varphi \otimes \mathrm{d}\varphi\right)\right] \,. \tag{6.3}$$

We now consider this exterior region to be a subset of the unphysical spacetime M, with ψ given by the canonical inclusion map. This unphysical spacetime can be extended to the whole coordinate range $\rho < R^{-1}$, with $I = \{\rho = 0\}$ being the boundary of $\tilde{M}$ in M. Further, M can be equipped with a metric

$$g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \mathrm{d}u \otimes \mathrm{d}\rho + \mathrm{d}\rho \otimes \mathrm{d}u - \rho^2(1 - R\rho)\mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \,\mathrm{d}\varphi \otimes \mathrm{d}\varphi \,. \tag{6.4}$$

One easily checks that this metric has everywhere Lorentzian signature. This can be seen by explicitly writing the orthonormal basis

$$\partial_u + \frac{\rho^2(1-R\rho)-1}{2}\partial_\rho \,, \quad \partial_u + \frac{\rho^2(1-R\rho)+1}{2}\partial_\rho \,, \quad \partial_\vartheta \,, \quad \frac{1}{\sin\vartheta}\partial_\varphi \,, \tag{6.5}$$

which is well-defined on all of $\rho < R^{-1}$. Further, defining $\Omega = \rho$, we see that $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$ and that Ω is positive on $\tilde{M}$. On the boundary I, we have $\Omega = 0$, while its gradient satisfies

$$n_\mu \mathrm{d}x^\mu = \partial_\mu \Omega \, \mathrm{d}x^\mu = \mathrm{d}\rho \neq 0 \,. \tag{6.6}$$

Together with the inverse of the unphysical metric, which reads

$$g^{\mu\nu}\partial_\mu \otimes \partial_\nu = \partial_u \otimes \partial_\rho + \partial_\rho \otimes \partial_u + \rho^2(1-R\rho)\partial_\rho \otimes \partial_\rho + \partial_\vartheta \otimes \partial_\vartheta + \frac{1}{\sin^2\vartheta}\partial_\varphi \otimes \partial_\varphi \,, \tag{6.7}$$

we thus see that indeed

$$g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega = \rho^2(1-R\rho) \,, \tag{6.8}$$

which vanishes at the boundary I, where $\rho = 0$. Raising one index with the metric gives the vector field

$$n^\mu \partial_\mu = \partial_u + \rho^2(1-R\rho)\partial_\rho \,, \tag{6.9}$$

which restricts to ∂_u on I, and so is indeed tangent to I. On $\mathscr{I}$, we can use the coordinates $(x^a) = (u, \vartheta, \varphi)$, and in these coordinates the tangent null vector field is thus given by

$$\mathfrak{n}^a \partial_a = \partial_u \,, \tag{6.10}$$

while the pullback of the metric reads

$$\mathfrak{g}_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b = \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \, \mathrm{d}\varphi \otimes \mathrm{d}\varphi \,, \tag{6.11}$$

which is indeed degenerate of signature $(0, +, +)$, and satisfies $\mathfrak{n}^a \mathfrak{g}_{ab} = 0$.

6.2 Minkowski spacetime

A limiting case of the Schwarzschild spacetime given above is Minkowski spacetime. It can be obtained by taking the limit $R \to 0$. Note that this limit is well defined both for the physical metric $\tilde{g}_{\mu\nu}$ and the coordinate transformation, where

$$\lim_{R\to 0} u = t - r - \lim_{R\to 0} R \ln\left(\frac{r}{R} - 1\right) = t - r \,, \tag{6.12}$$

which can easily be verified from

$$\lim_{R\to 0} R \ln\left(\frac{r}{R} - 1\right) = \lim_{k\to\infty} \frac{1}{k} \ln(kr - 1) = 0 \,. \tag{6.13}$$

The physical metric thus reads

$$\tilde{g}_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \frac{1}{\rho^2}\left[\mathrm{d}u \otimes \mathrm{d}\rho + \mathrm{d}\rho \otimes \mathrm{d}u - \rho^2 \mathrm{d}u \otimes \mathrm{d}u + \left(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \, \mathrm{d}\varphi \otimes \mathrm{d}\varphi\right)\right] \,, \tag{6.14}$$

and is defined for $\rho > 0$. The unphysical metric

$$g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \mathrm{d}u \otimes \mathrm{d}\rho + \mathrm{d}\rho \otimes \mathrm{d}u - \rho^2 \mathrm{d}u \otimes \mathrm{d}u + \left(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \, \mathrm{d}\varphi \otimes \mathrm{d}\varphi\right) \tag{6.15}$$

is well-defined and of Lorentzian signature for all $\rho \in \mathbb{R}$, as can be seen from the orthonormal basis

$$\partial_u + \frac{\rho^2 - 1}{2}\partial_\rho \,, \quad \partial_u + \frac{\rho^2 + 1}{2}\partial_\rho \,, \quad \partial_\vartheta \,, \quad \frac{1}{\sin\vartheta}\partial_\varphi \,. \tag{6.16}$$

Here we are in particular interested which asymptotic symmetries are generated by the ten Killing vector fields of the Minkowski spacetime. In Cartesian coordinates (t, x, y, z), they are given by the generators of translations

$$T_0 = \partial_t\,, \quad T_1 = \partial_x\,, \quad T_2 = \partial_y\,, \quad T_3 = \partial_z\,, \tag{6.17}$$

the rotation generators

$$R_1 = z\partial_y - y\partial_z\,, \quad R_2 = x\partial_z - z\partial_x\,, \quad R_3 = y\partial_x - x\partial_y\,, \tag{6.18}$$

as well as the Lorentz boosts

$$L_1 = x\partial_t + t\partial_x\,, \quad L_2 = y\partial_t + t\partial_y\,, \quad L_3 = z\partial_t + t\partial_z\,. \tag{6.19}$$

In the first step, we write these generators in spherical coordinates, so that they are given by

$$T_0 = \partial_t\,, \tag{6.20a}$$

$$T_1 = \sin\vartheta\cos\varphi\partial_r + \frac{\cos\vartheta\cos\varphi}{r}\partial_\vartheta - \frac{\sin\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{6.20b}$$

$$T_2 = \sin\vartheta\sin\varphi\partial_r + \frac{\cos\vartheta\sin\varphi}{r}\partial_\vartheta + \frac{\cos\varphi}{r\sin\vartheta}\partial_\varphi\,, \tag{6.20c}$$

$$T_3 = \cos\vartheta\partial_r - \frac{\sin\vartheta}{r}\partial_\vartheta\,, \tag{6.20d}$$

$$R_1 = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{6.20e}$$

$$R_2 = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{6.20f}$$

$$R_3 = -\partial_\varphi\,, \tag{6.20g}$$

$$L_1 = r\sin\vartheta\cos\varphi\partial_t + t\left(\sin\vartheta\cos\varphi\partial_r + \frac{\cos\vartheta\cos\varphi}{r}\partial_\vartheta - \frac{\sin\varphi}{r\sin\vartheta}\partial_\varphi\right)\,, \tag{6.20h}$$

$$L_2 = r\sin\vartheta\sin\varphi\partial_t + t\left(\sin\vartheta\sin\varphi\partial_r + \frac{\cos\vartheta\sin\varphi}{r}\partial_\vartheta + \frac{\sin\varphi}{r\cos\vartheta}\partial_\varphi\right)\,, \tag{6.20i}$$

$$L_3 = r\cos\vartheta\partial_t + t\left(\cos\vartheta\partial_r - \frac{\sin\vartheta}{r}\partial_\vartheta\right)\,. \tag{6.20j}$$

In the next step, we transform these vector fields to the coordinates $(u, \rho, \vartheta, \varphi)$, and obtain

$$T_0 = \partial_u\,, \tag{6.21a}$$

$$T_1 = -\sin\vartheta\cos\varphi(\partial_u + \rho^2\partial_\rho) + \rho\cos\vartheta\cos\varphi\partial_\vartheta - \rho\frac{\sin\varphi}{\sin\vartheta}\partial_\varphi\,, \tag{6.21b}$$

$$T_2 = -\sin\vartheta\sin\varphi(\partial_u + \rho^2\partial_\rho) + \rho\cos\vartheta\sin\varphi\partial_\vartheta + \rho\frac{\cos\varphi}{\sin\vartheta}\partial_\varphi\,, \tag{6.21c}$$

$$T_3 = -\cos\vartheta(\partial_u + \rho^2\partial_\rho) - \rho\sin\vartheta\partial_\vartheta\,, \tag{6.21d}$$

$$R_1 = \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{6.21e}$$

$$R_2 = -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi\,, \tag{6.21f}$$

$$R_3 = -\partial_\varphi\,, \tag{6.21g}$$

$$L_1 = -(1+u\rho)\left[\sin\vartheta\cos\varphi\left(\frac{u}{1-u\rho}\partial_u + \rho\partial_\rho\right) - \cos\vartheta\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\sin\vartheta}\partial_\varphi\right]\,, \tag{6.21h}$$

$$L_2 = -(1+u\rho)\left[\sin\vartheta\sin\varphi\left(\frac{u}{1-u\rho}\partial_u + \rho\partial_\rho\right) - \cos\vartheta\sin\varphi\partial_\vartheta - \frac{\sin\varphi}{\cos\vartheta}\partial_\varphi\right]\,, \tag{6.21i}$$

$$L_3 = -(1+u\rho)\left[\cos\vartheta\left(\frac{u}{1-u\rho}\partial_u + \rho\partial_\rho\right) + \sin\vartheta\partial_\vartheta\right]\,. \tag{6.21j}$$

Finally, we extend the vector fields to $\rho \leq 0$, and restrict them to I by setting $\rho = 0$. As expected, one finds that they are indeed tangent to I, having no ∂_ρ component. On $\mathscr{I}$, with coordinates (u, ϑ, φ), they are given by

$$
\begin{aligned}
\mathfrak{T}_0 &= \partial_u \,, && (6.22a)\\
\mathfrak{T}_1 &= -\sin\vartheta\cos\varphi\partial_u \,, && (6.22b)\\
\mathfrak{T}_2 &= -\sin\vartheta\sin\varphi\partial_u \,, && (6.22c)\\
\mathfrak{T}_3 &= -\cos\vartheta\partial_u \,, && (6.22d)\\
\mathfrak{R}_1 &= \sin\varphi\partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta}\partial_\varphi \,, && (6.22e)\\
\mathfrak{R}_2 &= -\cos\varphi\partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta}\partial_\varphi \,, && (6.22f)\\
\mathfrak{R}_3 &= -\partial_\varphi \,, && (6.22g)\\
\mathfrak{L}_1 &= -u\sin\vartheta\cos\varphi\partial_u + \cos\vartheta\cos\varphi\partial_\vartheta - \frac{\sin\varphi}{\sin\vartheta}\partial_\varphi \,, && (6.22h)\\
\mathfrak{L}_2 &= -u\sin\vartheta\sin\varphi\partial_u + \cos\vartheta\sin\varphi\partial_\vartheta + \frac{\sin\varphi}{\cos\vartheta}\partial_\varphi \,, && (6.22i)\\
\mathfrak{L}_3 &= -u\cos\vartheta\partial_u - \sin\vartheta\partial_\vartheta \,. && (6.22j)
\end{aligned}
$$

We now discuss the structure of these vector fields more closely. First, we pose the question which of them correspond to supertranslations, i.e., which lie in the kernel of the induced metric $\mathfrak{g}_{ab}$. These are exactly those vector fields which are proportional to the coordinate vector field ∂_u, hence those generated by the translations $\mathfrak{T}_0, \ldots, \mathfrak{T}_3$. Studying the angular dependence of these vector fields, we find that $\mathfrak{T}_0$ is constant on the sphere, hence it corresponds to the spherical harmonic of the lowest degree $l = 0$. The spatial translations $\mathfrak{T}_1, \ldots, \mathfrak{T}_3$ are similarly represented by spherical harmonics of degree $l = 1$. The remaining vector fields have also angular components, i.e., they act non-trivially on the sphere. Taking the Lie derivative of the metric (6.11), one finds that it is given by

$$
(\mathcal{L}_{\mathfrak{R}_1}\mathfrak{g})_{ab} = (\mathcal{L}_{\mathfrak{R}_2}\mathfrak{g})_{ab} = (\mathcal{L}_{\mathfrak{R}_3}\mathfrak{g})_{ab} = 0 \,, \tag{6.23}
$$

so that the generators of rotations act as Killing symmetries on the metric, while the Lorentz boosts satisfy

$$
(\mathcal{L}_{\mathfrak{L}_1}\mathfrak{g})_{ab} = -2\sin\vartheta\cos\varphi\,\mathfrak{g}_{ab} \,, \quad (\mathcal{L}_{\mathfrak{L}_2}\mathfrak{g})_{ab} = -2\sin\vartheta\sin\varphi\,\mathfrak{g}_{ab} \,, \quad (\mathcal{L}_{\mathfrak{L}_3}\mathfrak{g})_{ab} = -2\cos\vartheta\,\mathfrak{g}_{ab} \,, \tag{6.24}
$$

which shows that they are conformal Killing symmetries. Neither depend on u, and so they belong to the BMS group of asymptotic symmetries.