Linearized Einstein equations and gravitational waves

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October 26, 2022

1 Linearized Einstein equations

In the following, we will assume a coordinate system (x^μ) to be chosen, in which we can describe the metric tensor as a linear perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.1}$$

around the flat Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Further, we will consider the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{1.2}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.3}$$

is the Einstein tensor. We have already seen in a previous lecture that for the linear perturbation around the Minkowski metric, we can write the Einstein tensor in the form

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho{}_\nu + \partial_\rho\partial_\nu h^\rho{}_\mu - \partial_\mu\partial_\nu h^\rho{}_\rho - \Box h_{\mu\nu} + \eta_{\mu\nu}\Box h^\rho{}_\rho - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma}\right)\,, \tag{1.4}$$

where indices are raised and lowered with the Minkowski background metric, and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembert operator. We now aim to simplify this equation. First, it is helpful to introduce the trace-reversed metric perturbation[1]

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h^\rho{}_\rho \eta_{\mu\nu}\,. \tag{1.5}$$

It is trace-reversed in the sense that under this operation, the trace changes its sign:

$$\bar{h}^\mu{}_\mu = h^\mu{}_\mu - \frac{1}{2} h^\nu{}_\nu \delta^\mu_\mu = h^\mu{}_\mu - 2h^\nu{}_\nu = -h^\mu{}_\mu\,. \tag{1.6}$$

From this we see that the operation of trace-reversal is its own inverse, so we can get the original perturbation back:

$$\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}^\rho{}_\rho\eta_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h^\rho{}_\rho\eta_{\mu\nu} + \frac{1}{2} h^\rho{}_\rho\eta_{\mu\nu} = h_{\mu\nu}\,. \tag{1.7}$$

By inserting this into the Einstein tensor (1.4), we find that it takes the form

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\mu\bar{h}^\rho{}_\nu + \partial_\rho\partial_\nu\bar{h}^\rho{}_\mu - \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} - \Box\bar{h}_{\mu\nu}\right)\,. \tag{1.8}$$

We see that in these new variables, the two terms proportional to the trace of the field variable are absent.

2 Gauge transformation

To further simplify the Einstein tensor (1.8), we now consider an infinitesimal coordinate transformation

$$x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x)\,. \tag{2.1}$$

[1] Note that the bar here denotes trace reversal, not the background value! It is a common notation.

For a general metric $g_{\mu\nu}$, this coordinate transformation induces the transformation

$$\begin{aligned} g_{\mu\nu}(x) &= g'_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} \\ &= \left[g'_{\alpha\beta}(x) + \xi^\gamma(x) \partial_\gamma g'_{\alpha\beta}(x)\right] \left[\delta^\alpha_\mu + \partial_\mu \xi^\alpha(x)\right] \left[\delta^\beta_\nu + \partial_\nu \xi^\beta(x)\right] + \mathcal{O}(\xi^2) \\ &= g'_{\mu\nu} + \xi^\gamma \partial_\gamma g'_{\mu\nu} + \partial_\mu \xi^\gamma g'_{\gamma\nu} + \partial_\nu \xi^\gamma g'_{\mu\gamma} + \mathcal{O}(\xi^2) \\ &= g'_{\mu\nu} + (\mathcal{L}_\xi g')_{\mu\nu} + \mathcal{O}(\xi^2)\,. \end{aligned} \tag{2.2}$$

Here from the last line we have omitted the argument x, since all tensor fields are taken at x. The second term in the last line is called the Lie derivative.
We then return to the metric (1.1), which is a linear perturbation around the Minkowski metric, and we take a transformed metric of the same form

$$g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}\,, \tag{2.3}$$

also as a perturbation around the Minkowski metric. We will assume that both metric perturbations and coordinate transformation are small, so that we keep only terms which are linear in either of them, and neglect any terms of the form $\xi^2, h\xi, h^2 \ldots$ Then the gauge transformation reads

$$\begin{aligned} \eta_{\mu\nu} + h_{\mu\nu} &= \eta_{\mu\nu} + h'_{\mu\nu} + \xi^\gamma \partial_\gamma(\eta_{\mu\nu} + h'_{\mu\nu}) + \partial_\mu \xi^\gamma(\eta_{\gamma\nu} + h'_{\gamma\nu}) + \partial_\nu \xi^\gamma(\eta_{\mu\gamma} + h'_{\mu\gamma}) \\ &= \eta_{\mu\nu} + h'_{\mu\nu} + \partial_\mu \xi^\gamma \eta_{\gamma\nu} + \partial_\nu \xi^\gamma \eta_{\mu\gamma} \\ &= \eta_{\mu\nu} + h'_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\,. \end{aligned} \tag{2.4}$$

Here we have used the fact that the Minkowski metric is constant, $\partial_\gamma \eta_{\mu\nu} = 0$, and also used the Minkowski metric for lowering the indices of ξ. Let us also note the transformation of the trace-reversed metric perturbation:

$$\begin{aligned} \bar{h}_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2} h^\rho{}_\rho \eta_{\mu\nu} \\ &= h'_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2}\left(h'^\rho{}_\rho + 2\partial_\rho \xi^\rho\right) \eta_{\mu\nu} \\ &= \bar{h}'_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial_\rho \xi^\rho \eta_{\mu\nu}\,. \end{aligned} \tag{2.5}$$

Finally, let us insert this result into the Einstein tensor (1.8), and study how much it changes under the transformation. For convenience, we include a factor 2 here to cancel the $\frac{1}{2}$:

$$\begin{aligned} 2(G_{\mu\nu} - G'_{\mu\nu}) &= \partial_\rho \partial_\mu (\partial^\rho \xi_\nu + \partial_\nu \xi^\rho - \partial_\sigma \xi^\sigma \delta^\rho_\nu) + \partial_\rho \partial_\nu (\partial^\rho \xi_\mu + \partial_\mu \xi^\rho - \partial_\sigma \xi^\sigma \delta^\rho_\mu) \\ &\quad - \eta_{\mu\nu} \partial_\rho \partial_\sigma (\partial^\rho \xi^\sigma + \partial^\sigma \xi^\rho - \partial_\tau \xi^\tau \eta^{\rho\sigma}) - \Box(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial_\rho \xi^\rho \eta_{\mu\nu}) \\ &= \Box \partial_\mu \xi_\nu + \partial_\mu \partial_\nu \partial_\rho \xi^\rho - \partial_\mu \partial_\nu \partial_\sigma \xi^\sigma + \Box \partial_\nu \xi_\mu + \partial_\mu \partial_\nu \partial_\rho \xi^\rho - \partial_\mu \partial_\nu \partial_\sigma \xi^\sigma \\ &\quad - \eta_{\mu\nu} \Box \partial_\rho \xi^\rho - \Box \partial_\mu \xi_\nu - \Box \partial_\nu \xi_\mu + \Box \partial_\rho \xi^\rho \eta_{\mu\nu} \\ &= 0 \end{aligned} \tag{2.6}$$

Hence, we see that if $\bar{h}_{\mu\nu}$ is a solution to the linearized Einstein equations, so is $\bar{h}'_{\mu\nu}$.

3 Harmonic gauge

We now make use of the gauge freedom we found above in order to find a gauge, i.e., a choice of the coordinate system, in which the Einstein tensor takes an even simpler form. We will assume that $h_{\mu\nu}$ is given in this special coordinate system, while $h'_{\mu\nu}$ is given in an arbitrary coordinate system. The aim is to transform (1.8) into the form of a wave equation. This would be achieved if $\bar{h}_{\mu\nu}$ would satisfy the condition

$$\partial_\rho \partial_\mu \bar{h}^\rho{}_\nu + \partial_\rho \partial_\nu \bar{h}^\rho{}_\mu - \eta_{\mu\nu} \partial_\rho \partial_\sigma \bar{h}^{\rho\sigma} = 0. \tag{3.1}$$

Taking a closer look at this equation, we see that a sufficient condition for this is given by

$$\partial^\mu \bar{h}_{\mu\nu} = 0\,. \tag{3.2}$$

We transform this condition to the usual metric perturbation by trace-reversal:

$$\begin{aligned} 0 &= \partial^\mu \bar{h}_{\mu\nu} \\ &= \partial^\mu \left(h_{\mu\nu} - \frac{1}{2} h^\rho{}_\rho \eta_{\mu\nu} \right) \\ &= \partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^\mu{}_\mu \,. \end{aligned} \tag{3.3}$$

This condition is known as the harmonic gauge condition. To see that this is really a condition on the coordinate system we use, one can approach it from a different direction. Consider a scalar function f defined on spacetime. Application of the d'Alembert operator (which shall now be the full d'Alembert operator of the metric $g_{\mu\nu}$ yields

$$\Box f = g^{\mu\nu} \nabla_\mu \nabla_\nu f = g^{\mu\nu} (\partial_\mu \partial_\nu f - \Gamma^\rho{}_{\nu\mu} \partial_\rho f) \,. \tag{3.4}$$

A function which solves the differential equation $\Box f = 0$ is called a *harmonic* function. We now aim to find coordinate (x^α) such that each component x^α is a harmonic function. Hence, let us assume that the function f is given by the coordinate x^α. Then its partial derivatives satisfy

$$\partial_\mu x^\alpha = \delta^\alpha_\mu \,, \quad \partial_\mu \partial_\nu x^\alpha = 0 \,. \tag{3.5}$$

We now insert this into the equation (3.4), and further assume that the metric is a linear perturbation (1.1). Then we find that the coordinate behaves as a harmonic function if and only if

$$\begin{aligned} 0 &= g^{\mu\nu} (\partial_\mu \partial_\nu x^\alpha - \Gamma^\rho{}_{\nu\mu} \partial_\rho x^\alpha) \\ &= -g^{\mu\nu} \Gamma^\rho{}_{\nu\mu} \delta^\alpha_\rho \\ &= -g^{\mu\nu} \Gamma^\alpha{}_{\nu\mu} \\ &= -\frac{1}{2} g^{\mu\nu} g^{\alpha\rho} (\partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\nu\mu}) \\ &= -\frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\rho} (\partial_\nu h_{\rho\mu} + \partial_\mu h_{\nu\rho} - \partial_\rho h_{\nu\mu}) \\ &= -\partial_\mu h^{\mu\alpha} + \frac{1}{2} \partial^\alpha h^\mu{}_\mu \,, \end{aligned} \tag{3.6}$$

where we used the Minkowski metric in the last line to move indices. We see that this is exactly the harmonic gauge condition (3.3). Hence, we can always achieve this condition to hold by choosing coordinates such that they are harmonic.

4 Plane waves

We will now turn our attention to vacuum solutions of the linearized Einstein equations

$$G_{\mu\nu} = 0 \,. \tag{4.1}$$

Hence, working in the harmonic gauge, we are looking for solutions of the wave equation

$$\Box \bar{h}_{\mu\nu} = 0 \,. \tag{4.2}$$

A particularly simple class of solution is given by plane waves. To motivate these solutions, let us consider the Fourier transform

$$\bar{h}_{\mu\nu}(x) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4 k \, \hat{\bar{h}}_{\mu\nu}(k) e^{-ik_\mu x^\mu} \,, \tag{4.3}$$

with inverse

$$\hat{\bar{h}}_{\mu\nu}(k) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4 x \, \bar{h}_{\mu\nu}(x) e^{ik_\mu x^\mu} \,. \tag{4.4}$$

Applying the d'Alembert operator yields

$$0 = \Box \bar{h}_{\mu\nu}(x) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4 k \, \hat{\bar{h}}_{\mu\nu}(k) (-k_\mu k^\mu) e^{-ik_\mu x^\mu} \,. \tag{4.5}$$

Hence, a solution of the wave equation is given as a linear combination of plane waves $e^{-ik_\mu x^\mu}$, whose wave covector satisfies $k_\mu k^\mu = 0$. This is the most simple example of a gravitational wave, and we see that it propagates at the speed of light.