Newtonian limit and Euler equations

Manuel Hohmann

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1 Dynamics of a point mass

We start by studying the dynamics of a point mass of mass m. Using coordinates (x^μ), its trajectory or worldline $\tau \mapsto \gamma^\mu(\tau)$ parametrized by an arbitrary curve parameter τ. Its dynamics is governed by the action

$$S[\gamma] = -m \int \sqrt{-g_{\mu\nu}(\gamma(\tau))\dot{\gamma}^\mu(\tau)\dot{\gamma}^\nu(\tau)}\mathrm{d}\tau\,. \tag{1.1}$$

Note that this action is independent under the choice of the parametrization: it does not change of we replace τ by another parameter τ' which is given by a monotonous function $\tau'(\tau)$. This allows us to choose it such that the trajectory is parametrized by its arc length, i.e., such that

$$g_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu = -1\,, \tag{1.2}$$

where here and from now on we suppress the arguments τ and $\gamma(\tau)$ in our notation. Under this choice of parametrization the equations of motion derived from the action above take the familiar form of the geodesic equation

$$\ddot{\gamma}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{\gamma}^\nu\dot{\gamma}^\rho = 0\,. \tag{1.3}$$

In order to derive Newton's equations of motion, we assume a slowly moving test mass in a weak, time-independent gravitational field. To make these assumptions precise, we assume that the coordinates are given by Cartesian coordinates $(x^\mu) = (x^0, x^i)$, where x^0 is the time coordinate and (x^i) are spatial coordinates. Defining the components of the velocity as

$$v^i = \frac{\mathrm{d}\gamma^i}{\mathrm{d}\gamma^0} = \frac{\mathrm{d}\gamma^i}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\gamma^0}{\mathrm{d}\tau}\right)^{-1} = \frac{\dot{\gamma}^i}{\dot{\gamma}^0}\,, \tag{1.4}$$

the assumption of a slowly moving test mass is expressed by $|v^i| \ll 1$, where we have normalized the speed of light as $c = 1$. Further, we assume that the metric is given by a perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.5}$$

around the Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)\,. \tag{1.6}$$

Our aim is now to calculate the acceleration

$$a^i = \frac{\mathrm{d}^2\gamma^i}{(\mathrm{d}\gamma^0)^2} = \frac{\ddot{\gamma}^i}{(\dot{\gamma}^0)^2}\,. \tag{1.7}$$

Expanding the geodesic equation into space and time components, we find

$$\ddot{\gamma}^i = -\Gamma^i{}_{00}(\dot{\gamma}^0)^2 - 2\Gamma^i{}_{0j}\dot{\gamma}^0\dot{\gamma}^j - \Gamma^i{}_{jk}\dot{\gamma}^j\dot{\gamma}^k \approx -\Gamma^i{}_{00}(\dot{\gamma}^0)^2\,, \tag{1.8}$$

where in the approximation we have made use of the assumption $|\dot{\gamma}^i| \ll |\dot{\gamma}^0|$ of slow motion to keep only the leading order term in the expansion. Hence, we can write the acceleration as

$$a^i = -\Gamma^i{}_{00} = \frac{1}{2}\delta^{ij}\partial_j h_{00}\,, \tag{1.9}$$

where also on the right hand side we have kept only the lowest order term in $h_{\mu\nu}$ and further used the assumption that the gravitational field is time-independent, in order to neglect the time derivative $\partial_0 h_{0j}$. Finally, identifying the metric perturbation with the Newtonian potential U as

$$h_{00} = 2U\,, \tag{1.10}$$

we see that we find Newton's equations of motion

$$a_i = \partial_i U \tag{1.11}$$

for a slowly moving point mass in a weak, time-independent gravitational field.

2 Energy-momentum and Euler equations

We continue by studying the dynamics of a perfect fluid, which is described by the energy-momentum tensor

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}\,, \tag{2.1}$$

and thus characterized by its density ρ, pressure p and four-velocity u^μ at every spacetime point. Here the four-velocity is normalized such that

$$g_{\mu\nu}u^\mu u^\nu = -1\,, \tag{2.2}$$

As for the point mass discussed above, we will assume that the fluid is moving slowly in a chosen Cartesian coordinate system, so that we have

$$v^i = \frac{u^i}{u^0}\,, \quad |v^i| \ll 1\,. \tag{2.3}$$

Under this assumption, the normalization condition becomes

$$-1 = g_{00}(u^0)^2 + 2g_{0i}u^0u^i + g_{ij}u^iu^j = (u^0)^2\left(-1 + h_{00} + 2h_{0i}v^i + v^2 + h_{ij}v^iv^j\right) \approx -(u^0)^2\,, \tag{2.4}$$

giving $u^0 \approx 1$ and thus $u^i \approx v^i$ at the leading order, which will turn out to be sufficient for our following derivation. Further, we assume that the pressure is of the same order of magnitude as the kinetic energy density, and so

$$p \sim \rho v^2 \ll \rho\,. \tag{2.5}$$

Under these assumptions, the energy-momentum tensor has the leading order components

$$T^{00} \approx \rho\,, \quad T^{0i} \approx \rho v^i\,, \quad T^{ij} \approx \rho v^i v^j + p\delta^{ij}\,. \tag{2.6}$$

The dynamics of the fluid is described by the energy-momentum conservation law

$$0 = \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\rho\mu}T^{\rho\nu} + \Gamma^\nu{}_{\rho\mu}T^{\mu\rho}\,, \tag{2.7}$$

which we now decompose into its space and time components. We start with the time component, which reads at leading order

$$0 \approx \partial_0 T^{00} + \partial_i T^{i0} \approx \partial_0\rho + \partial_i(\rho v^i)\,. \tag{2.8}$$

Here we have neglected all terms involving Christoffel symbols, since they contain derivatives of $h_{\mu\nu}$. We see that the result resembles the continuity equation of a classical fluid. For the spatial part, we assume that the kinetic and potential energies of the fluid are or the same order of magnitude, and hence

$$h_{00} = 2U \sim v^2\,. \tag{2.9}$$

At the lowest order, we thus retain one Christoffel symbol, and find

$$0 \approx \partial_0 T^{0i} + \partial_j T^{ji} + \Gamma^i{}_{00}T^{00} = \partial_0(\rho v^i) + \partial_j(\rho v^i v^j + p\delta^{ij}) - \rho\partial^i U\,. \tag{2.10}$$

We can further use the continuity equation (2.8) to eliminate $\partial_0\rho$ and find

$$0 \approx \rho\partial_0 v^i + \rho v^j\partial_j v^i + \partial^i p - \rho\partial^i U\,. \tag{2.11}$$

This is the Euler equation of motion for a fluid. We can also rewrite these two equations by introducing the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_0 + v^i \partial_i \tag{2.12}$$

along the flow lines of the fluid. Then the continuity equation (2.8) becomes

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \partial_i v^i = 0\,, \tag{2.13}$$

while the Euler equation (2.11) reads

$$\rho \frac{\mathrm{d}v^i}{\mathrm{d}t} + \partial^i p - \rho \partial^i U = 0\,. \tag{2.14}$$

3 Newtonian limit of general relativity

We finally take a look at the weak field, slow motion expansion of the gravitational field equations of general relativity, given by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{3.1}$$

It turns out to be simpler to trace-reverse this equation first. Taking the trace on both sides, one finds

$$R - 2R = 8\pi G T\,, \tag{3.2}$$

and so $R = -8\pi G T$. We can thus shift the second term from the left hand side to the right hand side, after which we obtain

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)\,. \tag{3.3}$$

In the following, only the time component of this equation will be relevant. On the left hand side, a weak field approximation yields

$$R_{00} = -\frac{1}{2} \partial^i \partial_i h_{00} = -\triangle U\,, \tag{3.4}$$

where we have also assumed that we can neglect time derivatives, and introduced the Laplace operator. On the right hand side, the leading order is given by

$$T_{00} - \frac{1}{2} T g_{00} = \frac{1}{2} \rho\,. \tag{3.5}$$

We thus find

$$\triangle U = -4\pi G \rho\,, \tag{3.6}$$

which is the well-known Poisson equation for the Newtonian gravitational potential.