The parametrized post-Newtonian formalism

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1 Preliminaries

In the following we use Cartesian coordinates $(x^\mu) = (t, x^i)$ and denote partial derivatives with respect to these coordinates by

$$A_{,\mu} = \frac{\partial A}{\partial x^\mu}\,, \quad A_{,0} = \frac{\partial A}{\partial t}\,, \quad A_{,i} = \frac{\partial A}{\partial x^i}\,. \tag{1.1}$$

The total time derivative is given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + v^i\frac{\partial A}{\partial x^i} = A_{,0} + v^i A_{,i}\,. \tag{1.2}$$

2 Post-Newtonian bookkeeping

Purpose of the parametrized post-Newtonian (PPN) formalism[1] is to provide a universal procedure for testing gravity theories using observations in the solar system. Necessary condition to apply the formalism to a gravity theory is that the theory models the spacetime geometry by a metric $g_{\mu\nu}$ and that the motion of test masses within this geometry follows geodesics. The gravitational field equations of the theory in question determine the metric $g_{\mu\nu}$ depending on the distribution of gravitating source matter, which in turn determines the motion of test masses. Both the matter distribution in the solar system and the motion of its constituents can be measured. The combination of these measurements then provides a test for the gravitational field equations of the theory in question.
We assume that all masses in the solar system move at small velocities $|\vec{v}| \ll c \equiv 1$. The action for the motion of a test mass m_0 then takes the form

$$S = -m_0\int\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\mathrm{d}t = -m_0\int\sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^iv^j}\mathrm{d}t\,. \tag{2.1}$$

This expression has the form of a perturbative expansion in the velocity $\vec{v}$. We further assume that the gravitational field is weak, so that we can approximate the metric in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.2}$$

as an expansion around the flat Minkowski metric. It follows from the structure of the action (2.1) that perturbations of the component g_{00} have a stronger influence on the motion of test masses than g_{ij}, since the latter is suppressed by a factor v^2. The perturbation $h_{\mu\nu}$ is thus further divided into velocity orders

$$h_{\mu\nu} = \overset{1}{h}_{\mu\nu} + \overset{2}{h}_{\mu\nu} + \overset{3}{h}_{\mu\nu} + \overset{4}{h}_{\mu\nu} + \mathcal{O}(5)\,, \tag{2.3}$$

where each term $\overset{n}{h}_{\mu\nu} \sim |\vec{v}|^n$ is of order $\mathcal{O}(n)$. It turns out that in the Newtonian limit the only relevant and non-vanishing term in this expansion is $\overset{2}{h}_{00}$, while in the post-Newtonian limit we consider here also $\overset{2}{h}_{ij}, \overset{3}{h}_{0i}, \overset{4}{h}_{00}$ are necessary.
We further assume that the gravitational field equations are of the form

$$K_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{2.4}$$

[1] Here we use the formalism as given in [1]. A modified version is given in [3], which uses different conventions for the matter variables and PPN potentials.

where $K_{\mu\nu}$ is a geometric curvature tensor which depends on the metric $g_{\mu\nu}$ and vanishes for a flat Minkowski metric. Through the perturbative expansion (2.2) and (2.3) it decomposes into terms $K^{(n)}_{\mu\nu}$ of velocity order $\mathcal{O}(n)$. We thus need a similar decomposition of the energy-momentum tensor, for which we assume a perfect fluid form

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + p g^{\mu\nu} \tag{2.5}$$

with rest energy density ρ, internal energy density $\rho\Pi$, pressure p and four-velocity u^μ. Based on their values in the solar system we assign velocity orders $\rho \sim \Pi \sim \mathcal{O}(2)$ and $p \sim \mathcal{O}(4)$. Together with the velocity components $v^i = u^i/u^0$ we then find the decomposition of the energy-momentum tensor in the form

$$T_{00} = \rho\left(1 + \Pi + v^2 - \overset{2}{h}_{00}\right) + \mathcal{O}(6)\,, \tag{2.6a}$$
$$T_{0j} = -\rho v_j + \mathcal{O}(5)\,, \tag{2.6b}$$
$$T_{ij} = \rho v_i v_j + p\delta_{ij} + \mathcal{O}(6)\,. \tag{2.6c}$$

Since all changes of metric components and the matter source over time are induced by motions of the source matter with velocity $|v|$ we further weight all time derivatives with another factor $\mathcal{O}(1)$.

3 Metric ansatz

In order to solve generic gravitational field equations of the form (2.4) one uses the metric ansatz

$$\overset{2}{h}_{00} = 2\alpha U\,, \tag{3.1a}$$
$$\overset{2}{h}_{ij} = 2\gamma U\delta_{ij}\,, \tag{3.1b}$$
$$\overset{3}{h}_{0i} = -\frac{1}{2}(3 + 4\gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i\,, \tag{3.1c}$$
$$\begin{aligned}\overset{4}{h}_{00} = {}& -2\beta U^2 - 2\xi\Phi_W + (2 + 2\gamma + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 + 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)\Phi_2 \\ & + 2(1 + \zeta_3)\Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 - (\zeta_1 - 2\xi)\mathcal{A}\,.\end{aligned} \tag{3.1d}$$

The quantities $U, V_\alpha, W_\alpha, \Phi_W, \Phi_1, \ldots, \Phi_4, \mathcal{A}$ are the PPN potentials, which are Poisson-like integrals over the source matter distribution of the form

$$\chi = -\int d^3x'\rho'|\vec{x} - \vec{x}'| \sim \mathcal{O}(2)\,, \tag{3.2a}$$
$$U = \int d^3x' \frac{\rho'}{|\vec{x} - \vec{x}'|} \sim \mathcal{O}(2)\,, \tag{3.2b}$$
$$U_{ij} = \int d^3x' \frac{\rho'}{|\vec{x} - \vec{x}'|^3}(x_i - x'_i)(x_j - x'_j) = \chi_{,ij} - \frac{1}{2}\triangle\chi\delta_{ij} \sim \mathcal{O}(2)\,, \tag{3.2c}$$
$$V_i = \int d^3x' \frac{\rho' v'_i}{|\vec{x} - \vec{x}'|} \sim \mathcal{O}(3)\,, \tag{3.2d}$$
$$W_i = \int d^3x' \frac{\rho' v'_j(x_i - x'_i)(x_j - x'_j)}{|\vec{x} - \vec{x}'|^3} \sim \mathcal{O}(3)\,, \tag{3.2e}$$
$$\Phi_1 = \int d^3x' \frac{\rho' v'^2}{|\vec{x} - \vec{x}'|} \sim \mathcal{O}(4)\,, \tag{3.2f}$$
$$\Phi_2 = \int d^3x' \frac{\rho' U'}{|\vec{x} - \vec{x}'|} \sim \mathcal{O}(4)\,, \tag{3.2g}$$
$$\Phi_3 = \int d^3x' \frac{\rho' \Pi'}{|\vec{x} - \vec{x}'|} \sim \mathcal{O}(4)\,, \tag{3.2h}$$
$$\Phi_4 = \int d^3x' \frac{p'}{|\vec{x} - \vec{x}'|} \sim \mathcal{O}(4)\,, \tag{3.2i}$$
$$\mathcal{A} = \int d^3x' \frac{\rho' \left[v'_i(x_i - x'_i)\right]^2}{|\vec{x} - \vec{x}'|^3} \sim \mathcal{O}(4)\,, \tag{3.2j}$$

$$\mathcal{B} = \int \mathrm{d}^3x' \frac{\rho'}{|\vec{x} - \vec{x}'|}(x_i - x'_i)\frac{\mathrm{d}v_i}{\mathrm{d}t} \sim \mathcal{O}(4)\,, \tag{3.2k}$$

$$\Phi_W = \int \mathrm{d}^3x' \int \mathrm{d}^3x'' \rho'\rho'' \frac{x_i - x'_i}{|\vec{x} - \vec{x}'|^3}\left(\frac{x'_i - x''_i}{|\vec{x} - \vec{x}''|} - \frac{x_i - x''_i}{|\vec{x}' - \vec{x}''|}\right) \sim \mathcal{O}(4)\,, \tag{3.2l}$$

where we used the notation $\rho' = \rho(t, \vec{x}')$ and similar for matter variables ρ, Π, p, v_i or PPN potentials evaluated at points $\vec{x}'$ or $\vec{x}''$. They appear as generic solutions to the field equations of gravity theories, since they satisfy the Poisson-like equations

$$\begin{gathered}
\triangle\triangle\chi = 8\pi\rho\,, \quad \triangle\triangle\mathcal{A} = 8\pi(\rho v_a v_b)_{,ab} - 4\pi\triangle(\rho v^2)\,, \quad \triangle\triangle\mathcal{B} = 8\pi[\triangle p - (U_{,a}\rho)_{,a}]\,, \\
\triangle\Phi_1 = -4\pi\rho v^2\,, \quad \triangle\Phi_2 = -4\pi\rho U\,, \quad \triangle\Phi_3 = -4\pi\rho\Pi\,, \quad \triangle\Phi_4 = -4\pi p\,, \\
\triangle U = -4\pi\rho\,, \quad \triangle V_a = -4\pi\rho v_a\,, \quad \triangle\Phi_W = 4\pi\rho U - 4U_{,a}U_{,a} + 2U_{,ab}\chi_{,ab}\,.
\end{gathered} \tag{3.3}$$

The constant α, which corresponds to the effective gravitational constant, is conventionally set to 1. The remaining constants $\beta, \gamma, \alpha_1, \ldots, \alpha_3, \zeta_1, \ldots, \zeta_4, \xi$ are the PPN parameters, which are characteristics of the theory under consideration. They are determined by a perturbative solution of the field equations (2.4). For general relativity they take the values

$$\gamma = \beta = 1\,, \quad \alpha_1 = \ldots = \alpha_3 = \zeta_1 = \ldots = \zeta_4 = \xi = 0\,. \tag{3.4}$$

By experimentally probing the spacetime geometry in the solar system through various effects these PPN parameters can also be measured. Comparing the theoretical values obtained from a particular theory with the measured values then allows testing the viability of this theory.

4 Physical interpretation of the PPN parameters

The coefficients of the terms in the metric (3.1) are chosen as linear combinations of PPN parameters in a particular way so that the PPN parameters can be interpreted by different physical effects.

4.1 γ

The PPN parameter γ describes how much spatial curvature is produced per unit mass. This spatial curvature can be measured, for example, by the deflection of light. A light ray passing by a mass m at a distance d is deflected by the angle

$$\delta\theta = (1 + \gamma)(1 + \cos\theta_0)\frac{m}{d}\,, \tag{4.1}$$

where θ_0 is the angular separation between the observed light ray and the line of sight towards the deflecting mass.
Another method is to measure the time delay of a light or radio signal on its way from Earth to a planet or spacecraft and back, typically using the solar mass $m = M_\odot$. In solar centered coordinates, $\vec{x}_\odot = 0$, the two-way travel time of a radio signal is given by

$$\Delta t = 2|\vec{x}_\oplus - \vec{x}_r| + \delta t = 2|\vec{x}_\oplus - \vec{x}_r| + 2(1 + \gamma)m\ln\frac{(r_\oplus + \vec{x}_\oplus \cdot \hat{n})(r_r - \vec{x}_r \cdot \hat{n})}{d^2}\,, \tag{4.2}$$

where $\vec{x}_\oplus$ is the position of the Earth, $\vec{x}_r$ is the position of the reflector, $r_\oplus = |\vec{x}_\oplus|$ and $r_r = |\vec{x}_r|$ are their distances from the Sun, $\hat{n}$ is the direction of the photon on its return flight and d is the distance at which the signal passes by the Sun. The time delay δt is maximal when the reflector is in superior conjunction, i.e., straight behind the Sun, in which case

$$\delta t = 2(1 + \gamma)m\ln\frac{4r_\oplus r_r}{d^2}\,. \tag{4.3}$$

This method has been used by the Cassini experiment, in which $d = 1.6r_\odot$ and $r_r = 8.43\mathrm{AU}$. The resulting value of γ obtained from this experiment is $\gamma - 1 = (2.1 \pm 2.3) \cdot 10^{-5}$ [2].

4.2 β

The PPN parameter β measures the non-linearity in the Newtonian law of gravity. It can be measured, for example, by the perihelion precession of Mercury. The integrated perihelion shift over one orbital period of Mercury is given by

$$\Delta\tilde{\omega} = \frac{6\pi m}{p}\left[\frac{2+2\gamma-\beta}{3} + \frac{2\alpha_1 - \alpha_2 + \alpha_3 - 2\zeta_2}{6}\frac{\mu}{m} + J_2\frac{R^2}{2mp}\right], \tag{4.4}$$

where m is the solar mass, μ is the mass of Mercury, p is the perihelion distance of Mercury from the Sun, J_2 is the solar quadrupole moment and R is the solar radius. The second term can be neglected since

$$\frac{\mu}{m} = \frac{M_{☿}}{M_{\odot}} \approx 2 \cdot 10^{-7}. \tag{4.5}$$

Currently the best bound obtained from experiments is given by $|2\gamma - \beta - 1| < 3 \cdot 10^{-3}$ [2].

4.3 ξ

The PPN parameter ξ introduces a term into the Lagrangian of an n-body system of the form

$$L_\xi = -\frac{\xi}{2}\sum_{i,j}\frac{m_i m_j}{r_{ij}^3}\vec{r}_{ij}\cdot\left[\sum_k m_k\left(\frac{\vec{r}_{jk}}{r_{ik}} - \frac{\vec{r}_{ik}}{r_{jk}}\right)\right]. \tag{4.6}$$

As a consequence, the Newtonian law for the interaction between two bodies acquires a dependence on their location relative to other gravitating bodies. This term therefore violates local position independence.

4.4 Violation of Lorentz invariance

The metric (3.1) is in general not invariant under local Lorentz transformations [1]. Transformation of this metric to a system which is moving with a relative velocity $\vec{w}$ introduces terms into the metric which explicitly depend on $\vec{w}$. However, these terms have coefficients which are linear combinations of the parameters $\alpha_1, \alpha_2, \alpha_3$. They therefore vanish for $\alpha_1 = \alpha_2 = \alpha_3 = 0$. These parameters thus measure the violation of Lorentz invariance.

4.5 Violation of total energy-momentum conservation

The local covariant energy-momentum conservation $\nabla_\mu T^{\mu\nu}$ has the post-Newtonian approximation

$$0 = \nabla_\mu T^{\mu 0} = \rho_{,0} + (\rho v_i)_{,i} + \mathcal{O}(5), \tag{4.7a}$$

$$0 = \nabla_\mu T^{\mu i} = \rho\frac{\mathrm{d}v_i}{\mathrm{d}t} + p_{,i} - \rho U_{,i} + \mathcal{O}(6). \tag{4.7b}$$

These conservation laws are satisfied independent of the theory of gravity, because they are a consequence of the diffeomorphism invariance of the matter action.
In contrast, global conservation laws concern the conservation of the total energy-momentum of a gravitating system in an asymptotically flat spacetime. These depend both on the matter and gravity theories. In the post-Newtonian approximation the conservation of the total energy-momentum is given only if the local conservation laws can be integrated to global conservation laws. The integrability condition involves the PPN parameters. It can be shown that the local conservation laws are integrable only for $\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$ [1]. These parameters thus measure the violation of total energy-momentum conservation.

References

[1] C. M. Will, *Theory and experiment in gravitational physics*, Cambridge University Press 1993.

[2] C. M. Will, Living Rev. Rel. **17** (2014) 4 [arXiv:1403.7377 [gr-qc]].

[3] C. M. Will, *Theory and experiment in gravitational physics*, Cambridge University Press 2018.