Perturbations of connections, tetrads and Finsler functions

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December 17, 2020

1 Connections and metric-affine geometry

So far we have considered the metric $g_{\mu\nu}$ as the field which defines the spacetime geometry, possibly together with other tensor fields, and performed a perturbative expansion of these tensor fields. However, there exists also the possibility to consider an affine connection as a fundamental field which defines the spacetime geometry, which is thus not a tensor field. Also in this case one can perform a perturbative expansion, which we shall study now.

1.1 Definition

In Riemannian geometry, which is the geometric foundation of general relativity, we are used to encountering the Levi-Civita connection of the metric $g_{\mu\nu}$, whose coefficients are given by

$$\overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho})\,, \tag{1.1}$$

where we now use a circle on $\overset{\circ}{\Gamma}{}^{\mu}{}_{\nu\rho}$ and all of its derived quantities, in order to distinguish it from other connections which we will encounter in this lecture. The coefficients of the Levi-Civita connection are also called the Christoffel symbols, and they define the covariant derivative of a vector field X^μ as

$$\overset{\circ}{\nabla}_\mu X^\nu = \partial_\mu X^\nu + \overset{\circ}{\Gamma}{}^{\nu}{}_{\rho\mu}X^\rho\,. \tag{1.2}$$

In general, we can understand an *affine connection* as a geometric object, which defines the covariant derivative of a tensor field, such that this becomes another tensor field. For a general connection, we retain the same form of the covariant derivative

$$\hat{\nabla}_\mu X^\nu = \partial_\mu X^\nu + \hat{\Gamma}^{\nu}{}_{\rho\mu}X^\rho\,, \tag{1.3}$$

but with arbitrary coefficients, which we now write as $\hat{\Gamma}^{\mu}{}_{\nu\rho}$. In order for $\hat{\nabla}_\mu X^\nu$ to transform as a tensor field under coordinate transformations $(x^\mu) \mapsto (\tilde{x}^\mu)$, the connection coefficients must transform as

$$\hat{\Gamma}^{\mu}{}_{\nu\rho} = \hat{\Gamma}'^{\alpha}{}_{\beta\gamma}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x'^\nu}\frac{\partial x'^\gamma}{\partial x'^\rho} + \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial^2 x'^\alpha}{\partial x'^\nu \partial x'^\rho}\,, \tag{1.4}$$

and so they *do not* constitute the components of a tensor field, due to the appearance of an inhomogeneous term. We can, however, obtain several tensor fields from a connection. The most familiar one, as it appears in general relativity, is the curvature. For a general connection, it is given by

$$\hat{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu\hat{\Gamma}^{\rho}{}_{\sigma\nu} - \partial_\nu\hat{\Gamma}^{\rho}{}_{\sigma\mu} + \hat{\Gamma}^{\rho}{}_{\tau\mu}\hat{\Gamma}^{\tau}{}_{\sigma\nu} - \hat{\Gamma}^{\rho}{}_{\tau\nu}\hat{\Gamma}^{\tau}{}_{\sigma\mu}\,. \tag{1.5}$$

Another important tensor field is the torsion, defined by

$$\hat{T}^{\mu}{}_{\nu\rho} = \hat{\Gamma}^{\mu}{}_{\rho\nu} - \hat{\Gamma}^{\mu}{}_{\nu\rho}\,. \tag{1.6}$$

Finally, if one also has a metric in addition to the connection, one enters the realm of *metric-affine geometry*. In this case one may define another tensor field, called the nonmetricity, given by

$$\hat{Q}_{\rho\mu\nu} = \hat{\nabla}_\rho g_{\mu\nu}\,. \tag{1.7}$$

In general, none of these tensors vanish. The Levi-Civita connection, however, is a special case; it is the unique affine connection, which satisfies the conditions

$$\mathring{T}^{\mu}{}_{\nu\rho} = \mathring{\Gamma}^{\mu}{}_{\rho\nu} - \mathring{\Gamma}^{\mu}{}_{\nu\rho} \tag{1.8}$$

and

$$\mathring{Q}_{\rho\mu\nu} = \mathring{\nabla}_{\rho} g_{\mu\nu} \tag{1.9}$$

of vanishing torsion and nonmetricity. Finally, given a metric, one may decompose the coefficients $\hat{\Gamma}^{\mu}{}_{\nu\rho}$ uniquely in the form

$$\hat{\Gamma}^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} + \hat{K}^{\rho}{}_{\mu\nu} + \hat{L}^{\rho}{}_{\mu\nu}\,, \tag{1.10}$$

where we have introduced the contortion tensor

$$\hat{K}^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(\hat{T}_{\nu}{}^{\mu}{}_{\rho} + \hat{T}_{\rho}{}^{\mu}{}_{\nu} - \hat{T}^{\mu}{}_{\nu\rho}\right)\,, \tag{1.11}$$

as well as the disformation tensor

$$\hat{L}^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(\hat{Q}^{\mu}{}_{\nu\rho} - \hat{Q}_{\nu}{}^{\mu}{}_{\rho} - \hat{Q}_{\rho}{}^{\mu}{}_{\nu}\right)\,. \tag{1.12}$$

Note that these two tensor fields are defined only if both a metric and an affine connection are present, since the metric is required in order to raise and lower indices on the torsion and nonmetricity. For the Levi-Civita connection, they vanish,

$$\mathring{K}^{\rho}{}_{\mu\nu} = \mathring{L}^{\rho}{}_{\mu\nu} = 0\,, \tag{1.13}$$

which is obvious from the definition.

1.2 Perturbations

We now consider an affine connection, whose coefficients are given as a linear perturbation

$$\hat{\Gamma}^{\mu}{}_{\nu\rho} = \bar{\hat{\Gamma}}^{\mu}{}_{\nu\rho} + \delta\hat{\Gamma}^{\mu}{}_{\nu\rho}\,, \tag{1.14}$$

where $\bar{\hat{\Gamma}}^{\mu}{}_{\nu\rho}$ denote the coefficients of a background connection, while $\delta\hat{\Gamma}^{\mu}{}_{\nu\rho}$ are given as the difference of connection coefficients, and so they form the components of a tensor field. One can now easily calculate the perturbations of the derived tensor fields which we showed above. The most straightforward is the torsion (1.6), for which we find

$$\delta\hat{T}^{\mu}{}_{\nu\rho} = \delta\hat{\Gamma}^{\mu}{}_{\rho\nu} - \delta\hat{\Gamma}^{\mu}{}_{\nu\rho}\,. \tag{1.15}$$

We then continue with the curvature (1.5). Since here we consider only linear perturbations, we find

$$\delta\hat{R}^{\rho}{}_{\sigma\mu\nu} = \bar{\hat{\nabla}}_{\mu}\delta\hat{\Gamma}^{\rho}{}_{\sigma\nu} - \bar{\hat{\nabla}}_{\nu}\delta\hat{\Gamma}^{\rho}{}_{\sigma\mu} + \bar{\hat{T}}^{\omega}{}_{\mu\nu}\delta\hat{\Gamma}^{\rho}{}_{\sigma\omega}\,. \tag{1.16}$$

Finally, we come to the nonmetricity. Here we assume that also the metric is given in the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{1.17}$$

as a linear perturbation around a background metric $\bar{g}_{\mu\nu}$. Considering the perturbations of the metric and the affine connection to be independent of each other, and keeping only terms up to the linear order in either perturbation, we then find that the perturbation of the nonmetricity is given by

$$\delta\hat{Q}_{\rho\mu\nu} = \bar{\hat{\nabla}}_{\rho}\delta g_{\mu\nu} - \delta\hat{\Gamma}^{\sigma}{}_{\mu\rho}\bar{g}_{\sigma\nu} - \delta\hat{\Gamma}^{\sigma}{}_{\nu\rho}\bar{g}_{\mu\sigma}\,. \tag{1.18}$$

Similar rules can also be derived for the covariant derivative of other tensor fields.

1.3 Gauge transformations

We now perform an infinitesimal coordinate transformation of the form

$$x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x)\,, \tag{1.19}$$

where we assume that the coefficients ξ^μ of the gauge transforming vector field are sufficiently small to retain the linear perturbation of the geometry. From the transformed connection coefficients, we define the perturbation by subtracting the same background as for the original connection, hence

$$\hat{\Gamma}'^\mu{}_{\nu\rho} = \hat{\bar{\Gamma}}^\mu{}_{\nu\rho} + \delta\hat{\Gamma}'^\mu{}_{\nu\rho}\,. \tag{1.20}$$

We can insert this perturbation and the coordinate change in the transformation (1.4) and find that, at linear order in ξ^μ, the change of the connection is given by

$$\begin{aligned}\delta\hat{\Gamma}^\mu{}_{\nu\rho} - \delta\hat{\Gamma}'^\mu{}_{\nu\rho} &= (\mathcal{L}_\xi\hat{\bar{\Gamma}})^\mu{}_{\nu\rho} \\ &= \xi^\sigma\partial_\sigma\hat{\bar{\Gamma}}^\mu{}_{\nu\rho} - \partial_\sigma\xi^\mu\hat{\bar{\Gamma}}^\sigma{}_{\nu\rho} + \partial_\nu\xi^\sigma\hat{\bar{\Gamma}}^\mu{}_{\sigma\rho} + \partial_\rho\xi^\sigma\hat{\bar{\Gamma}}^\mu{}_{\nu\sigma} + \partial_\nu\partial_\rho\xi^\mu \\ &= \hat{\bar{\nabla}}_\rho\hat{\bar{\nabla}}_\nu\xi^\mu - \xi^\sigma\hat{\bar{R}}^\mu{}_{\nu\rho\sigma} - \hat{\bar{\nabla}}_\rho(\xi^\sigma\hat{\bar{T}}^\mu{}_{\nu\sigma})\,.\end{aligned} \tag{1.21}$$

Here we find the Lie derivative of the connection coefficients, as their change under an infinitesimal coordinate transformation. Note that this is again a tensor field [Yan57], which can be seen by the last line, where we have expressed it in terms of the tensors we introduced earlier. This is consistent with the left hand side of the equation above, since the difference between two tensor fields must again be a tensor field.

2 Tetrads and teleparallel geometry

Another possible description of geometry, which makes use of a different connection as laid out above, is used in teleparallel theories of gravity [AP13]. In the conventional formulation of these theories, one uses a tetrad and a spin connection as fundamental fields, and then studies their perturbation. We will now see hoe to relate these approaches.

2.1 Definition

The definition of teleparallel geometry we use here will follow the discussion of metric-affine geometry above, and is known as the Palatini approach. Hence, we will assume that the fundamental fields are a metric $g_{\mu\nu}$ and an affine connection, whose coefficients we will now denote as $\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho}$ to distinguish it from the most general connection which we introduced above. The teleparallel connection is characterized by two additional conditions, namely vanishing curvature, and

$$\overset{\bullet}{R}{}^\rho{}_{\sigma\mu\nu} = \partial_\mu\overset{\bullet}{\Gamma}{}^\rho{}_{\sigma\nu} - \partial_\nu\overset{\bullet}{\Gamma}{}^\rho{}_{\sigma\mu} + \overset{\bullet}{\Gamma}{}^\rho{}_{\tau\mu}\overset{\bullet}{\Gamma}{}^\tau{}_{\sigma\nu} - \overset{\bullet}{\Gamma}{}^\rho{}_{\tau\nu}\overset{\bullet}{\Gamma}{}^\tau{}_{\sigma\mu} \equiv 0\,, \tag{2.1}$$

as well as compatibility with the metric, hence vanishing nonmetricity

$$\overset{\bullet}{Q}{}_{\rho\mu\nu} = \overset{\bullet}{\nabla}{}_\rho g_{\mu\nu} \equiv 0\,. \tag{2.2}$$

While this definition is suitable for most use cases, one mostly finds a different approach in the literature. One fundamental field in this approach is the tetrad $\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu$, where $a = 0,\ldots,3$ is a Lorentz index. Hence, one may interpret the tetrad as a collection of one-forms, labeled by a. One further demands that at every point of the spacetime manifold these one-forms constitute a basis of the cotangent space; hence, there exists an inverse $e_a = e_a{}^\mu\partial_\mu$ which is uniquely determined from θ^a by the condition

$$e_a{}^\mu\theta^b{}_\mu = \delta^b_a\,, \quad e_a{}^\mu\theta^a{}_\nu = \delta^\mu_\nu\,. \tag{2.3}$$

Demanding this basis to be orthonormal with respect to the metric,

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}\,, \tag{2.4}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric, then uniquely determines the metric in terms of the tetrad as

$$g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu\theta^b{}_\nu\,. \tag{2.5}$$

The second fundamental field used in this formulation of teleparallel geometry is the spin connection $\overset{\bullet}{\omega}{}^a{}_b = \overset{\bullet}{\omega}{}^a{}_{b\mu}\mathrm{d}x^\mu$. It can be related to the affine connection by the "tetrad postulate", which states that the total derivative

$$\partial_\mu\theta^a{}_\nu + \overset{\bullet}{\omega}{}^a{}_{b\mu}\theta^b{}_\nu - \overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu}\theta^a{}_\rho = 0 \tag{2.6}$$

should vanish. This means that the affine connection is determined from the tetrad and the spin connection as

$$\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho} = e_a{}^\mu\left(\partial_\rho\theta^a{}_\nu + \overset{\bullet}{\omega}{}^a{}_{b\rho}\theta^b{}_\nu\right)\,. \tag{2.7}$$

This relation also poses certain restrictions on the spin connection. The condition (2.1) of vanishing curvature means that the spin connection must satisfy

$$\overset{\bullet}{R}{}^a{}_{b\mu\nu} = \partial_\mu\overset{\bullet}{\omega}{}^a{}_{b\nu} - \partial_\nu\overset{\bullet}{\omega}{}^a{}_{b\mu} + \overset{\bullet}{\omega}{}^a{}_{c\mu}\overset{\bullet}{\omega}{}^c{}_{b\nu} - \overset{\bullet}{\omega}{}^a{}_{c\nu}\overset{\bullet}{\omega}{}^c{}_{b\mu} \equiv 0\,, \tag{2.8}$$

while the metric compatibility (2.2) means that it must be antisymmetric,

$$\overset{\bullet}{Q}{}_{\mu ab} = -\eta_{ac}\overset{\bullet}{\omega}{}^c{}_{b\mu} - \eta_{cb}\overset{\bullet}{\omega}{}^c{}_{a\mu} \equiv 0\,. \tag{2.9}$$

Given a tetrad $\theta^a{}_\mu$ and a spin connection $\overset{\bullet}{\omega}{}^a{}_{b\mu}$ satisfying the conditions (2.8) and (2.9), these uniquely define the metric and affine connection via the relations (2.5) and (2.5). The converse, however, is not true. Any other tetrad $\theta'^a{}_\mu$ yields the same metric (2.5), if and only if these tetrads are related by a local Lorentz transformation $\Lambda^a{}_b$ via

$$\theta'^a{}_\mu = \Lambda^a{}_b\theta^b{}_\mu\,, \tag{2.10}$$

where $\Lambda^b{}_b$ must satisfy

$$\eta_{ab}\Lambda^a{}_c\Lambda^b{}_d = \eta_{cd}\,. \tag{2.11}$$

Once the tetrad is chosen, also the spin connection is uniquely determined from the tetrad and the affine connection as

$$\overset{\bullet}{\omega}{}^a{}_{b\mu} = e_b{}^\nu(\overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu}\theta^a{}_\rho - \partial_\mu\theta^a{}_\nu)\,, \tag{2.12}$$

which is simply another possibility to rearrange the tetrad postulate (2.6). It then follows that choosing another tetrad $\theta'^a{}_\mu$, which is related to the tetrad $\theta^a{}_\mu$ via the Lorentz transformation (2.10), one obtains the same affine connection $\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho}$ if and only if one also transforms the spin connection as

$$\overset{\bullet}{\omega}{}'^a{}_{b\mu} = \Lambda^a{}_c(\Lambda^{-1})^d{}_b\,\overset{\bullet}{\omega}{}^c{}_{d\mu} + \Lambda^a{}_c\,\partial_\mu(\Lambda^{-1})^c{}_b\,. \tag{2.13}$$

Hence, the freedom of choosing the tetrad and spin connection to represent a given metric and teleparallel affine connection amounts exactly to local Lorentz transformations. Finally, it follows from the flatness and metric compatibility of the spin connection that (at least locally) one can always choose a local Lorentz transformation such that the spin connection vanishes, $\overset{\bullet}{\omega}{}^a{}_{b\mu} \equiv 0$. This choice is known as the *Weitzenböck gauge*. In the following, we will be working in this Weitzenböck gauge for simplicity. In this case the tetrad is determined up to a global Lorentz transformation, and it is the only fundamental field, which in turn defined the metric and affine connection.

2.2 Perturbations

Following the introduction above, we now consider a tetrad $\theta^a{}_\mu$ which is given as a perturbation around a background tetrad $\bar{\theta}^a{}_\mu$ as

$$\theta^a{}_\mu = \bar{\theta}^a{}_\mu + \delta\theta^a{}_\mu\,. \tag{2.14}$$

We could now develop the perturbation theory for the perturbation $\delta\theta^a{}_\mu$, as we have done for other fields. However, in practice it turns out that it is more convenient to express the perturbation through another field, which is given by

$$\tau_{\mu\nu} = \eta_{ab}\bar{\theta}^a{}_\mu\delta\theta^b{}_\nu\,. \tag{2.15}$$

Note that since the tetrad perturbation $\delta\theta^a{}_\mu$ has 16 independent components, the same holds for $\tau_{\mu\nu}$, i.e., all of its components are independent. To see its usefulness, we calculate the perturbation of the metric (2.5), and find that it is given by

$$\delta g_{\mu\nu} = 2\tau_{(\mu\nu)} \,. \tag{2.16}$$

This can be used for calculating the linear perturbation of the Levi-Civita connection. As we have seen in a previous lecture, the latter can most easily be expressed as the covariant derivative of the metric perturbation, and thus reads

$$\delta\overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\overset{\circ}{\nabla}}_\mu\delta g_{\sigma\nu} + \bar{\overset{\circ}{\nabla}}_\nu\delta g_{\mu\sigma} - \bar{\overset{\circ}{\nabla}}_\sigma\delta g_{\mu\nu}\right) = \bar{g}^{\rho\sigma}\left(\bar{\overset{\circ}{\nabla}}_\mu\tau_{(\sigma\nu)} + \bar{\overset{\circ}{\nabla}}_\nu\tau_{(\mu\sigma)} - \bar{\overset{\circ}{\nabla}}_\sigma\tau_{(\mu\nu)}\right) . \tag{2.17}$$

For the teleparallel affine connection (2.5) we find the formula

$$\delta\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho} = \bar{\overset{\bullet}{\nabla}}_\rho\tau^\mu{}_\nu \,, \tag{2.18}$$

where the index of $\tau^\mu{}_\nu$ has been raised with the background metric. Hence, we see that we can easily express the perturbations of relevant geometric objects with the perturbation field $\tau_{\mu\nu}$. Also we now see how this perturbation can be related to the metric-affine description of teleparallel geometry which we started from, based on a metric $g_{\mu\nu}$ and a flat, metric compatible affine connection $\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho}$. Recall that for a general connection, the perturbation of the curvature is given by the relation (1.16), so that this expression must vanish in the case of a teleparallel affine connection. Indeed, we find

$$\begin{aligned}
\delta\overset{\bullet}{R}{}^\rho{}_{\sigma\mu\nu} &= \bar{\overset{\bullet}{\nabla}}_\mu\delta\overset{\bullet}{\Gamma}{}^\rho{}_{\sigma\nu} - \bar{\overset{\bullet}{\nabla}}_\nu\delta\overset{\bullet}{\Gamma}{}^\rho{}_{\sigma\mu} + \bar{\overset{\bullet}{T}}{}^\omega{}_{\mu\nu}\delta\overset{\bullet}{\Gamma}{}^\rho{}_{\sigma\omega} \\
&= \bar{\overset{\bullet}{\nabla}}_\mu\bar{\overset{\bullet}{\nabla}}_\nu\tau^\rho{}_\sigma - \bar{\overset{\bullet}{\nabla}}_\nu\bar{\overset{\bullet}{\nabla}}_\mu\tau^\rho{}_\sigma + \bar{\overset{\bullet}{T}}{}^\omega{}_{\mu\nu}\bar{\overset{\bullet}{\nabla}}_\omega\tau^\rho{}_\sigma \\
&= \overset{\bullet}{R}{}^\rho{}_{\omega\mu\nu}\tau^\omega{}_\sigma - \overset{\bullet}{R}{}^\omega{}_{\sigma\mu\nu}\tau^\rho{}_\omega \\
&= 0 \,,
\end{aligned} \tag{2.19}$$

since the curvature of the teleparallel connection vanishes. Similarly, for the nonmetricity we use the relation (1.18) to find the perturbation

$$\begin{aligned}
\delta\overset{\bullet}{Q}{}_{\rho\mu\nu} &= \bar{\overset{\bullet}{\nabla}}_\rho\delta g_{\mu\nu} - \delta\overset{\bullet}{\Gamma}{}^\sigma{}_{\mu\rho}\bar{g}_{\sigma\nu} - \delta\overset{\bullet}{\Gamma}{}^\sigma{}_{\nu\rho}\bar{g}_{\mu\sigma} \\
&= 2\bar{\overset{\bullet}{\nabla}}_\rho\tau_{(\mu\nu)} - \bar{\overset{\bullet}{\nabla}}_\rho\tau_{\nu\mu} - \bar{\overset{\bullet}{\nabla}}_\rho\tau_{\mu\nu} \\
&= 0 \,,
\end{aligned} \tag{2.20}$$

so that the perturbation retains also the vanishing nonmetricity.

2.3 Gauge transformations

As we have seen for tensor fields and connections, we can also perform a gauge transformation of a perturbation in the teleparallel geometry. To obtain a formula for the different expressions which we have studied above, recall that the geometry (in the Weitzenböck gauge) is defined by the tetrad. Hence, we must study how it transforms under infinitesimal diffeomorphisms, generated by a vector field ξ^μ. Since it is a tensor field, constituted by one-forms, it must transform with the Lie derivative

$$\theta^a{}_\mu = \theta'^a{}_\mu + (\mathcal{L}_\xi\bar{\theta})^a{}_\mu \,, \tag{2.21}$$

where only terms of at most linear order in the tetrad perturbation and the vector field ξ^μ have been considered; hence, the tetrad has been replaced by the background tetrad $\bar{\theta}^a{}_\mu$ in the second term. Writing the transformed tetrad as a perturbation of the same background tetrad,

$$\theta'^a{}_\mu = \bar{\theta}^a{}_\mu + \delta\theta'^a{}_\mu \,, \tag{2.22}$$

we find that the transformation of the perturbation is given by

$$\delta\theta^a{}_\mu - \delta\theta'^a{}_\mu = (\mathcal{L}_\xi\bar{\theta})^a{}_\mu = \xi^\nu\partial_\nu\bar{\theta}^a{}_\mu + \partial_\mu\xi^\nu\bar{\theta}^a{}_\nu \,, \tag{2.23}$$

using the formula for the Lie derivative of a one-form. Lowering and transforming the Lorentz index with the background geometry, and replacing the partial derivatives acting on the background tetrad and the vector field by coefficients of the teleparallel affine connection and covariant derivatives, we find that the perturbation tensor field changes as

$$\begin{aligned}\tau_{\mu\nu} - \tau'_{\mu\nu} &= \eta_{ab}\bar{\theta}^a{}_\mu(\delta\theta^b{}_\nu - \delta\theta'^b{}_\nu)\\ &= \eta_{ab}\bar{\theta}^a{}_\mu(\xi^\rho\partial_\rho\bar{\theta}^b{}_\nu + \partial_\nu\xi^\rho\bar{\theta}^b{}_\rho)\\ &= \bar{g}_{\mu\rho}\left(\overset{\bullet}{\bar{\Gamma}}{}^\rho{}_{\nu\sigma}\xi^\sigma + \partial_\nu\xi^\rho\right)\\ &= \overset{\bullet}{\bar{\nabla}}_\nu\xi_\mu - \overset{\bullet}{\bar{T}}_{\mu\nu}{}^\rho\xi_\rho\\ &= \overset{\circ}{\bar{\nabla}}_\nu\xi_\mu + \overset{\bullet}{\bar{K}}_{\mu\nu}{}^\rho\xi_\rho\,.\end{aligned} \tag{2.24}$$

From this result, it is now easy to derive the transformation of the metric perturbation, which is given by

$$\delta g_{\mu\nu} - \delta g'_{\mu\nu} = 2(\tau_{(\mu\nu)} - \tau'_{(\mu\nu)}) = 2\overset{\circ}{\bar{\nabla}}_{(\mu}\xi_{\nu)}\,, \tag{2.25}$$

which gives the well-known formula, as one may have expected. Similarly, the connection perturbation transforms as

$$\delta\overset{\bullet}{\Gamma}{}^\mu{}_{\nu\rho} - \delta\overset{\bullet}{\Gamma}{}'^\mu{}_{\nu\rho} = \overset{\bullet}{\bar{\nabla}}_\rho(\tau^\mu{}_\nu - \tau'^\mu{}_\nu) = \overset{\bullet}{\bar{\nabla}}_\rho\overset{\bullet}{\bar{\nabla}}_\nu\xi^\mu - \overset{\bullet}{\bar{\nabla}}_\rho(\overset{\bullet}{\bar{T}}{}^\mu{}_{\nu\sigma}\xi^\sigma)\,, \tag{2.26}$$

which is simply the gauge transformation (1.21) of a connection with vanishing curvature.

3 Finsler functions and Finsler geometry

Finally, one may describe the geometry of spacetime also by considering fields which are not defined on the spacetime manifold M itself, but on its tangent bundle TM. This is the realm of Finsler geometry [BM07]. Here we can only cover a few basic elements.

3.1 Definition

In general relativity and various other gravity theories, it is assumed that the action for a point particle is given by its proper time, which in turn is defined via the geodesic length

$$\int_{t_1}^{t_2}\sqrt{|g_{\mu\nu}(x(t))\dot{x}^\mu(t)\dot{x}^\nu(t)|}\mathrm{d}t\,, \tag{3.1}$$

in terms of the metric $g_{\mu\nu}$. The basic idea of Finsler geometry is to generalize this length functional to be of the form

$$\int_{t_1}^{t_2} F(x(t), \dot{x}(t))\,\mathrm{d}t\,, \tag{3.2}$$

where F is called the *Finsler function*. It is a function which depends both on the position and velocity of the point particle, and hence may be regarded as a function on the tangent bundle TM of the spacetime manifold M. In order for the integral (3.2) to be independent of the parametrization of the point particle trajectory, one must demand

$$F(x, \lambda\dot{x}) = \lambda F(x, \dot{x}) \tag{3.3}$$

for all $\lambda > 0$. In other words, F must be a 1-homogeneous function on the tangent bundle.

3.2 Perturbations

As with the other geometric objects we studied so far, we can consider a perturbation of the Finsler function which is of the form

$$F(x, \dot{x}) = \bar{F}(x, \dot{x}) + \delta F(x, \dot{x})\,, \tag{3.4}$$

where we restrict ourselves to linear perturbations. Since both F and $\bar{F}$ are Finsler functions, and hence 1-homogeneous by definition, it follows that

$$\delta F(x, \lambda\dot{x}) = F(x, \lambda\dot{x}) - \bar{F}(x, \lambda\dot{x}) = \lambda[F(x, \dot{x}) - \bar{F}(x, \dot{x})] = \lambda\delta F(x, \dot{x})\,, \tag{3.5}$$

so that also the perturbation δF must be 1-homogeneous.

3.3 Gauge transformations

We finally pose the question how the perturbation of the Finsler function changes under infinitesimal coordinate transformations on the spacetime manifold M. For this purpose it is helpful to note that the velocity $\dot{x}$ behaves as a tangent vector, and so under a coordinate transformation $x \mapsto x'$ it transforms as

$$\dot{x}^{\mu} \mapsto \dot{x}'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \dot{x}^{\nu} \,. \tag{3.6}$$

Hence, under the infinitesimal coordinate change generated by a vector field ξ^{μ} we have

$$\dot{x}'^{\mu} = \dot{x}^{\mu} + \dot{x}^{\nu} \partial_{\nu} \xi^{\mu} \,. \tag{3.7}$$

We then apply this transformation to the Finsler function. At linear perturbation order, we then find the change of the perturbation as usual by the Lie derivative of the background. In the case of the Finsler function and the infinitesimal coordinate change given above, this takes the form

$$\delta F(x, \dot{x}) - \delta F'(x, \dot{x}) = \xi^{\mu} \frac{\partial}{\partial x^{\mu}} \bar{F}(x, \dot{x}) + \dot{x}^{\nu} \partial_{\nu} \xi^{\mu} \frac{\partial}{\partial \dot{x}^{\mu}} \bar{F}(x, \dot{x}) \,. \tag{3.8}$$

Of course, the question arises whether this transformation is compatible with the condition that the perturbation of the Finsler function is 1-homogeneous. This is indeed the case. It follows from Euler's homogeneous function theorem that $\partial \bar{F} / \partial x^{\mu}$ is 1-homogeneous, while $\partial \bar{F} / \partial \dot{x}^{\mu}$ is 0-homogeneous. Since the last term also comes with a factor $\dot{x}^{\nu}$, which is again 1-homogeneous, and ξ^{μ} is 0-homogeneous, since it is a vector field on the base manifold M and does not depend on the velocity, one finds that the whole expression is 1-homogeneous, as required.

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