Production of gravitational waves

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November 9, 2020

1 Inhomogeneous wave equation

The starting point for our derivation is the Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{1.1}$$

where the Einstein tensor is expressed as

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} \tag{1.2}$$

in the harmonic gauge

$$\partial^{\mu}\bar{h}_{\mu\nu} = 0\,, \tag{1.3}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h^{\rho}{}_{\rho}\eta_{\mu\nu} \tag{1.4}$$

is the trace-reversed metric perturbation. Hence, the equation we aim to solve is the inhomogeneous wave equation

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{1.5}$$

There are different possible ways to solve this equation. The most straightforward is to employ the method of Green's functions. The correct choice of the Green's function depends on the properties of the source and the type of solution one aims to obtain. Here we will make the following assumptions:

1. The source is contained in a compact region of space.
2. There is only outgoing and no incoming radiation.

Under these assumptions, the inhomogeneous wave equation (1.5) can be solved by using the retarded Green's function. The solution then reads

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G\int \mathrm{d}^3x'\,\frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}\,. \tag{1.6}$$

This integral converges if we assume that the source is located inside a compact domain.

2 Fourier transformation

For practical purposes it is often more convenient to consider the Fourier transform of the metric perturbation and energy-momentum tensor. Here we will consider only a Fourier transform of the time domain, which for any function $f(t)$ can be defined as

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}t\,f(t)e^{i\omega t}\,, \tag{2.1}$$

with inverse given by

$$f(t) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega\,\hat{f}(\omega)e^{-i\omega t}\,. \tag{2.2}$$

Applied to the metric perturbation, this yields the relation

$$\begin{aligned}
\hat{\bar{h}}_{\mu\nu}(\omega,\vec{x}) &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}t\,\bar{h}_{\mu\nu}(t,\vec{x})e^{i\omega t}\\
&= \frac{4G}{\sqrt{2\pi}}\int \mathrm{d}t\int \mathrm{d}^3x'\,\frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}e^{i\omega t}\\
&= \frac{4G}{\sqrt{2\pi}}\int \mathrm{d}\tilde{t}\int \mathrm{d}^3x'\,\frac{T_{\mu\nu}(\tilde{t},\vec{x}')}{|\vec{x}-\vec{x}'|}e^{i\omega(\tilde{t}+|\vec{x}-\vec{x}'|)}\\
&= \frac{4G}{\sqrt{2\pi}}\int \mathrm{d}^3x'\,\frac{e^{i\omega|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\int \mathrm{d}\tilde{t}\,T_{\mu\nu}(\tilde{t},\vec{x}')e^{i\omega\tilde{t}}\\
&= 4G\int \mathrm{d}^3x'\,\frac{e^{i\omega|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\hat{T}_{\mu\nu}(\omega,\vec{x}')\\
&\approx 4G\frac{e^{i\omega r}}{r}\int \mathrm{d}^3x'\hat{T}_{\mu\nu}(\omega,\vec{x}')\,,
\end{aligned} \tag{2.3}$$

Here we have performed various simplifications and transformations. First, we have made use of the solution (1.6) for the metric perturbation using the Green's function method. Then, we have defined a new time coordinate

$$\tilde{t} = t - |\vec{x}-\vec{x}'|\,, \tag{2.4}$$

and changed the integral from t to $\tilde{t}$. In the next step we have reordered the integrals over time and space, and grouped under the time integral only those terms which depend on time. This integral turns out to be simply the Fourier transform of the energy-momentum tensor $T_{\mu\nu}$, so we replaced it with $\hat{T}_{\mu\nu}$. Finally, we made the approximation

$$|\vec{x}-\vec{x}'| \approx r\,, \tag{2.5}$$

where r is the distance between the observer at $\vec{x}$ and the source at $\vec{x}'$, which is based on the assumption that the extension of the source is small compared to the distance to the observer, so that all points of the source will be at approximately the same distance r.

3 The quadrupole formula

In the following, we will switch to upper indices again, as it will turn out to be more convenient. Recall that in deriving the inhomogeneous wave equation (1.5) we imposed the harmonic gauge condition

$$0 = \partial_\nu\bar{h}^{\nu\mu} = \partial_0\bar{h}^{0\mu} + \partial_i\bar{h}^{i\mu}\,. \tag{3.1}$$

Applying the Fourier transform to this equation, we find the relation

$$\begin{aligned}
\partial_i\hat{\bar{h}}^{i\mu}(\omega,\vec{x}) &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}t\,\partial_i\bar{h}^{i\mu}(t,\vec{x})e^{i\omega t}\\
&= -\frac{1}{\sqrt{2\pi}}\int \mathrm{d}t\,\partial_0\bar{h}^{0\mu}(t,\vec{x})e^{i\omega t}\\
&= -\frac{1}{2\pi}\int \mathrm{d}t\,e^{i\omega t}\int \mathrm{d}\omega'\,\hat{\bar{h}}^{0\mu}(\omega',\vec{x})\partial_0 e^{-i\omega' t}\\
&= \frac{1}{2\pi}\int \mathrm{d}t\,e^{i\omega t}\int \mathrm{d}\omega'\,i\omega'\hat{\bar{h}}^{0\mu}(\omega',\vec{x})e^{-i\omega' t}\\
&= i\omega\hat{\bar{h}}^{0\mu}(\omega,\vec{x})\,,
\end{aligned} \tag{3.2}$$

Hence, once we know $\hat{\bar{h}}^{i\mu}$, we can easily obtain

$$\hat{\bar{h}}^{0\mu}(\omega,\vec{x}) = \frac{1}{i\omega}\partial_i\hat{\bar{h}}^{i\mu}(\omega,\vec{x})\,. \tag{3.3}$$

Starting from first setting the index ν to a spatial index j, we obtain the mixed components, and then we apply the same formula again with $\mu = 0$ to calculate the time-time components. Hence, we only need to calculate $\hat{\bar{h}}^{ij}$, and can then derive all other components. To achieve this, we calculate

$$\hat{\bar{h}}^{ij}(\omega, \vec{x}) = 4G\frac{e^{i\omega r}}{r}\int \mathrm{d}^3x'\hat{T}^{ij}(\omega, \vec{x}')\,. \tag{3.4}$$

Our aim is now to simplify this integral. For this purpose, note that

$$\partial_k(x^i\hat{T}^{kj}) = \delta^i_k\hat{T}^{kj} + x^i\partial_k\hat{T}^{kj} = \hat{T}^{ij} + x^i\partial_k\hat{T}^{kj}\,, \tag{3.5}$$

using $\partial_k x^i = \delta^i_k$. Hence, we can replace the integral by writing

$$\int \mathrm{d}^3x'\hat{T}^{ij}(\omega, \vec{x}') = \int \mathrm{d}^3x'\left[\partial'_k(x'^i\hat{T}^{kj}(\omega, \vec{x}')) - x'^i\partial'_k\hat{T}^{kj}(\omega, \vec{x}')\right]\,, \tag{3.6}$$

where we must take the derivative ∂'_k with respect to x'^k. Now the first term is a boundary term, and so can be replaced by a boundary integral; however, since we assumed that the source is located inside a compact spatial domain, this integral vanishes, since at the boundary there is vacuum. We are thus left with only the second part. Here we use the energy-momentum conservation, which takes at the lowest perturbation order the form

$$\partial_\nu T^{\nu\mu} = 0\,. \tag{3.7}$$

Note that it has essentially the same form as the harmonic gauge condition (3.1), and so we can draw the analogous conclusion

$$\partial_k\hat{T}^{kj}(\omega, \vec{x}) = i\omega\hat{T}^{0j}(\omega, \vec{x})\,. \tag{3.8}$$

Hence, we have the formula

$$\int \mathrm{d}^3x'\hat{T}^{ij}(\omega, \vec{x}') = -i\omega\int \mathrm{d}^3x'x'^i\hat{T}^{0j}(\omega, \vec{x}') = -i\omega\int \mathrm{d}^3x'x'^{(i}\hat{T}^{j)0}(\omega, \vec{x}')\,, \tag{3.9}$$

where the last operation, symmetrizing the indices i and j, is valid since we started from a symmetric tensor $\hat{T}^{ij}$, and we could have applied the same transformation to the index j instead of i. Now we can essentially use the same method, realizing that

$$\partial_k(x^ix^j\hat{T}^{0k}) = x^j\hat{T}^{0i} + x^i\hat{T}^{0j} + x^ix^j\partial_k\hat{T}^{0k}\,, \tag{3.10}$$

to replace the integral by

$$\begin{aligned}\int \mathrm{d}^3x'\hat{T}^{ij}(\omega, \vec{x}') &= -\frac{i\omega}{2}\int \mathrm{d}^3x'\left[x'^i\hat{T}^{j0}(\omega, \vec{x}') + x'^j\hat{T}^{i0}(\omega, \vec{x}')\right]\\ &= -\frac{i\omega}{2}\int \mathrm{d}^3x'\left[\partial'_k(x'^ix'^j\hat{T}^{k0}(\omega, \vec{x}')) - x'^ix'^j\partial'_k\hat{T}^{k0}(\omega, \vec{x}')\right]\\ &= -\frac{\omega^2}{2}\int \mathrm{d}^3x'x'^ix'^j\hat{T}^{00}(\omega, \vec{x}')\,.\end{aligned} \tag{3.11}$$

The integral defined the *quadrupole tensor*

$$\hat{I}^{ij}(\omega) = \int \mathrm{d}^3x'x'^ix'^j\hat{T}^{00}(\omega, \vec{x}')\,, \quad I^{ij}(\tilde{t}) = \int \mathrm{d}^3x'x'^ix'^jT^{00}(\tilde{t}, \vec{x}')\,. \tag{3.12}$$

With its help, the metric perturbation (3.4) finally takes the form

$$\hat{\bar{h}}_{ij}(\omega, \vec{x}) = -2G\omega^2\frac{e^{i\omega r}}{r}\hat{I}_{ij}(\omega)\,. \tag{3.13}$$

By performing the inverse Fourier transform, we can absorb the factor $-\omega^2$ into a second time derivative, as well as $e^{i\omega r}$ by transforming to the retarded time, following the previously used steps in reverse. Thus, we finally arrive at the quadrupole formula

$$\bar{h}_{ij}(t, \vec{x}) = \frac{2G}{r}\ddot{I}_{ij}(t - r)\,. \tag{3.14}$$

4 The binary system

We now assume a particular source of gravitational waves, namely a symmetric binary system consisting of two equal, point-like objects of identical mass M, each on a circular orbit of radius R (so that their distance is $2R$), with angular frequency Ω, given by Kepler's law

$$\Omega^2 = \frac{GM}{4R^3} \,. \tag{4.1}$$

The positions of these two masses are therefore given by

$$\vec{x}_A(t) = -\vec{x}_B(t) = R \cdot (\cos(\Omega t), \sin(\Omega t), 0) \,, \tag{4.2}$$

where we assume the orbit in the $x^3 = 0$ plane around the origin of the coordinate system. Assuming point masses, the mass density is given by

$$T^{00}(t, \vec{x}) = \rho = M\delta(x^3)[\delta(x^1 - \cos(\Omega t))\delta(x^2 - \sin(\Omega t)) + \delta(x^1 + \cos(\Omega t))\delta(x^2 + \sin(\Omega t))] \,. \tag{4.3}$$

The quadrupole tensor (3.12) is thus given by

$$I_{ij}(t) = MR^2 \cdot \begin{pmatrix} 1 + \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1 - \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{4.4}$$

The metric perturbation (3.14) therefore is given by

$$\bar{h}_{ij}(t) = -\frac{8GM\Omega^2 R^2}{r} \cdot \begin{pmatrix} \cos(2\Omega(t - r)) & \sin(2\Omega(t - r)) & 0 \\ \sin(2\Omega(t - r)) & -\cos(2\Omega(t - r)) & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{4.5}$$