Combinatory Completeness in Structured Multicategories

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Abstract. We give a general notion of combinatory completeness with respect to a faithful cartesian club and use it systematically to obtain characterisations of a number of different kinds of applicative system. Each faithful cartesian club determines a notion of structured multicategory, with the different notions of structured multicategory obtained in this way giving different notions of polynomial over an applicative system, which in turn give different notions of combinatory completeness. We obtain the classical characterisation of combinatory algebras as combinatory complete applicative systems as a specific instance.

Keywords: Category Theory · Combinatory Logic · Categorical Logic

1 Introduction

Combinatory logic was introduced, essentially independently, by Schönfinkel and Curry [8,23], and has played a fundamental role in the development of logic and computer science during the century that followed (see e.g., [4]).

A *combinatory algebra* is an algebraic model of combinatory logic [2]. One defines an *applicative system* to be a set A of *combinators* together with a (total) function $\bullet : A \times A \to A$ called *application*. Adopting the usual conventions, we treat application as a left-associative infix binary operation whose symbol is often omitted, so that $xy = x \bullet y = \bullet(x, y)$ and $xyz = (xy)z$. Next, we define:

- a B combinator to be some $\mathsf{B} \in A$ such that $\mathsf{B}xyz = x(yz)$ for all $x, y, z \in A$.
- a C combinator to be some $\mathsf{C} \in A$ such that $\mathsf{C}xyz = xzy$ for all $x, y, z \in A$.
- a K combinator to be some $\mathsf{K} \in A$ such that $\mathsf{K}xy = x$ for all $x, y \in A$.
- a W combinator to be some $\mathsf{W} \in A$ such that $\mathsf{W}xy = xyy$ for all $x, y \in A$.
- an I combinator to be some $\mathsf{I} \in A$ such that $\mathsf{I}x = x$ for all $x \in A$.

We call applicative systems which have some subset H of these combinators H-algebras, so that for example a BI-algebra is an applicative system with a B and I combinator, and a BCI-algebra is a BI-algebra with a C combinator. Then in particular a combinatory algebra is defined to be a BCKWI-algebra[1].

[1] It is of course possible to define combinatory algebras using fewer combinators, and in fact the I combinator is redundant in our presentation since it is obtainable from K and W. We feel that this larger combinator basis is better suited to our aims herein.

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Given an applicative system $(A, \bullet)$ one defines a *polynomial* in variables $x_1, \ldots, x_n$ to be one of: a variable x_i where $1 \leq i \leq n$; an element $a \in A$; or $t \bullet s$ where t and s are polynomials in $x_1, \ldots, x_n$. A polynomial t is said to be *computable* in case there exists $a \in A$ such that for all $b_1, \ldots, b_n \in A$ we have:

$$ab_1 \cdots b_n = t[b_1, \ldots, b_n/x_1, \ldots, x_n]$$

For example, if the applicative system in question has a W combinator, then the polynomial $x_1 x_2 x_2$ is computable. An applicative system in which every polynomial is computable is said to be *combinatory complete*. This turns out to be equivalent to being a combinatory algebra, as in:

Theorem 1 (After [14], Chapter 6). *An applicative system* $(A, \bullet)$ *is combinatory complete if and only if it is a combinatory algebra.*

In fact, to obtain a combinatory algebra it is enough to ask that only the *regular* polynomials are computable, where a polynomial is said to be regular in case it contains only variables. For example, if $a \in A$ then $x_1(x_2x_3)$ and x_2a are both polynomials in variables x_1, x_2, x_3, but the former is regular while the latter is not. For many classes of polynomial, computability of the regular polynomials implies computability of all polynomials, so that they express the same notion of combinatory completeness. However, this is not true for all such classes. We mention this here because the distinction will appear in our development.

A natural question is whether there exist analogues of Theorem 1 characterising applicative systems in which only some subset of the distinguished elements of a combinatory algebra need exist. For example, it is known that an applicative system is a BCI-algebra if and only if every *linear* polynomial is computable, where a polynomial t in variables $x_1, \ldots, x_n$ is said to be linear in case each variable $x_1, \ldots, x_n$ occurs exactly once in t (see e.g., [15,25]). As far as we are aware, no satisfying answer to the wider question exists in the literature.

In this paper we seek to improve the situation by giving a general notion of combinatory completeness and using it to obtain a number of combinatory completeness results in a systematic fashion. Specifically, we obtain combinatory completeness results characterising applicative systems with B and I combinators together with most subsets of the combinators C, K, and W.

Central to our approach is the notion of faithful cartesian club [24], which is a sort of well-behaved subcategory of the category **Fun** of functions between sets $\underline{n} = \{1, \ldots, n\}$ (i.e., the skeleton of the category of finite sets and functions). Every faithful cartesian club $\mathfrak{S}$ determines a notion of structured multicategory whose instances are called $\mathfrak{S}$-multicategories. We work with a more abstract notion of applicative system, in which the carrier A becomes an object of some ambient $\mathfrak{S}$-multicategory $\mathcal{M}$ and application becomes a morphism $\bullet \in \mathcal{M}(A, A; A)$.

The morphisms of the smallest sub-$\mathfrak{S}$-multicategory $\mathcal{M}$ containing the application morphism play the role of regular polynomials, and we say that an applicative system is *weakly* $\mathfrak{S}$-combinatory complete when every such morphism is computable in an appropriate sense. Similarly, the morphisms of the smallest sub-$\mathfrak{S}$-multicategory of $\mathcal{M}$ containing the application morphism and all of the generalised elements $a \in \mathcal{M}(;A)$ of the carrier play the role of the full collection of polynomials, and when every such morphism is computable we say that the applicative system in question is $\mathfrak{S}$-combinatory complete.

The definition of $\mathsf{B}, \mathsf{C}, \mathsf{K}, \mathsf{W}$, and I combinators in an applicative system is easily adapted to applicative systems in $\mathfrak{S}$-multicategories, with the caveat that for certain combinators to be expressible the faithful cartesian club $\mathfrak{S}$ must contain certain functions. Specifically, while the definitions of the B and I combinator make sense in an $\mathfrak{S}$-multicategory for any $\mathfrak{S}$, to express the C combinator we require that $\mathfrak{S}$ contains the bijections, with the K and W combinators requiring $\mathfrak{S}$ to contain the monotone injections and monotone surjections, respectively.

Our combinatory completeness results are summarised in Fig. 1. The entries of the first column indicate subcategories of **Fun** that form faithful cartesian clubs $\mathfrak{S}$, which we will usually refer to by the short names given in the second column. For example, **Inj** is the wide subcategory of **Fun** that contains only injective functions as morphisms, and **Id** is the wide subcategory of **Fun** containing only identity functions. The third column tells us what a weakly $\mathfrak{S}$-combinatory complete applicative system in an $\mathfrak{S}$-multicategory is. For example, the second row states that an applicative system in a **Bij**-multicategory is weakly **Bij**-combinatory complete if and only if it is a BCI-algebra. The final column indicates whether or not weak $\mathfrak{S}$-combinatory completeness implies $\mathfrak{S}$-combinatory completeness. For example, we see that weak **Bij**-combinatory completeness implies **Bij**-combinatory completeness, but that this is not the case for **Minj**-combinatory completeness. Explicitly, we have that **Minj**-combinatory completeness implies that one has a BKI-algebra, but have only a partial converse since from a BKI-algebra we obtain only weak **Minj**-combinatory completeness.

Club $\mathfrak{S}$	Short Name	Characterises	Only Weak
Identities	**Id**	BI-algebras	Yes
Bijections	**Bij**	BCI-algebras	No
Monotone Injections	**Minj**	BKI-algebras	Yes
Injections	**Inj**	BCKI-algebras	No
Surjections	**Srj**	BCWI-algebras	No
Functions	**Fun**	BCKWI-algebras	No

Fig. 1. Table of combinatory completeness results.

If we restrict our attention to the **Fun**-multicategory Set with sets as objects and with functions $f : A_1 \times \cdots \times A_n \to B$ as morphisms $f \in \mathsf{Set}(A_1, \ldots, A_n; B)$ then we recover the classical notion of applicative system and of the $\mathsf{B}, \mathsf{C}, \mathsf{K}, \mathsf{W}$, and I combinators. All of our combinatory completeness results specialise to the classical setting. For example, we obtain Theorem 1 as an instance of the fact that an applicative system in a **Fun**-multicategory is **Fun**-combinatory complete if and only if it is a BCKWI-algebra. In this way, our results apply to the classical notion of applicative system as a set equipped with a binary operation.

We present our results in Sect. 4. Sections 2 and 3 introduce the notions of faithful cartesian club and structured multicategory, respectively, and are largely adapted from the work of Shulman [24]. We conclude in Sect. 5.

1.1 Related Work

While the idea of a multicategory has arisen independently a number of times, the work of Lambek [19] is most closely aligned with our purposes here. A more in-depth discussion of the multiple origins of the notion of multicategory may be found in Leinster's treatment [21, Chapter 2], which is also an excellent reference in general. The sequent calculus presentation of multicategories originates with Lambek [19,20], but see also the work of Szabo [27,28] on the subject.

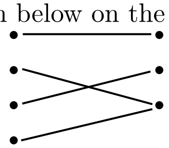
The precise notion of structured multicategory we consider in this paper is due to Shulman [24], as is the attendant notion of faithful cartesian club. Faithful cartesian clubs are an instance of the more general notion of club introduced by Kelly [16,17]. See also the work of Crutwell and Shulman on generalised multicategories [7]. We found the treatment of structured multicategories in the work of Krämer and Mahaman [18] to be helpful.

The work presented here grew out of an interest in the work of Cockett and Hofstra on Turing categories [5], which contains a combinatory completeness result for *partial* combinatory algebras that is similar in spirit to the results presented herein. Perhaps more directly relevant is the work of Longo and Moggi [22], which contains a version of Theorem 1 internal to categories with finite products. We have also been inspired in part by Hasegawa's work on multicategories arising from extensional applicative systems [12].

2 Faithful Cartesian Clubs

The notion of faithful cartesian club revolves around the category of finite ordinals and functions, which we introduce now. For $n \in \mathbb{N}$, write $\underline{n} = \{1, \dots, n\}$. Let **Fun** be the category with natural numbers as objects, and with morphisms $\mathbf{a} : n \to m$ given by functions $\mathbf{a} : \underline{n} \to \underline{m}$. Composition and identities are given by function composition and identity functions, respectively.

Morphisms of **Fun** can be understood intuitively as "dot and line" diagrams. For example, the morphism indicated below on the right corresponds to the diagram below on the left:



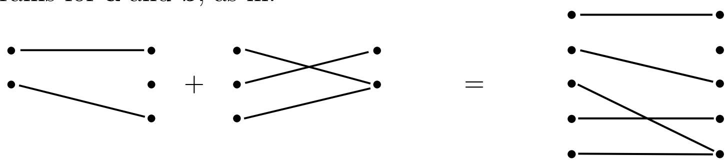
$$\mathbf{a}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 2 & \text{if } x = 3 \\ 3 & \text{otherwise} \end{cases} \qquad \mathbf{a} : 4 \to 3$$

The set $\underline{n}$ is depicted as a sequence of n dots, with the uppermost dot corresponding to $1 \in \underline{n}$, the dot below it corresponding to $2 \in \underline{n}$, and so on. The action of the function is indicated by lines connecting each element of the domain to the element of the codomain that it is mapped to. We note that in general such diagrams indicate a *relation* between finite ordinals, but may nonetheless be used to discuss functions.

We will be interested in the monoidal category structure $(\mathbf{Fun}, +, 0)$ on **Fun** in which $+$ is defined on objects as the usual addition of natural numbers, and is defined on morphisms $\mathbf{a} : n \to m$ and $\mathbf{b} : h \to k$ as in:

$$(\mathbf{a}+\mathbf{b})(x) = \begin{cases} \mathbf{a}(x) & \text{if } x \leq n \\ \mathbf{b}(x-n)+m & \text{if } x > n \end{cases}$$

In terms of dot and line diagrams, $\mathbf{a}+\mathbf{b}$ is obtained by vertically "concatenating" the diagrams for $\mathbf{a}$ and $\mathbf{b}$, as in:



It so happens that $(\mathbf{Fun}, +, 0)$ is cocartesian monoidal, so that $\sum_{i\in\underline{n}} k_i$ is a coproduct of $k_1, \ldots, k_n \in \mathbb{N}$. We will respectively write

$$\amalg_j^{(k_i)_{i\in\underline{n}}} : k_j \to (\textstyle\sum_{i\in\underline{n}} k_i) \qquad \text{and} \qquad \langle \mathbf{a}_1 \mid \ldots \mid \mathbf{a}_n \rangle : (\textstyle\sum_{i\in\underline{n}} k_i) \to m$$

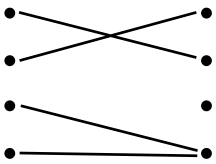
for the coproduct injections and the copairing of the family $(\mathbf{a}_i : k_i \to m)_{i\in\underline{n}}$. In particular, for each object n of **Fun** we have $n = (\sum_{i\in\underline{n}} 1)$, and so any morphism $\mathbf{a} : m \to n$ can be written as a copairing

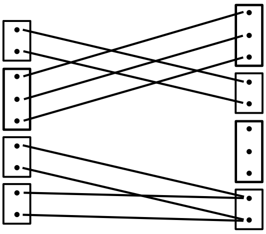
$$\mathbf{a} = \left\langle \amalg_{\mathbf{a}(1)}^{(1)_{i\in\underline{n}}} \middle| \cdots \middle| \amalg_{\mathbf{a}(m)}^{(1)_{i\in\underline{n}}} \right\rangle$$

of coproduct injections. The fact that every morphism of **Fun** can be written in this way is helpful in defining an operation called the *wreath product*. Specifically, for each $\mathbf{a} : m \to n$ in **Fun** and $k_1, \ldots, k_n \in \mathbb{N}$ we define their wreath product to be a morphism $\mathbf{a} \wr (k_1, \ldots, k_n) : (\sum_{j\in\underline{m}} k_{\mathbf{a}(j)}) \to (\sum_{i\in\underline{n}} k_i)$ of **Fun** as in:

$$\mathbf{a} \wr (k_1, \ldots, k_n) = \left\langle \amalg_{\mathbf{a}(1)}^{(1)_{i\in\underline{n}}} \middle| \cdots \middle| \amalg_{\mathbf{a}(m)}^{(1)_{i\in\underline{n}}} \right\rangle \wr (k_1, \ldots, k_n) := \left\langle \amalg_{\mathbf{a}(1)}^{(k_i)_{i\in\underline{n}}} \middle| \cdots \middle| \amalg_{\mathbf{a}(m)}^{(k_i)_{i\in\underline{n}}} \right\rangle$$

The above definition is somewhat opaque. Fortunately, the effect of the wreath product is easily understood when we consider it in terms of dot and line diagrams. There the diagram representing $\mathbf{a} \wr (k_1, \ldots, k_n)$ is obtained from the diagram representing $\mathbf{a}$ by "thickening" it in amounts given by the k_i. First one thickens the codomain by replacing each dot $i \in \underline{n}$ with k_i separate dots, so that the new codomain is $\sum_{i\in\underline{n}} k_i$, and thickens the domain by replacing each dot $j \in \underline{m}$ with $k_{\mathbf{a}(j)}$ separate dots, so that the new domain is $\sum_{j\in\underline{m}} k_{\mathbf{a}(j)}$. Finally one thickens the line leaving each dot $j \in \underline{m}$ into $k_{\mathbf{a}(j)}$ parallel lines, connecting the $k_{\mathbf{a}(j)}$ points that have replaced $j \in \underline{m}$ to the $k_{\mathbf{a}(j)}$ points that have replaced $\mathbf{a}(j) \in \underline{n}$. For example, if $\mathbf{a} : 4 \to 4$ is represented by the diagram below left, then $\mathbf{a} \wr (3, 2, 3, 2) : 9 \to 10$ is represented by the diagram below right.





We are now ready to give the central definition of this section:

Definition 1 ([24]). *A* faithful cartesian club *is a subcategory* $\mathfrak{S}$ *of* **Fun** *such that:*

- $\mathfrak{S}$ *contains all the objects of* **Fun** *(i.e., it is a* wide *subcategory).*
- *If morphisms* $\mathbf{a}$ *and* $\mathbf{b}$ *are in* $\mathfrak{S}$ *then so is* $\mathbf{a}+\mathbf{b}$ *(i.e., it is closed under* $+$*).*
- *If* $\mathbf{a} : m \to n$ *is in* $\mathfrak{S}$ *then so is* $\mathbf{a} \wr (k_1, \ldots, k_n)$ *for all* $k_1, \ldots, k_n$ *(i.e., it is closed under* $\wr$*).*

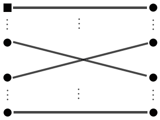
Club $\mathfrak{S}$	Consists of	Generated by
Id	identities	–
Bij	bijections	τ
Minj	monotone injections	δ
Inj	injections	τ, δ
Srj	surjections	τ, σ
Fun	functions	τ, σ, δ

Fig. 2. Some faithful Cartesian clubs and their relationship to the face, degeneracy, and transposition maps.

The faithful Cartesian clubs that will be relevant to us here are listed in Fig. 2. These clubs can be understood in terms of certain special classes of morphism in **Fun**. Specifically, we consider:

- *transpositions* $\tau_i^n : n \to n$ for all $n > 1$ and $1 \leq i < n$ defined as in:

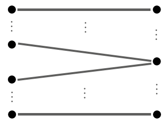
$$\tau_i^n(x) = \begin{cases} x+1 & \text{if } x = i \\ x-1 & \text{if } x = i+1 \\ x & \text{otherwise} \end{cases}$$



so τ_i^n is the permutation that swaps the ith and $(i+1)$st elements.

– *degeneracy maps* $\sigma_i^n : n+1 \to n$ for all n and $1 \le i \le n$ defined as in:

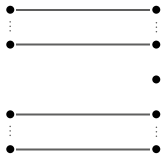
$$\sigma_i^n(x) = \begin{cases} x & \text{if } x \le i \\ x-1 & \text{if } x > i \end{cases}$$



so σ_i^n is the monotone surjection that merges the ith and $(i+1)$st elements.

– *face maps* $\delta_i^n : n-1 \to n$ for all $n \ge 1$ and $1 \le i \le n$, defined as in:

$$\delta_i^n(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } x \ge i \end{cases}$$



so δ_i^n is the monotone injection that skips the ith element of the codomain.

Each of these clubs is generated by certain sorts of special morphism together with identity morphisms through composition. **Fun** itself is generated by all of the face, degeneracy, and transposition maps, **Bij** is generated by the transposition maps, and so on (see e.g., [3,10]). The other characterisations are given in the third column of Fig. 2.

Note that not all possible combinations of generators occur in Fig. 2. In particular, the monotone surjections and monotone functions are not closed under the wreath product, and so do not form a faithful cartesian club. In particular, $\sigma_1^1 \wr (2)$ is not monotone.

3 Structured Multicategories

In this section we define multicategories, define $\mathfrak{S}$-multicategories for a faithful cartesian club $\mathfrak{S}$, and recall their connection to the sequent calculus. Multicategories are similar to categories, the primary difference being that while the domain of a morphism in a category consists of a single object, the domain of a morphism in a multicategory consists of a finite sequence of objects. Explicitly:

Definition 2 ([21]). *A* multicategory $\mathcal{M}$ *consists of the following data:*

– *A set* $\mathcal{M}_0$ *whose elements are called the* objects *of* $\mathcal{M}$*.*
– *For each* $n \in \mathbb{N}$ *and* $A_1, \ldots, A_n, B \in \mathcal{M}_0$*, a set* $\mathcal{M}(A_1, \ldots, A_n; B)$ *of* morphisms*. Any* $f \in \mathcal{M}(A_1, \ldots, A_n; B)$ *is said to* have arity n*.*
– *For each* $k_1, \ldots, k_n \in \mathbb{N}$*,* $A_1, \ldots, A_n, B \in \mathcal{M}_0$*, and* $\Gamma_1, \ldots, \Gamma_n \in \mathcal{M}_0^*$*, a* composition *operation:*

$$\mathcal{M}(A_1, \ldots, A_n; B) \times \mathcal{M}(\Gamma_1; A_1) \times \cdots \times \mathcal{M}(\Gamma_n; A_n) \xrightarrow{\circ} \mathcal{M}(\Gamma_1, \ldots, \Gamma_n; B)$$

which we will usually write infix as in $f \circ (g_1, \ldots, g_n) = \circ(f, g_1, \ldots, g_n)$*.*
– *For each* $A \in \mathcal{M}_0$*, an* identity *morphism* $1_A \in \mathcal{M}(A; A)$*.*

This data must be such that:

– *Composition is associative. That is, we have:*

$$f \circ (g_1 \circ (h_1^1, \ldots, h_1^{k_1}), \ldots, g_n \circ (h_n^1, \ldots, h_n^{k_n}))$$
$$= (f \circ (g_1, \ldots, g_n)) \circ (h_1^1, \ldots, h_1^{k_1}, \ldots, h_n^1, \ldots, h_n^{k_n})$$

whenever f, g_i, h_i^j *are morphisms for which the composites make sense.*

– *Identity morphisms are unital. That is, we have:*

$$f \circ (1_{A_1}, \ldots, 1_{A_n}) = f = 1_B \circ f$$

for every $f \in \mathcal{M}(A_1, \ldots, A_n; B)$.

If A is an object of a multicategory $\mathcal{M}$ we write A^n to indicate the sequence $A, \ldots, A$ consisting of n copies of A. Similarly, if $f \in \mathcal{M}(\Gamma; A)$ we write f^n to indicate the sequence consisting of n copies of f. For example, if $g \in \mathcal{M}(A^n; B)$ then we may write $g \circ (f^n)$ to indicate the composite $g \circ (f, \ldots, f)$. It is important to note that morphisms of a multicategory $\mathcal{M}$ may have arity 0, in which case their domain is the empty sequence as in $\mathcal{M}(; B)$.

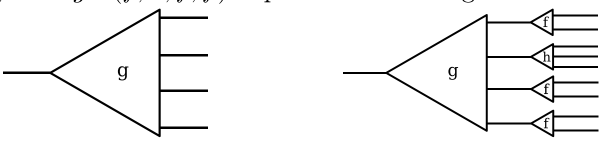
Definition 3 ([24]). *Let* $\mathfrak{S}$ *be a faithful cartesian club. An* $\mathfrak{S}$-multicategory *is a multicategory* $\mathcal{M}$ *together with an operation:*

$$\mathcal{M}(A_{\mathbf{a}(1)}, \ldots, A_{\mathbf{a}(m)}; B) \xrightarrow{[-]\mathbf{a}} \mathcal{M}(A_1, \ldots, A_n; B)$$

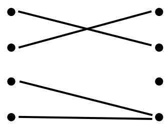
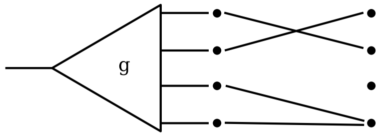
for each $\mathbf{a} : m \to n$ *in* $\mathfrak{S}$*, such that:*

– $[[f]\mathbf{a}]\mathbf{b} = [f](\mathbf{b} \circ \mathbf{a})$
– $[f]1_{\underline{n}} = f$
– $g \circ ([f_1]\mathbf{a}_1, \ldots, [f_n]\mathbf{a}_n) = [g \circ (f_1, \ldots, f_n)](\mathbf{a}_1 + \cdots + \mathbf{a}_n)$
– $[g]\mathbf{a} \circ (f_1, \ldots, f_n) = [g \circ (f_{\mathbf{a}(1)}, \ldots, f_{\mathbf{a}(m)})](\mathbf{a} \wr (k_1, \ldots, k_n))$ *where each* k_i *is the arity of* f_i.

A good way to understand morphisms in multicategories is through their string diagrams. For example a morphism g of arity 4 in some multicategory is pictured below left. Composites may be depicted by merging input and output wires. For example if in addition to g we have morphisms f and h of arity 2 and 3 respectively, then $g \circ (f, h, f, f)$ is pictured below right.



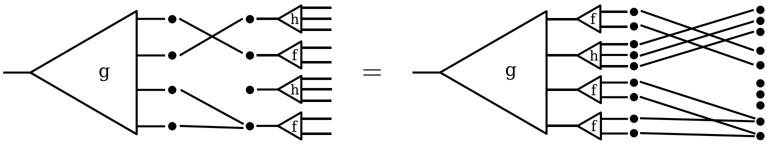
In $\mathfrak{S}$-multicategories, it is convenient to depict the action of a given $\mathbf{a} \in \mathfrak{S}$ by juxtaposing the dot and line diagram for $\mathbf{a}$ with the string diagram depicting f. For example, if our morphism g inhabits a **Fun**-multicategory and $\mathbf{a} : 4 \to 4 \in$ **Fun** is the function depicted below left, then the morphism $[g]\mathbf{a}$ is pictured below right.

It can be helpful to consider the axioms of an $\mathfrak{S}$-multicategory from this perspective. In particular, the axiom concerning whiskering tells us that:

$$[g]\mathbf{a} \circ (h, f, h, f) = [g \circ (f, h, f, f)](\mathbf{a} \wr (3, 2, 3, 2))$$

which is pictured as in:



While some of the notions of structured multicategory we consider are a little exotic, we note that **Id**-multicategories are just multicategories, that a **Bij**-multicategory is what is usually called a *symmetric multicategory*, and that a **Fun**-multicategory is what is usually called a *cartesian multicategory*.

3.1 Sequent Calculus

Structured multicategories enjoy a deep connection to intuitionistic sequent calculus. Let Σ be a (multi-sorted) signature in which operation symbols are typed as in $f : A_1, \ldots, A_n \vdash B$ where $A_1, \ldots, A_n, B$ are generating sorts. Then a *term* over Σ is a sequent that is derivable via the following inference rules:

$$\frac{}{x : A \vdash x : A}\ \textsc{var} \qquad \frac{(\Gamma_i \vdash t_i : A_i)_{i \in \{1,\ldots,n\}} \qquad (f : A_1, \ldots, A_n \vdash B) \in \Sigma}{\Gamma_1, \ldots, \Gamma_n \vdash f(t_1, \ldots, t_n) : B}\ \textsc{op}$$

Crucially, for a sequent $\Gamma \vdash t : B$ to be considered well-formed, the context Γ must not contain any repeated variables. This means that, for example, when we write $\Gamma_1, \ldots, \Gamma_n$, it is implied that the variables in the Γ_i are disjoint.

Terms over a signature form a multicategory, with identity morphisms given by the VAR rule and with composition given by substitution. More precisely, the composition operation is given by the following admissible inference rule:

$$\frac{(\Gamma_i \vdash t_i : A_i)_{i \in \{1,\ldots,n\}} \qquad x_1 : A_1, \ldots, x_n : A_n \vdash t : B}{\Gamma_1, \ldots, \Gamma_n \vdash t[t_1, \ldots, t_n / x_1, \ldots, x_n] : B}\ \textsc{comp}$$

That this satisfies the equations of a multicategory follows from certain elementary properties of substitution. For example, for any $\Gamma \vdash t : B$ the right-unitality law ($1_B \circ f = f$) holds as in:

$$(\Gamma \vdash x[t/x] : B) = (\Gamma \vdash t : B)$$

We omit the redundant information when writing such equations, so that for example $\Gamma \vdash x[t/x] = t : B$ is an equivalent way of expressing the above equation.

The multicategory of terms over a signature is in fact the *free* multicategory over that signature, in the sense that this construction gives the left adjoint of an adjunction between a category of signatures and the category of multicategories. The right adjoint maps a multicategory $\mathcal{M}$ to the $\mathcal{M}_0$-sorted signature with an operation symbol $f : A_1, \ldots, A_n \vdash B$ for each $f \in \mathcal{M}(A_1, \ldots, A_n; B)$. The counit of the adjunction gives a morphism from the multicategory of terms over this signature into $\mathcal{M}$, and quotienting the terms by the equations that hold in the image of this morphism yields a sequent calculus presentation of $\mathcal{M}$. It follows that we may reason about morphisms in any multicategory by means of sequent calculus, interpreting composition as substitution and identities as variables.

To extend this to structured multicategories requires an additional structural inference rule. For a given faithful cartesian club $\mathfrak{S}$ we ask that:

$$\frac{\mathbf{a} : m \to n \in \mathfrak{S} \qquad x_1 : A_{a(1)}, \ldots, x_m : A_{a(m)} \vdash t : B}{x_1 : A_1, \ldots, x_n : A_n \vdash [t]\mathbf{a} : B}\ \text{ACT}$$

where the term $[t]\mathbf{a}$ is defined as in:

$$[t]\mathbf{a} = \begin{cases} x_{\mathbf{a}(i)} & \text{if } t = x_i \text{ is a variable} \\ f([t_1]\mathbf{a}, \ldots, [t_n]\mathbf{a}) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Note that $[t]\mathbf{a}$ is a meta-level operation, just like substitution. Now the derivable sequents form the free $\mathfrak{S}$-multicategory, and as before one obtains a sequent calculus presentation for arbitrary $\mathfrak{S}$-multicategories.

The effect of the ACT rule is to allow the variables of the context to be used more flexibly in terms, with the degree of flexibility depending on $\mathfrak{S}$. In the sequent calculus for multicategories, derivable sequents $\Gamma \vdash t : B$ have the property that the variables of the context Γ occur exactly once in t, in exactly the same order they appear in Γ. For example, we can derive $x_1 : A_1, x_2 : A_2 \vdash f(x_1, x_2) : B$ but not $x_1 : A_1, x_2 : A_2 \vdash f(x_2, x_1) : B$ or $x_1 : A_1 \vdash f(x_1, x_1) : B$. Similarly, while we can derive $x_1 : A_1 \vdash g(x_1) : B$, we cannot derive $x_1 : A_1, x_2 : A_2 \vdash g(x_1) : B$. In a **Fun**-multicategory all of these are possible as in:

$$\frac{\tau_1^2 : 2 \to 2 \in \mathbf{Fun} \qquad x_1 : A_1, x_2 : A_2 \vdash f(x_1, x_2) : B}{x_1 : A_1, x_2 : A_2 \vdash [f(x_1, x_2)]\tau_1^2 = f(x_2, x_1) : B}$$

$$\frac{\sigma_1^1 : 2 \to 1 \in \mathbf{Fun} \qquad x_1 : A_1 \vdash f(x_1, x_2) : B}{x_1 : A_1 \vdash [f(x_1, x_2)]\sigma_1^1 = f(x_1, x_1) : B}$$

$$\frac{\delta_1^2 : 1 \to 2 \in \mathbf{Fun} \qquad x_1 : A_1 \vdash g(x_1) : B}{x_1 : A_1, x_2 : A_2 \vdash [g(x_1)]\delta_1^2 = g(x_1) : B}$$

This is a good way to think about the different notions of polynomial obtained from the different faithful cartesian clubs. Different sorts of combinatory completeness correspond to different sorts of restriction on the use of variables in terms. Formally, terms represent morphisms in some $\mathfrak{S}$-multicategory, and the choice of $\mathfrak{S}$ determines how variables may be used.

4 Combinatory Completeness in Structured Multicategories

In this section we give general notions of combinatory completeness and use them to characterise certain applicative systems, as summarised in Fig. 1. We begin with the notion of applicative system:

Definition 4. *Let $\mathcal{M}$ be a multicategory. An* applicative system $(A, \bullet)$ *in $\mathcal{M}$ consists of an object $A \in \mathcal{M}_0$ together with a morphism $\bullet \in \mathcal{M}(A, A; A)$, called* application*.*

When working with applicative systems in multicategories, we will tend to favour the sequent calculus syntax discussed in Sect. 3, and when doing so will adopt the usual syntactic conventions for working with application discussed in Sect. 1, so that for example $xyz = (xy)z = (x \bullet y) \bullet z$. Moreover, when we are working with an applicative system $(A, \bullet)$ in a multicategory $\mathcal{M}$, all of the relevant morphisms are elements of $\mathcal{M}(A^n; A)$ for some $n \in \mathbb{N}$. This allows us to omit the types from our sequents, since everything has type A. For example we may write $x_1, x_2 \vdash f(x_1, x_2)$ instead of $x_1 : A, x_2 : A \vdash f(x_1, x_2) : A$, which helps to make things less cluttered. We will moreover allow ourselves to use variable names beyond x_i. While this can be made fully formal, we refrain from doing so here. The interested reader may consult the appendix of [24], which follows the "nominal" approach of Gabbay and Pitts [9].

If $(A, \bullet)$ is an applicative system in $\mathcal{M}$ then we define an *iterated application* operation $\bullet^n \in \mathcal{M}(A, A^n; A)$ for each $n \in \mathbb{N}$ as in $\bullet^0 = 1_A$ and $\bullet^{n+1} = \bullet \circ (\bullet^n, 1_A)$. Note that $\bullet^1 = \bullet \circ (1_A, 1_A) = \bullet$, and that in the sequent calculus notation $\bullet^n$ becomes $x, x_1, \ldots, x_n \vdash x x_1 \cdots x_n$. This facilitates the following definition:

Definition 5. *Let $\mathcal{M}$ be a multicategory and let $(A, \bullet)$ be an applicative system in $\mathcal{M}$. We say that $f \in \mathcal{M}(A^n; A)$ is* $(A, \bullet)$-computable *in case there exists some $a \in \mathcal{M}(; A)$ in $\mathcal{M}$ such that $\bullet^n \circ (a, 1_A, \ldots, 1_A) = f$ or, equivalently, such that $x_1, \ldots, x_n \vdash a x_1 \cdots x_n = f(x_1, \ldots, x_n)$.*

The morphisms $a \in \mathcal{M}(; A)$ play the role of elements of A, and are all $(A, \bullet)$-computable as in $\bullet^0 \circ (a) = 1_A \circ (a) = a$.

Next, every faithful cartesian club gives two notions of polynomial as follows:

Definition 6. *Let $\mathfrak{S}$ be a faithful cartesian club, and let $(A, \bullet)$ be an applicative system in an $\mathfrak{S}$-multicategory $\mathcal{M}$. We define:*

- *the* regular $\mathfrak{S}$-polynomials over $(A, \bullet)$*, written $\mathfrak{S}(A, \bullet)$, to be the smallest sub-$\mathfrak{S}$-multicategory of $\mathcal{M}$ containing $\bullet \in \mathcal{M}(A, A; A)$.*
- *the* $\mathfrak{S}$-polynomials over $(A, \bullet)$*, written $\mathfrak{S}[A, \bullet]$ to be the smallest sub-$\mathfrak{S}$-multicategory of $\mathcal{M}$ containing $\bullet \in \mathcal{M}(A, A; A)$ and also containing each morphism $a \in \mathcal{M}(; A)$ of $\mathcal{M}$.*

Adapting the classical notion of combinatory completeness, we obtain:

Definition 7. *Let $\mathfrak{S}$ be a faithful cartesian club, $\mathcal{M}$ be an $\mathfrak{S}$-multicategory, and $(A, \bullet)$ be an applicative system in $\mathcal{M}$. We say that $(A, \bullet)$ is:*

- weakly $\mathfrak{S}$-combinatory complete *in case every regular $\mathfrak{S}$-polynomial over $(A, \bullet)$ is $(A, \bullet)$-computable.*
- $\mathfrak{S}$ combinatory complete *in case every $\mathfrak{S}$-polynomial over $(A, \bullet)$ is $(A, \bullet)$-computable.*

We proceed to establish our results, beginning with the simplest. We define:

Definition 8. *Let $\mathcal{M}$ be a multicategory (i.e., an* **Id***-multicategory), and let $(A, \bullet)$ be an applicative system in $\mathcal{M}$.*

- *A* B combinator *for $(A, \bullet)$ is $\mathsf{B} \in \mathcal{M}(; A)$ such that $\bullet^3 \circ (\mathsf{B}, 1_A, 1_A, 1_A) = \bullet \circ (1_A, \bullet)$, or equivalently $x_1, x_2, x_3 \vdash \mathsf{B}x_1x_2x_3 = x_1(x_2x_3)$.*
- *An* I combinator *for $(A, \bullet)$ is a morphism $\mathsf{I} \in \mathcal{M}(; A)$ such that $\bullet \circ (\mathsf{I}, 1_A) = 1_A$, or equivalently $x \vdash \mathsf{I}x = x$.*

If $(A, \bullet)$ has both a B and I combinator, we say that it is a BI-algebra*.*

Lemma 1. *Let $(A, \bullet)$ be a BI-algebra in a multicategory $\mathcal{M}$. We define a morphism $\mathsf{B}^n \in \mathcal{M}(; A)$ for each $n \in \mathbb{N}$ as follows: $\mathsf{B}^0 = \mathsf{I}$, $\mathsf{B}^1 = \mathsf{B}$, and $\mathsf{B}^{n+1} = \mathsf{BB}(\mathsf{B}^n)$ for $n \geq 1$. Then for all $n \in \mathbb{N}$ we have $\bullet^{n+1} \circ (\mathsf{B}^n, 1_A, 1_A, 1_A^n) = \bullet \circ (1_A, \bullet^n)$, or equivalently, $b, a, x_1, \ldots, x_n \vdash \mathsf{B}^n b a x_n \cdots x_1 = b(a x_n \cdots x_1)$.*

Proof. By induction on $n \in \mathbb{N}$. The base cases are when $n = 0$, in which case we have $b, a \vdash \mathsf{B}^0 ba = \mathsf{I}ba = ba$, and when $n = 1$, in which case we have $b, a, x_1 \vdash \mathsf{B}^1 bax_1 = \mathsf{B}bax_1 = b(ax_1)$. For the inductive case, suppose that we have $b, a, x_n, \ldots, x_1 \vdash \mathsf{B}^n bax_n \cdots x_1 = b(ax_n \cdots x_1)$. Then we also have:

$$
\begin{aligned}
&b, a, x_{n+1}, x_n, \ldots, x_1 \vdash \mathsf{B}^{n+1} bax_{n+1}x_n \cdots x_1 = \mathsf{BB}(\mathsf{B}^n)bax_{n+1}x_n \cdots x_1 \\
&= \mathsf{B}(\mathsf{B}^n b)ax_{n+1}x_n \cdots x_1 = (\mathsf{B}^n b)(ax_{n+1})x_n \cdots x_1 = \mathsf{B}^n b(ax_{n+1})x_n \cdots x_1 \\
&= b((ax_{n+1})x_n \cdots x_1) = b(ax_{n+1}x_n \cdots x_1)
\end{aligned}
$$

The claim follows by induction.

Now our combinatory completeness result for BI-algebras is as follows:

Theorem 2. *Let $(A, \bullet)$ be an applicative system in a multicategory $\mathcal{M}$. Then $(A, \bullet)$ is weakly* **Id***-combinatory complete if and only if it is a BI-algebra.*

Proof. Suppose $(A, \bullet)$ is weakly **Id**-combinatory complete. Then $\bullet \circ (1_A, \bullet)$ and 1_A are $(A, \bullet)$-computable, since they are both morphisms of $\mathbf{Id}(A, \bullet)$. That is, there exist $\mathsf{B} \in \mathcal{M}(; A)$ and $\mathsf{I} \in \mathcal{M}(; A)$ such that $\bullet^3 \circ (B, 1_A, 1_A, 1_A) = \bullet \circ (1_A, \bullet)$ and $\bullet \circ (\mathsf{I}, 1_A) = 1_A$. It follows that $(A, \bullet)$ is a BI-algebra.

The converse is somewhat more involved. First, define a *binary bracketing* (see e.g., [26]) to be some $\mathcal{B}$ generated by the following grammar:

$$\mathcal{B} ::= \Box \mid (\mathcal{B}\mathcal{B})$$

and define the *length* of a binary bracketing to be the number of occurrences of $\Box$ it contains, so that for example $(\Box(\Box\Box))$ and $((\Box\Box)(\Box\Box))$ are binary bracketings of length 3 and 4, respectively.

Given a binary bracketing $\mathcal{B}$ of length n and a family of morphisms $(\Gamma_i \vdash t_i)_{i\in\{1,\dots,n\}}$ (where everything is of type A), we obtain a morphism $\Gamma_1,\dots,\Gamma_n \vdash \mathcal{B}(t_1,\dots,t_n)$, where the term $\mathcal{B}(t_1,\dots,t_n)$ is obtained by substituting t_i for the ith occurrence of $\Box$ in $\mathcal{B}$, ordered from left to right. For example, $\Box(t) = t$, $(\Box(\Box\Box))(t_1,t_2,t_3) = (t_1(t_2t_3))$ and $((\Box\Box)(\Box\Box))(t_1,t_2,t_3,t_4) = ((t_1t_2)(t_3t_4))$.

Notice that the morphisms of $\mathbf{Id}(A,\bullet)$ are precisely the morphisms of $\mathcal{M}$ of the form $x_1,\dots,x_n \vdash \mathcal{B}(x_1,\dots,x_n)$ where $\mathcal{B}$ is a binary bracketing of length n. This means that in order to establish the converse it suffices to show that whenever $(A,\bullet)$ is a BI-algebra, every such morphism is $(A,\bullet)$-computable.

Given a binary bracketing $\mathcal{B}$ of length n, for each $1 \le i \le n$ we obtain a new binary bracketing $i \triangleright \mathcal{B}$ of length $n+1$ by replacing the ith instance of $\Box$ in $\mathcal{B}$ by $(\Box\Box)$. For example, we have $1 \triangleright (\Box(\Box\Box)) = ((\Box\Box)(\Box\Box))$, $2 \triangleright (\Box(\Box\Box)) = (\Box((\Box\Box)\Box))$, and $3 \triangleright (\Box(\Box\Box)) = (\Box(\Box(\Box\Box)))$. Observe that for every binary bracketing $\mathcal{B}$ of length $n+1$ there exists a binary bracketing $\mathcal{B}'$ of length n and $1 \le i \le n$ such that $\mathcal{B} = i \triangleright \mathcal{B}'$.

We proceed to show that if $(A,\bullet)$ is a BI-algebra then for every binary bracketing $\mathcal{B}$ of length n the morphism $x_1,\dots,x_n \vdash \mathcal{B}(x_1,\dots,x_n)$ is $(A,\bullet)$-computable. We do this by induction on n. The base case is when $n=1$, in which case $\mathcal{B}$ is $\Box$, which is $(A,\bullet)$-computable as in $x \vdash \Box(x) = x = \mathsf{I}x$. For the inductive case, suppose that the claim holds for all binary bracketings of length at most n, and let $\mathcal{B}$ be a binary bracketing of length $n+1$. Then for some binary bracketing $\mathcal{B}'$ of length n and $1 \le i \le n$ we have $\mathcal{B} = i \triangleright \mathcal{B}'$. Moreover, our inductive hypothesis gives that $x_1,\dots,x_n \vdash \mathcal{B}'(x_1,\dots,x_n)$ is $(A,\bullet)$-computable, which is to say that for some $a \in \mathcal{M}(;A)$ we have $x_1,\dots,x_n \vdash ax_1\cdots x_n = \mathcal{B}'(x_1,\dots,x_n)$. Now Lemma 1 gives that the morphism corresponding to $\mathcal{B}$ is $(A,\bullet)$-computable:

$$
\begin{aligned}
&x_1,\dots,x_{i-1},y_1,y_2,x_{i+1},\dots,x_n \vdash \mathsf{B}^{i-1}\mathsf{B}ax_1\cdots x_{i-1}y_1y_2x_{i+1}\cdots x_n\\
&= \mathsf{B}(ax_1\cdots x_{i-1})y_1y_2x_{i+1}\cdots x_n = ax_1\cdots x_{i-1}(y_1y_2)x_{i+1}\cdots x_n\\
&= \mathcal{B}'(x_1,\dots,x_{i-1},(y_1y_2),x_{i+1},\dots,x_n)\\
&= (i \triangleright \mathcal{B}')(x_1,\dots,x_{i-1},y_1,y_2,x_{i+1},\dots,x_n)\\
&= \mathcal{B}(x_1,\dots,x_{i-1},y_1,y_2,x_{i+1},\dots,x_n)
\end{aligned}
$$

The claim follows.

We proceed to consider the other combinators:

Definition 9. *Let $\mathfrak{S}$ be a faithful cartesian club, $\mathcal{M}$ be an $\mathfrak{S}$-multicategory, and $(A,\bullet)$ be an applicative system in $\mathcal{M}$.*

– *If $\mathfrak{S}$ contains the transpositions, we define a* C combinator *for $(A,\bullet)$ to be a morphism $\mathsf{C} \in \mathcal{M}(;A)$ such that $\bullet^3 \circ (\mathsf{C}, 1_A, 1_A, 1_A) = [\bullet^2]\tau_2^3$, or in the sequent calculus notation $x_1,x_2,x_3 \vdash \mathsf{C}x_1x_2x_3 = x_1x_3x_2$.*

– *If* $\mathfrak{S}$ *contains the degeneracy maps, we define a* W combinator *for* $(A, \bullet)$ *to be a morphism* $\mathsf{W} \in \mathcal{M}(;A)$ *such that* $\bullet^2 \circ (\mathsf{W}, 1_A, 1_A) = [\bullet^3]\sigma_2^2$*, or in the sequent calculus notation* $x_1, x_2 \vdash \mathsf{W}x_1x_2 = x_1x_2x_2$*.*
– *If* $\mathfrak{S}$ *contains the face maps, we define a* K combinator *for* $(A, \bullet)$ *to be a morphism* $\mathsf{K} \in \mathcal{M}(;A)$ *such that* $\bullet^2 \circ (\mathsf{K}, 1_A, 1_A) = [1_A]\delta_2^2$*, or in the sequent calculus notation* $x_1, x_2 \vdash \mathsf{K}x_1x_2 = x_1$*.*

As in the classical case, we say that an applicative system is, for example, a BCI-algebra in case it has a B, C, and I combinator. We prove a technical lemma:

Lemma 2. *Let* $\mathfrak{S}$ *be a faithful cartesian club,* $\mathcal{M}$ *be an* $\mathfrak{S}$*-multicategory, and* $(A, \bullet)$ *be* BI*-algebra in* $\mathcal{M}$*. Then we have:*

1. *If* $\mathfrak{S}$ *contains the transpositions and* $(A, \bullet)$ *has a* C *combinator, then whenever* $f \in \mathcal{M}(A^n; A)$ *is* $(A, \bullet)$*-computable so is* $[f]\tau_i^n$ *for* $1 \le i < n$*.*
2. *If* $\mathfrak{S}$ *contains the degeneracy maps and* $(A, \bullet)$ *has a* W *combinator, then whenever* $f \in \mathcal{M}(A^{n+1}; A)$ *is* $(A, \bullet)$*-computable so is* $[f]\sigma_i^n$ *for* $1 \le i \le n$*.*
3. *If* $\mathfrak{S}$ *contains the face maps and* $(A, \bullet)$ *has a* K *combinator, then whenever* $f \in \mathcal{M}(A^{n-1}; A)$ *is* $(A, \bullet)$*-computable so is* $[f]\delta_i^n$ *for* $1 \le i \le n$*.*

Proof. 1. Since $f \in \mathcal{M}(A^n; A)$ is $(A, \bullet)$-computable we have $a \in \mathcal{M}(;A)$ such that:

$$x_1, \ldots, x_i, x_{i+1}, \ldots, x_n \vdash ax_1 \cdots x_i x_{i+1} \cdots x_n = f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n)$$

Then using Lemma 1 we have:

$$\begin{aligned} &x_1, \ldots, x_i, x_{i+1}, \ldots, x_n \vdash \mathsf{B}^{i-1}\mathsf{C}ax_1 \cdots x_i x_{i+1} \cdots x_n \\ &= \mathsf{C}(ax_1 \cdots)x_i x_{i+1} \cdots x_n = ax_1 \cdots x_{i+1} x_i \cdots x_n \\ &= f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) = [f]\tau_i^n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) \end{aligned}$$

and the claim follows.

2. Since $f \in \mathcal{M}(A^{n+1}; A)$ is $(A, \bullet)$-computable we have $a \in \mathcal{M}(;A)$ such that:

$$x_1, \ldots, x_{n+1} \vdash ax_1 \cdots x_{n+1} = f(x_1, \ldots, x_{n+1})$$

Then using Lemma 1 we have:

$$\begin{aligned} &x_1, \ldots, x_i, \ldots, x_n \vdash \mathsf{B}^{i-1}\mathsf{W}ax_1 \cdots x_i \cdots x_n = \mathsf{W}(ax_1 \cdots)x_i \cdots x_n \\ &= ax_1 \cdots x_i x_i \cdots x_n = f(x_1, \ldots, x_i, x_i, \ldots, x_n) = [f]\sigma_i^n(x_1, \ldots, x_i, \ldots, x_n) \end{aligned}$$

and the claim follows.

3. Since $f \in \mathcal{M}(A^{n-1}; A)$ is $(A, \bullet)$-computable we have $a \in \mathcal{M}(;A)$ such that:

$$x_1, \ldots, x_n \vdash ax_1 \cdots x_{n-1} = f(x_1, \ldots, x_{n-1})$$

Then using Lemma 1 we have:

$$\begin{aligned} &x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n \vdash \mathsf{B}^{i-1}\mathsf{K}ax_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_n \\ &= \mathsf{K}(ax_1 \cdots x_{i-1})x_i x_{i+1} \cdots x_n = ax_1 \cdots x_{i-1} x_{i+1} \cdots x_n \\ &= f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = [f]\delta_i^n(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \end{aligned}$$

and the claim follows.

The other results concerning weak combinatory completeness follow easily:

Theorem 3. *Let $\mathfrak{S}$ be a faithful cartesian club, $\mathcal{M}$ be an $\mathfrak{S}$-multicategory, and $(A, \bullet)$ be an applicative system in $\mathcal{M}$. Then we have:*

1. *If $\mathfrak{S}$ contains the bijections, then $(A, \bullet)$ is weakly* **Bij***-combinatory complete if and only if it is a* BCI*-algebra.*
2. *If $\mathfrak{S}$ contains the monotone injections, then $(A, \bullet)$ is weakly* **MInj***-combinatory complete if and only if it is a* BKI*-algebra.*
3. *If $\mathfrak{S}$ contains the injections, then $(A, \bullet)$ is weakly* **Inj***-combinatory complete if and only if it is a* BCKI*-algebra.*
4. *If $\mathfrak{S}$ contains the surjections, then $(A, \bullet)$ is weakly* **Srj***-combinatory complete if and only if it is a* BCWI*-algebra.*
5. *If $\mathfrak{S}$ is* **Fun***, then $(A, \bullet)$ is weakly* **Fun***-combinatory complete if and only if it is a* BCKWI*-algebra.*

Proof. We give the proof for BCWI-algebras. The other cases are similar. Suppose that $(A, \bullet)$ is weakly **Srj**-combinatory complete. Then $(A, \bullet)$ is weakly **Id**-combinatory complete, and Theorem 2 gives that it is a BI-algebra. Moreover, $[\bullet^2]\tau_2^3$ and $[\bullet^3]\sigma_2^2$ are in $\mathbf{Srj}(A, \bullet)$, and so they are $(A, \bullet)$-computable, which is to say that $(A, \bullet)$ has a C and W combinator, and is therefore a BCWI-algebra.

For the converse, suppose that $(A, \bullet)$ is a BCWI-algebra. We must show that every morphism of $\mathbf{Srj}(A, \bullet)$ is $(A, \bullet)$-computable. Theorem 2 gives that every morphism of $\mathbf{Id}(A, \bullet)$ is $(A, \bullet)$-computable. We know **Srj** is generated by transpositions and degeneracy maps, so it suffices to show that the $(A, \bullet)$-computable morphisms are closed under the action of such maps, but this is part of Lemma 2.

The final thing we require in order to establish the results of Fig. 1 is some relationship between the two notions of combinatory completeness. We have:

Lemma 3. *Let $\mathfrak{S}$ be a faithful cartesian club that contains the bijections, let $\mathcal{M}$ be an $\mathfrak{S}$-multicategory, and let $(A, \bullet)$ be a* BCI*-algebra in $\mathcal{M}$. Then $(A, \bullet)$ is $\mathfrak{S}$-combinatory complete if and only if it is weakly $\mathfrak{S}$-combinatory complete.*

Proof. It suffices to show that for $n \geq 1$, if $x_1, \ldots, x_n \vdash f(x_1, \ldots, x_n) \in \mathcal{M}(A^n; A)$ is $(A, \bullet)$-computable then so is the morphism:

$$x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \vdash f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$$

for any $b \in \mathcal{M}(; A)$ and $1 \leq i \leq n$. Indeed, if we have $x_1, \ldots, x_n \vdash ax_1 \cdots x_n = f(x_1, \ldots, x_n)$ for some $a \in \mathcal{M}(; A)$ then using Lemma 1 we have:

$$\begin{aligned} x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n &\vdash \mathsf{B}^i(\mathsf{CI}b)ax_1 \cdots x_{i-1}x_{i+1} \cdots x_n \\ &= \mathsf{CI}b(ax_1 \cdots x_{i-1})x_{i+1} \cdots x_n = \mathsf{I}(ax_1 \cdots x_{i-1})bx_{i+1} \cdots x_n \\ &= ax_1 \cdots x_{i-1}bx_{i+1} \cdots x_n = f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) \end{aligned}$$

Now, if $(A, \bullet)$ is weakly $\mathfrak{S}$-combinatory complete then every morphism of $\mathfrak{S}(A, \bullet)$ is $(A, \bullet)$-computable. We have seen above that $(A, \bullet)$-computable morphisms are closed under precomposition with morphisms of $\mathcal{M}(; A)$ in arbitrary positions, and it follows that every morphism of $\mathfrak{S}[A, \bullet]$ is $(A, \bullet)$-computable, which is to say that $(A, \bullet)$ is $\mathfrak{S}$-combinatory complete. Obviously $\mathfrak{S}$-combinatory completeness implies weak $\mathfrak{S}$-combinatory completeness, and the claim follows.

5 Concluding Remarks

We have introduced a general notion of combinatory completeness parameterised by a faithful cartesian club, and have used it systematically to obtain characterisations of a number of different kinds of applicative system, summarised in Fig. 1. Our work subsumes the classical characterisation of combinatory algebras as combinatory complete applicative systems (Theorem 1), working both in more general settings and with other notions of completeness.

We imagine two primary directions for future work. The first is to obtain a more abstract characterisation of (weak) $\mathfrak{S}$-combinatory completeness. While in this paper we have focused on the connection between combinatory completeness and the existence of specific combinators, there should also be a connection between combinatory completeness and the existence of categorical structure on the computable morphisms of the applicative system in question. In particular this seems to be rather tricky in the cases where weak $\mathfrak{S}$-combinatory completeness does not imply $\mathfrak{S}$-combinatory completeness.

The second is to extend our approach to capture other sorts of applicative system. For example, in this paper the BWI-algebras and BKWI-algebras are conspicuously missing. More broadly we would like to be able to express combinatory completeness of *partial* combinatory algebras, and also more exotic kinds of applicative system, such as the linear combinatory algebras of Abramsky et al. [1], the monadic combinatory algebras of Cohen et al. [6], the braided and ribbon combinatory algebras of Hasegawa et al. [11,13], and the various sorts of planar combinatory algebra considered by Tomita [29–31]. A related question is for which notions of generalized multicategory (see e.g., [7]) the concepts of applicative system and combinatory completeness make sense, which may lead to any number of variations on the theme of combinatory algebra.

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References

1. Abramsky, S., Haghverdi, E., Scott, P.: Geometry of interaction and linear combinatory algebras. Math. Struct. Comput. Sci. **12**, 625–665 (2002)
2. Barendregt, H.P.: The Lambda Calculus: Its Syntax and Semantics. Elsevier, New York (1981)
3. Burroni, A.: Higher-dimensional word problems with applications to equational logic. Theoret. Comput. Sci. **115**(1), 43–62 (1993)

4. Cardone, F., Hindley, J.R.: History of lambda-calculus and combinatory logic. Handbook History Logic **5**, 723–817 (2006)
5. Cockett, J., Hofstra, P.: Introduction to turing categories. Ann. Pure Appl. Logic **156**, 183–209 (2008)
6. Cohen, L., Grunfeld, A., Kirst, D., Miquey, É.: From partial to monadic: combinatory algebra with effects. LIPIcs **337**, 14:1–14:22 (2025)
7. Cruttwell, G.S., Shulman, M.A.: A unified framework for generalized multicategories. Theory Appl. Categ. **24**(21), 580–655 (2010)
8. Curry, H.B.: Grundlagen der kombinatorischen logik. Am. J. Math. **52**(4), 789–834 (1930)
9. Gabbay, M., Pitts, A.: A new approach to abstract syntax involving binders. In: Proceedings. 14th Symposium on Logic in Computer Science (Cat. No. PR00158), pp. 214–224. IEEE (1999)
10. Grandis, M.: Finite sets and symmetric simplicial sets. Theory Appl. Categ. [Electronic Only] **8**, 244–252 (2001)
11. Hasegawa, M.: A braided lambda calculus. In: Dal Lago, U., de Paiva, V. (eds.) Proceedings of the Second Joint International Workshop on Linearity & Trends in Linear Logic and its Applications (Linearity & TLLA 2020). Electronic Proceedings in Theoretical Computer Science (EPTCS), vol. 353, pp. 94–108. EPTCS, — (2021). https://doi.org/10.4204/EPTCS.353.5
12. Hasegawa, M.: The internal operads of combinatory algebras. Electron. Notes Theoret. Inform. Comput. Sci. **1** (2023)
13. Hasegawa, M., Lechenne, S.: Braids, twists, trace and duality in combinatory algebras. In: Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2024. Association for Computing Machinery, New York (2024). https://doi.org/10.1145/3661814.3662098
14. Curry, H.B., Robert Feys, W.C.: Combinatory Logic. North-Holland Publishing Company, W.C. (1958)
15. Hoshino, N.: Linear realizability. In: International Workshop on Computer Science Logic, pp. 420–434. Springer (2007)
16. Kelly, G.M.: An abstract approach to coherence. In: Coherence in Categories, pp. 106–147. Springer (1972)
17. Kelly, G.M.: Many-variable functorial calculus. I. In: Coherence in Categories, pp. 66–105. Springer (1972)
18. Krähmer, U., Mahaman, M.: Clones from comonoids. arXiv preprint arXiv:2109.09634 (2021)
19. Lambek, J.: Deductive systems and categories II. standard constructions and closed categories. In: Category Theory, Homology Theory and their Applications I: Proceedings of the Conference held at the Seattle Research Center of the Battelle Memorial Institute, June 24–July 19, 1968 Volume One, pp. 76–122. Springer (1969)
20. Lambek, J.: Multicategories revisited. Contemp. Math. **92**, 217–239 (1989)
21. Leinster, T.: Higher Operads, Higher Categories. Cambridge University Press (2004). preprint version available at arXiv:math.CT/0305049
22. Longo, G., Moggi, E.: A category-theoretic characterization of functional completeness. Theoret. Comput. Sci. **70**(2), 193–211 (1990)
23. Schönfinkel, M.: Über die bausteine der mathematischen logik. Math. Ann. **92**(3), 305–316 (1924)
24. Shulman, M.: Categorical logic from a categorical point of view. Technical reports, AARMS Summer School (2016). https://mikeshulman.github.io/catlog/catlog.pdf

25. Simpson, A.K.: Reduction in a linear lambda-calculus with applications to operational semantics. In: Proceedings of the 16th International Conference on Rewriting Techniques and Applications (RTA 2005). LNCS, vol. 3467, pp. 219–234. Springer (2005). https://doi.org/10.1007/978-3-540-32033-3_17
26. Stanley, R.P.: Hipparchus, Plutarch, Schröder, and Hough. Am. Math. Mon. **104**(4), 344–350 (1997)
27. Szabo, M.E.: A categorical equivalence of proofs. Notre Dame J. Formal Logic **15**(2), 177–191 (1974)
28. Szabo, M.: Proof-theoretical investigations in catagorical algebra. Ph.D. thesis. McGill University (1971)
29. Tomita, H.: Realizability Without Symmetry. In: Baier, C., Goubault-Larrecq, J. (eds.) 29th EACSL Annual Conference on Computer Science Logic (CSL 2021). Leibniz International Proceedings in Informatics (LIPIcs), vol. 183, pp. 38:1–38:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany (2021). https://doi.org/10.4230/LIPIcs.CSL.2021.38. https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.CSL.2021.38
30. Tomita, H.: Planar realizability via left and right applications. In: Manea, F., Simpson, A. (eds.) 30th EACSL Annual Conference on Computer Science Logic (CSL 2022). Leibniz International Proceedings in Informatics (LIPIcs), vol. 216, pp. 35:1–35:17. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany (2022). https://doi.org/10.4230/LIPIcs.CSL.2022.35. https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.CSL.2022.35
31. Tomita, H.: Categorical realizability for non-symmetric closed structures. arXiv preprint arXiv:2307.04119 (2023)