A Simple Categorical Calculus of Interacting Processes

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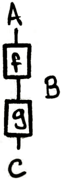
Categories:

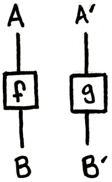
$f \leftrightsquigarrow$ 

$fg \leftrightsquigarrow$ 

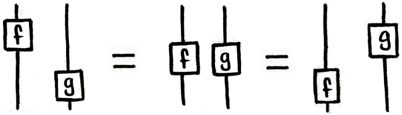
$1_A \leftrightsquigarrow$ 

Objects, morphisms, composition, identities.

(Strict) Monoidal Categories:

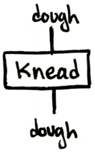
$f \otimes g \quad \leftrightsquigarrow$ 

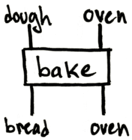
Tensor product carries monoid stucture. Monoid of objects.



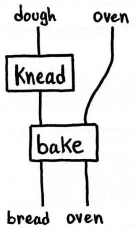
Tensor is functorial, naturality axioms.

“Resource-Theoretic” interpretation.



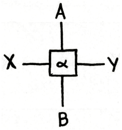


Composite processes.



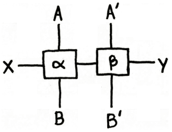
(Single-Object Strict) Double Categories:

$$\begin{tikzcd} * \arrow[r, "A"] \arrow[d, "X"'] \arrow[dr, phantom, "\alpha"] & * \arrow[d, "Y"] \\ * \arrow[r, "B"'] & * \end{tikzcd} \qquad \leftrightsquigarrow$$

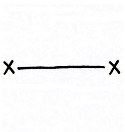


Horizontal and vertical edge monoids, cells.

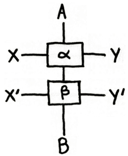
$\alpha|\beta \quad \leftrightsquigarrow$



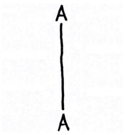
$id_X \quad \leftrightsquigarrow$



$\dfrac{\alpha}{\beta} \quad \leftrightsquigarrow$



$1_A \quad \leftrightsquigarrow$



Horizontal and vertical composition, identities.

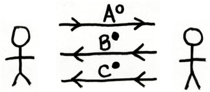
The free cornering $\ulcorner\!\mathbb{A}\!\lrcorner$ of a monoidal category $\mathbb{A}$ has (Part 1):

Horizontal edge monoid is $(\mathbb{A}_0, \otimes, I)$.

Vertical edge monoid is $(\mathbb{A}^{\circ\bullet}, \otimes, I)$, as in: (also monoid equations)

$$\frac{A \in \mathbb{A}_0}{A^\circ \in \mathbb{A}^{\circ\bullet}} \qquad \frac{A \in \mathbb{A}_0}{A^\bullet \in \mathbb{A}^{\circ\bullet}} \qquad \frac{}{I \in \mathbb{A}^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}^{\circ\bullet} \qquad W \in \mathbb{A}^{\circ\bullet}}{U \otimes W \in \mathbb{A}^{\circ\bullet}}$$

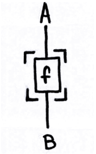
Interpretation: $\mathbb{A}$-valued exchanges. $A^\circ \otimes B^\bullet \otimes C^\bullet \in \mathbb{A}^{\circ\bullet}$ is:



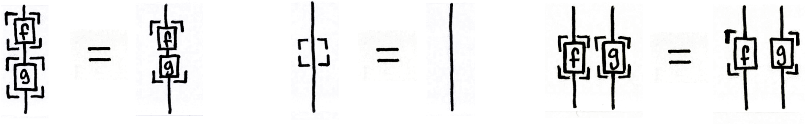
The free cornering $\ulcorner \mathbb{A} \lrcorner$ of a monoidal category $\mathbb{A}$ has (Part 2):

For each $f : A \to B$ in $\mathbb{A}$, a cell:

$$\ulcorner f \lrcorner \qquad \leftrightsquigarrow$$

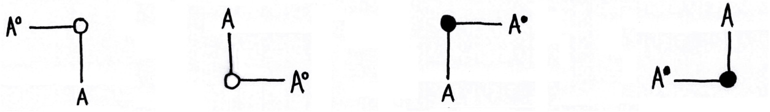


Subject to equations:

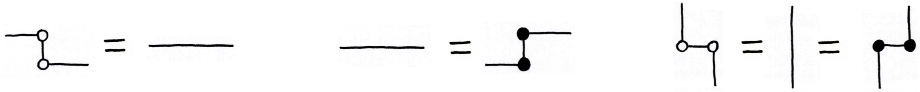


The free cornering ${}^{\ulcorner}_{\llcorner}\mathbb{A}^{\urcorner}_{\lrcorner}$ of a monoidal category $\mathbb{A}$ has (Part 3):

Corner cells for each object A of $\mathbb{A}$:

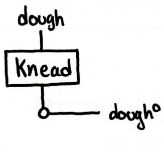


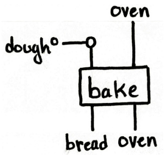
Subject to equations:

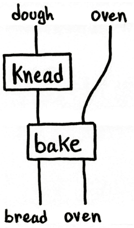


Interpretation: resource-passing.

Transform input (top) to output (bottom) by participating in exchanges (left/right).







Horizontal composition $\leftrightsquigarrow$ interaction.

The free cornering is cool/fun, but it is *static*.

We would like a dynamic version. A *calculus*.

This is rather tricky! Failed attempts.

What works well: virtual double categories!

Instead of a monoidal category, start with a multicategory $\mathcal{M}$.

Multicategories and monoidal categories are almost the same thing:

$$\begin{tikzcd} \mathsf{Mul} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Mon} \arrow[l, bend left, "U"] \end{tikzcd}$$

. . . so this should be uncontroversial.

Multicategories $\leftrightsquigarrow$ Intuitionistic Sequent Calculus

Identities and other morphisms:

$$\frac{}{x : A \vdash x : A}\ \text{VAR} \qquad\qquad \frac{(\Gamma_i \vdash t_i : A_i)_{i=1}^{n} \qquad f \in \mathcal{M}(A_1, .., A_n; B)}{\Gamma_1,..,\Gamma_n \vdash f(t_1,..,t_n) : B}\ \text{OP}$$

Composition via cut (admissible):

$$\frac{(\Gamma_i \vdash t_i : A_i)_{i=1}^{n} \qquad x_1{:}A_1,..,x_n{:}A_n \vdash t : B}{\Gamma_1,..,\Gamma_n \vdash t[t_1,..,t_n/x_1,..,x_n] : B}\ \text{CUT}$$

Modulo equations, every multicategory is presentable in this way.

Terms of our calculus will be (derivations of) sequents:

$$x_1{:}A_1,..,x_n{:}A_n \Vdash t : (U, B, W)$$

"t transforms resources of type $A_1 .. A_n$ into a resource of type B by particupating in the exchanges U/W along the left/right boundary."

The term formation rules of our calculus are:

$$\frac{\Gamma \vdash v : A}{\Gamma \Vdash [v] : (\lambda, A, \lambda)}\ \textsc{seq}$$

$$\frac{(\Gamma_i \Vdash t_i : (U_{i-1}, A_i, U_i))_{i=1}^{n} \qquad x_1{:}A_1,..,x_n{:}A_n \Vdash s : (V, B, W)}{\Gamma_1,..,\Gamma_n \Vdash \mathsf{let}\ x_1,..,x_n \downarrow (t_1 \mid \cdots \mid t_n)\ \mathsf{in}\ s : (U_0 V, B, U_n W)}\ \textsc{let}$$

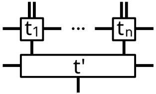
$$\frac{\Delta \Vdash t : (U, B, W) \qquad \Gamma \vdash v : A}{\Delta, \Gamma \Vdash \mathsf{putR}(v, t) : (U, B, A^{\circ} W)}\ \textsc{putR} \qquad \frac{x : A, \Gamma \Vdash t : (U, B, W)}{\Gamma \Vdash \mathsf{getL}(x.t) : (A^{\circ} U, B, W)}\ \textsc{getL}$$

$$\frac{\Gamma, x : A \Vdash t : (U, B, W)}{\Gamma \Vdash \mathsf{getR}(x.t) : (U, B, A^{\bullet} W)}\ \textsc{getR} \qquad \frac{\Gamma \vdash v : A \qquad \Delta \Vdash t : (U, B, W)}{\Gamma, \Delta \Vdash \mathsf{putL}(v, t) : (A^{\bullet} U, B, W)}\ \textsc{putL}$$

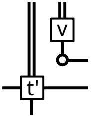
Interpreted as cells of $\ulcorner F(\mathcal{M}) \urcorner$:

$[\![[v]]\!] \quad \leftrightsquigarrow$

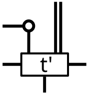
$[\![\mathsf{let}\ x_1, .., x_n \downarrow (t_1 \mid \cdots \mid t_n)\ \mathsf{in}\ t']\!] \quad \leftrightsquigarrow$



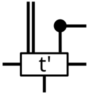
$[\![\mathsf{putR}(v, t')]\!] \quad \leftrightsquigarrow$



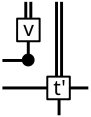
$[\![\mathsf{getL}(x.t')]\!] \quad \leftrightsquigarrow$



$[\![\mathsf{getR}(x.t')]\!] \quad \leftrightsquigarrow$



$[\![\mathsf{putL}(v, t')]\!] \quad \leftrightsquigarrow$



The generating rewrites of our calculus are:

[R0] let $x_1,..,x_n \downarrow ([v_1] \mid \cdots \mid [v_n])$ in $t \;\rightarrow\; t[v_1,..,v_n/x_1,..,x_n]$

[R1] let $x_1,..,x_n \downarrow (\overline{a} \mid \mathsf{putR}(v,t) \mid \mathsf{getL}(y.s) \mid \overline{b})$ in $r \;\rightarrow\;$ let $x_1,..,x_n \downarrow (\overline{a} \mid t \mid s[v/y] \mid \overline{b})$ in r

[R2] let $x_1,..,x_n \downarrow (\overline{a} \mid \mathsf{getR}(y.t) \mid \mathsf{putL}(v,s) \mid \overline{b})$ in $r \;\rightarrow\;$ let $x_1,..,x_n \downarrow (\overline{a} \mid t[v/y] \mid s \mid \overline{b})$ in r

[R3] let $x_1,..,x_n \downarrow (\mathsf{putL}(v,t) \mid \overline{a})$ in $r \;\rightarrow\; \mathsf{putL}(v,$ let $x_1,..,x_n \downarrow (t \mid \overline{a})$ in $r)$

[R4] let $x_1,..,x_n \downarrow (\mathsf{getL}(y.t) \mid \overline{a})$ in $r \;\rightarrow\; \mathsf{getL}(y.$let $x_1,..,x_n \downarrow (t \mid \overline{a})$ in $r)$

[R5] let $x_1,..,x_n \downarrow (\overline{a} \mid \mathsf{putR}(v,t))$ in $r \;\rightarrow\; \mathsf{putR}(v,$ let $x_1,..,x_n \downarrow (\overline{a} \mid t)$ in $r)$

[R6] let $x_1,..,x_n \downarrow (\overline{a} \mid \mathsf{getR}(y.t))$ in $r \;\rightarrow\; \mathsf{getR}(y.$let $x_1,..,x_n \downarrow (\overline{a} \mid t)$ in $r)$

[R7] $\mathsf{putR}(v, \mathsf{putL}(w,t)) \;\rightarrow\; \mathsf{putL}(w, \mathsf{putR}(v,t))$

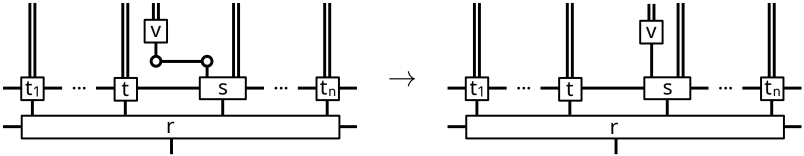
[R8] $\mathsf{putR}(v, \mathsf{getL}(x.t)) \;\rightarrow\; \mathsf{getL}(x.\mathsf{putR}(v,t))$ when x does not occur in v

[R9] $\mathsf{getR}(x.\mathsf{putL}(v,t)) \;\rightarrow\; \mathsf{putL}(v, \mathsf{getR}(x.t))$ when x does not occur in v

[R10] $\mathsf{getR}(x.\mathsf{getL}(y.t)) \;\rightarrow\; \mathsf{getL}(y.\mathsf{getR}(x.t))$

Rules **[R.2]** and **[R.3]** are where the interaction happens.

In the image of our interpretation **[R3]** becomes:



The side conditions on rules **[R.2]** and **[R.3]** are unnecessary if we work only with well-typed terms.

This is confluent and terminating.

Normal forms contain no let-bindings (cut elimination).

Terms modulo convertibility ($\overset{*}{\leftrightarrow}$) form a (single-object) virtual double category $[\mathcal{M}]$.

Virtual double categories are to double categories as multicategories are to monoidal categories:

$$\begin{tikzcd} \mathsf{Mul} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Mon} \arrow[l, bend left, "U"] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{VDC} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{DBL} \arrow[l, bend left, "U"] \end{tikzcd}$$

Our interpretation of terms as cells of the free cornering gives a functor of virtual double categories:

$$[\![-]\!] : [\mathcal{M}] \to U\left(\ulcorner F(\mathcal{M}) \urcorner \right)$$

Thanks for listening!

Future Work:

- Clone-style version of the calculus.
- More structure. Protocol choice, iteration.
- Toy programming language.
- ...