Partial and Relational Algebraic Theories

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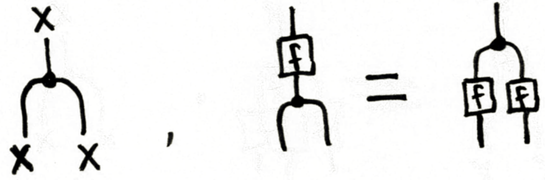
(Categories and Companions 2021)

The Plan:

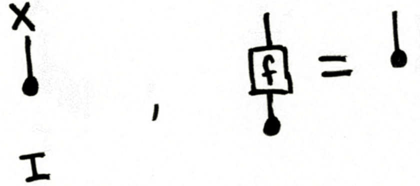
- String diagrams for finite products
- Algebraic theories
- String diagrams for partial maps
- Partial algebraic theories
- String diagrams for relations
- Relational algebraic theories

Categories with finite products are the same thing as symmetric monoidal categories with:

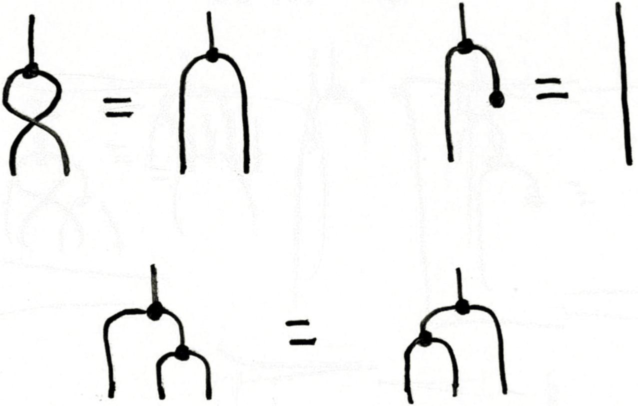
- Copying:



- Deleting:



Copying and deleting form a commutative comonoid:



And must satisfy a coherence condition.

An <u>algebraic theory</u> is a category $\mathbb{X}$ with finite products.

A <u>model</u> of $\mathbb{X}$ is a functor

$$F : \mathbb{X} \longrightarrow \underline{\mathrm{Set}}$$

that preserves finite products.

For example, models of the theory of commutative monoids are commutative monoids.

E.g. Theory of Commutative Monoids

$-\circ-/2 \leftrightsquigarrow$  $\qquad e/0 \leftrightsquigarrow$ 

$$(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3)$$

$\leftrightsquigarrow$  $=$ 

$x_1 \circ x_2 = x_2 \circ x_1 \leftrightsquigarrow$  $=$ 

$x_1 \circ e = x_1 \leftrightsquigarrow$  $=$ 

Composition is substitution

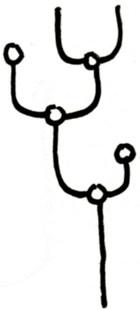
$(x_1 \circ x_2) \circ x_1 \leftrightsquigarrow$ 

$x_1 \circ x_2 \leftrightsquigarrow$ 

$$(x_1 \circ x_2) \circ x_1 \begin{bmatrix} x_1 \mapsto e \\ x_2 \mapsto x_1 \circ x_2 \end{bmatrix}$$

$$=$$

$$(e \circ (x_1 \circ x_2)) \circ e$$

$\leftrightsquigarrow$  $=$ 

A <u>model morphism</u> is a natural transformation

$$\alpha : F \longrightarrow G$$

For the theory of commutative monoids, model morphisms are precisely monoid homomorphisms.

Categories of models and model morphisms are called <u>Varieties</u>.

What about algebraic theories whose operations are <u>partial</u> functions?

i.e., models in $\underline{\mathrm{Par}}$ instead of $\underline{\mathrm{Set}}$.

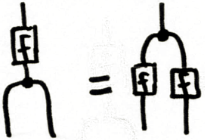
Classical Syntax $\leftrightsquigarrow$ Finite products.

$\underline{\mathrm{Par}}$ does not have finite products.

... So classical syntax isn't going to work!

What structure does Par have?

- Copying: 

, 

, $x \longmapsto (x,x)$

- Restriction: 

, $x \longmapsto *$

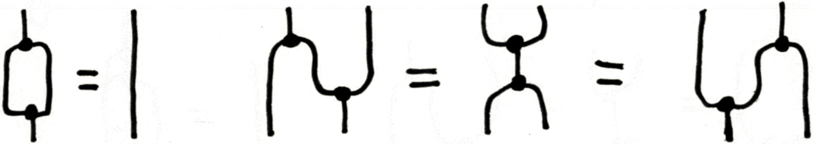
- Sameness: 

, $(x,y) \longmapsto \begin{cases} x & \text{if } x=y \\ \uparrow & \text{otherwise} \end{cases}$

Copying and Restriction form a Comm. Comonoid.

Sameness is Commutative and associative.

Copying and Sameness are Special Frobenius.

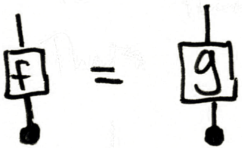


Sensible Coherence Conditions are also Satisfied.

The arrow

Corresponds to the domain of definition of f.

Thus, the equation



Expresses that f and g have the same domain of definition.

A symmetric monoidal category with this structure is a <u>partial algebraic theory</u>.

A <u>model</u> is a symmetric monoidal functor

$$F : \mathbb{X} \to \underline{\mathrm{Par}}$$

that preserves copying, restriction, and sameness.

A <u>model morphism</u> is a lax transformation

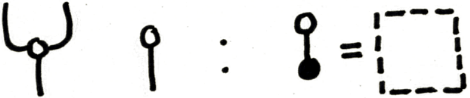
$$\alpha : F \to G$$

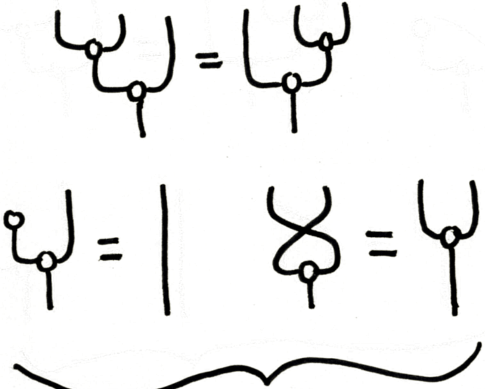
That is, a family $\alpha_X : FX \to GX$ with

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\leq"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

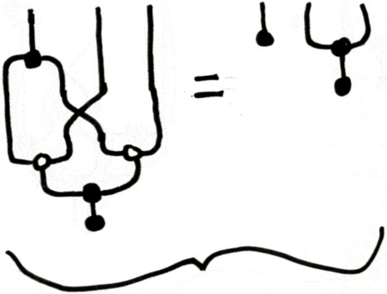
Where $\leq$ is the extension ordering. ($\sqsubseteq$ for partial functions)

For example, the theory of <u>Separation algebras</u> is the free such category generated by





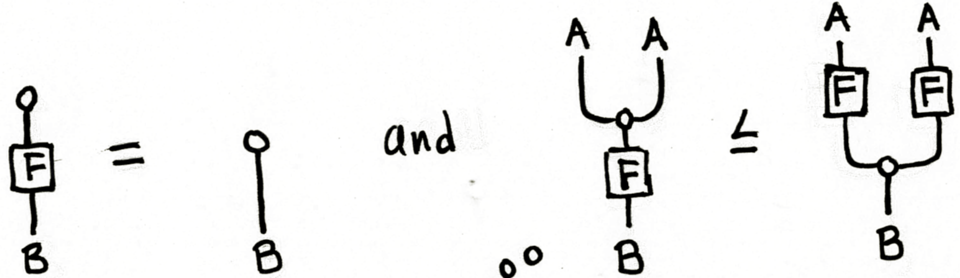
Partial Commutative monoid



Cancellativity

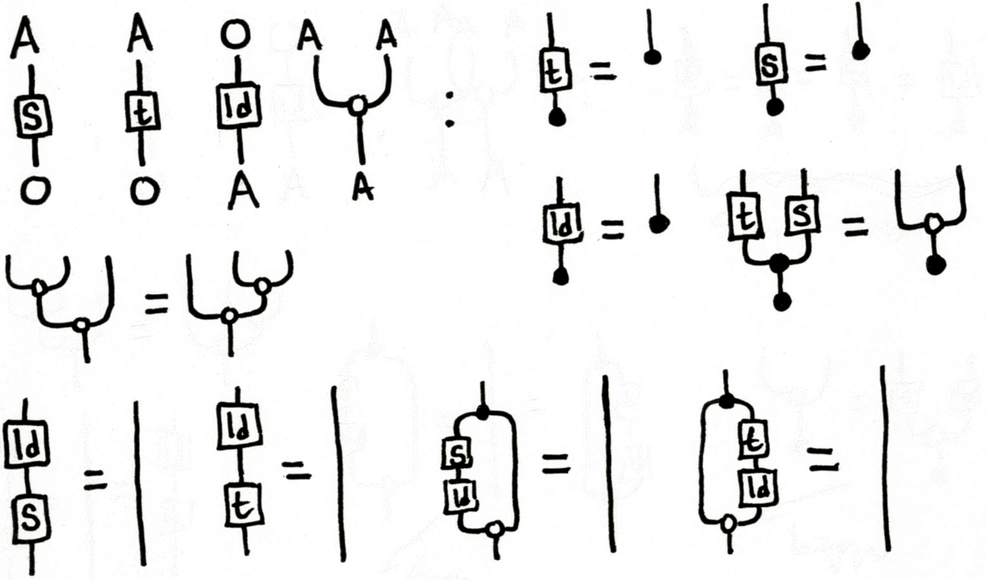
Models are separation algebras.

Model morphisms $A \xrightarrow{F} B$ are total functions with



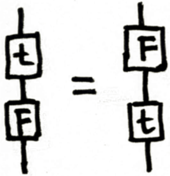
If $a \circ b \downarrow$ then $F(a) \circ F(b) \downarrow$
and $F(a \circ b) = F(a) \circ F(b)$.

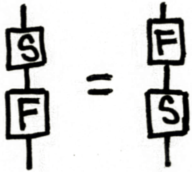
Categories are captured by the 2-sorted theory:

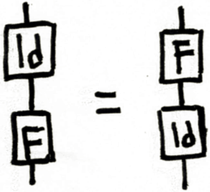


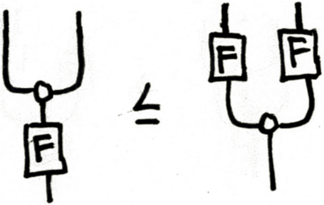
Models are (small) Categories.

Model morphisms $A \xrightarrow{F} B$ are functors.









If f,g are composable, then so are F(f), F(g), and F(f);F(g) = F(f;g)

Theorem: The Categories that arise as models and model morphisms of partial theories are precisely the locally finitely presentable Categories.

Corollary: Two partial theories present the Same LFP Category If and only If splitting their restriction idempotents yields equivalent Categories.

What about algebraic theories whose operations are <u>relations</u>?

i.e., models in <u>Rel</u> instead of <u>Set</u>.

Once again, classical syntax is unsuitable.

Again, we will use string diagrams instead.

What structure does **Rel** have?

δ_X: 

$\{(a,(a,a)) \mid a \in X\}$

ε_X: 

$\{(a,*) \mid a \in X\}$

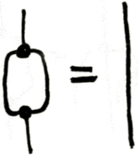
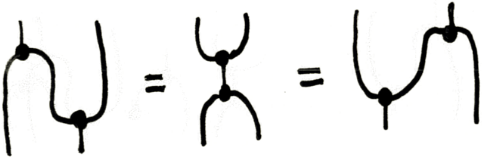
μ_X: 

$\{((a,a),a) \mid a \in X\}$

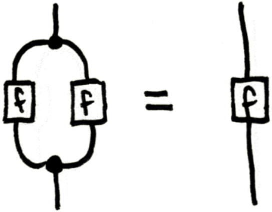
η_X: 

$\{(*,a) \mid a \in X\}$

This structure forms a Special Frobenius algebra:

- $(\delta_X, \varepsilon_X)$ a Commutative Comonoid
- (μ_X, η_X) a Commutative Monoid
-  and 

Further, for every arrow f we have:



A symmetric monoidal category with this structure is a *relational algebraic theory.*

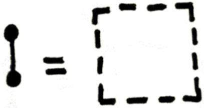
A *model* is a symmetric monoidal functor

$$F : \mathbb{X} \longrightarrow \underline{\mathrm{Rel}}$$

that preserves the Frobenius algebra structure.

Model *morphisms* are again lax transformations.

For example, the theory of <u>nonempty sets</u> is the free such Category on one generating object, subject to the equation:

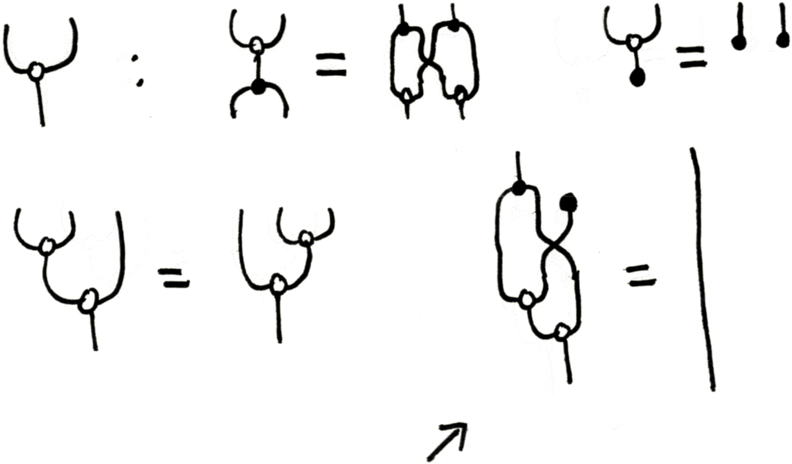


Models are nonempty sets:

$$\eta_X \varepsilon_X = \{(*,*) \mid \exists x \in X.\ (*,x) \in \eta_X \wedge (x,*) \in \varepsilon_X\}$$
$$= \{(*,*) \mid \exists x \in X\}$$

Model morphisms are functions.

The theory of Regular semigroups is generated by:



$\forall a \in S\ \exists x \in S.\ axa = a$ — "Regularity"

Theorem: The categories that arise as models and model morphisms of relational theories are precisely the "definable" categories.

Kuber & Rosický 2018

Corollary: Two relational theories present the same definable category if and only if splitting their symmetric idempotents (partial equivalence relations) yields equivalent categories.

$$
\begin{tikzcd}
 & & \mathsf{CAT}_{\mathsf{split}} \arrow[dr, "\mathsf{FinProd}(-{,}\mathsf{Set})", "\sim"'] & \\
\mathsf{CAT} \arrow[urr, "\mathsf{Split}_{\mathsf{all}}"] \arrow[rr, "\cong"] \arrow[d, dashed, "+\mu"] & & \mathsf{FinProd} \arrow[r, "\mathsf{FinProd}(-{,}\mathsf{Set})"'] \arrow[d, "\mathsf{eq}"'] & \mathsf{VAR}^{\mathsf{op}} \arrow[d, hook] \\
\mathsf{PAT} \arrow[r, "\mathsf{Split}_{\mathsf{cor}}"] \arrow[d, dashed, "+\eta"] & \mathsf{PAT}_{\mathsf{tab}} \arrow[r, "\mathsf{Total}", "\sim"'] \arrow[d, dashed, "?"] & \mathsf{LEX} \arrow[r, "\mathsf{LEX}(-{,}\mathsf{Set})", "\sim"'] \arrow[d, "\mathsf{reg/lex}"'] & \mathsf{LFP}^{\mathsf{op}} \arrow[d, hook, dashed, "?"] \\
\mathsf{RAT} \arrow[r, "\mathsf{Split}_{\mathsf{cor}}"] \arrow[dr, "\mathsf{Split}_{\mathsf{per}}"'] & \mathsf{RAT}_{\mathsf{tab}} \arrow[r, "\mathsf{Map}", "\sim"'] \arrow[d, "\mathsf{Split}_{\mathsf{eq}}"] & \mathsf{REG} \arrow[r, "\mathsf{REG}(-{,}\mathsf{Set})"] \arrow[d, "\mathsf{ex/reg}"'] & \mathsf{DEF}^{\mathsf{op}} \\
 & \mathsf{RAT}_{\mathsf{eff}} \arrow[r, "\mathsf{Map}", "\sim"'] & \mathsf{EX} \arrow[ur, "\sim", "\mathsf{REG}(-{,}\mathsf{Set})"'] &
\end{tikzcd}
$$

References

[1] I. Di Liberti, F. Loregian, C. Nester, P. Sobociński. Functorial Semantics for Partial Theories. *Proc. ACM Program. Lang.* POPL 2021. doi: 10.1145/3434338

[2] F. Bonchi, D. Pavlovic, P. Sobociński. Functorial Semantics for Relational Theories. `https://arxiv.org/abs/1711.08699`.

[3] C. Nester. A Variety Theorem for Relational Universal Algebra. `https://arxiv.org/abs/2105.04958`.