Realizability and Turing Categories

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We can construct a model of first-order intuitionistic arithmetic out of the partial recursive functions (Kleene 1945)

Instead of truth values, propositions are modelled as subsets of $\mathbb{N}$, which we say *realize* them. These act as constructive evidence that a proposition holds.

This model is sound, but not complete. Some propositions are realizable, but not provable in the intuitionistic deductive system.

We define the set of realizers $[\![\varphi]\!] \subseteq \mathbb{N}$ for a proposition φ by

$[\![\varphi]\!] = \mathbb{N}$ if φ is a true atomic formula, e.g. $4 = 4$

$[\![\varphi]\!] = \{\}$ if φ is a false atomic formula, e.g. $3 = 4$

$$[\![\varphi \wedge \psi]\!] = \{\langle n, m \rangle \mid n \in [\![\varphi]\!], m \in [\![\psi]\!]\}$$

$$[\![\varphi \vee \psi]\!] = \{\langle 0, n \rangle \mid n \in [\![\varphi]\!]\} \cup \{\langle 1, n \rangle \mid n \in [\![\psi]\!]\}$$

$$[\![\varphi \Rightarrow \psi]\!] = \{n \mid \forall m \in [\![\varphi]\!].\phi_n(m) \in [\![\psi]\!]\}$$

$$[\![\exists x \varphi]\!] = \{\langle n, m \rangle \mid n \in [\![\varphi[m/x]]\!]\}$$

$$[\![\forall x \varphi]\!] = \{n \mid \forall m \in \mathbb{N}.\phi_n(m) \in [\![\varphi[m/x]]\!]\}$$

A similar approach can be used to construct a realizabilty model for topos logic. This is called the *effective topos* (Hyland 1982)

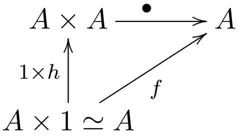
We can construct this sort of *realizability topos* for any partial combinatory algebra*, not just the one given by the partial recursive functions. (Hyland, Pitts, Johnstone, ...)

These are pretty cool. Applications in programming language semantics.

*: Any partial combinatory algebra *on sets*.
(Cockett & Hofstra 2008)

A *partial applicative system* in a cartesian restriction category $\mathbb{X}$ consists of an object A and a map $\bullet : A \times A \to A$. (That's it!)

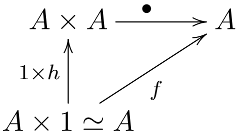
We say a map $f : A \to A$ of $\mathbb{X}$ is A-*computable* in case there is a total map $h : 1 \to A$ such that



A partial applicative system is *combinatory complete* in case the A-computable maps form a cartesian restriction subcategory of $\mathbb{X}$.

Such a partial applicative system is called a *partial combinatory algebra* (PCA).

A *Turing category* is a cartesian restriction category with a *Turing object*. That is, a universal object A together with an application map $\bullet : A \times A \to A$ such that for every map $f : A \to A$ there is a total map $h : 1 \to A$ such that

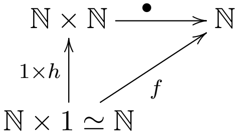


Think of a Turing category as a bunch of computable maps, with the Turing object representing the “data” that we want to compute with.

We can do computability theory in every Turing category. There is a universality theorem, parameter theorem, and so on.

For example, the partial recursive functions give a Turing category that embeds into the category of sets and partial functions, **ptl**.

The Turing object is $\mathbb{N}$, and the application map $\bullet : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $\bullet(m, n) = \phi_n(m)$. Then, for the nth partial recursive function f, the total map $h : 1 \to \mathbb{N}$ such that



is the map $\{* \mapsto n\}$.

This example is caled *Kleene's first model* of computation.

We want to construct realizability models in which the realizers come from an arbitrary Turing category, not necessarily a subcategory of sets and partial functions, **ptl**.

To that end, we work with a cartesian restriction functor

$$F : \mathbb{A} \to \mathbb{X}$$

where $\mathbb{A}$ is a Turing category.

If the codomain of F is **ptl** *and* $F : \mathbb{A} \to \mathbf{ptl}$ is an inclusion, the construction yields the usual realizability tripos.

If the codomain of F is **ptl**, the construction yields the generalized reazliabiliy tripos of (Birkedal 2002).

Let $\mathbb{A}$ be a Turing category, $\mathbb{X}$ be a cartesian restriction category, and $F : \mathbb{A} \to \mathbb{X}$ be a cartesian restriction functor.

An *assembly* is a restriction idempotent $\varphi : \mathcal{O}(F(A) \times X)$ in $\mathbb{X}$ where A is an object of $\mathbb{A}$, and X is an object of $\mathbb{X}$.

A *morphism of assemblies* $f : \varphi \to \psi$ for $\varphi : \mathcal{O}(F(A) \times X)$, $\psi : \mathcal{O}(F(B) \times Y)$ is a map $f : X \to Y$ of $\mathbb{X}$ which is *tracked* by some map $\gamma : A \to B$ of $\mathbb{A}$. That is

(i) $\varphi(F(\gamma) \times f) = \varphi(F(\gamma) \times f)\psi$

(ii) $\overline{\varphi(1 \times f)} = \overline{\varphi(F(\gamma) \times f)}$

Assemblies and their morphisms form a cartesian restriction category, denoted $\mathsf{asm}(F)$.

For example, when F is the inclusion of Kleene's first model into **ptl**, an assembly $\varphi : \mathcal{O}(\mathbb{N} \times X)$ defines a relation $\varphi \subseteq \mathbb{N} \times X$, which we view as a map $\varphi : X \to \mathcal{P}(\mathbb{N})$.

In this case, a morphism of assemblies $f : \varphi \to \psi$ for $\varphi : \mathcal{O}(\mathbb{N} \times X)$, $\psi : \mathcal{O}(\mathbb{N} \times Y)$ is a map $f : X \to Y$ such that there exists a partial recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall x \in X.(b \in \varphi(x) \wedge f(x)\downarrow) \Rightarrow (\gamma(b)\downarrow \wedge \gamma(b) \in \psi(f(x)))$$

The category of total maps of asm(F) is the usual category of assemblies constructed from Kleene's first model.

For a Turing category $\mathbb{A}$ and cartesian restriction functor $F : \mathbb{A} \to \mathbb{X} \ldots$

If $\mathbb{X}$ is a cartesian restriction category, then asm(F) is a cartesian restriction category.

If $\mathbb{X}$ is a discrete range restriction category, then asm(F) is a range restriction category.

If $\mathbb{X}$ is a discrete cartesian closed restriction category, then asm(F) is a locally cartesian closed range restriction category.

A *discrete cartesian closed restriction category* is a cartesian closed restriction category in which, for each object X, the diagonal map $\Delta : X \to X \times X$ has a partial inverse.

For example, **ptl** is a discrete cartesian closed restriction category.

However, **ptl** is a *partial topos* (Curien & Obtulowicz 1989). It has *more* logical structure than a discrete cartesian closed restriction category.

Our realizability tripos construction only requires the codomain of

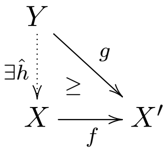
$$F : \mathbb{A} \to \mathbb{X}$$

to be a discrete cartesian closed restriction category.

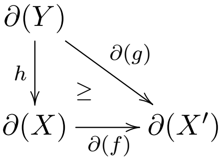
Let $\partial : \mathbb{E} \to \mathbb{X}$ be a restriction functor.

An arrow $f : X \to X'$ of $\mathbb{E}$ is *prone* in case whenever we have maps $g : Y \to X'$ in $\mathbb{E}$ and $h : \partial(Y) \to \partial(X)$ in $\mathbb{X}$ such that $h\partial(f) \geq \partial(g)$, there exists a map $\hat{h} : Y \to X$ such that $\hat{h}f \geq g$, $\partial\hat{h} \leq h$, and for any other map $k : Y \to X$ with these properties, $\hat{h} \leq k$.

In $\mathbb{E}$:



In $\mathbb{X}$:



$\partial : \mathbb{E} \to \mathbb{X}$ is a *latent fibration* in case, for each map $f : A \to \partial(X)$ of $\mathbb{X}$, there is prone arrow above f with codomain X.

Latent fibrations are the correct notion of fibration for restriction categories.

A latent fibration $\partial : \mathbb{E} \to \mathbb{X}$ is *total* if it reflects total maps. That is, if $\overline{\partial(f)} = 1$ implies $\overline{f} = 1$.

From any total latent fibration $\partial : \mathbb{E} \to \mathbb{X}$, we can construct a fibration in the usual sense, $\partial_t : \mathbb{E} \to \text{total}(\mathbb{X})$, whose fibers are those of ∂.

Now, for a cartesian restriction functor $F : \mathbb{A} \to \mathbb{X}$, $\mathbb{A}$ a Turing category, $\mathbb{X}$ a cartesian restriction category, there is a forgetful restriction functor

$$\partial : \mathsf{asm}(F) \to \mathbb{X}$$

This turns out to be a total latent fibration. The prone map above $f : X \to \partial(\psi)$ for $\psi : \mathcal{O}(F(B) \times Y)$ is

$$\overline{(1 \times f)\psi} \xrightarrow{f} \psi$$

If $\mathbb{X}$ is a discrete cartesian closed restriction category, then ∂_t is a tripos. The *realizability tripos*.

We can also do this construction when $\mathbb{A}$ has only part of the structure of a Turing category to obtain a series of *realizability pretriposes* in the sense of (Birkedal 2002)

Let $\mathbb{X}$ be a restriction category.

Define the category $\mathcal{R}(\mathbb{X})$ by

objects: pairs (X, e) where X an object of $\mathbb{X}$, $e : \mathcal{O}(X)$

maps: a map $(X, e) \xrightarrow{f} (X', e')$ is a map $f : X \to X'$ of $\mathbb{X}$ satisfying $e \leq \overline{fe'}$.

composition: as in $\mathbb{X}$

identities: the identity on (X, e) is 1_X

There is a forgetful restriction functor

$$\mathcal{O} : \mathcal{R}(\mathbb{X}) \to \mathbb{X}$$

which also reflects total maps.

In fact $\mathcal{O} : \mathcal{R}(\mathbb{X}) \to \mathbb{X}$ is a total latent fibration.

The prone map above $f : X \to \mathcal{O}(X', e')$ is

$$(X, \overline{fe'}) \xrightarrow{f} (X', e')$$

If $\mathbb{X}$ is a discrete cartesian closed restriction category then the corresponding fibration $\mathcal{O}_t$ is a tripos.

That's new. (Cockett & Hofstra unpublished notes)

Given a tripos $p : \mathbb{E} \to \mathbb{X}$, we can use the internal language to construct a topos whose objects are partial equivalence relations on objects of $\mathbb{X}$, and whose maps $f : X \to Y$ are relations that are

Strict: $\forall_{xy}(f(x,y) \Rightarrow x \sim x \wedge y \sim y)$
Relational: $\forall_{xx'yy'}(f(x,y) \wedge x \sim x' \wedge y \sim y' \Rightarrow f(x',y'))$
Deterministic: $\forall_{xyy'}(f(x,y) \wedge f(x,y') \Rightarrow y \sim y')$
Total: $\forall_x(x \sim x \Rightarrow \exists_y(f(x,y)))$

If we remove the requirement that a relation is total, the resulting category is a *partial topos*. Since every discrete cartesian closed restriction category determines a tripos, this means it also determines a partial topos. More generally, every tripos determines a partial topos.

Thanks for listening!