Protocol Choice and Iteration for the Free Cornering

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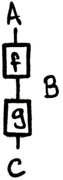
Categories:

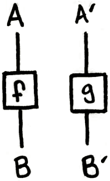
$f \leftrightsquigarrow$ 

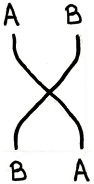
$fg \leftrightsquigarrow$ 

$1_A \leftrightsquigarrow$ 

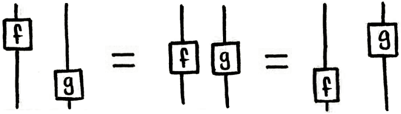
Objects, morphisms, composition, identities.

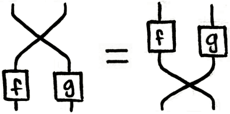
Symmetric Monoidal Categories:

$f \otimes g \rightsquigarrow$ 

$\sigma_{A,B} \rightsquigarrow$ 

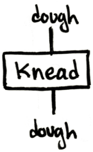
Tensor product and braiding. Monoid of objects.

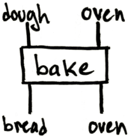




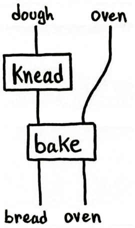
Naturality Conditions.

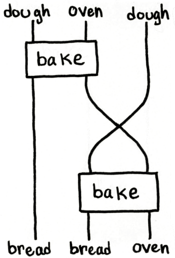
“Resource-Theoretic” interpretation.



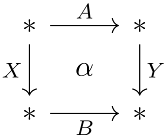


Composite processes.

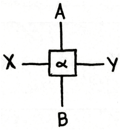




(Single-Object) Double Categories:

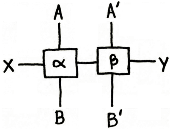


$\leftrightsquigarrow$

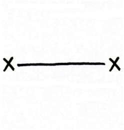


Horizontal and vertical edge monoids, cells.

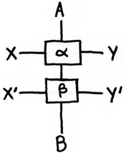
$\alpha|\beta \leftrightsquigarrow$



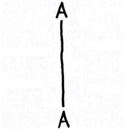
$id_X \leftrightsquigarrow$



$\frac{\alpha}{\beta} \leftrightsquigarrow$



$1_A \leftrightsquigarrow$



Horizontal and vertical composition, identities.

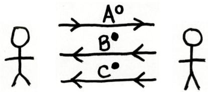
The free cornering $\ulcorner \mathbb{A} \lrcorner$ of a monoidal category $\mathbb{A}$ has (Part 1):

Horizontal edge monoid is $(\mathbb{A}_0, \otimes, I)$.

Vertical edge monoid is $(\mathbb{A}^{\circ\bullet}, \otimes, I)$, as in: (also monoid equations)

$$\frac{A \in \mathbb{A}_0}{A^\circ \in \mathbb{A}^{\circ\bullet}} \qquad \frac{A \in \mathbb{A}_0}{A^\bullet \in \mathbb{A}^{\circ\bullet}} \qquad \frac{}{I \in \mathbb{A}^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}^{\circ\bullet} \qquad W \in \mathbb{A}^{\circ\bullet}}{U \otimes W \in \mathbb{A}^{\circ\bullet}}$$

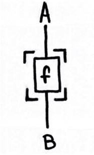
Interpretation: $\mathbb{A}$-valued exchanges. $A^\circ \otimes B^\bullet \otimes C^\bullet \in \mathbb{A}^{\circ\bullet}$ is:



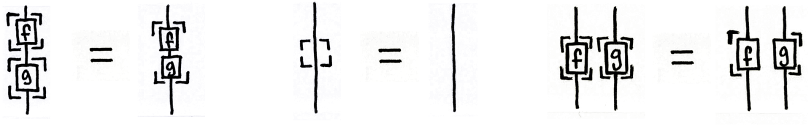
The free cornering $\ulcorner\mathbb{A}\urcorner$ of a monoidal category $\mathbb{A}$ has (Part 2):

For each $f : A \to B$ in $\mathbb{A}$, a cell:

$\ulcorner f \urcorner \quad \leftrightsquigarrow$



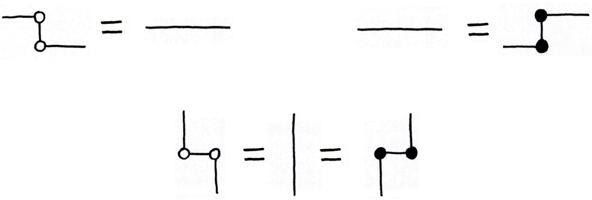
Subject to equations:



The free cornering $\ulcorner\mathbb{A}\urcorner$ of a monoidal category $\mathbb{A}$ has (Part 3):

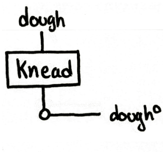
Corner cells for each object A of $\mathbb{A}$:

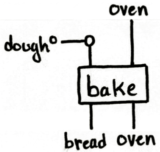
Subject to equations:

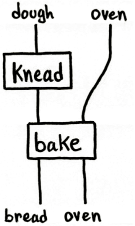


Interpretation: resource-passing.

Transform input (top) to output (bottom) by participating in exchanges (left/right).







Horizontal composition $\leftrightsquigarrow$ interaction.

Alternatively, elements of $\mathbb{A}^{\circ\bullet}$ are *protocol types!*

Q: Is this a good foundation for interactive programming?

A: Not yet. $\mathbb{A}^{\circ\bullet}$ is disappointing as a system of protocol types.

Goal: Support for more protocols. Choice. Iteration.

For choice, vertical edge monoid becomes (plus monoid equations):

$$\frac{A \in \mathbb{A}_0}{A^\circ \in \mathbb{A}^{\circ\bullet}_\oplus} \qquad \frac{A \in \mathbb{A}_0}{A^\bullet \in \mathbb{A}^{\circ\bullet}_\oplus} \qquad \frac{}{I \in \mathbb{A}^{\circ\bullet}_\oplus} \qquad \frac{U \in \mathbb{A}^{\circ\bullet}_\oplus \qquad W \in \mathbb{A}^{\circ\bullet}_\oplus}{U \otimes W \in \mathbb{A}^{\circ\bullet}_\oplus}$$

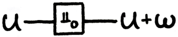
$$\frac{U \in \mathbb{A}^{\circ\bullet}_\oplus \qquad W \in \mathbb{A}^{\circ\bullet}_\oplus}{U + W \in \mathbb{A}^{\circ\bullet}_\oplus} \qquad \frac{U \in \mathbb{A}^{\circ\bullet}_\oplus \qquad W \in \mathbb{A}^{\circ\bullet}_\oplus}{U \times W \in \mathbb{A}^{\circ\bullet}_\oplus}$$

Interpretation:

- $U + W$ is the protocol in which the *left* participant chooses one of U or W, which then happens.
- $U \times W$ is the protocol in which the *right* participant chooses one of U or W, which then happens.

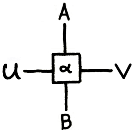
E.g., $(A^\circ \otimes B^\bullet) + (A^\circ \times B^\bullet)$

For $U + W$, we require *injections*:



Moreove, we require that:

$\forall$



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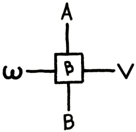
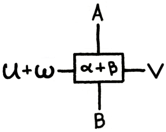
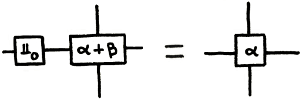
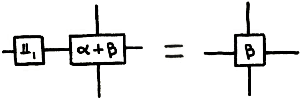


$\exists!$



satisfying:

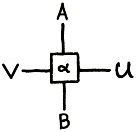




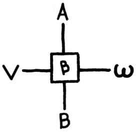
For $U \times W$ we require *projections*:

Moreove, we require that:

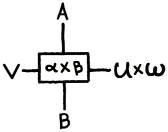
$\forall$



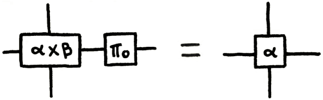
and

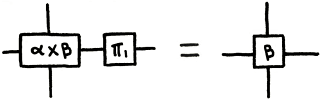


$\exists!$

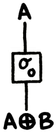


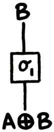
satisfying:





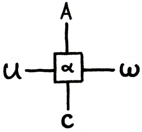
Let $\mathbb{A}$ have distributive binary coproducts $A \oplus B$ with injections:





Then we can require cells as in:

$\forall$



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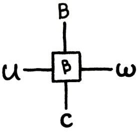
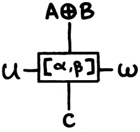


$\exists!$



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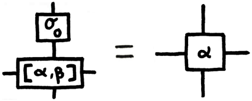
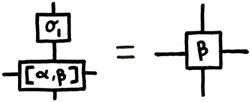




Case statements / "sequential choice".

For iteration, vertical edge monoid becomes:

$$\frac{A \in \mathbb{A}_0}{A^\circ \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{A \in \mathbb{A}_0}{A^\bullet \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{}{I \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}_*^{\circ\bullet} \qquad W \in \mathbb{A}_*^{\circ\bullet}}{U \otimes W \in \mathbb{A}_*^{\circ\bullet}}$$

$$\frac{U \in \mathbb{A}_*^{\circ\bullet} \qquad W \in \mathbb{A}_*^{\circ\bullet}}{U + W \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}_*^{\circ\bullet} \qquad W \in \mathbb{A}_*^{\circ\bullet}}{U \times W \in \mathbb{A}_*^{\circ\bullet}}$$

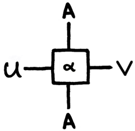
$$\frac{U \in \mathbb{A}_*^{\circ\bullet}}{U^+ \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}_*^{\circ\bullet}}{U^\times \in \mathbb{A}_*^{\circ\bullet}}$$

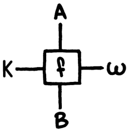
subject to (in addition to monoid equations):

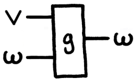
$$U^+ = I + (U \otimes U^+) \qquad U^\times = I \times (U \otimes U^\times)$$

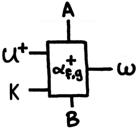
Interpretation: given by the equations.

For U^+, we require:

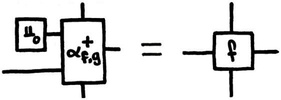
$\forall$ 

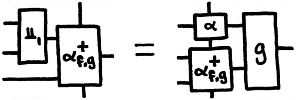
and 

and 

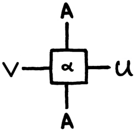
$\exists!$ 

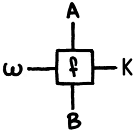
satisfying:

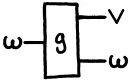


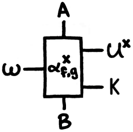


For $U^\times$, we require:

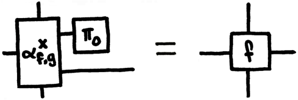
$\forall$ 

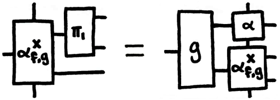
and 

and 

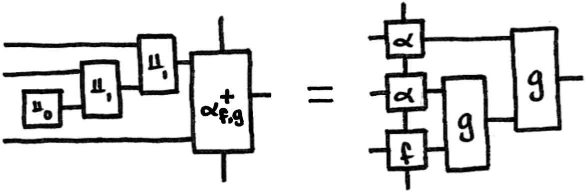
$\exists!$ 

satisfying:





For example:



For convenience, given $\alpha : (U \,{}^{A}_{A}\, W)$ we define:

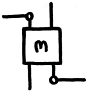
$$\alpha^{+} = \alpha^{+}_{(1_A | \rotatebox[origin=c]{180}{\pi}_0), \rotatebox[origin=c]{180}{\pi}_1} : (U^{+} \,{}^{A}_{A}\, W^{+})$$

yielding the unique cell such that:

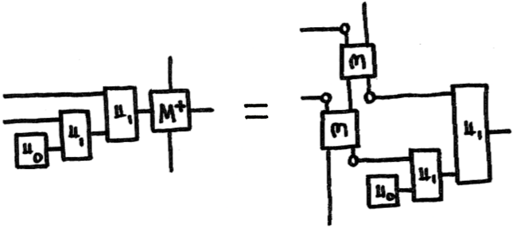
$$\rotatebox[origin=c]{180}{\pi}_0 \mid \alpha^{+} = 1_A \mid \rotatebox[origin=c]{180}{\pi}_0 \qquad\qquad \rotatebox[origin=c]{180}{\pi}_1 \mid \alpha^{+} = \frac{\alpha}{\alpha^{+}} \mid \rotatebox[origin=c]{180}{\pi}_1$$

A *Mealy machine* in $\mathbb{A}$ is a morphism $m : S \otimes A \to B \otimes S$.

This gives a cell:

$M =$ 

Then M^+ is the process that m is meant to define:



Nice mathematical properties:

Monoidal double category.

Protocol choice a lot like (co)products.

Protocol iteration defines a (co)monad.

There is a naturally occurring model of all this.

Let $\mathbb{C}$ be cartesian closed and define $\mathsf{S}(\mathbb{C})$ to have:

1. Horizontal edge monoid $(\mathbb{C}_0, \otimes, I)$
2. Vertical edge monoid $(\mathbb{C}^{\mathbb{C}}, \circ, 1_C)$ (only strong endofunctors)
3. Cells $\alpha : \mathsf{S}(\mathbb{C})(U \, {}^{A}_{B} \, W)$ are strong natural transformations

$$\alpha_X : UX \otimes A \to W(X \otimes B)$$

---

1. $\mathsf{S}(\mathbb{C})$ models $\ulcorner\mathbb{C}\urcorner$ (structure-preserving double functor).
2. If $\mathbb{C}$ distributive, $\mathsf{S}(\mathbb{C})$ models cornering with choice.
3. If $\mathbb{C}$ is the category of containers over Set, $\mathsf{S}(\mathbb{C})$ models cornering with iteration.

Furture Work:

Sequential iteration ("Active" iteration).

Term logic. Dynamics. Coherence questions.

Connections to monadic effects. More on $\mathsf{S}(\mathbb{C})$.

Questions about double categories. What structure is this?

. . . Implementation?

Some References

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Runners for Interleaving Algebraic Effects. Voorneveld. 2022.

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