# Elgot Categories and Abacus Programs

Chad Nester
Tallinn University of Technology

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# The Plan

- **Structured Monoidal Categories**
  ... review of rig categories etc.
- **Categorical Representation of Partial Functions**
  ... review of (strong) representability etc.
- **Elgot Categories**
  ... represent the partial recursive functions
- **Abacus Programs**
  ... define an initial elgot category (and more)

# Structured Monoidal Categories

Our monoidal categories will all be *strict*, for the sake of concision.

Composition is written *correctly*, in diagrammatic order.

$$\frac{f : A \to B \qquad g : B \to C}{fg : A \to C}$$

String diagrams are read *top to bottom*, like everything else.

A *cocartesian monoidal category* is a tuple

$$(\mathbb{C}, (+, 0, \sigma^{+}), (\eta, \mu))$$

such that:

- $(\mathbb{C}, +, 0, \sigma^{+})$ is a symmetric monoidal category.
- $\mu_A : A + A \to A$ and $\eta_A : 0 \to A$ are monoidal natural transformations.
- For all objects $A$, $(A, \mu_A, \eta_A)$ is a commutative monoid in $\mathbb{C}$.

$\mu \quad \leftrightsquigarrow$  $\qquad \eta \quad \leftrightsquigarrow$ 

Coproducts. Copairing $[f, g] = (f + g)\mu$. Injections $\iota_0 = (1 + \eta)$.

A *traced monoidal category* is a tuple

$$(\mathbb{C}, (\oplus, 0, \sigma^{\oplus}), \mathsf{Tr})$$

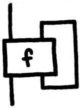
such that:

- $(\mathbb{C}, \oplus, 0, \sigma^{\oplus})$ is a symmetric monoidal category.
- Tr is a family of operations

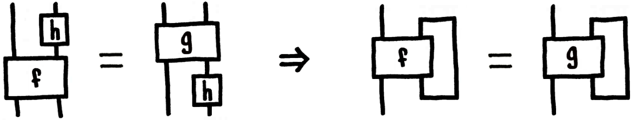
$$\mathsf{Tr}^{C}_{A,B} : \mathbb{C}(A \oplus C, B \oplus C) \to \mathbb{C}(A, B)$$

  which must satisfy a number of equations.

$$\mathsf{Tr}^{C}_{A,B}(f) \quad \leftrightsquigarrow$$



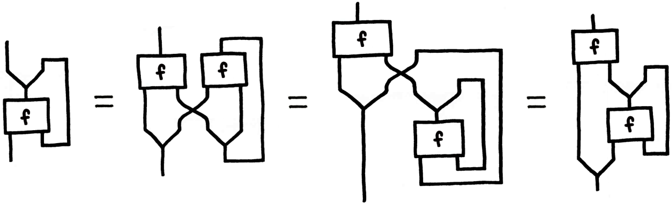
A traced monoidal category is called *uniform* (or *enzymatic*) in case for all appropriately typed $f, g, h$ we have:




For example, in a cocartesian setting uniformity implies:

A cocartesian monoidal category is traced if and only if it admits a parameterised fixed point operator.

Defining $(-)^\dagger : \mathbb{C}(A, X + A) \to \mathbb{C}(A, X)$ by $f^\dagger = \mathsf{Tr}^A_{A,X}(\mu_A f)$ gives a parameterised fixed point operator. $f^\dagger = f[1_X, f^\dagger]$ as in:



A *rig category* is a tuple

$$(\mathbb{C}, (\oplus, 0, \sigma^{\oplus}), (\otimes, I), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^{l}, \delta^{r}))$$

such that:

- $(\mathbb{C}, \oplus, 0, \sigma^{\oplus})$ is a symmetric strict monoidal category.
- $(\mathbb{C}, \otimes, I)$ is a monoidal category.
- $\lambda^{\bullet}$ and $\rho^{\bullet}$ are natural isomorphisms with components

$$\lambda^{\bullet}_{A} : 0 \otimes A \to 0 \qquad \rho^{\bullet}_{A} : A \otimes 0 \to 0$$

- $\delta^{l}$ and $\delta^{r}$ are natural isomorphisms with components

$$\delta^{l}_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C)$$

$$\delta^{r}_{A,B,C} : (A \oplus B) \otimes C \to (A \otimes C) \oplus (B \otimes C)$$

Satisfying a number of coherence axioms. (22 axioms!)

We write $\otimes$ as juxtaposition for objects of rig categories.

For example:

$$ABC = A \otimes B \otimes C \qquad AB + AC = (A \otimes B) + (A \otimes C)$$

$$IA = I \otimes A \qquad (A + B)(C + D) = (A + B) \otimes (C + D)$$

$$\delta^l_{A,B,C} : A(B + C) \to AB + AC$$

And so on. We don't do this for morphisms.

A *cocartesian rig category* is a tuple

$$(\mathbb{C}, (\oplus, 0, \sigma^{\oplus}), (\mu, \eta), (\otimes, I), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^l, \delta^r))$$

such that:

- $(\mathbb{C}, (+, 0, \sigma^{+}), (\otimes, I), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^l, \delta^r))$ is a rig category.
- $(\mathbb{C}, (+, 0, \sigma^{+}), (\mu, \eta))$ is a cocartesian monoidal category.

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A *(uniform) traced rig category* is a tuple

$$(\mathbb{C}, (\oplus, 0, \sigma^{\oplus}), \mathsf{Tr}, (\otimes, I), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^l, \delta^r))$$

such that:

- $(\mathbb{C}, (\oplus, 0, \sigma^{\oplus}), (\otimes, I), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^l, \delta^r))$ is a rig category.
- $(\mathbb{C}, (\oplus, 0, \sigma^{\oplus}), \mathsf{Tr})$ is a (uniform) traced monoidal category.

## Categorical Representation of Partial Functions

A *natural numbers algebra* $(N, z, s)$ in a monoidal category $(\mathbb{C}, \otimes, I)$ is a diagram:

$$I \xrightarrow{z} N \xleftarrow{s} N$$

Every natural numbers algebra defines a *numeral* $\underline{n} : I \to N$ in $\mathbb{C}$ for each $n \in \mathbb{N}$ as in $\underline{0} = z$ and $\underline{n+1} = \underline{n}s$.

A natural numbers algebra is called[1] *strong* in case:

$$\underline{n} = \underline{m} \quad \Rightarrow \quad n = m$$

[1]This is new terminology.

Let $(N, z, s)$ be a natural numbers algebra in $(\mathbb{C}, \otimes, I)$.

Let $f : \mathbb{N}^n \to \mathbb{N}$ be a partial function, and let $\underline{f} : N^n \to N$ be a morphism of $\mathbb{C}$.

We say that $\underline{f}$ *represents* $f$ in case $\forall k_1, \ldots, k_n \in \mathbb{N}$ we have:

$$f(k_1, \ldots, k_n) = k \quad \Rightarrow \quad (\underline{k_1} \otimes \cdots \otimes \underline{k_n})\underline{f} = \underline{k}$$

Representability is about interpretation.

Let $(N, z, s)$ be a natural numbers algebra in $(\mathbb{C}, \otimes, I)$.

Let $f : \mathbb{N}^n \to \mathbb{N}$ be a partial function, and let $\underline{f} : N^n \to N$ be a morphism of $\mathbb{C}$.

We say that $\underline{f}$ *strongly represents* $f$ in case $\forall k_1, \ldots, k_n \in \mathbb{N}$ we have:

$$f(k_1, \ldots, k_n) = k \quad \Leftrightarrow \quad (\underline{k_1} \otimes \cdots \otimes \underline{k_n})\underline{f} = \underline{k}$$

Strong representability is about definition.

$(\mathbb{N}, 0, +1)$ is a natural numbers algebra in $(\text{Par}, \times, 1)$.

Let $f : \mathbb{N}^n \to \mathbb{N}$ be a partial function, and let $g : \mathbb{N}^n \to \mathbb{N}$ be a partial function.

We say that $g$ is *Kleene equal* to $f$ in case $\forall k_1, \ldots, k_n \in \mathbb{N}$ we have:

$$f(k_1, \ldots, k_n) = k \quad \Leftrightarrow \quad g(k_1, \ldots, k_n) = k$$

So strong representability is generalised Kleene equality $(f \simeq g)$.

Let $(N, z, s)$ be a strong natural numbers algebra in $(\mathbb{C}, \otimes, I)$

Then any $f : N^n \to N$ of $\mathbb{C}$ *defines* a partial function $\overline{f} : \mathbb{N}^n \to \mathbb{N}$ as in:

$$\overline{f}(k_1, \ldots, k_n) = \begin{cases} k \text{ if } (\underline{k_1} \otimes \cdots \otimes \underline{k_n})f = \underline{k} \\ \uparrow \text{ otherwise} \end{cases}$$

Moreover, $f$ strongly represents $\overline{f}$.

Each $f$ strongly represents exactly one partial function.

A *weak left natural numbers object* in $(\mathbb{C}, \otimes, I)$ is a natural numbers algebra $(N, z, s)$ with the property that for any diagram $B \xrightarrow{b} A \xleftarrow{a} A$ there exists a morphism $h : N \otimes B \to A$ such that:

$$\begin{array}{ccccc} B & \xrightarrow{z \otimes 1_B} & N \otimes B & \xleftarrow{s \otimes 1_B} & N \otimes B \\ {\scriptstyle 1_B}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ B & \xrightarrow[b]{} & A & \xleftarrow[a]{} & A \end{array}$$

**Theorem (Thibault, Lambek & Scott, Paré & Román)**

*Let $(\mathbb{C}, \otimes, I)$ be a monoidal category, and let $(N, z, s)$ be a weak left natural numbers object in $\mathbb{C}$. Then all primitive recursive functions are representable in $\mathbb{C}$ relative to $(N, z, s)$.*

In a cocartesian rig category . . .

A natural numbers algebra is a morphism $\iota : I + N \to N$.

A *natural numbers coalgebra* is a morphism $\gamma : N \to I + N$.

We say $\gamma$ is *weakly final* in case for any $\beta : A \to I + A$ there exists $h : A \to N$ such that:

$$\begin{array}{ccc} A & \xrightarrow{\beta} & I + A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_I + h} \\ N & \xrightarrow[\gamma]{} & I + N \end{array}$$

## Theorem (Plotkin)

*In a cocartesian rig category* $\mathbb{C}$*, suppose that:*

- $\iota : I + N \to N$ *is an isomorphism.*
- $(N, \iota)$ *is a weak left natural numbers object.*
- $(N, \iota^{-1})$ *is a weakly final natural numbers coalgebra.*

*Then all partial recursive functions are representable in* $\mathbb{C}$ *relative to* $(N, z, s)$ *where* $[z, s] = \iota$*.*

## Theorem (Plotkin)

*In a cocartesian rig category* $\mathbb{C}$, *suppose that:*

- $\iota : I + N \to N$ *is an isomorphism.*
- $(N, \iota)$ *is a weak left natural numbers object.*
- $(N, \iota^{-1})$ *is a weakly final natural numbers coalgebra.*
- $z \neq zs : I \to N$ *where* $[z, s] = \iota$ *(iff* $(N, \iota)$ *strong).*
- *Every partial function that is strongly representable in* $\mathbb{C}$ *is partial recursive.*

*Then all partial recursive functions are strongly representable in* $\mathbb{C}$ *relative to* $(N, z, s)$ *where* $[z, s] = \iota$*.*

# Elgot Categories

A *pre-Elgot Category* is a tuple

$$(\mathbb{C}, (+, 0, \sigma^{+}), (\mu, \eta), \mathsf{Tr}, (\otimes, I), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^{l}, \delta^{r}), (N, \iota))$$

in which:

- $(\mathbb{C}, (+, 0, \sigma^{+}), (\eta, \mu), (\otimes, I), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^{l}, \delta^{r}))$ is a cocartesian rig category.
- $(\mathbb{C}, (+, 0, \sigma^{+}), \mathsf{Tr}, (\otimes, I), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^{l}, \delta^{r}))$ is a traced rig category.
- $N$ is an object of $\mathbb{C}$ with $\iota : I + N \stackrel{\sim}{\to}$ an isomorphism.

An *Elgot category* is a pre-Elgot category in which the underlying traced rig category is uniform.

**Lemma**

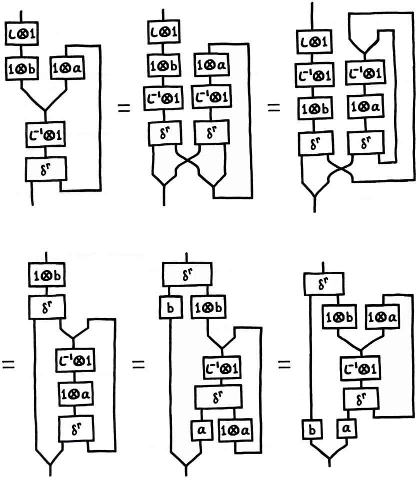
*In any pre-Elgot category, $(N, \iota)$ is a weak left natural numbers object.*

It suffices to show for for any $B \xrightarrow{b} A \xleftarrow{a} A$ we have $h : NB \to A$ such that:

$$\begin{array}{ccc}
(I+N)B & \xrightarrow{\iota \otimes 1_B} & NB \\
\Big\downarrow{\scriptstyle \delta^r_{I,N,B}} & & \Big\downarrow{\scriptstyle h} \\
B + NB & \xrightarrow[{[b, ha]}]{} & A
\end{array}$$

Let $h = \mathrm{Tr}^{NA}_{NB,A}([1_N \otimes b, 1_N \otimes a](\iota^{-1} \otimes 1_A)\delta^r_{I,N,A})$

Then $(\iota \otimes 1_B)h = \delta^r_{I,N,B}[b, ha]$ as in:



### Lemma

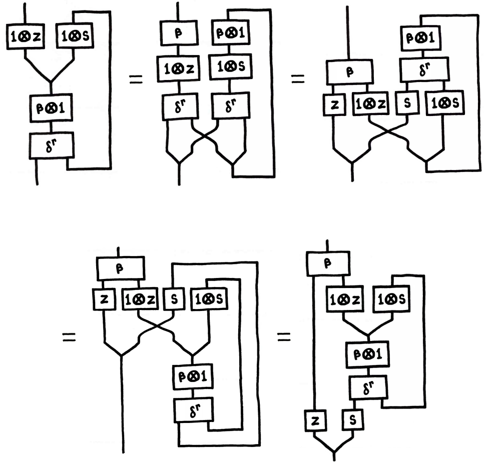
*In any pre-Elgot category, $(N, \iota^{-1})$ is a weakly final natural numbers coalgebra.*

It suffices to show that for any $\beta : A \to I + A$ we have $h : A \to N$ such that:

$$\begin{array}{ccc} A & \xrightarrow{\beta} & I + A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_I + h} \\ N & \xleftarrow[\iota]{} & I + N \end{array}$$

Let $h = \mathrm{Tr}^{AN}_{A,N}([1_A \otimes z, 1_A \otimes s](\beta \otimes 1_N)\delta^r_{I,A,N})$

Then $h = \beta(1_I + h)\iota$ as in:



So we have:

**Theorem**

*The partial recursive functions are representable in any pre-Elgot category.*

and moreover:

**Theorem**

*The partial recursive functions are strongly representable in any pre-Elgot category such that:*

- $(N, \iota)$ *is a strong natural numbers algebra.*
- *Every partial function that is strongly representable is partial recursive.*

# Abacus Programs

**Start**
↓

↓
**Stop**

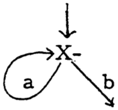
↓
$X^+$
↓

↓
$Y^-$
↙ ↘

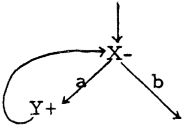
(If $Y \neq \emptyset$, take one pebble away and go to the left; else go to the right)

FIGURE 2. Abacus instructions

EXAMPLE 1.




EXAMPLE 2.



Next, a category $\mathbb{A}$ of abacus programs. This construction owes much to the work of Bonchi, Di Giorgio, and Santamaria.

Objects are *polynomials* over the set $\{N\}$. i.e., $\mathbb{A}_0 = (\{N\}^*)^*$.

$I \in \{N\}^*$ is the empty sequence.

$0 \in (\{N\}^*)^*$ is the empty sequence of sequences.

Concatenation of sequences $U, V \in \{N\}^*$ is written $UV \in \{N\}^*$.

Concatenation of sequences of sequences $P, Q \in (\{N\}^*)^*$ is written $P + Q \in (\{N\}^*)^*$.

Singleton sequences of sequences are called *monomials*.

The following are all monomials:

$$I \qquad N \qquad NN \qquad NNN \qquad NNNNNNN$$

The following are all polynomials that are not monomials:

$$0 \qquad NN + N \qquad NNN + NNNN$$

Important: $INN = NN = NIN$ and
$0 + NN + N = NN = N = NN + 0 + N$ etc.

The morphisms of $\mathbb{A}$ are given as follows:

$$\frac{U, V \text{ monomial}}{\mathsf{succ}_{U,V} : UNV \to UNV} \qquad \frac{U, V, \text{ monomial}}{\mathsf{zero}_{U,V} : UV \to UNV}$$

$$\frac{U, V \text{ monomial}}{\mathsf{pred}_{U,V} : UNV \to UV + UNV} \qquad \frac{U \text{ monomial}}{1_U : U \to U} \qquad \frac{U \text{ monomial}}{\eta_U : 0 \to U}$$

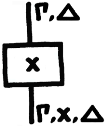
$$\frac{U \text{ monomial}}{\mu_U : U + U \to U} \qquad \frac{U, W \text{ monomial}}{\sigma^+_{U,W} : U + W \to W + U} \qquad \frac{}{1_0 : 0 \to 0}$$

$$\frac{f : P \to Q \qquad g : Q \to R}{fg : P \to R} \qquad \frac{f : P \to Q \qquad g : R \to S}{f + g : P + R \to Q + S}$$

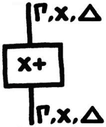
$$\frac{W \text{ monomial} \qquad P, Q \text{ polynomial} \qquad f : P + W \to Q + W}{\mathsf{Tr}^W_{P,Q}(f) : P \to Q}$$

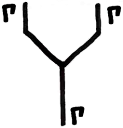
Morphisms of $\mathbb{A}$ are (modified) abacus programs:

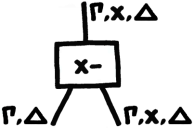
$1_U$ $\leftrightsquigarrow$ 


$\mathsf{zero}_{U,V}$ $\leftrightsquigarrow$ 


$\eta_U$ $\leftrightsquigarrow$ 


$\mathsf{succ}_{U,V}$ $\leftrightsquigarrow$ 


$\mu_U$ $\leftrightsquigarrow$ 


$\mathsf{pred}_{U,V}$ $\leftrightsquigarrow$ 

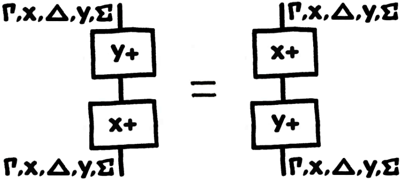
Arrows of $\mathbb{A}$ are subject to equations ensuring that:

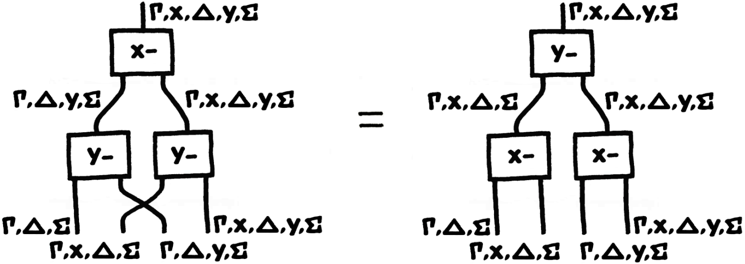
- $(\mathbb{A}, (+, 0, \sigma^+), (\eta, \mu), \mathsf{Tr})$ is a cocartesian monoidal category.
- $(\mathbb{A}, (+, 0, \sigma^+), \mathsf{Tr})$ is a uniform traced monoidal category.

Along with a few equations concerning succ,zero, and pred:

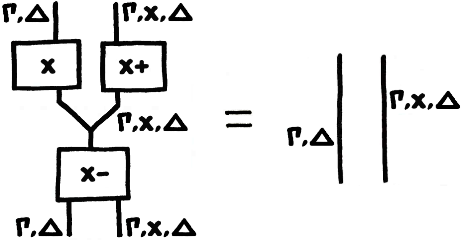
- **[N1]** $(\mathsf{zero}_{U,V} + \mathsf{succ}_{U,V})\mu_{UNV}\mathsf{pred}_{U,V} = 1_{UV+UNV}$
- **[N2]** $\mathsf{pred}_{U,V}(\mathsf{zero}_{U,V} + \mathsf{succ}_{U,V})\mu_{UNV} = 1_{UNV}$
- **[N3]** $\mathsf{succ}_{U,VNW}\mathsf{succ}_{UNV,W} = \mathsf{succ}_{UNV,W}\mathsf{succ}_{U,VNW}$
- ...
- **[N9]** $\mathsf{succ}_{UNV,W}\mathsf{pred}_{U,VNW} = \mathsf{pred}_{U,VNW}(\mathsf{succ}_{UV,W} + \mathsf{succ}_{UNV,W})$
- **[N10]** $\mathsf{zero}_{UNV,W}\mathsf{pred}_{U,VNW} = \mathsf{pred}_{U,VW}(\mathsf{zero}_{UV,W} + \mathsf{zero}_{UNV,W})$
- **[N11]** $\mathsf{pred}_{U,VNW}(\mathsf{pred}_{UV,W} + \mathsf{pred}_{UNV,W}) = \mathsf{pred}_{UNV,W}(\mathsf{pred}_{U,VW} + \mathsf{pred}_{U,VNW})(1_{UVW} + \sigma^+_{UNVW,UVNW} + 1_{UNVNW})$

Equations **[N3]** through **[N9]** say things like:



Equation **[N1]** says:




And equation **[N2]** is the "converse".

**Theorem**

*$\mathbb{A}$ is an Elgot category.*

**Theorem**

*$\mathbb{A}$ is an initial Elgot category.*

**Lemma**

- $\underline{0} \neq \underline{1}$ *in* $\mathbb{A}$.
- *If* $f : \mathbb{N}^n \to \mathbb{N}^n$ *is strongly representable in* $\mathbb{A}$ *then it is partial recursive.*

**Theorem**

*$\mathbb{A}$ strongly represents all (and only) the partial recursive functions.*

The only hard part is the construction of the multiplicative monoidal category structure on $\mathbb{A}$.

**On Objects**: On monomials $UV$ is sequence concatenation. For $U$ monomial and $Q$ polynomial define $UQ$ by induction on $Q$:

$$U0 = 0 \qquad\qquad U(V + Q) = UV + UQ$$

Now for polynomials $P, Q$ define $PQ$ by induction on $P$:

$$0Q = 0 \qquad\qquad (U + P)Q = UQ + PQ$$

This works. For all polynomials $P, Q, R$ we have:

$$(PQ)R = P(QR) \qquad IP = P \qquad PI = P$$

$$0P = 0 \qquad P0 = 0 \qquad (P+Q)R = PR + QR$$

And also for monomials $U$ we have:

$$U(Q+R) = UQ + UR$$

The rig category structure on $\mathbb{A}$ will be *right strict*. That is, $\delta^r, \lambda^\bullet, \rho^\bullet$ will be identities.

Monomial Whiskering. Let $U$ be monomial and define $(U \ltimes -) : \mathbb{A} \to \mathbb{A}$ and $((- \rtimes U) : \mathbb{A} \to \mathbb{A}$ as in:

- $U \ltimes \mathsf{succ}_{V,W} = \mathsf{succ}_{UV,W}$
- $U \ltimes \mathsf{zero}_{V,W} = \mathsf{zero}_{UV,W}$
- $U \ltimes \mathsf{pred}_{V,W} = \mathsf{pred}_{UV,W}$
- $U \ltimes 1_V = 1_{UV}$
- $U \ltimes \eta_V = \eta_{UV}$
- $U \ltimes \mu_V = \mu_{UV}$
- $U \ltimes 1_0 = 1_0$
- $U \ltimes \sigma^+_{V,W} = \sigma^+_{UV,UW}$
- $U \ltimes fg = (U \ltimes f)(U \ltimes g)$
- $U \ltimes f + g = (U \ltimes f) + (U \ltimes g)$
- $U \ltimes \mathsf{Tr}^W_{P,Q} = \mathsf{Tr}^{UW}_{UP,UQ}(U \ltimes f)$

- $\mathsf{succ}_{V,W} \rtimes U = \mathsf{succ}_{V,WU}$
- $\mathsf{zero}_{V,W} \rtimes U = \mathsf{zero}_{V,WU}$
- $\mathsf{pred}_{V,W} \rtimes U = \mathsf{pred}_{V,WU}$
- $1_V \rtimes U = 1_{VU}$
- $\eta_V \rtimes U = \eta_{VU}$
- $\mu_V \rtimes U = \mu_{VU}$
- $1_0 \rtimes U = 1_0$
- $\sigma^+_{V,W} \rtimes U = \sigma^+_{VU,WU}$
- $fg \rtimes U = (f \rtimes U)(g \rtimes U)$
- $f + g \rtimes U = (f \rtimes U) + (g \rtimes U)$
- $\mathsf{Tr}^W_{P,Q} \rtimes U = \mathsf{Tr}^{WU}_{PU,QU}(f \rtimes U)$

Define $\delta^l_{P,Q,R} : P(Q+R) \to PQ + PR$ by induction on $P$:

$$\delta^l_{0,Q,R} = 1_0$$

$$\delta^l_{U+P,Q,R} = (1_{UQ+UR} + \delta^l_{P,Q,R})(1_{UQ} + \sigma^+_{UR,PQ} + 1_{PR})$$

Now define $(P \ltimes -)$ and $(- \rtimes P)$ by induction on $P$:

$$0 \ltimes f = 1_0 \qquad U + P \ltimes f = (U \ltimes f) + (P \ltimes f)$$

and

$$f \rtimes 0 = 1_0 \qquad f \rtimes U + P = \delta^l_{R,U,P}((f \rtimes U) + (f \rtimes P))(\delta^l_{S,U,P})^{-1}$$

Then we have (after much proof by induction):

**Lemma**

*For all* $f : R \to S$ *in* $\mathbb{A}$ *and all polynomials* $P, Q$ *of* $\mathbb{A}$*:*

- $I \ltimes f = f$
- $f \rtimes I = f$
- $PQ \ltimes f = P \ltimes (Q \ltimes f)$
- $f \rtimes PQ = (f \rtimes P) \rtimes Q$
- $(P \ltimes f) \rtimes Q = P \ltimes (f \rtimes Q)$

**Lemma**

*For all* $f : P \to Q$ *and* $g : R \to S$ *of* $\mathbb{A}$*,*
$(f \rtimes R)(Q \ltimes g) = (P \ltimes g)(f \rtimes S)$*.*

And it follows that:

**Lemma**

*Define* $f \otimes g = (f \rtimes R)(Q \ltimes g)$ *for* $f : P \to Q$ *and* $g : R \to S$. *Then* $(\mathbb{A}, \otimes, I)$ *is a (strict) monoidal category.*

**Lemma**

*The* $\delta^l$ *make* $\mathbb{A}$ *into a (right strict) rig category.*

**Theorem**

$\mathbb{A}$ *is an Elgot category.*

Some conjectures about $\mathbb{A}$:

- The multiplicative structure of $\mathbb{A}$ is a distributive restriction category.
- $\mathbb{A}$ *uniquely* strongly represents all and only the partial recursive functions.
- $\mathbb{A}$ is isomorphic (perhaps equivalent) to the (rig) category of partial recursive functions.

Some questions, future work:

- What about dynamics?
- Relationship to Turing categories.
- Free cornering with choice and iteration of $\mathbb{A}$ as a notion of *interactive* computability. Write a programming language?