# Networks and flows
# Ford-Fulkerson algorithm

Let $G = (V, E)$ be a directed graph. For vertex $v \in V$ we can define its *indegree (sisendaste)* $\overrightarrow{\deg}(v)$ and *outdegree (väljundaste)* $\overleftarrow{\deg}(v)$.

If $\overrightarrow{\deg}(v) = 0$ or $\overleftarrow{\deg}(v) = 0$, the vertex $v$ is called *source (lähe)* or *sink (suue)* of graph $G$, respectively.

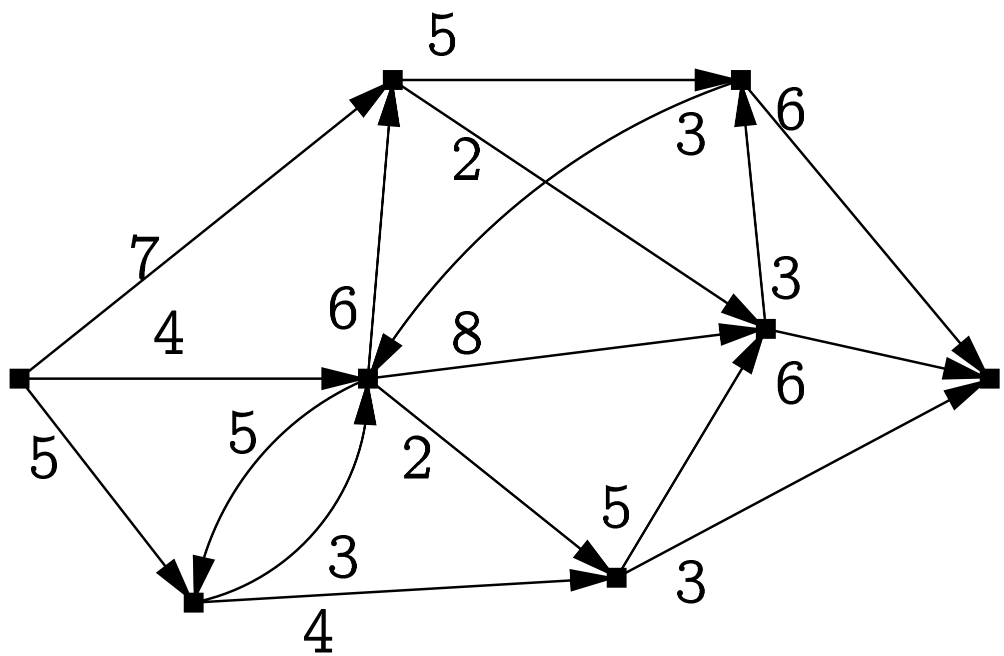
*Capacity* of graph $G$ is a function $\psi : E \longrightarrow \mathbb{R}_+$.

The quantities

$$\overrightarrow{\deg_\psi}(v) = \sum_{e \in E\,\mathcal{E}(e)=(u,v)} \psi(e) \text{ and } \overleftarrow{\deg_\psi}(v) = \sum_{e \in E\,\mathcal{E}(e)=(v,u)} \psi(e)$$

are called *$\psi$-indegree* and *$\psi$-outdegree* of vertex $v \in V$, respectively.

*Network (võrk)* is a pair $(G, \psi)$ where $G$ is a directed graph and $\psi$ its capacity.


5
6
3
2
7
3
6
4
8
5
5
2
6
5
3
3
4

**Proposition.** The sums of all $\psi$-indegrees and $\psi$-outdegrees of graph $G$ are equal.

Proof.

$$\sum_{v \in V} \overrightarrow{\deg_\psi}(v) = \sum_{v \in V} \sum_{\substack{e \in E \\ \mathcal{E}(e)=(u,v)}} \psi(e) = \sum_{e \in E} \psi(e) = \sum_{v \in V} \sum_{\substack{e \in E \\ \mathcal{E}(e)=(v,u)}} \psi(e) = \sum_{v \in V} \overleftarrow{\deg_\psi}(v) \quad .$$

$\square$

Let $(G, \psi)$ be a network. We assume that $G$ has exactly one source $s$ and exactly one sink $t$.

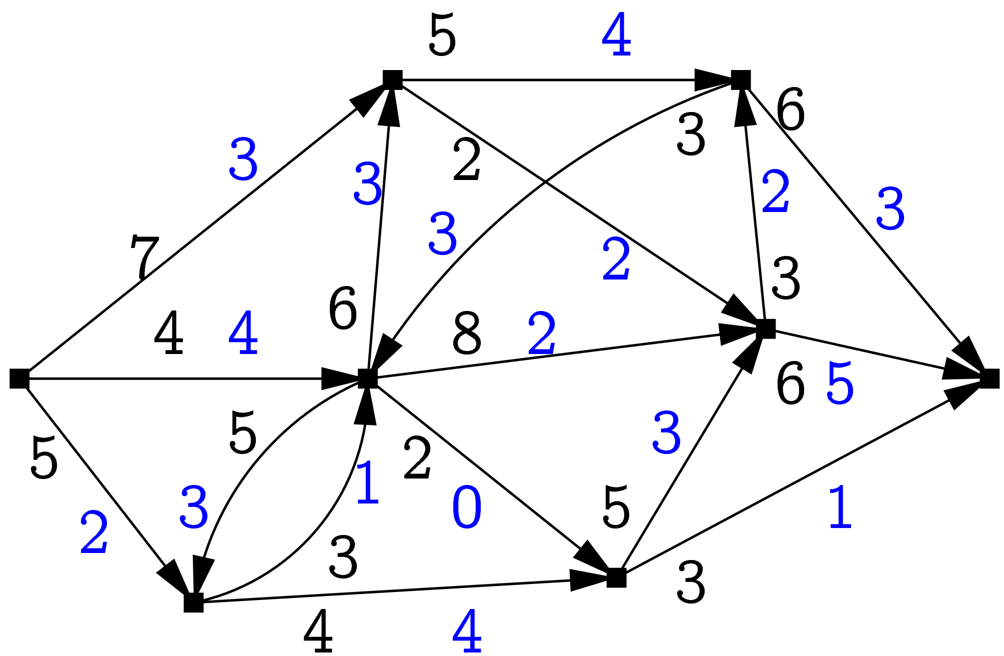
*Flow (voog)* on the network $(G, \psi)$ is a function $\varphi : E \longrightarrow \mathbb{R}_+$, such that

- $\varphi(e) \leq \psi(e)$ for every $e \in E$.
- $\overrightarrow{\deg_\varphi}(v) = \overleftarrow{\deg_\varphi}(v)$ for every $v \in V \backslash \{s, t\}$.

The previous proposition implies $\overleftarrow{\deg_\varphi}(s) = \overrightarrow{\deg_\varphi}(t)$. This quantity is called *value (väärtus)* of the flow $\varphi$ and denoted $|\varphi|$.

The flow is *maximal* if its value is the largest possible.

We assume that $G = (V, E)$ has no loops nor multiple directed arcs, i.e. $E \subseteq V \times V$.



5
4
6
3
3
3
2
3
2
3
7
3
2
6
4
4
8
2
6
5
5
2
5
3
1
0
3
5
2
3
1
3
4
4
3

**Proposition.** Consider a network $(G, \psi)$ with $G = (V, E)$. Let $V = V_s \dot\cup V_t$, such that $s \in V_s$ and $t \in V_t$. Let

$$\Phi(V_s, V_t) = \sum_{e \in E \cap (V_s \times V_t)} \varphi(e) - \sum_{e \in E \cap (V_t \times V_s)} \varphi(e) \ .$$

Then $\Phi(V_s, V_t)$ is equal to the value of $\varphi$.

Proof. Induction over $|V_s|$.

*Base.* If $|V_s| = 1$ then $V_s = \{s\}$. The set $V_s \times V_t$ contains all the arcs originating from $s$ and $V_t \times V_s = \emptyset$.

*Step.* Let the claim hold for some sets $V_s$ and $V_t$. Let $x \in V_t \setminus \{t\}$, $V_s' = V_s \cup \{x\}$ and $V_t' = V_t \setminus \{x\}$. It is enough to prove $\Phi(V_s, V_t) = \Phi(V_s', V_t')$.

$\Phi(V_s, V_t)$ :

| $V \times V$ | $V_s$ | $x$ | $V_t'$ |
|---|---|---|---|
| $V_s$ | | $+\varphi$ | $+\varphi$ |
| $x$ | $-\varphi$ | | |
| $V_t'$ | $-\varphi$ | | |

$\Phi(V_s', V_t')$ :

| $V \times V$ | $V_s$ | $x$ | $V_t'$ |
|---|---|---|---|
| $V_s$ | | | $+\varphi$ |
| $x$ | | | $+\varphi$ |
| $V_t'$ | $-\varphi$ | $-\varphi$ | |

$$\Phi(V_s, V_t) - \Phi(V_s', V_t') =$$

| $V \times V$ | $V_s$ | $x$ | $V_t'$ |
|---|---|---|---|
| $V_s$ | | $+\varphi$ | |
| $x$ | $-\varphi$ | | $-\varphi$ |
| $V_t'$ | | $+\varphi$ | |

$$= \overrightarrow{\deg}_\varphi(x) - \overleftarrow{\deg}_\varphi(x) = 0 \ . \qquad \square$$

A *cut (lõige)* in the network $(G, \psi)$ (where $G = (V, E)$) is such an arc set $L \subseteq E$ that every directed path from source to sink uses some arc from the set $L$.

Alternatively: $L \subseteq E$ is a cut, if there are no directed paths from $s$ to $t$ in the graph $(V, E \backslash L)$.

*Capacity (läbisaskevõime)* of $L$ is the quantity $\psi(L) = \sum_{e \in L} \psi(e)$.

The cut is *minimal* if its capacity is the smallest possible.

**Theorem (Ford and Fulkerson).** The value of all maximal flows in a network is equal to the capacity of all the minimal cuts.

Proof. Let $(G, \psi)$ be a network with $G = (V, E)$, source $s$ and sink $t$. We will show that

I. The value of no flow is larger than the capacity of any cut.

II. For any maximal flow $\varphi$ there exists a cut with capacity $|\varphi|$.

*Part I* Let $\varphi$ be a flow and $L$ a cut.

Let $V_s \subseteq V$ be the set of such nodes $v$ that there exists a directed path from $s$ to $v$ without using any arc from $L$. Let $V_t = V \backslash V_s$. Since $E \cap (V_s \times V_t) \subseteq L$, we have

$$\psi(L) \geq \sum_{e \in E \cap (V_s \times V_t)} \psi(e) \geq \sum_{e \in E \cap (V_s \times V_t)} \varphi(e) \geq \Phi(V_s, V_t) = |\varphi| \ .$$

*Part II* Let $\varphi$ be a maximal flow.

Let $V_s \subseteq V$ be the set of all vertices $v$ such that:
There exists an *undirected* path $s = v_0 \overset{e_1}{\text{———}} v_1 \overset{e_2}{\text{———}} \cdots \overset{e_m}{\text{———}} v_m = v$, such that

- If $e_i = (v_{i-1}, v_i)$ then $\varphi(e_i) < \psi(e_i)$.
- If $e_i = (v_i, v_{i-1})$ then $\varphi(e_i) > 0$.

We say that the flow between $v_{i-1}$ and $v_i$ is *unsaturated (küllastamata)*.

Such a path is called *augmenting (suurendav)*.

Let $V_t = V \setminus V_s$. We will show that $t \in V_t$. Indeed, if $t \in V_s$ then $\varphi$ is not maximal:

Let $s = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_m} v_m = t$ be some augmenting path. Define positive real numbers $\delta_i$ as follows:

$$\delta_i = \begin{cases} \psi(e_i) - \varphi(e_i), & \text{if } e_i = (v_{i-1}, v_i) \\ \varphi(e_i), & \text{if } e_i = (v_i, v_{i-1}) \end{cases} .$$

Let $\varepsilon = \min\limits_i \delta_i$ and let $\varphi'$ be the following flow:

$$\varphi'(e) = \begin{cases} \varphi(e), & \text{if } e \notin \{e_1, \ldots, e_m\} \\ \varphi(e) + \varepsilon, & \text{if } e = e_i = (v_{i-1}, v_i) \\ \varphi(e) - \varepsilon, & \text{if } e = e_i = (v_i, v_{i-1}) \end{cases} .$$

Then $\varphi'$ is a flow and $|\varphi'| = |\varphi| + \varepsilon$.

Construction of the sets $V_s$ and $V_t$ gives:

- If $e \in E \cap (V_s \times V_t)$ then $\varphi(e) = \psi(e)$.
- If $e \in E \cap (V_t \times V_s)$ then $\varphi(e) = 0$.

Let $L = E \cap (V_s \times V_t)$. Then $L$ is a cut and $\psi(L) = |\varphi|$. $\square$

**Algorithm for finding a maximal flow (Ford-Fulkerson).**
Let $(G, \psi)$ be a network with $G = (V, E)$.

Let $\varphi$ be some initial flow on the network $(G, \psi)$, say, $\forall e :$ $\varphi(e) = 0$.

Repeat:

1. Find an augmenting path $s = v_0 \overset{e_1}{\text{———}} v_1 \overset{e_2}{\text{———}} \cdots \overset{e_m}{\text{———}} v_m = t$.
   If there is no such path then stop and output $\varphi$.

2. Construct $\varphi'$ as described 2 slides ago.

3. Assign $\varphi := \varphi'$.

The augmenting path is found traversing the graph in some manner.

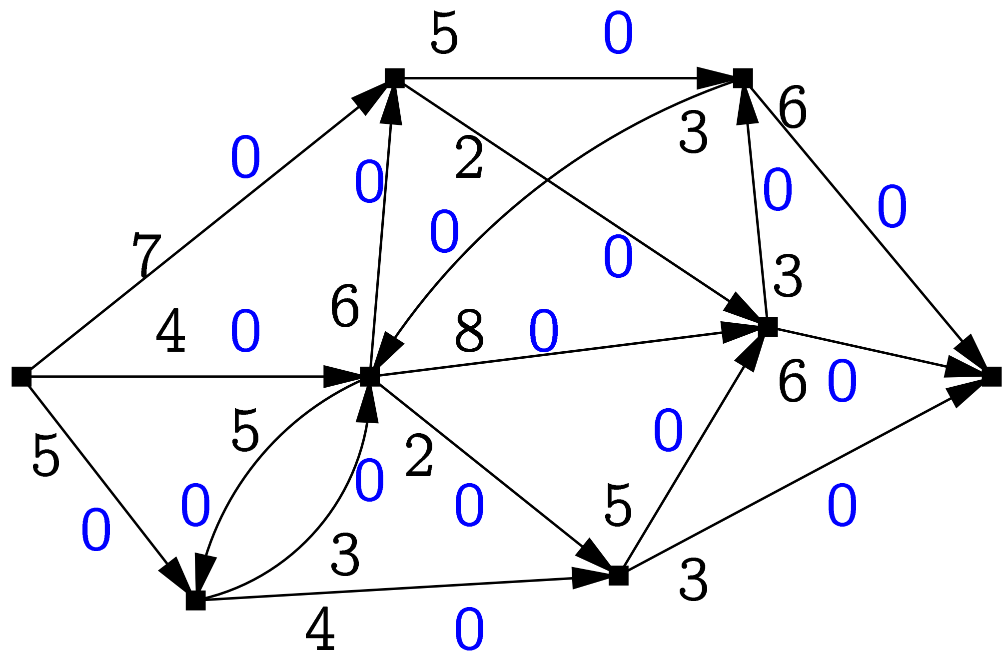
**Theorem.** Ford-Fulkerson algorithm finds a maximal flow.

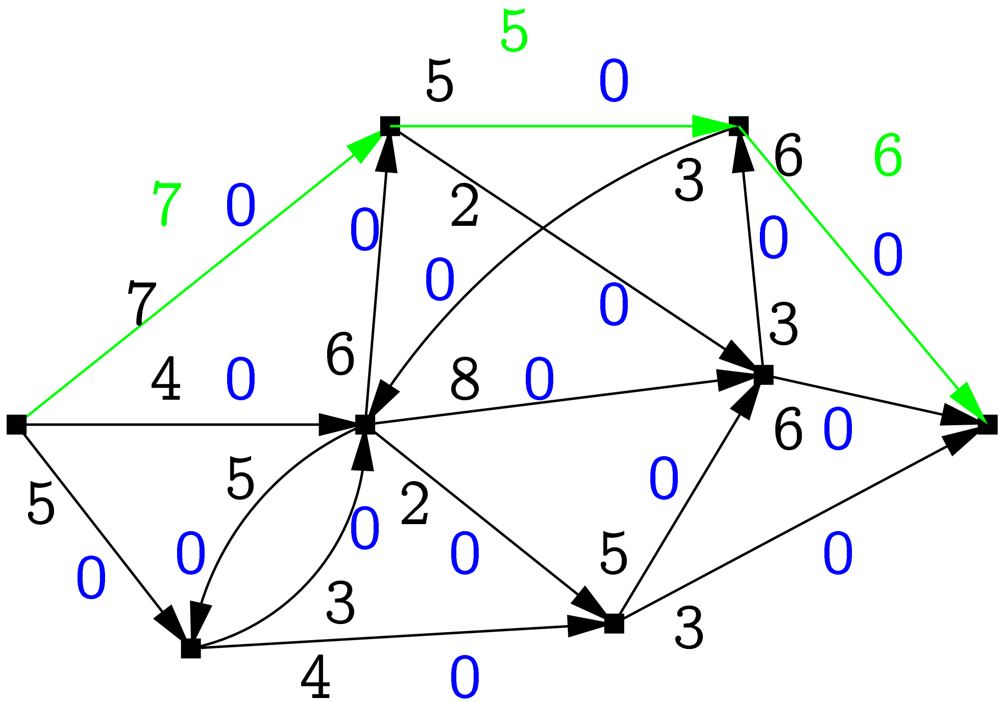
*Proof.* The algorithm obviously outputs a flow. We need to prove that it does not stop before a maximal flow is found.

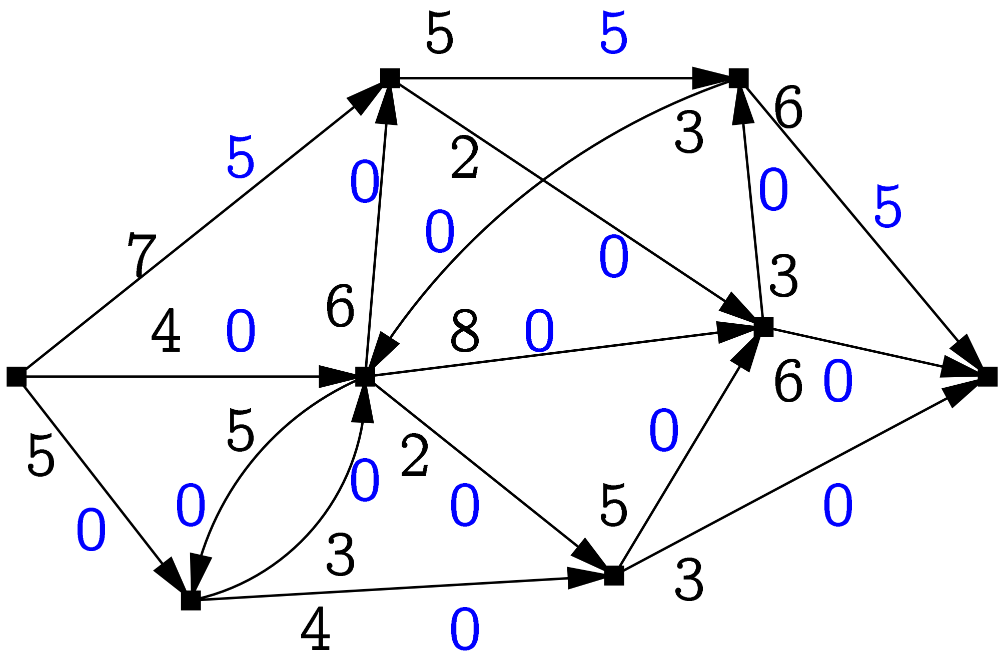
We will show that if $\varphi$ is not a maximal flow then there exists an augmenting path $s \rightsquigarrow t$ for it.

Let $V_s$ be the set of vertices $v$ such that there exists an augmenting path from $s$ to $v$ and let $V_t = V \setminus V_s$. Assume that $t \in V_t$.

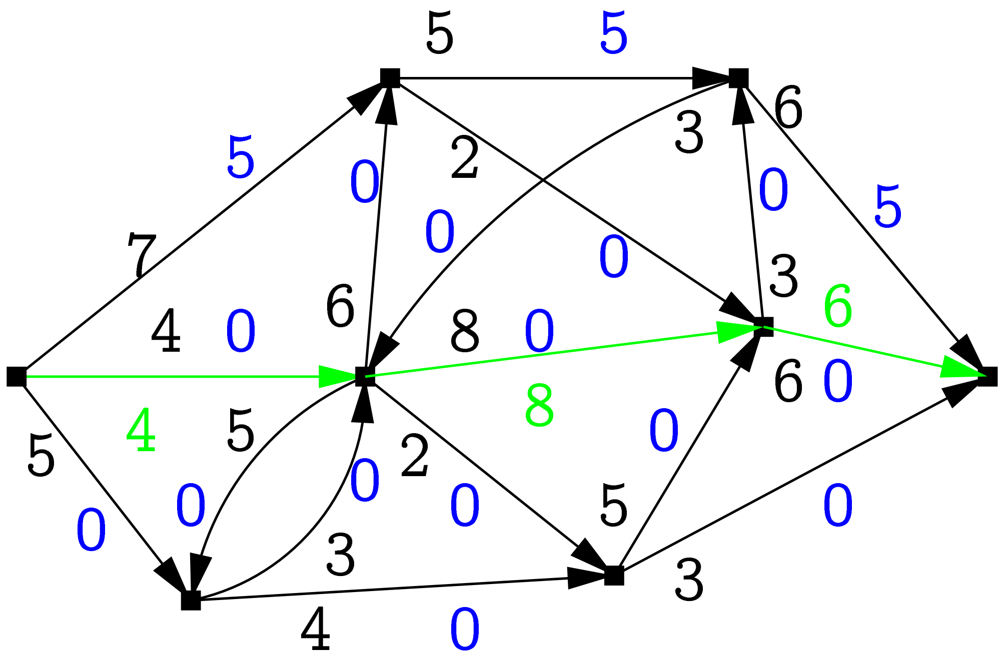
Similarly to the proof of the previous theorem we get that $L = E \cap (V_s \times V_t)$ is a cut and $\psi(L) = |\varphi|$. Thus $\varphi$ must be maximal. $\square$

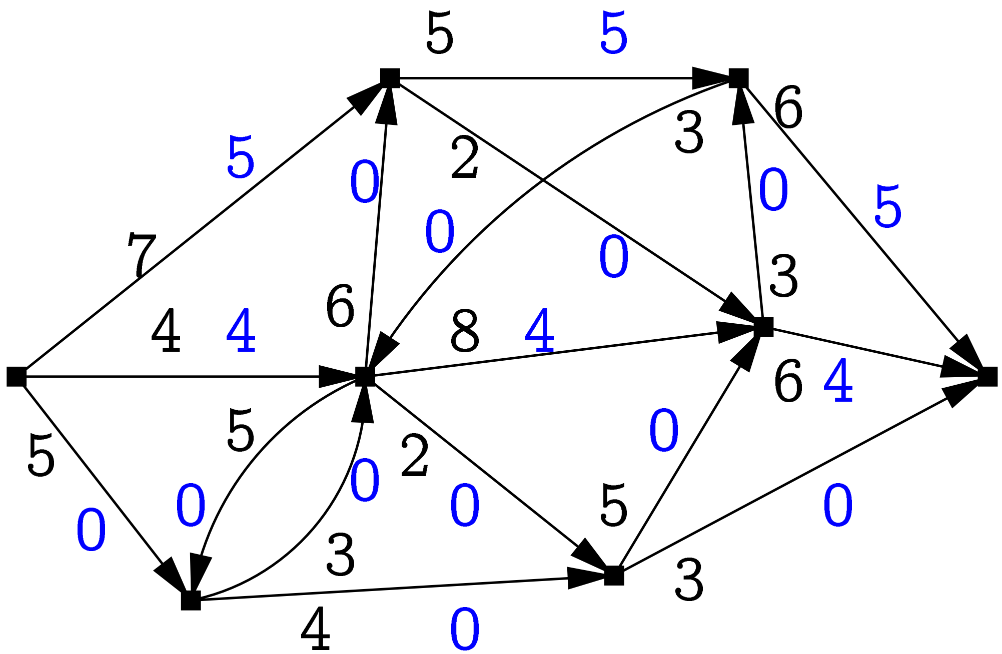

5
0
6
3
2
0
7
0
0
0
0
3
6
4
0
8
0
5
6
0
0
2
5
0
0
0
0
5
3
3
4
0


5
5
0
6
6
3
7
0
2
0
0
0
0
7
3
4
0
6
8
0
6
0
5
5
2
0
0
0
0
5
3
0
3
4
0

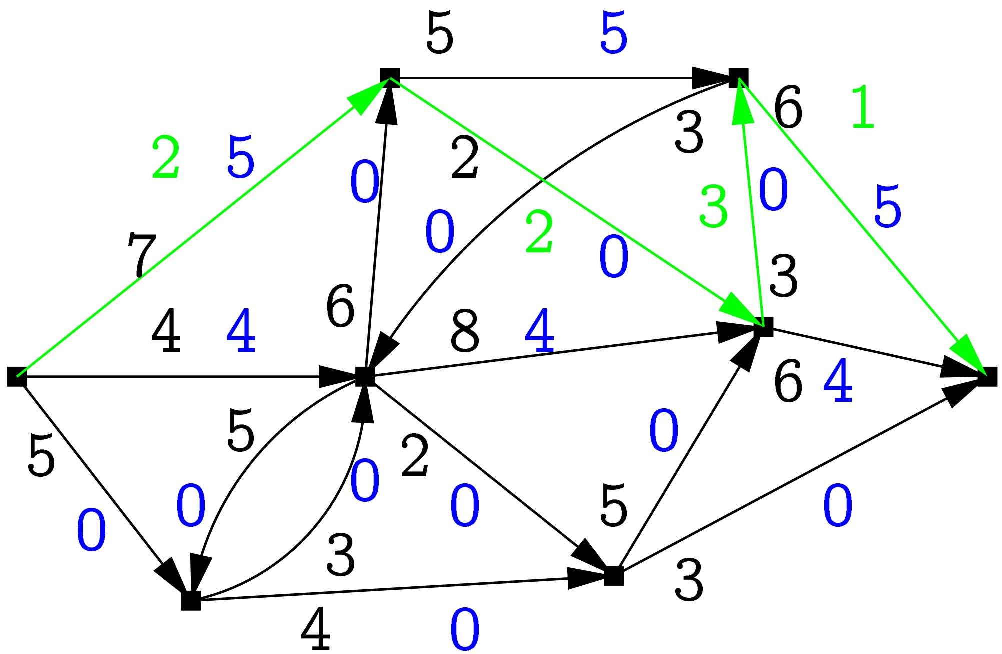


5
5
6
3
5
7
0
2
0
0
0
5
3
6
4
0
8
0
6
0
5
5
2
0
0
0
0
5
0
3
0
3
4
0

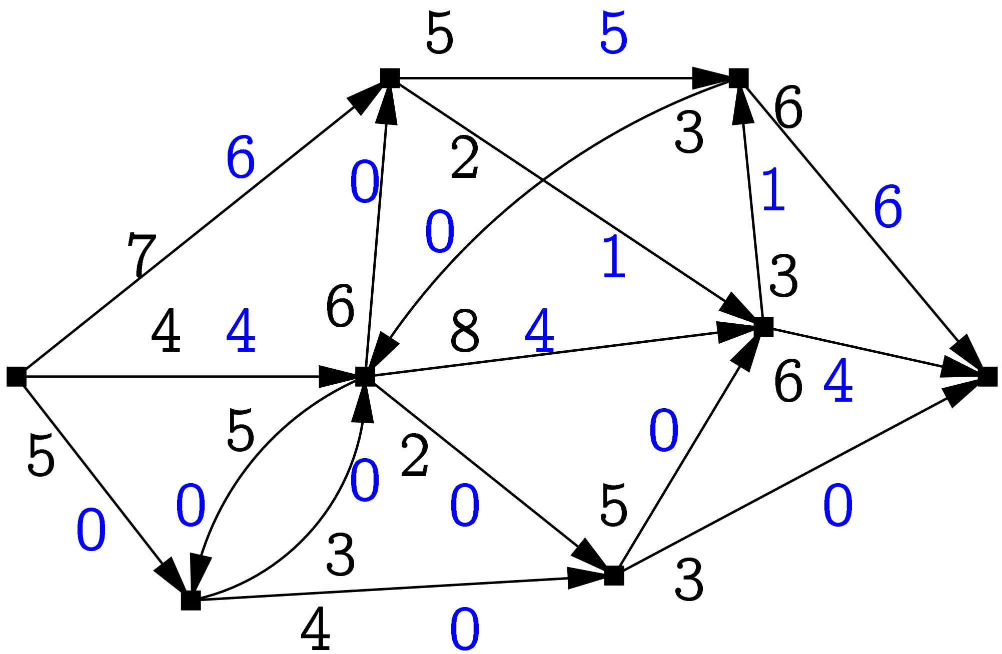

5
5
6
3
2
5
7
0
0
0
5
0
3
6
4
0
6
8
0
6
0
4
5
8
5
0
0
2
0
5
0
0
3
0
3
4
0

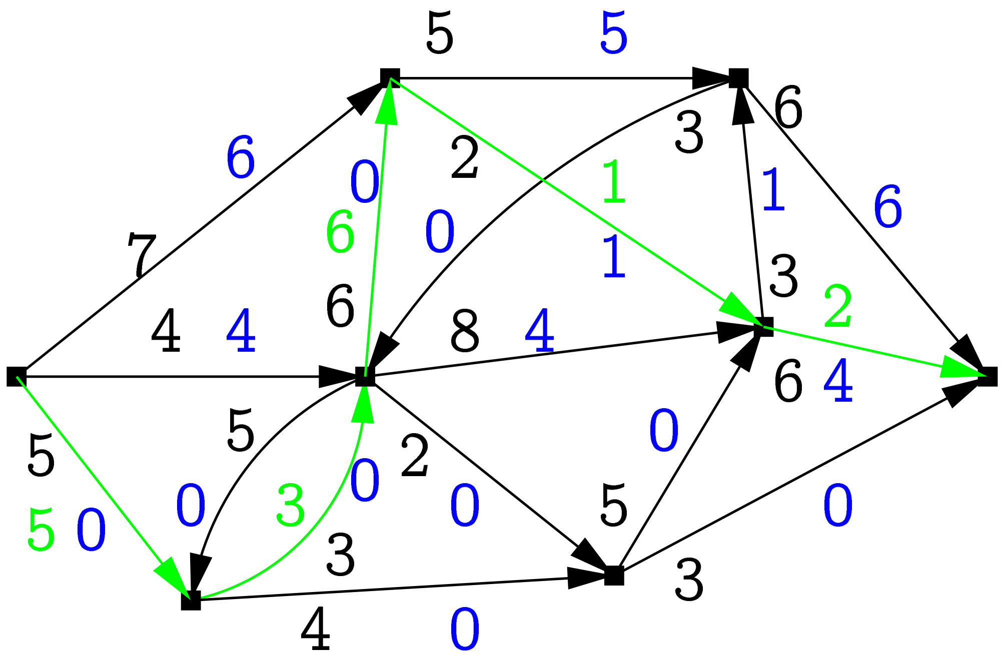


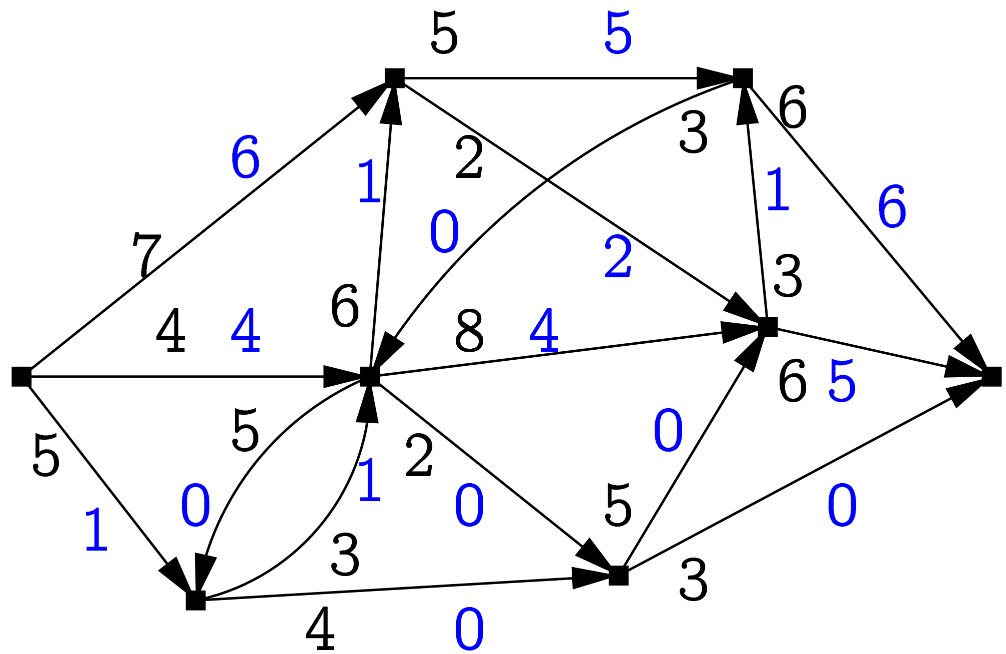
5
5
6
3
5
2
0
0
5
0
7
0
3
6
4
4
8
4
6
4
5
0
5
2
0
0
0
5
0
0
3
3
4
0



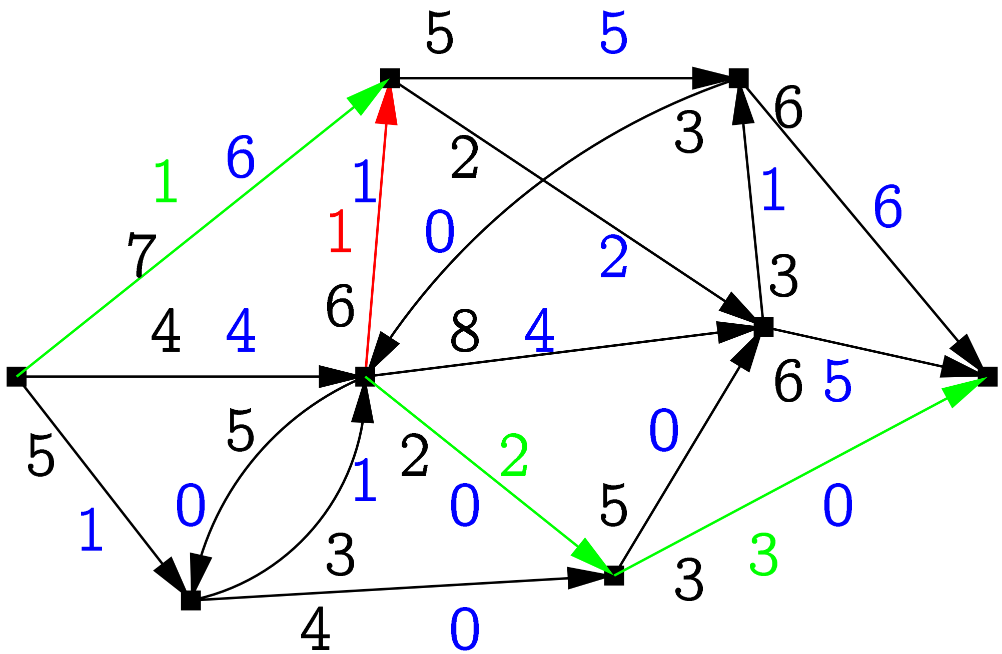
5
5
6
1
2
5
3
2
0
0
3
0
5
7
2
0
3
6
4
4
8
4
6
4
5
5
2
0
0
0
0
0
5
3
0
3
4
0

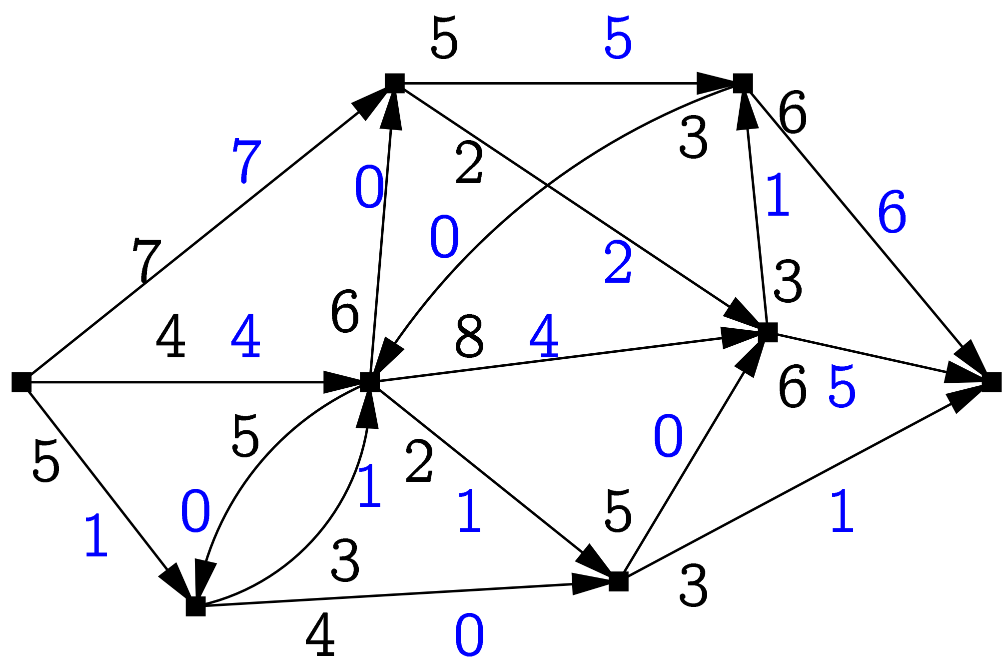

5
5
6
6
3
2
0
0
1
7
1
6
3
6
4
4
8
4
6
4
5
5
0
0
2
0
0
5
3
0
3
4
0


5
5
6
3
2
1
6
0
7
6
0
1
1
6
3
4
4
8
4
2
6
4
5
5
0
2
0
0
5
0
3
0
5
0
3
3
4
0

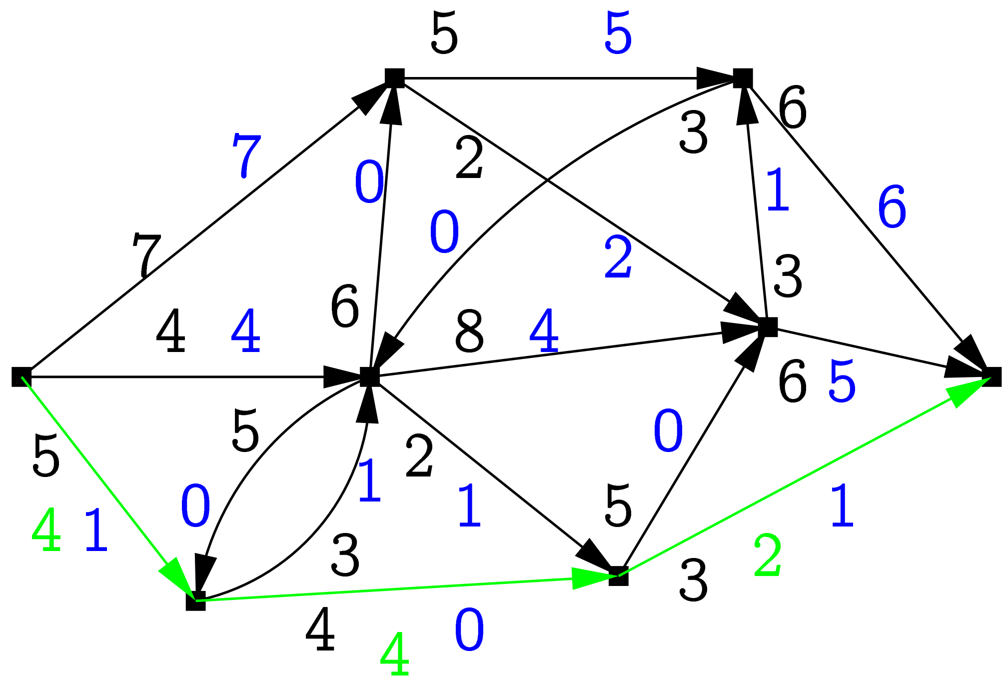


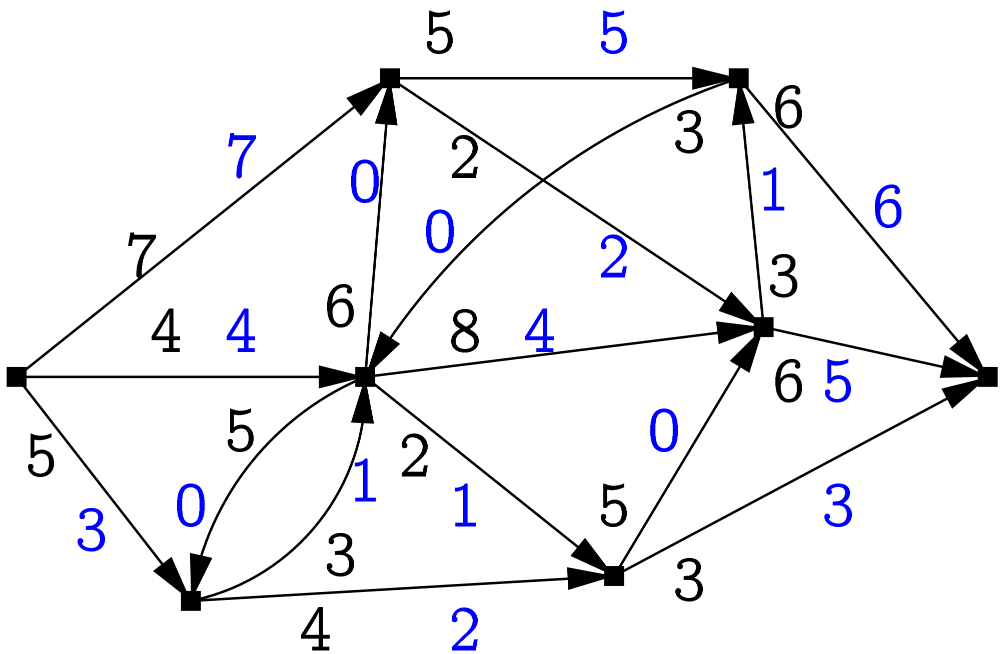
5
5
6
6
3
2
1
7
1
0
6
2
3
4
4
6
8
4
6
5
5
5
0
2
1
0
0
1
5
3
0
3
4
0


5
5
6
1
6
3
2
1
1
0
1
6
7
2
3
6
4
4
8
4
6
5
5
0
2
5
1
0
2
1
0
5
0
3
3
3
0
4
0

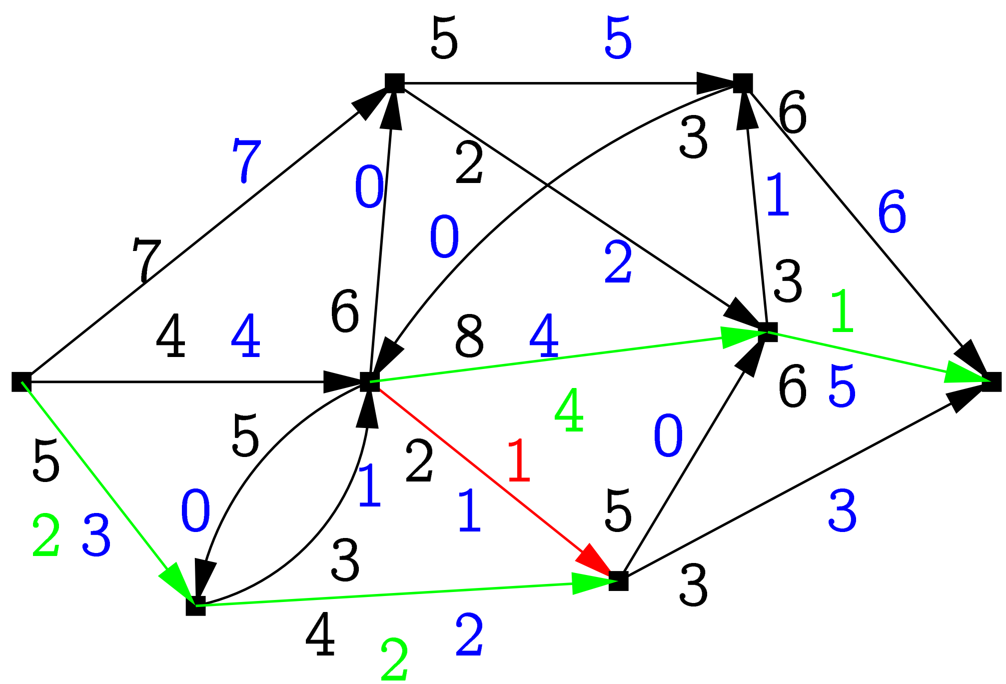


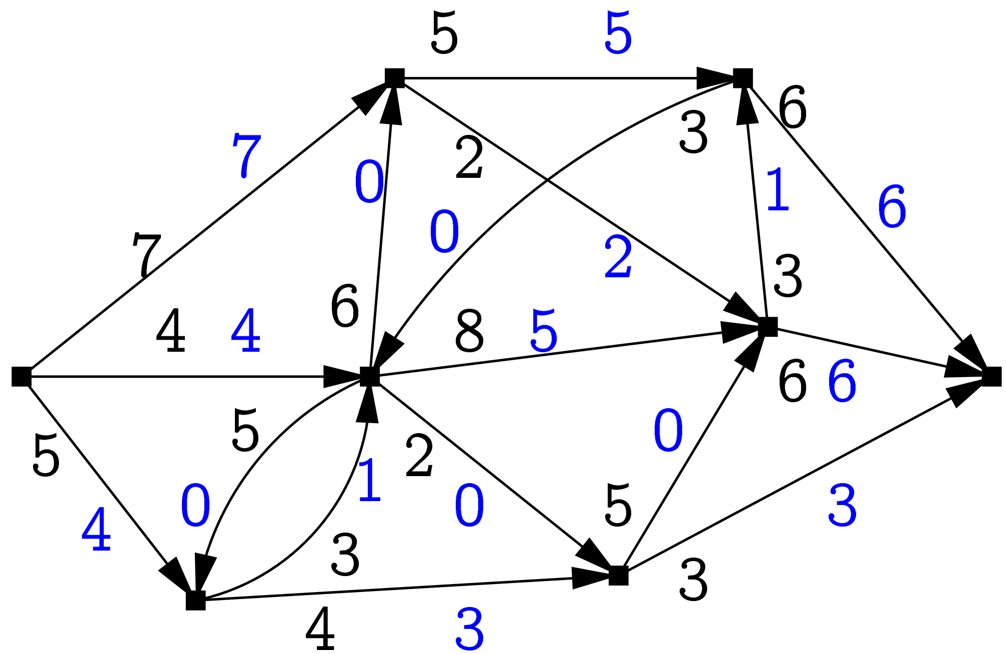
5
5
6
3
2
7
0
1
6
0
7
2
3
6
4
4
8
4
6
5
5
0
5
2
5
0
1
1
1
1
5
3
3
4
0


5
5
6
7
0
2
3
0
1
6
7
2
3
6
4
4
8
4
6
5
5
5
0
2
0
1
5
4
1
1
3
1
3
2
4
4
0



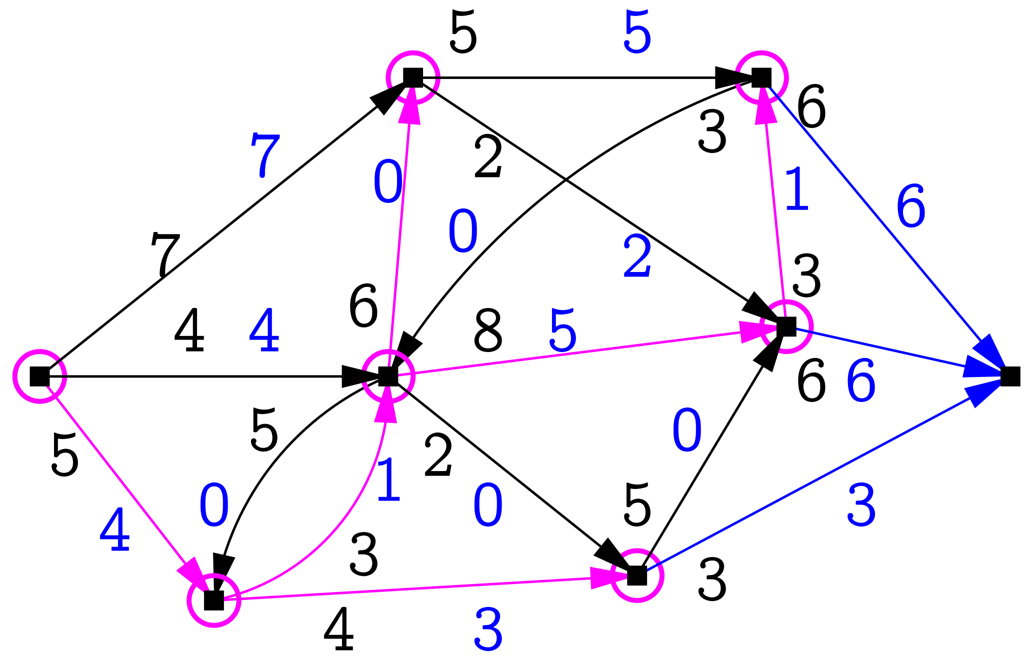
5
5
7
6
3
0
2
0
1
6
7
2
3
6
4
4
6
8
4
5
5
0
6
5
2
0
1
1
5
3
3
3
4
2


5
5
6
3
7
2
0
1
6
0
7
2
3
1
6
4
4
8
4
6
5
4
5
0
2
1
5
1
2
3
0
3
5
3
3
4
2
2



5
5
6
7
0
2
3
0
1
6
7
2
3
6
4
4
8
5
6
6
5
5
0
2
0
1
0
5
4
3
3
3
4
3

there is an augmenting path to these vertices




minimum cut: with circle → without circle

**Finding the augmenting path:**

Let $V_s = \{s\}$, $W = \{s\}$.

While $W \neq \emptyset$ and $t \notin V_s$ do:

1. Somehow choose $v \in W$. Remove it from the set $W$.
2. For each $e \in E$ and vertex $v$ incident with it: if the flow between $v$ and $e$'s other endpoint $w$ is unsaturated and $w \notin V_s$ then
   (a) Add $w$ to sets $V_s$ and $W$.
   (b) Remember that $v$ is the vertex "preceding" $w$.

If $t \notin V_s$ then there is no augmenting path. If $t \in V_s$ one can construct an augmenting path moving from $t$ by "preceding" vertices to $s$.

**Proposition.** If capacities of all the edges are integers then the main cycle of the algorithm is run at most $|\varphi|$ times where $\varphi$ is a maximal flow.

*Proof.* Each iteration increases the value of the flow. Since our computations do not introduce non-integers, each increase has to be at least by 1. $\square$

We will now assume that the augmenting path is found using breadth-first traversal of the graph (Edmonds-Karp algorithm).

The augmenting path $s = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_m} v_m = t$ found will have the following property:
For each $i$, the path $s = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_i} v_i$ is the shortest augmenting path from source to $v_i$.

Let $(G, \psi)$ be a network with $G = (V, E)$ and let $\varphi$ be a flow on it. Denote the length of the shortest path from source to $v \in V$ as $\delta_\varphi(v)$.

**Proposition.** Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be the sequence of flows generated during the maximal flow finding algorithm. Then for each $v \in V$ the sequence $\delta_{\varphi_i}(v)$ is non-decreasing.

Proof. Consider the flows $\varphi_n$ and $\varphi_{n+1}$ in this sequence and let $B = \{v \mid \delta_{\varphi_{n+1}}(v) < \delta_{\varphi_n}(v)\}$. Assume that $B$ is not empty and let $v \in B$ be such that $\delta_{\varphi_{n+1}}(v)$ is the smallest possible.

Let $P'$ be the shortest augmenting path from source to $v$ w.r.t the flow $\varphi_{n+1}$. Let $u$ be the vertex preceding $v$ on this path. Since $\delta_{\varphi_{n+1}}(u) < \delta_{\varphi_{n+1}}(v)$, we have $u \notin B$.

Consider the flow $\varphi_n$ between the vertices $u$ and $v$.

If $\varphi_n$ is unsaturated between the vertices $u$ and $v$ then

$$\delta_{\varphi_n}(v) \leq \delta_{\varphi_n}(u) + 1 \leq \delta_{\varphi_{n+1}}(u) + 1 = \delta_{\varphi_{n+1}}(v)$$

and $v \notin B$, a contradiction.

If $\varphi_n$ is saturated between the vertices $u$ and $v$ then let $P_n$ be the augmenting path from source to sink that was used to generate $\varphi_{n+1}$ from $\varphi_n$.

In $\varphi_{n+1}$, the flow between $u$ and $v$ becomes unsaturated. Thus, in the path $P_n$ there exists an edge $\cdots — v — u — \cdots$. According to the properties of $P_n$ we get $\delta_{\varphi_n}(v) = \delta_{\varphi_n}(u) - 1$. Consequently,

$$\delta_{\varphi_n}(v) = \delta_{\varphi_n}(u) - 1 \leq \delta_{\varphi_{n+1}}(u) - 1 = \delta_{\varphi_{n+1}}(v) - 2 < \delta_{\varphi_{n+1}}(v)$$

and $v \notin B$, a contradiction. $\square$

**Theorem.** Edmonds-Karp algorithm makes at most $(|V| - 2) \cdot |E|$ iterations.

Proof. Consider the $n$th iteration of the algorithm. On this iteration, the augmenting path $P_n : s = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_m} v_m = t$ is constructed. Call the pair of vertices $(v_{i-1}, v_i)$ *critical* if the respective quantity $\delta_i$ (showing how much the flow between $v_{i-1}$ and $v_i$ must be changed to make it saturated) is minimal (i.e. $\delta_i = \varepsilon$).

Each iteration has a critical pair of vertices. On the next iteration it becomes saturated.

Let's count the number of iterations where a pair $(u, v)$ can be critical. If it is critical on the $n$th iteration, we have $\delta_{\varphi_n}(v) = \delta_{\varphi_n}(u) + 1$.

To make $(u,v)$ again critical on iteration number $n' > n$, there must exist another augmenting path $P_{n'}$ containing the arc $\cdots — v — u — \cdots$. Then

$$\delta_{\varphi_{n'}}(u) = \delta_{\varphi_{n'}}(v) + 1 \geq \delta_{\varphi_n}(v) + 1 = \delta_{\varphi_n}(u) + 2,$$

thus every time when $(u,v)$ is critical, $\delta_{\varphi}(u)$ has increased at least by 2.

The quantity $\delta_{\varphi}(u)$ can not exeed $|V| - 2$ (when $(u,v)$ is critical). Thus $(u,v)$ is critical at most $\frac{|V|-2}{2}$ times. The number of vertex pairs $(u,v)$ is at most $2 \cdot |E|$. $\square$

Let $(G, \psi)$ be a network. Also, let each edge $e$ of $G$ be assigned a *cost* $c(e) \in \mathbb{R}$ (possibly negative).

Let $\varphi$ be a flow on $(G, \psi)$ the *cost* of the flow is

$$c(\varphi) := \sum_{e \in E(G)} c(e)\varphi(e) \quad .$$

We are looking for a maximum flow with the minimum cost.

Let $(G, \psi)$, $G = (V, E)$ be a network (with any number of sources and sinks). $f : E \longrightarrow \mathbb{R}^+$ is a *circulation (ringlus)* on $(G, \psi)$, if

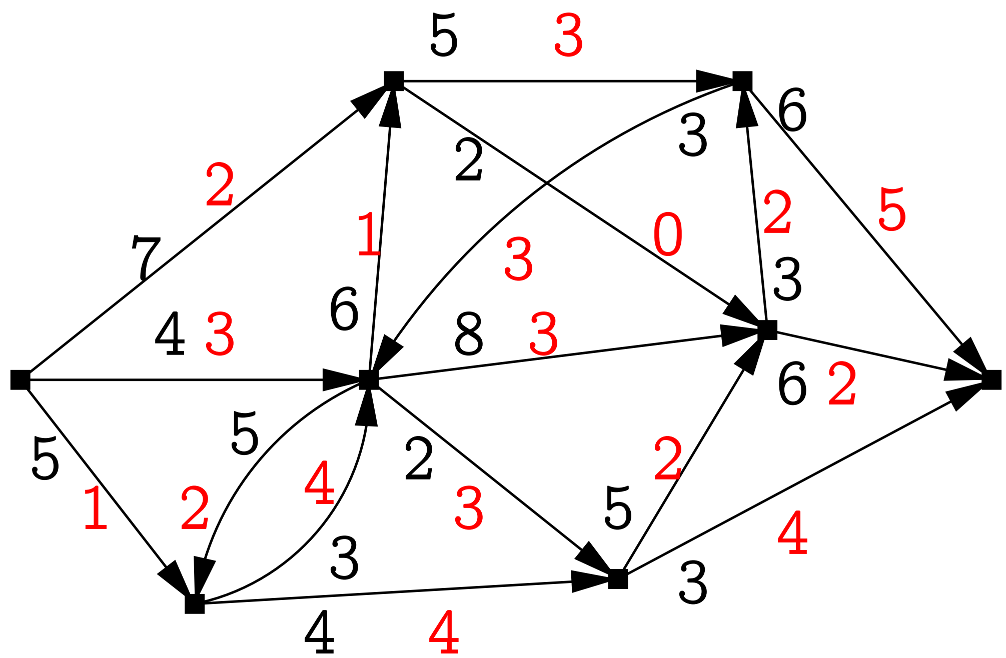
- $f(e) \leq \psi(e)$ for all $e \in E$.
- $\overrightarrow{\deg_f}(v) = \overleftarrow{\deg_f}(v)$ for all $v \in V$.

Let $c : E \longrightarrow \mathbb{R}^+$ give the costs of edges. We're looking for a minimum-cost circulation in $(G, \psi)$.

Finding the minimum-cost maximum flow can be reduced to finding the minimum-cost circulation.

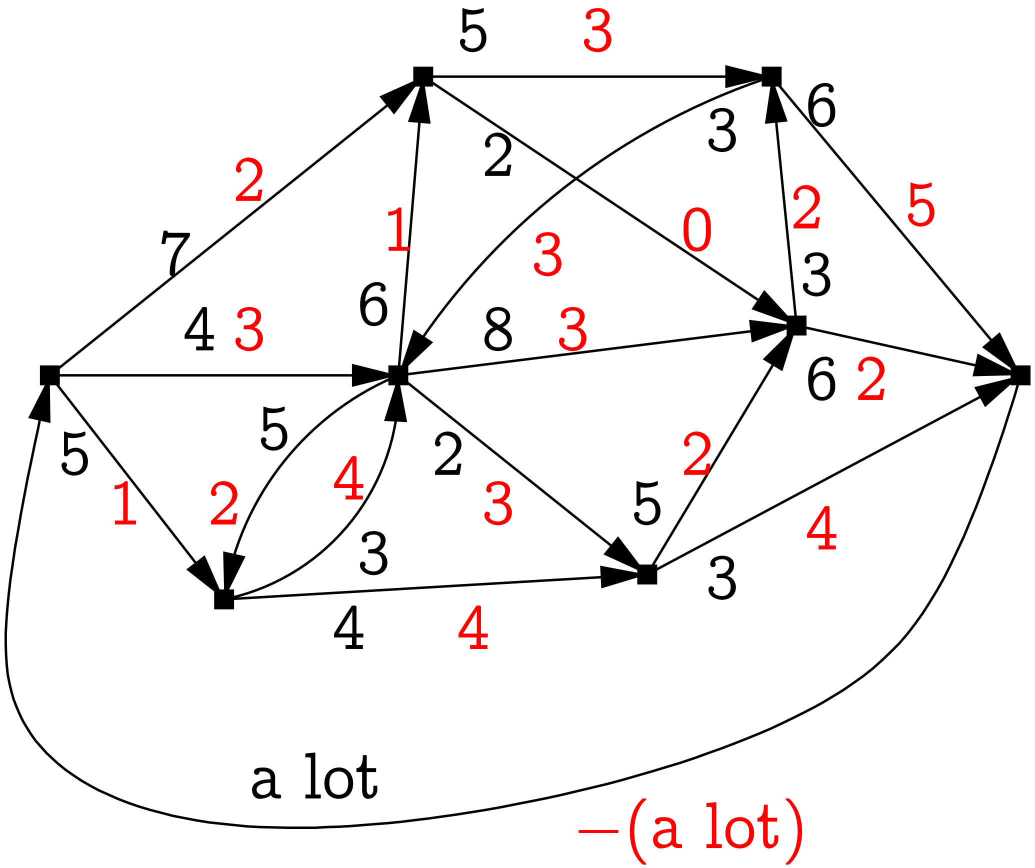
capacity

cost



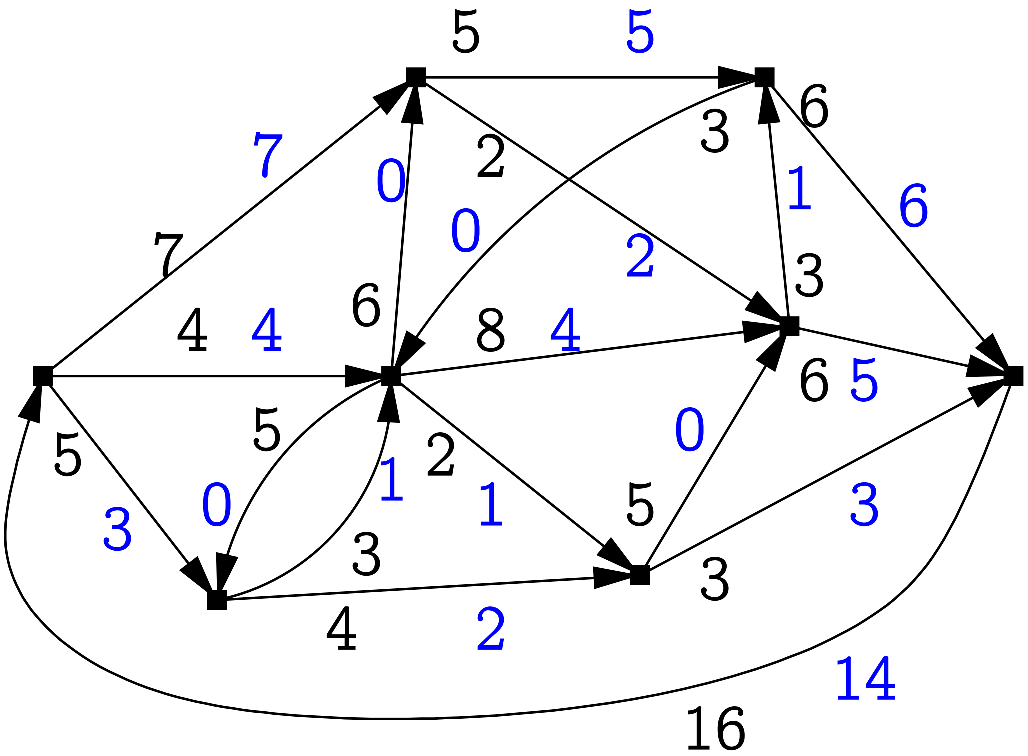
capacity

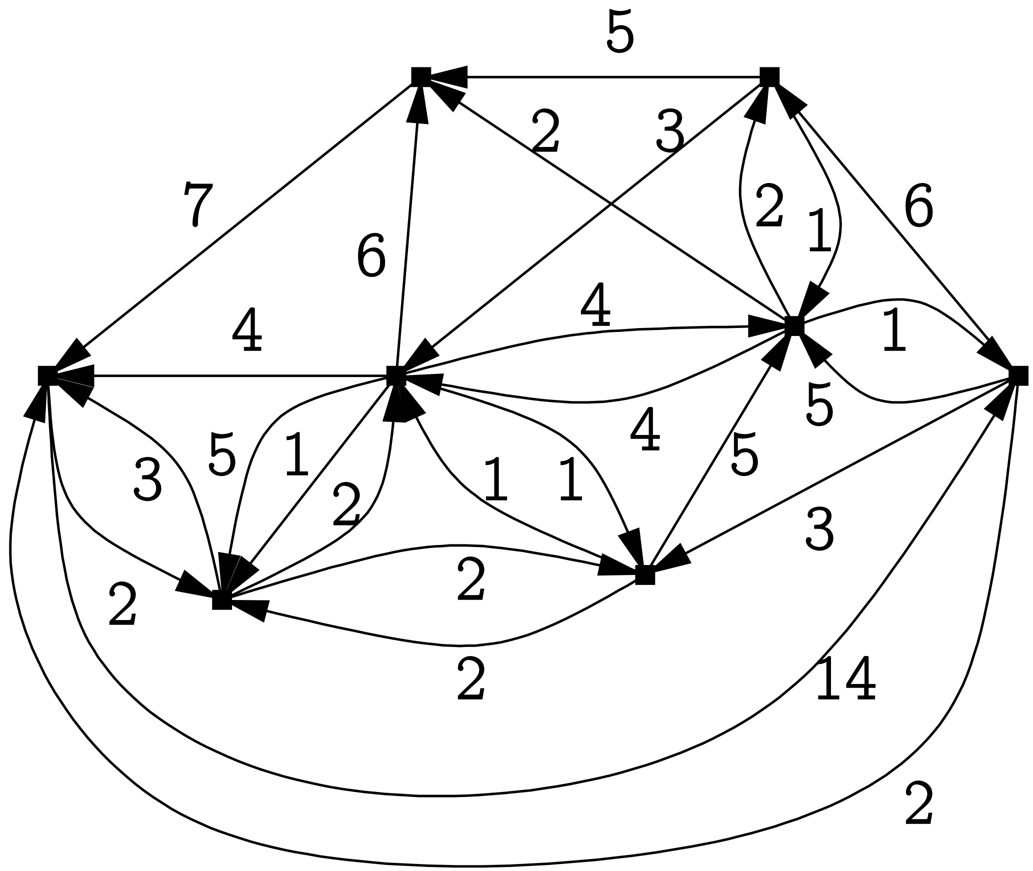
cost



Let $(G, \psi)$, $G = (V, E)$ be a network and $f$ a circulation on it. The *residual network (jääkvõrk)* $(G_f, \psi_f)$, made up of the *residual graph* $G = (V, E_f)$ and the *residual circulation* is defined as follows:

- For any $e = (u, v) \in E$:
  - If $f(e) < \psi(e)$, then $e^+ \in E_f$. Also, $\mathcal{E}(e^+) = (u, v)$ and $\psi_f(e^+) = \psi(e) - f(e)$.
  - If $f(e) > 0$, then $e^- \in E_f$. Also, $\mathcal{E}(e^-) = (v, u)$ and $\psi_f(e^-) = f(e)$.


5 5
7
0
2
3
6
7
0
2
1
6
3
6
4 4
8 4
6 5
5
0
5
3
2
1
0
1
3
5
3
4 2
3
14
16


5
7
2
3
6
2
1
6
4
4
1
5
4
3
5
1
1
2
1
5
3
2
2
2
14
2

Let $f$ be a circulation on $(G, \psi)$ and $f'$ a circulation on the residual network $(G_f, \psi_f)$. Define $(f + f') : E \longrightarrow \mathbb{R}$ by:

$$\forall e \in E: \ (f + f')(e) = f(e) + f'(e^+) - f'(e^-)$$

(if some edge $e^+$ or $e^-$ does not exist, then assume that $f'$ on it equals 0)

**Theorem.** $f + f'$, as defined above, is a circulation on $(G, \psi)$ (for any $f, f'$).

Proof. First show that $0 \leq (f + f')(e) \leq \psi(e)$ for any $e \in E$.

$$
\begin{aligned}
0 = f(e) - \psi_f(e^-) \leq f(e) - f'(e^-) \leq \\
f(e) + f'(e^+) - f'(e^-) \leq \\
f(e) + f'(e^+) \leq f(e) + \psi_f(e^+) = \psi(e),
\end{aligned}
$$

(again, $f'$, applied to a non-existing edge, gives 0)

Show that $\overrightarrow{\deg}_{f+f'}(v) = \overleftarrow{\deg}_{f+f'}(v)$ for any $v \in V$.

$$\overrightarrow{\deg}_{f+f'}(v) = \sum_{\substack{e \in E \\ \mathcal{E}(e)=(u,v)}} (f + f')(e) =$$

$$\sum_{\substack{e \in E \\ \mathcal{E}(e)=(u,v)}} \big(f(e) + f'(e^+) - f'(e^-)\big) =$$

$$\overrightarrow{\deg}_f(v) + \sum_{\substack{e \in E \\ \mathcal{E}(e)=(u,v)}} \big(f'(e^+) - f'(e^-)\big)$$

We have

$$\sum_{\substack{e\in E\\ \mathcal{E}(e)=(u,v)}} f'(e^+) + \sum_{\substack{e\in E\\ \mathcal{E}(e)=(v,w)}} f'(e^-) = \overrightarrow{\deg}_{f'}(v) = \overleftarrow{\deg}_{f'}(v) = \sum_{\substack{e\in E\\ \mathcal{E}(e)=(v,w)}} f'(e^+) + \sum_{\substack{e\in E\\ \mathcal{E}(e)=(u,v)}} f'(e^-) \ .$$

Thus

$$\sum_{\substack{e\in E\\ \mathcal{E}(e)=(u,v)}} \left(f'(e^+) - f'(e^-)\right) = \sum_{\substack{e\in E\\ \mathcal{E}(e)=(v,w)}} \left(f'(e^+) - f'(e^-)\right) \ .$$

$$\overrightarrow{\deg}_{f+f'}(v) = \overrightarrow{\deg}_f(v) + \sum_{\substack{e \in E \\ \mathcal{E}(e)=(u,v)}} \big(f'(e^+) - f'(e^-)\big) =$$

$$\overleftarrow{\deg}_f(v) + \sum_{\substack{e \in E \\ \mathcal{E}(e)=(v,w)}} \big(f'(e^+) - f'(e^-)\big) =$$

$$\sum_{\substack{e \in E \\ \mathcal{E}(e)=(v,w)}} \big(f(e) + f'(e^+) - f'(e^-)\big) =$$

$$\sum_{\substack{e \in E \\ \mathcal{E}(e)=(v,w)}} (f + f')(e) = \overleftarrow{\deg}_{f+f'}(v)$$

□

Let $f$ and $g$ be circulations on $(G, \psi)$, $G = (V, E)$. Let $(g - f) : E_f \longrightarrow \mathbb{R}$ be defined by: For any $e \in E$:

- if $g(e) \geq f(e)$, then $(g - f)(e^+) = g(e) - f(e)$ and $(g - f)(e^-) = 0$;
- if $g(e) < f(e)$, then $(g - f)(e^+) = 0$ and $(g - f)(e^-) = f(e) - g(e)$.

**Theorem.** $g - f$, as defined above, is a circulation on $(G_f, \psi_f)$.

**Theorem.** $f + (g - f) = g$.

Proofs are similar to the proof of the previous theorem.

Let $f$ be a circulation on $(G, \psi)$, $G = (V, E)$. Let $c : E \longrightarrow \mathbb{R}$ give the costs of the edges of $G$. Define the costs $c_f$ of edges of $(G_f, \psi_f)$ as follows:

$$c_f(e^+) = c(e)$$
$$c_f(e^-) = -c(e)$$

for any $e \in E$.

**Theorem.** Let $f$ be a circulation on $(G, \psi)$, $G = (V, E)$, and $f'$ a circulation on the residual network $(G_f, \psi_f)$. Let $c : E \longrightarrow \mathbb{R}$ give the costs of the edges of $G$. Then $c(f + f') = c(f) + c_f(f')$.

Proof: from the definition of $f + f'$.

**Lemma.** If the network $(G, \psi)$, $G = (V, E)$ (the costs of edges are given by $c$) has no cycles with negative costs, then the minimum-cost circulation on this network is the zero circulation.

Proof. Using mathematical induction over the cardinality of

$$\operatorname{supp} f = \{e \in E \,:\, f(e) > 0\},$$

show that any circulation $f$ has a non-negative cost.

*Base.* $|\operatorname{supp} f| = 0$. Then $c(f) = 0$.

*Step.* $|\operatorname{supp} f| = n > 0$. Let $V' \subseteq V$ be the set of all vertices, such that some edge $e$ with $f(e) > 0$ ends there. There are also edges $e$ with $f(e) > 0$ starting from all these vertices.

Graph $(V', \operatorname{supp} f)$ contains a directed cycle $C$. Let $\delta = \min_{e \in C} f(e)$. Define the following circulation $g$:

$$g(e) = \begin{cases} f(e), \text{ if } e \notin C \\ f(e) - \delta, \text{ if } e \in C \end{cases}.$$

Then $|\operatorname{supp} g| < |\operatorname{supp} f|$ and $c(g) = c(f) - \delta \cdot c(C)$. As $c(C) \geq 0$, then $c(f) \geq c(g) \geq 0$. $\square$

**Theorem.** Let $(G, \psi)$ be a network, $c$ the costs of its edges, and $f$ a circulation on it. $f$ is minimal-cost iff the network $(G_f, \psi_f)$ has no cycles of negative cost.

Proof. $\Rightarrow$. If $(G_f, c_f)$ had cycles of negative cost, then it also would contain a negative-cost circulation $f'$. But then $f + f'$ would have smaller cost than $f$.

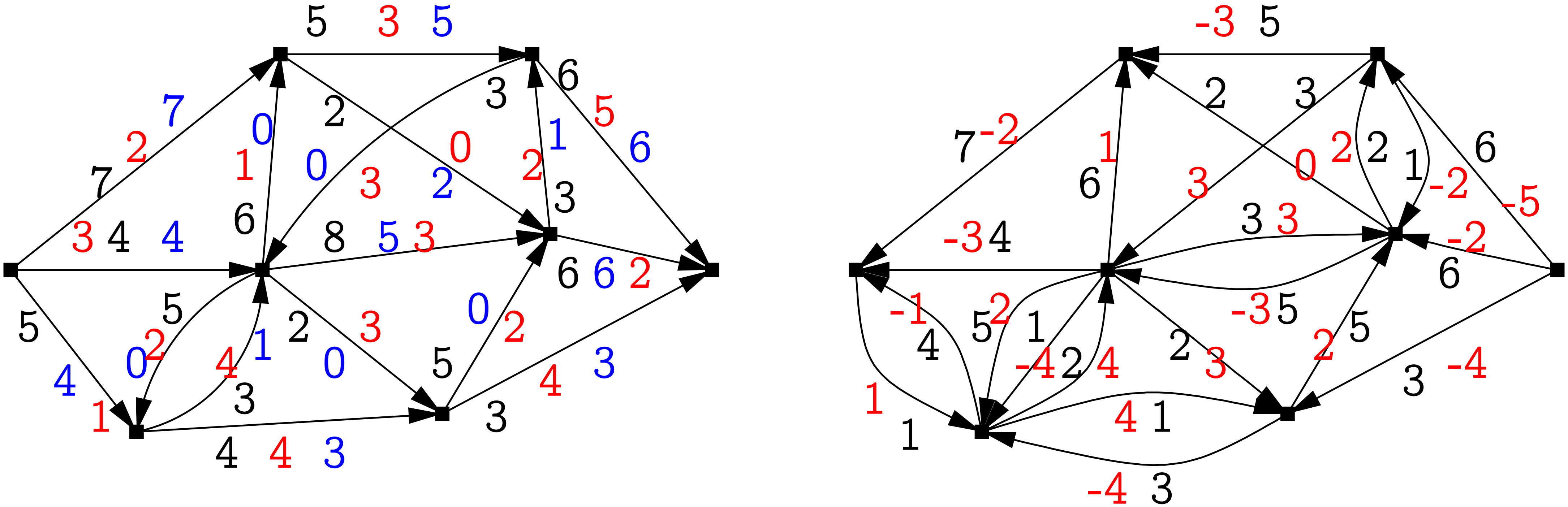
$\Leftarrow$. Let $f$ be a circulation on $(G, \psi)$ whose cost is not minimal. Let $f^*$ be a minimal-cost circulation on $(G, \psi)$. Then $f^* - f$ is a negative-cost circulation on $(G_f, \psi_f)$. Hence it also contains cycles of negative cost. $\square$

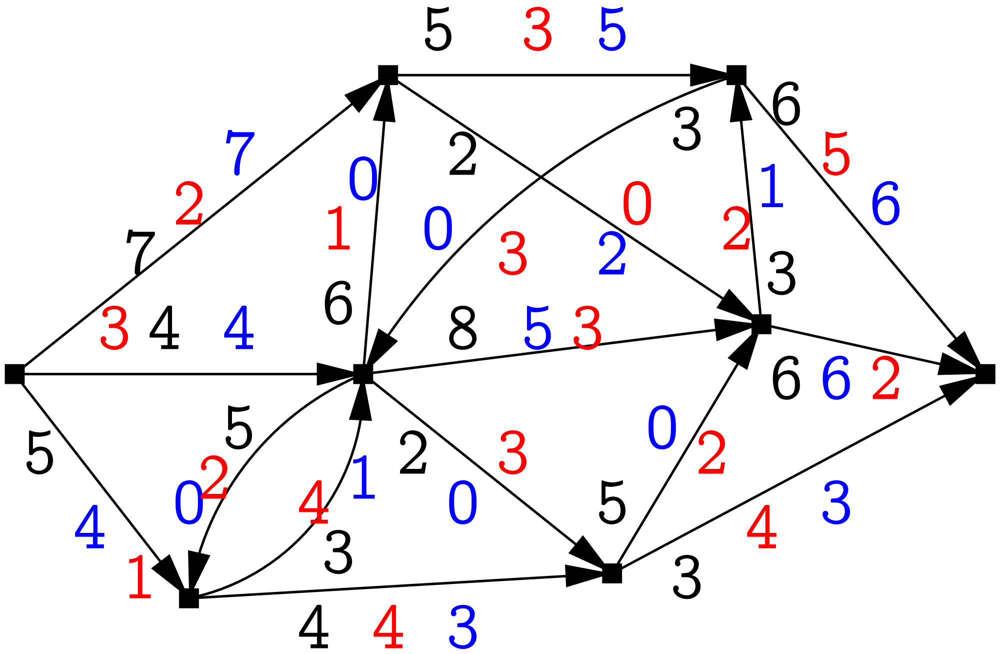
Algorithm to find a minimum-cost circulation in $(G, \psi)$ with the costs of edges $c$. Let $f$ be some initial circulation.

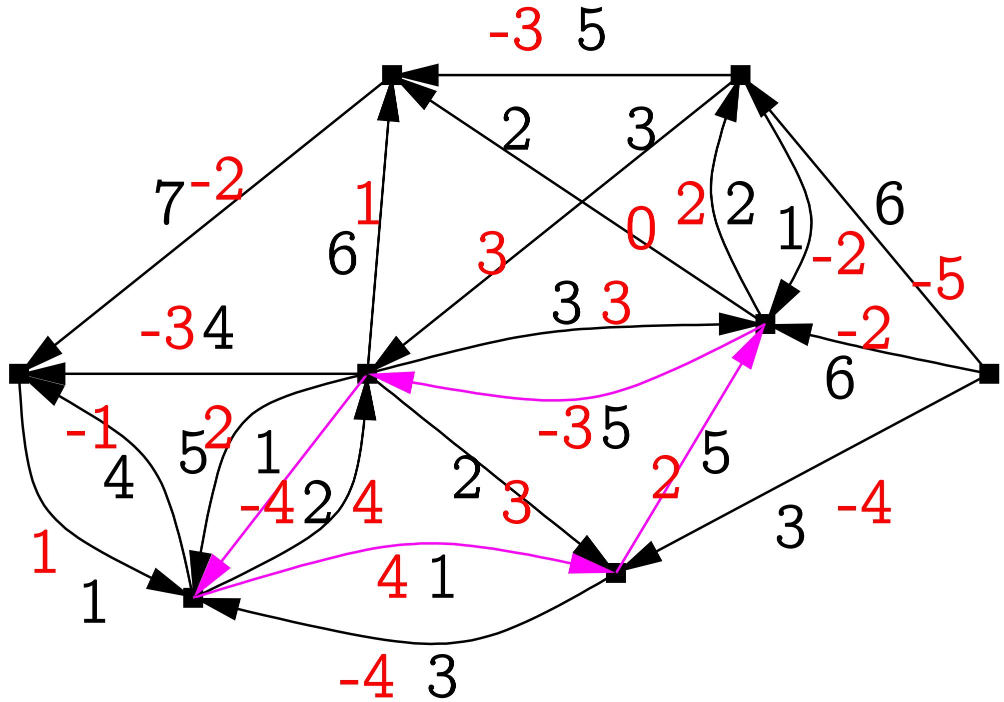
Repeat:

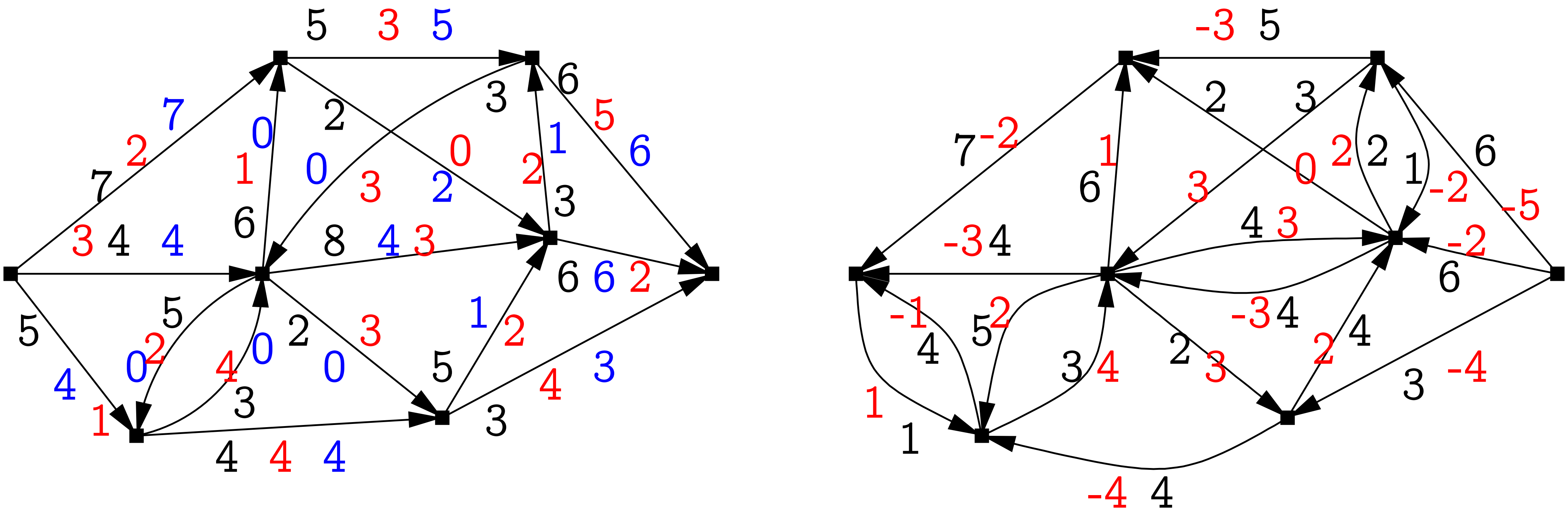
1. Find a negative-cost cycle $C$ in $(G_f, \psi_f)$. If such $C$ does not exist, then $f$ is a circulation of minimum cost.
2. Let $\delta = \min_{e^? \in C} \psi_f(e^?)$. Let $f'$ be a circulation in $(G_f, \psi_f)$, such that $f'(e^?) = \delta$ or $f'(e^?) = 0$, depending on whether $e^?$ lies on $C$ or not.
3. Set $f := f + f'$.

The initial $f$ may be found using e.g. the Ford-Fulkerson algorithm.


5 3 5
7 2 7
6 3 2 1
0 0 1 3 2
4 3 4 5 3 8
6 6 2
5 5 2 0
4 1 2 4 3 0 2
3 3 5
4 4 3
-3 5
7 -2 6 1 2 3
2 2 1 6
-3 4 3 3 0 -2 -5
3 3 -2 6
-1 2 1 -3 5
4 5 2 2 5
-4 2 4 3 2
1 4 1 3 -4
1
-4 3


5 3 5
3 6
7 2 2 0
2 5
1 0 0 1 2 3 6
7 1 2 3 3
3 4 4 6 8 5 3
6 6 2
5 5 2 3 0
2 0 2 1 0 3
4 4 0 5
1 3 4 3
4 4 3


-3 5
2 3
7 -2 1 2 2 1 6
6 3 0 -2 -5
-3 4 3 3 -2
6
-1 2 -3 5
4 5 1 2 5
-4 2 4 3 2 -4
1 3
4 1
1
-4 3



Bellman-Ford's algorithm may be used to find a cycle of negative cost.

Add an additional vertex $x$ to $G_f$. Add arcs of cost 0 from $x$ to all other vertices. Find the distances of all vertices from $x$ (length of edge $\equiv$ cost of edge); also find the shortest paths.

If something happens at the $|V|$-th iteration of the Bellman-Ford algorithm, then the back-pointers of all vertices (to previous vertices on shortest paths) will give us a cycle of negative length.