Discrete mathematics 2012

Practice session 1

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Summary of proof strategies

When constructing proofs and in logical argumentations (including non-mathematical) in general, we essentially need only a small number of logic rules that are presented as a summary in the following. A proving problem usually consists of two components: one component that we consider to be valid or known (premise), and second component that we wish to derive (conclusion). By constructing a proof we can start either from a premise or from a conclusion, expanding them according to the rules of logic, either sequentially or in parallel, until we get a gapless chain of inference. The following strategies can be used in every stage of a proof.

Proof strategies starting from the conclusion

Form of conclusion	What to do?
$A \to B$	Assume A and prove B OR use contrapositive proof: assume that B is not valid and prove that A is not valid.
$\neg A$	Use proof by contradiction: assume A and try to reach a contradiction.
$A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n$	Prove A_1, A_2, $\ldots$, A_n separately (in essence, prove n different claims)
$A_1 \vee A_2 \vee \ldots \vee A_n$	Select a suitable statement and prove that it is valid OR assume that all statements except the one are not valid and prove that the remaining statement is valid.
$A \leftrightarrow B$	Prove both $A \to B$ and $B \to A$.
$\forall x A(x)$	Select an arbitrary element x and prove $A(x)$.
$\exists x A(x)$	Select a value x_0 of x for which $A(x)$ is valid and prove that $A(x_0)$ is valid.

Proof strategies starting from the premise

Form of premise	What to do?
$A \to B$	If in addition it is known that A is valid, then conclude B, OR use contrapositive proof: if it is known that B is not valid, then conclude that A is not valid.
$\neg A$	If in course of a proof by contradiction it is proven that A is valid, then we obtain a necessary contradiction here.
$A_1 \mathbin{\&} A_2 \mathbin{\&} \ldots \mathbin{\&} A_n$	Conclude that each of the statements A_1, A_2, $\ldots$, A_n separately is valid.
$A_1 \vee A_2 \vee \ldots \vee A_n$	Consider cases: first case, when A_1 is valid; second case, when A_2 is valid etc. OR if it is known that some of the statements A_1, A_2, $\ldots$, A_n are not valid, then conclude that all remaining statements are valid.
$A \leftrightarrow B$	Replace by two premises: $A \to B$ and $B \to A$
$\forall x A(x)$	If an element c has been introduces previously, then we can conclude $A(c)$.
$\exists x A(x)$	Introduce a new notation, for example x_0, to denote the element for which $A(x_0)$ is valid.

Practice problems

Solutions of proving problems should be written down in the form of coherent text. If you use known facts (for example, set-theoretical identities) or definitions, then you should clearly refer them in the place you use them. If you consider necessary to use some representations (for example, characteristic functions or reducing to propositional calculus), then you should clearly point out, how the problem reduces to this representation.

Also in solutions of non-proving problems the answer should by accompanied with explanations that enable the reader to easily verify the correctness of your answer.

1. Let X, Y, Z be sets. Prove the equalities.

a) $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$

b) $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$

c) $X \setminus (X \setminus Y) = X \cap Y$

d) $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$

e) $X \setminus (Y \setminus Z) = (X \setminus Y) \cup (X \cap Z)$

2. Prove the inclusions.

a) $X' \setminus (Y \cup Z) \subseteq (X \cup Y)'$
b) $(X \cap Y) \setminus (X \cap Z) \subseteq Z'$
c) $(X' \cap Z) \cup (X \cap Y) \subseteq Y \cup (Z \cap Y')$
d) $(Y \setminus X) \cup (Y \setminus Z) \subseteq (X \cap Z)'$
e) $(X \cup Z) \setminus Y \subseteq ((X \cap Y) \cup (Y \cap Z))'$

3. Prove that the following statements are equivalent.

a) $X \subseteq Y$
b) $X \cup Y = Y$
c) $X \cap Y = X$
d) $X \setminus Y = \emptyset$
e) $X' \cup Y = U$

4. Prove that

a) if $X \subseteq Y$ and $Y' \cap Z \neq \emptyset$, then $X' \cap Z \neq \emptyset$
b) if $Y \cap Z = \emptyset$ and $X \cap Z' = \emptyset$, then $X \cap Y = \emptyset$

5. Let $f,\, g\colon X \to Y$ be functions. The images of the elements of the set $A \subseteq X$ by the function f form the set C and the images of the elements of the set $B \subseteq X$ by the function g form the set D

a) Does $C \cap D \neq \emptyset$ imply that $A \cap B \neq \emptyset$?
b) Is the statement in a) valid, if we assume that the functions f and g are injective?
c) Is the statement in a) valid, if we assume that the functions f and g are surjective?

6. Let X and Y be sets and $f\colon X \to Y$ and $g\colon Y \to X$ two functions such that $g(f(x)) = x$ for all $x \in X$. Prove that f is injective.

7. Determine whether a relation defined on the set X is an equivalence relation. If the relation is an equivalence relation, then describe the factor set of X by this relation.

a) $X = \mathbb{Z}$, $\varrho = \{(m, n)\colon |m| = n\}$
b) $X = \mathbb{R}$, $\varrho = \{(x, y)\colon x|y| = y|x|\}$

c) $X = \mathbb{R}$, $\varrho = \{(x, y)\colon x + |y + 1| = |x + 1| + y\}$

d) $X = \mathbb{R}$, $\varrho = \{(x, y)\colon \mathrm{round}(x) = \mathrm{round}(y)\}$, where round denotes rounding to the nearest integer

e) $X = \mathbb{R}$, $\varrho = \{(x, y)\colon \lfloor x - y \rfloor = 0\}$

f) $X = \mathbb{R} \times \mathbb{R}$, $\varrho = \{\big((a, b), (c, d)\big)\colon a - d = c - b\}$

8. Find a mistake in the following "proof".

Every symmetric and transitive relation is and equivalence relation.

Let $\varrho \in X \times X$ be an arbitrary symmetric and transitive relation on X. Choose an arbitrary element $x \in X$. Since the relation ϱ is symmetric, then always when $(x, y) \in \varrho$ also $(y, x) \in \varrho$. Because the relation ϱ is transitive, the inclusions $(x, y) \in \varrho$ and $(y, x) \in \varrho$ imply that $(x, x) \in \varrho$. Taking into account that the element x was arbitrary, we get that for any element x, $(x, x) \in \varrho$. Therefore, the relation ϱ is reflexive. Since ϱ is also symmetric and transitive, it is an equivalence relation.

9. Let $\varrho \subseteq X \times Y$ and σ, $\tau \subseteq Y \times Z$ be some relations.

a) Prove that $\varrho \circ (\sigma \cup \tau) = (\varrho \circ \sigma) \cup (\varrho \circ \tau)$

b) Prove that $\varrho \circ (\sigma \cap \tau) \subseteq (\varrho \circ \sigma) \cap (\varrho \circ \tau)$

c) The proof of b) is linear, i.e. it contains no branching. Sometimes, by reading such proofs "backwards" (reverting the direction of inference at every step), we can obtain the proof of the converse of the initial statement. Find the reason why in the current case, however, by reversing the proof in b) we obtain an incorrect proof.

d) By using the reason found in the previous sub-problem, construct an example which shows that in b) the reverse inclusion does not necessarily hold.

10. a) Prove that every reflexive and transitive relation ϱ satisfies the equality $\varrho \circ \varrho = \varrho$.

b) Find an example of a non-reflexive relation ϱ for which $\varrho \circ \varrho = \varrho$.

11. Let ϱ and σ be symmetric relations. Prove that the relation $\varrho \circ \sigma$ is symmetric if and only if $\varrho \circ \sigma = \sigma \circ \varrho$.

12. Find the kernels of the following mappings and describe the corresponding factor sets.

a) $f\colon \{1, 2, \dots, 100\} \to \mathbb{N}$, $f(n) = \text{cross-sum}(n)$

b) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(n) = n^2 - 3n + 1$

c) $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $f(x,y)=x^2+y^2$

d) $f\colon \mathbb{R}\to\mathbb{R}\times\mathbb{R}$, $f(x)=(\sin x,\cos x)$

e) $f\colon \mathcal{P}(\{1,2,3,4\})\to\mathcal{P}(\{1,2,3,4\})$, $f(X)=X\cap\{2,4\}$

Homework

From the following problems choose (at least) two and present their solutions. In the proofs, all essential details must be present; at the same time, the proof may not contain too much of unessential and unrelated material.

Submit your solutions electronically by the evening of the practice session in two weeks. Since these solutions will be used as a study material for other participants of this course, please do not put any personal information on your work; all necessary additional information can be written in the text box at your submission.

13. Prove in the form of coherent text that for any sets X, Y, Z,

$$(Z'\setminus X)\cup(Z'\setminus Y)=((X\cap Y)\cup Z)'.$$

14. Let X and Y be finite sets of equal cardinality. Let $f\colon X\to Y$ and $g\colon Y\to X$ be two functions, that for each $x\in X$ satisfy the condition $g(f(x))=x$. Prove that the function f in surjective.

15. Prove that a relation ϱ is an equivalence relation if and only if it satisfies the following two conditions at the same time:

a) ϱ is reflexive;

b) if $(x,y)\in\varrho$ and $(y,z)\in\varrho$, then $(z,x)\in\varrho$.

16. Let m, $n\in\mathbb{N}$. Consider the relation $\varrho=\{([x],[x])\colon x\in\mathbb{Z}\}\subseteq\mathbb{Z}_m\times\mathbb{Z}_n$, where $[x]$ denotes the equivalence class in $\mathbb{Z}_m$ or in $\mathbb{Z}_n$, respectively, containing the integer x. Find the condition between m and n at which this relation is a mapping. For m and n satisfying this condition, find the kernel of this mapping.