# Discrete mathematics 2012

Practice session 1

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## Practice problems

**1.** Which of the following relations are partial orders?

a) $X = \mathbb{Z},\ \varrho = \{(a, b) \colon |a| \leqslant |b|\}$
b) $X = \mathbb{Z}^+,\ \varrho = \{(m, n) \colon m \leqslant n^2\}$
c) $X = \mathcal{P}(\mathbb{N}),\ \varrho = \{(U, V) \colon U \setminus V = \emptyset\}$
d) $X = \mathbb{Z} \times \mathbb{N},\ \varrho = \{\big((a, b), (c, d)\big) \colon \frac{a}{b} \leqslant \frac{c}{d}\}$
e) $X = \mathbb{Z} \times \mathbb{N},\ \varrho = \{\big((a, b), (c, d)\big) \colon a \leqslant c \text{ ja } b \geqslant d\}$

**2.** Let $\varrho$ and $\sigma$ be partial orders on the set $X$. Prove that $\varrho \cap \sigma$ is also a partial order.

**3.** Find the transitive closures of the following relations.

a) $X = \mathbb{N},\ \varrho = \{(1, 3), (1, 4), (2, 5), (3, 4), (4, 2), (5, 2)\}$
b) $X = \mathbb{N},\ \varrho = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$
c) $X = \mathbb{Z},\ \varrho = \{(m, n) \colon m + 1 = n\}$
d) $X = \mathbb{Z},\ \varrho = \{(m, n) \colon |m + 1| = n\}$

**4.** Prove that the transitive closure of $\varrho$ and its reflexive transitive closure are related by the equality $\varrho^+ = \varrho \circ \varrho^*$.

**5.** Two relations $\varrho$ and $\sigma$ satisfy $\varrho \circ \sigma \subseteq \sigma \circ \varrho^+$. Prove that $\varrho^+ \circ \sigma \subseteq \sigma \circ \varrho^+$.

**6.** A graph whose vertices are all binary numbers of length $n$ and where two vertices are connected by an edge if and only if the corresponding binary numbers differ by exactly one binary digit, is called an *$n$-dimensional cube.*

a) Find the number of vertices of an $n$-dimensional cube.
b) Find the number of edges of an $n$-dimensional cube.

c) Prove that an $n$-dimensional cube is connected.

d) Find the largest distance between two vertices of an $n$-dimensional cube.

e) Does an $n$-dimensional cube contain bridges?

f) Does an $n$-dimensional cube contain cut vertices?

g) Prove that for every $k \leqslant n$ an $n$-dimensional cube contains a subgraph that is isomorphic to a $k$-dimensional cube.

h) Is an $n$-dimensional cube bipartite?

**7.** Let $G$ be a graph with the set of vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$. For each $i = 1, 2, \ldots, n$ let $G_i$ denote the subgraph of $G$ induced by the set of vertices $\{v_1, v_2, \ldots, v_i\}$. Prove that

$$\sum_{i=1}^{n} \deg_{G_i}(v_i) = |E(G)|.$$

**8.** Prove that for any graph $G$,

$$\sum_{v \in G} \deg_G(v)^2 = \sum_{e=uv \in E_G} \big(\deg_G(u) + \deg_G(v)\big).$$

**9.** Let $G$ be a graph and $v$ its vertex with minimal degree.

a) Prove that $G$ contains a path of length $\deg(v)$.

b) Assume that $G$ is connected but not complete. Prove that $G$ contains a path of length $\deg(v) + 1$.

**10.** a) Prove that if a graph contains a bridge, then it contains at least two vertices with odd degree.

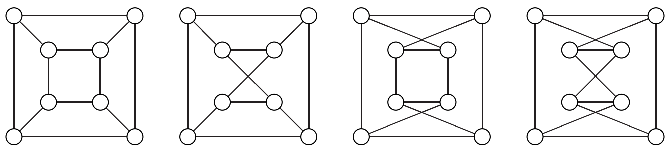
b) Given is a connected graph which contains a bridge $e$ and exactly two vertices $u$ and $v$ with odd degree. Prove that every path from $u$ to $v$ passes through the bridge $e$.

**11.** Prove that every graph contains a vertex that is not a cut vertex.

**12.** a) Each edge of the graph $K_6$ is colored either blue or red. Prove that in the colored graph there exits a triangle whose edges are all of the same color.

b) Each edge of the graph $K_5$ is colored either blue or red. Prove that in the colored graph there exits a cycle whose edges are all of the same color.

**13.** Which of the following graphs are isomophic?



**14.** Consider the graph $G$ for which $V(G) = \{1,2,3\} \times \{1,2\}$ and $E(G) = \big\{\big((a,b),(c,d)\big)\colon a+b+c+d \text{ is odd}\big\}$. Prove that $G$ is isomorphic to $K_{3,3}$.

**15.** Let $G$ be a graph with $V(G) = \mathcal{P}(\{x,y,z\})$ and $E(G) = \{\{A,B\}\colon A \subseteq B,$
$A \neq B\}$. Let $H$ be a graph with $V(H) = \{d\colon d \mid 30,\ d \in \mathbb{N}\}$ and $E(G) = \{\{a,b\}\colon a \mid b,\ a \neq b\}$. Prove that the graphs $G$ and $H$ are isomorphic.

## Homework

Choose (at least) two from the following problems and present their solutions.

**16.** Let $X$ be a set and $\varrho$ be a reflexive and transitive relation on $X$.

a) Prove that the relation $\sigma = \varrho \cap \varrho^{-1}$ is an equivalence relation on $X$.

b) Prove that the relation $\bar{\varrho} = \{(x/\sigma, y/\sigma)\colon x\ \varrho\ y\}$ is a partial order on $X/\sigma$.

**17.** In a simple graph with $2n$ vertices there are exactly two vertices with the same degree.

a) What is the degree of these two vertices?

b) Is the answer uniquely determined?

**18.** Let $G$ be a simple graph, where all vertices have the degree 3. Prove that $G$ contains a cycle of even length.

**19.** Let $A \subseteq \mathbb{N}$ be a finite set and let $G_A = (A, E)$ be a simple graph, where $rs \in E$ iff $\gcd(r,s) > 1$, for all $r,\ s \in A$. Prove that for every simple graph $G$ there exists a set $A$ such that $G$ is isomorphic to $G_A$.

**20.** For any natural number $k$ define a *$k$-oddity graph* $\mathcal{O}_k$, where $V(\mathcal{O}_k) = \{A \subseteq \{1,2,\ldots,2k+1\}\colon |A| = k\}$ and $E(\mathcal{O}_k) = \{\{A,B\}\colon A \cap B = \emptyset\}$. Prove that if $k \geqslant 3$, then the graph $\mathcal{O}_k$ contains a cycle of length 6 but does not contain a cycle of length less than 6.