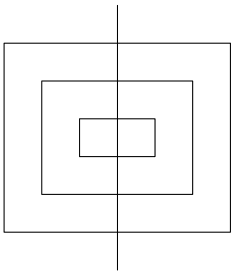
Discrete mathematics 2012

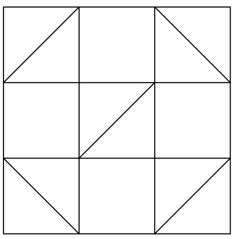
Practice session 1

Reimo Palm

Practice problems

1. Determine whether the following figures can be drawn without lifting one's pen from the paper or covering a line more than once.

a) 

b) 

2. a) Prove that if a connected graph has exactly two vertices of odd degree, then the graph contains an open Eulerian walk that begins in one of these vertices and ends in the other.

b) Let G be a connected graph that has $m > 0$ vertices of odd degree. Prove that the graph contains $m/2$ walks such that each edge of G lies on exactly one of these walks.

3. a) Prove that the vertices of every undirected graph G can be oriented in such a way that for each vertex $v \in G$, $|\overrightarrow{\deg}(v) - \overleftarrow{\deg}(v)| \leqslant 1$.

b) Find such orientation of edges in the Petersen graph shown in the problem 7c).

4. An exhibition room is divided into many intersecting corridors. Pictures are displayed on the walls of the corridors on both sides. When moving along a corridor, a visitor can either look at the pictures on one wall or go through the pictures on both walls. The room has one door. Prove that the visitor can enter the exhibition room, make a round-tour and exit in such a way that he has seen every picture exactly once.

5. Prove that for every connected simple graph G one of the following conditions holds.

a) G is Eulerian.

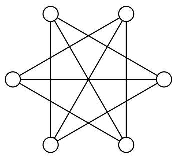
b) G can be obtained from an Eulerian graph by removing one vertex.

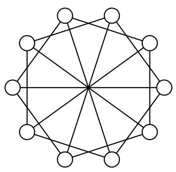
Does there exist a connected simple graph that satisfies both conditions at the same time?

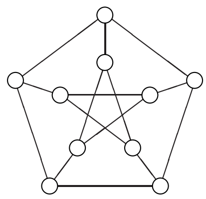
6. Arrange 4 zeros and 4 ones on a circle in such a way that when reading clockwise, the resulting ring of numbers contains every triple 000, 001, ..., 111 exactly once.

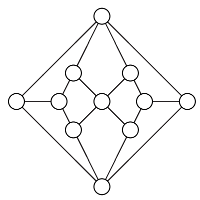
Hint. Consider a directed graph with vertices 000, 001, ..., 111 and find an Hamiltonian walk in it.

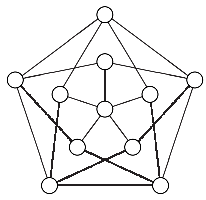
7. If the graph contains a Hamiltonian cycle, then find it. Otherwise prove that there is no Hamiltonian cycle.

a) 

b) 

c)  *Petersen graph*

d)  *Herschel graph*

e)  *Grötzsch graph*

8. Prove that for every $n \geqslant 1$, $K_{n,n+1}$ is not Hamiltonian.

9. Prove that for every $n \geqslant 2$, $K_{n,n}$ is Hamiltonian.

10. Prove that if each participant of a tea party has more acquaintances than strangers among other participiants, then all participants can be seated at the round table in such a way that everybody knows both his neighbours.

11. Let G be a graph with at least 4 vertices, such that for every three different vertices u, v, w, the subgraph induced by these vertices has at least two edges. Prove that G is Hamiltonian.

12. *Gray code* is an ordering of n bit binary numbers $0, 1, \ldots, 2^n - 1$, where each two consecutive numbers differ by exactly one bit.

a) Interpret an n-bit Gray code in terms of an n-dimensional cube (see the problems of previous practice).

b) Find one 3-bit Gray code.

c) Prove that for every integer n there exists an n-bit Gray code.

Homework

Choose at least two from the following problems and present their solutions in two weeks.

13. Let n be a positive integer and let $X_n = \{1, 2, \ldots, n\}$. Consider a graph G_n, where $V(G) = \{A \subseteq X_n \colon |A| = 2\}$ and $E(G) = \{\{A, B\} \colon A \cap B \neq \emptyset\}$. Find all positive integers n for which G_n is Eulerian.

14. A *symmetric difference* of graphs G_1 and G_2 on the same set of vertices is a graph $G = G_1 \Delta G_2$ with the set of vertices $V(G) = V(G_1) = V(G_2)$ and the set of edges $E(G) = E(G_1) \Delta E(G_2)$. Prove that if G_1 and G_2 are graphs, where every connected component is Eulerian, then also $G_1 \Delta G_2$ is a graph, where every connected component is Eulerian.

15. A *product* of graphs G and H is a graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and with the edge set $E(G \times H) = \{\{(u_1, v_1), (u_2, v_2)\} \colon u_1 = u_2,\ \{v_1, v_2\} \in E(G)$ or $v_1 = v_2,\ \{u_1, u_2\} \in E(G)\}$. Let both G and H be paths with n vertices. Prove that the graph $G \times H$ contains a Hamiltonian cycle if and only if n is even.

16. Prove that if one can remove some number m of vertices from a graph G in such a way that the remaining graph consists of more than m conncted components, then G is not Hamiltonian.