Discrete mathematics 2012

Practice session 1

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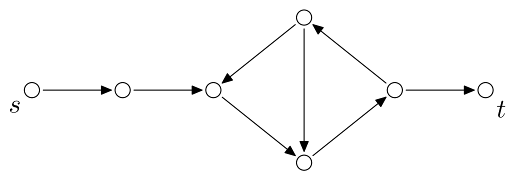
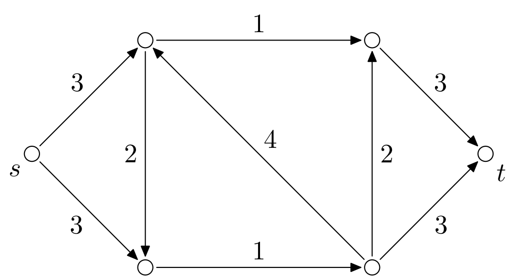
Practice problems

Transportation networks that are used to ship goods from production centres to markets, can be efficiently analyzed by representing them as a directed graphs with some additional structure. This week is devoted to the introduction to the corresponding mathematical theory. It has a broad range of applications and implications.

1. The figure below shows a graph, where the capacity of each arc is 3. Does this graph have a flow whose value on each arc is nonzero?



2. In the network

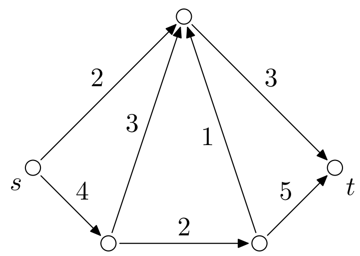


find

a) all integer flows;

b) the value of maximum flow.

3. In the network



find

a) all cuts;

b) the capacity of a minimum cut.

4. Find

a) a minimum cut in the problem 2;

b) a maximum flow in the problem 3.

5. Consider the function φ' on the flow (G, ψ), as defined in the graph's textbook (A. Buldas, P. Laud, J. Willemson. *Graafid.* Tartu 2008), page 46. Prove that φ' is a flow and that the value of the flow φ' is by ε larger than the value of the flow φ.

6. Let G be a digraph. Consider functions $\omega\colon E(G) \to \mathbb{R}$ defined on the edges of G. Let their sum and product with a scalar be defined as usual, i.e. the sum of functions $\omega_1\colon E(G) \to \mathbb{R}$ and $\omega_2\colon E(G) \to \mathbb{R}$ satisfies $(\omega_1 + \omega_2)(e) = \omega_1(e) + \omega_2(e)$, and the product of a function $\omega\colon E(G) \to \mathbb{R}$ with a real a satisfies $(a\varphi)(e) = a \cdot \varphi(e)$. Prove that the set of flows on the network (G, ψ) is convex, i.e if φ_1 and φ_2 are flows, then also $a\varphi_1 + (1 - a)\varphi_2$ is a flow for any real number $a \in [0, 1]$.

7. Prove that if X, $Y \subseteq V(G)$ are minimum cuts on the network (G, ψ), then also $X \cup Y$ and $X \cap Y$ are minimum cuts on the network (G, ψ).

8. Let (G, ψ) be a network with many sources and sinks. Let X and Y be the sets of its sources and sinks, respectively. Let φ be a flow in this network.

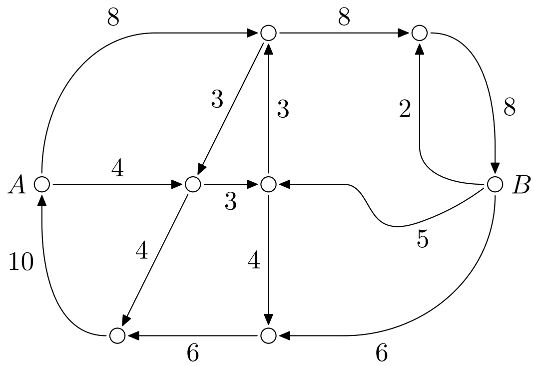
a) Prove that the net flow out of X and the net flow into Y are equal.

b) Define the value of the flow φ on the network (G, ψ).

c) Reduce the network with many sources and sinks to a network with one source and one sink by adding two vertices of certain capacities to the network (G, ψ).

d) Define the flow on this network with one source and one sink, corresponding to the flow φ.

9. The following schema shows bus routes between 8 main bus stations. The busses only drive in the directions of arrows. The capacities of arcs show the maximum number of busses per hour that can drive on each section. On each section there must be at least one bus per hour. A bus can transport at most 50 passengers. Find the maximum number of passengers in hour that can move from point A (sleeping district) to point B (city center).

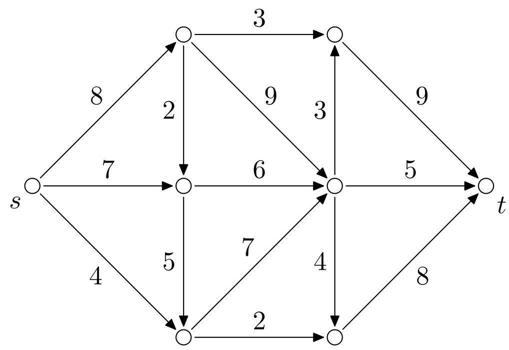


Note. This is a graph without sources and sinks. The total number of busses in the system must remain the same at every moment.

Homework

Choose at least two from the following problems and present their solutions.

10. Find a maximum flow and a minimum cut in the network



11. Let (G, ψ) be a network and φ a flow on it. Let $S \subset V(G)$ be a set such that $s \notin S$ and $t \notin S$, and $S' = V(G) \setminus S$. Prove that

$$\sum_{\substack{e \in E(G) \\ e \in S \times S'}} \varphi(e) = \sum_{\substack{e \in E(G) \\ e \in S' \times S}} \varphi(e).$$

12. Let (G, ψ) be a network, where $\psi(e) = 1$ for each $e \in E(G)$, and let s and t be the source and the sink of the network, respectively. Prove that the value of a maximum flow on this network equals to the number of pairwise edge-disjoint paths from s to t.