Discrete mathematics 2012

Practice session 1

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The current theme deals with a collection of theorems equivalent to each other, that are remarkably strong and rich. These include Ford-Fulkerson theorem, König-Egerváry theorem, Hall's theorem, König's theorem, Menger's theorem, Dilworth's theorem, Birkhoff–Von Neumann theorem, Berge's theorem, Tutte's theorem.

Practice problems

1. The cards of a 52 card deck are laid as a table with 13 rows and 4 columns. Prove that it is possible to choose one card from each row in such a way that these 13 cards include all possible card values.

Hint. $X = \{\text{card values}\}$, $Y = \{\text{rows}\}$, $E = \{\{x, y\}\colon \text{a card of value } x \text{ occurs in the row } y\}$.

2. Let $m \leqslant n$. A *latin rectangle* is an $m \times n$ matrix whose elements belong to the set$\{1, 2, \ldots, n\}$ and occur in each row and in each column of this matrix at most once. A *latin square* is a latin rectangle with $m = n$. Prove that every latin rectangle can be complemented to a latin square by adding $n - m$ rows.

Hint. Induction; $X = \{\text{matrix columns}\}$, $Y = \{1, 2, \ldots, n\}$, $E = \{\{i, j\}\colon \text{ column } i \text{ does not contain the number } j\}$.

3. A *transversal* of the sets A_1, A_2, ..., A_n is a set T, whose elements can be rearranged in such a way that $T = \{t_1, t_2, \ldots, t_n\}$, where for every i, $t_i \in A_i$. Prove that the sets A_1, A_2, ..., A_n have a transversal iff $|\cup_{i\in I} A_i| \geqslant |I|$ for every $I \subseteq \{1, 2, \ldots, n\}$.

Hint. $X = \{A_i\}$, $Y = \cup A_i$, $E = \{\{A, x\}\colon A \text{ contains } x\}$.

4. Does the set of all three-element subsets of the set $\{1, 2, 3, 4, 5\}$ have a transversal?

5. Prove the sufficiency part of Hall's theorem using Menger's theorem.

a) Add two vertices u and v to the graph G, one of which is connected to the base X and the other to the base Y.

b) Prove that the matching with the property mentioned in Hall's theorem exists if and only if the number of vertice-disjoint paths from s to t is $|X|$.

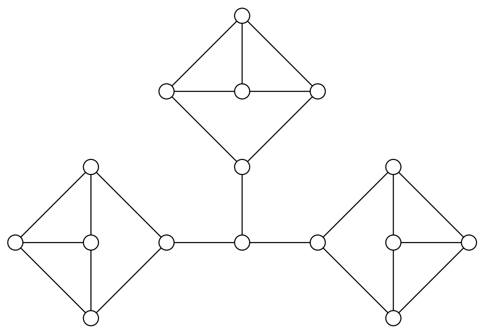
c) Deduce from Menger's theorem that this condition is fulfilled, if every vertex set S separating s and t in G consists of at least $|X|$ vertices.

d) Divide the vertices of the set S into two sets, corresponding to the bases X and Y.

e) Prove that $|X \setminus A| \leqslant |B|$, using the fact that S is a separating vertex set and that $X \setminus A$ satisfies the premises of sufficiency part of Hall's theorem.

f) Deduce that $|S| \geqslant |X|$.

6. Consider the *Sylvester graph*



a) Find a maximum matching in the Sylvester graph.

b) Find a minimum cover in the Sylvester graph.

c) Compare the results. Is here a contradition with some theorem?

d) Prove that the Sylvester graph does not contain a *perfect matching*, i.e. a matching that covers all vertices.

e) Prove that for every $r > 1$ there exists a regular simple graph with degree r that does not have a perfect matching.

Note. Sylvester graph is the smallest regular graph with odd degree that does not have a perfect matching.

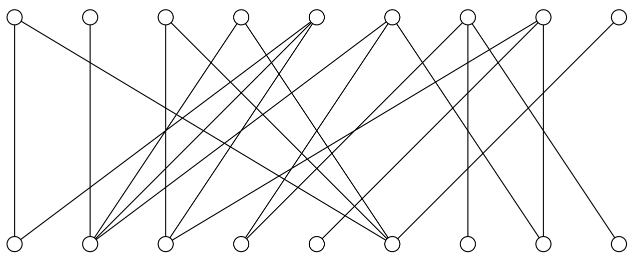
7. Find a maximum matching in the n-dimensional cube Q_n. How many edges does it have? Is it a perfect matching?

8. How many perfect matchings are there

a) in the graph $K_{n,n}$;

b) in the graph K_{2n}?

9. The executive director of a company wants to choose the best product from a number of products being developed for marketing. He orders every product to be analyzed by a team of two, consisting of a developer and a seller who write a report together. The teams are to be formed according to the following graph; every edge corresponds to a product and its endpoints to the developer and the seller (the sets A and M, respectively) who analyze this product. What is the minimum number of people who must be present at the final meeting such that every product will be covered by a report? (The report can be presented either by the developer or the seller.)



10. A production unit must complete n jobs 1, 2, …, n, each taking one day. The unit has two machines to do the jobs. The first machine does one job at a time and completes it in one day. The second machine does two jobs at a time and complete both of them in one day. The jobs are subject to a partial preference ordering $\leqslant$, where $i \leqslant j$ means that the job i must be completed before the job j can be started. The aim is to complete all jobs while minimizing $p_1 + p_2$, where p_i is the number of days during which the machine i is in use. Formulate this problem as a problem of finding a maximum matching in a suitably defined graph.

11. a) Let M be a matching in a graph G. Prove that the graph G has a maximum mathcing that covers all those vertices that are covered by matching M. *Hint.* Berge's theorem.

b) Deducte from this that every vertex in a connected nontrivial graph can be covered with some maximum matching.

c) Let $G = (X \cup Y, E)$ a bipartite graph, and let $A \subseteq X$ and $B \subseteq Y$. Prove that if the graph G has a matching that covers all vertices of

A, and a matching that covers all vertices of B, then it also has a matching that covers all vertices of $A \cup B$.

12. Let $G = (X \cup Y, E)$ be a bipartite graph. Prove that the number of edges in a maximum matching of G is $|X| - \max_{X' \subseteq X}\{|X'| - |N(X')|\}$.

Homework

Choose (at least) two from the following problems and present their solutions.

13. Let X be a set that has two partitions (i.e. representations as a union of pairwise disjoint nonempty sets) $X = \cup_{i=1}^{n} A_i = \cup_{i=1}^{n} B_i$, where $|A_i| = |B_i| = k$ for every $i = 1, 2, \ldots, n$. Prove that there exists a set $T \subseteq X$ that is a transversal for both the sets $\{A_1, A_2, \ldots, A_n\}$ and the sets $\{B_1, B_2, \ldots, B_n\}$.

Hint. $X = \{A_i\}$, $Y = \{B_i\}$, $E = \{\{A, B\} \colon A \cap B \neq \emptyset\}$.

14. Let G be a graph. Two players A ja B play the following game. At a move each player marks an unmarked vertex that (starting from the second move) is connected to a vertex marked by his opponent at the previous move. The players move alternatingly, A starts. The player who makes the last move wins.

a) Prove that if the graph G has a perfect matching, then the player B has a winning strategy.

b) Prove that if the graph G does not have a perfect matching, then the player A has a winning strategy.

15. a) Prove that the bipartite graph G has a perfect matching iff $|N(S)| \geqslant |S|$ for each $S \subseteq V(G)$.

b) Find an example showing that the previous statement does not remain valid, if the assumption about graph being bipartite is dropped.

16. Let A be a finite set, A_1, A_2, $\ldots$, A_n be its subsets and d_1, d_2, $\ldots d_n$ natural numbers. Prove that the following statements are equivalent to each other.

a) There exists pairwise disjoint sets D_1, D_2, $\ldots$, D_n such that for every $i = 1, 2, \ldots, n$, $D_i \subseteq A_i$ and $|D_i| = d_i$.

b) For every $I \subseteq \{1, 2, \ldots, n\}$

$$\Big|\bigcup_{i \in I} A_i\Big| \geqslant \sum_{i \in I} d_i.$$