Problem 1: Qubits

(a) Which of the following are valid quantum states:

\[
\begin{align*}
\left(\begin{array}{c} 0 \\ 1 \end{array}\right), & \quad \frac{1}{\sqrt{2}} \left(\begin{array}{c} 1 \\ -i \end{array}\right), & \quad \frac{1}{\sqrt{3}} \left(\begin{array}{c} 1 \\ 2 \end{array}\right), & \quad \left(\begin{array}{c} \sqrt{2} \\ \sqrt{3} \end{array}\right)
\end{align*}
\]

(b) For each of the valid quantum states from (a), answer the following: You perform a measurement (i.e., you ask “whether the state is a classical 0 or a classical 1”). What is the probability of answer 0 (i.e., yes), what is the probability of answer 1 (i.e., no)? What is the state after the measurement in each of those cases?

(c) Show that by applying a unitary transformation to a quantum state, no information is ever lost. More exactly, assume that a unitary transformation \(U\) is applied to a given quantum state \(\Psi\), resulting in a state \(\Phi\). Then show that there is another unitary transformation \(V\) (not depending on \(\Psi\) or \(\Phi\)) such that applying \(V\) to \(\Phi\) gives \(\Psi\) again.

(d) Assume that a photon is in the state \(\Psi = \left(\begin{array}{c} \alpha \\ \beta \end{array}\right)\). (Here the \(\alpha\) component corresponds to the vertical part and the \(\beta\) component to the horizontal part.) Let \(R\) be a rotation of angle \(\theta = \frac{\pi}{4}\). Let \(F\) denote a polarisation filter that lets only vertically polarised light through. Assume that the photon \(\Psi\) is first sent through \(R\) and then through \(F\). It turns out that in this setting, the photon is absorbed by \(F\) with probability 1.
Given these informations, what do you know about \( \alpha \)? (I.e., what are the possible values of \( \alpha \)?)

(e) What is wrong with the following approach:

Alice has a qubit \( |\Psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). She wants to initialise the qubit to \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). She knows that when measuring \( \Psi \), with probability \( \frac{1}{2} \) she get the measurement outcome 0 and the qubit will be in state \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Thus she repeatedly measures the qubit in the computational basis until she gets the outcome 0. Since the probability is \( \frac{1}{2} \) each time, the expected number of measurements until she gets her \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)-initialised qubit is 2.

(f) Which of the following are valid (unitary) transformations:

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}.
\]

Problem 2: Quantum Circuits

(a) What is the state after this quantum circuit?

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{X} \xrightarrow{H} \xrightarrow{X} \xrightarrow{H}
\]

Note that \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is the bit flip, and \( H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \) is the Hadamard transform.