Lecture February 17

- Quantum systems (n "classical poss")
- Quantum states ($|\psi\rangle \in \mathbb{C}^n$, $\| |\psi\rangle \| = 1$)
- Ops on Q states:
  - Unitary trafo (linear U with $U^\dagger U = I$)
  - Measurement

Measurements

Eg: $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ("0", "1") "Fully 0"

Ask: is it 0 or 1? $\longrightarrow$ Answer: 0   Afterwards: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|\psi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ "Fully 1"

$\longrightarrow$ Answer: 1   Afterwards: state $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\psi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \longrightarrow$ Answer: "0" with prob $\frac{1}{2} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; "1" with prob $\frac{1}{2} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

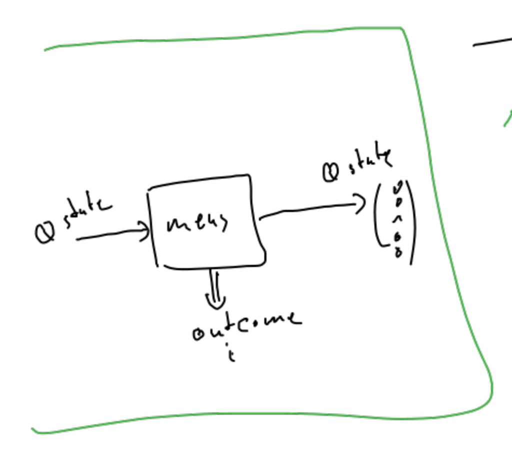
$|\psi\rangle = \begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix} \longrightarrow$ Answer: "0" with prob $|\sqrt{\frac{2}{3}}|^2 = \frac{2}{3} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; "1" —"— $|\sqrt{\frac{1}{3}}|^2 = \frac{1}{3} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

"Measurement in computational basis"

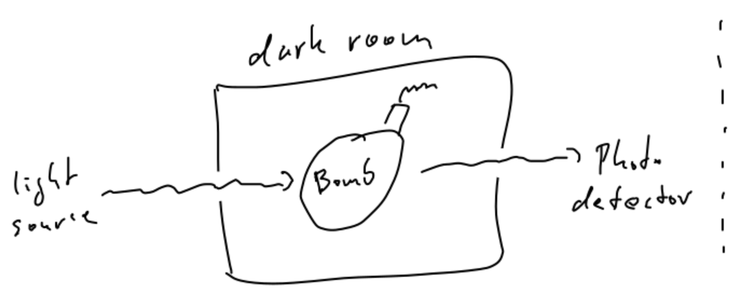
$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$. If we ask in which class. poss. is the state?

$\longrightarrow$ With prob $|\alpha_i|^2$ answer "i".

Afterwards: state is $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i-th pos



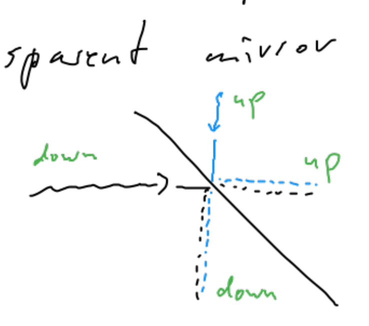
Elitzur-Vaidman bomb tester (93)



If photon hits bomb, bomb explodes!

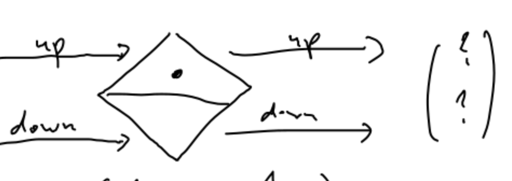
Main component: Beam splitter

Semi-transparent mirror



Incoming photon: $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ (up, down)

α = up-amplitude, β = down-amplitude



$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ → $\begin{pmatrix} ? \\ ? \end{pmatrix}$

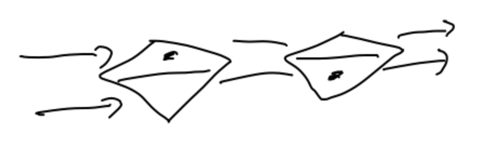
$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \leadsto \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \leadsto \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ $B\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ $B\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

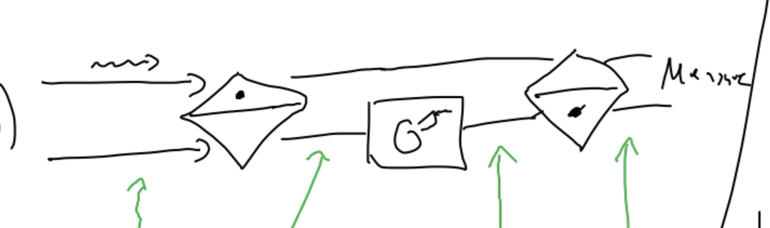
$B^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$



(Almost Mach-Zehnder interferometer)

$B^\dagger B = I$

Bomb tester

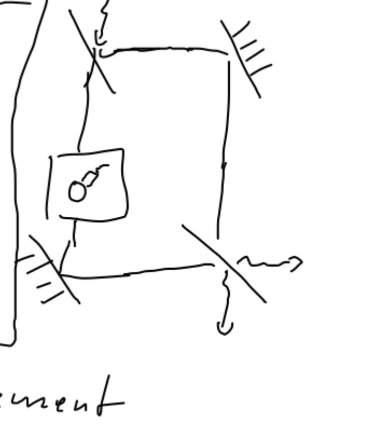


$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ → $B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ → $B^\dagger\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

Prob $\frac{1}{2}$: silence $\to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Prob $\frac{1}{2}$: boom

Outcome	without bomb	with bomb
up	1	1/4
down	0	1/4
boom	0	1/2



[bomb] $\hat{=}$ measurement

up $\to$ silence

down $\to$ boom

Final measurement (in case "no boom"):

up with prob. $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

down —"— $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

Measurements (ctd.)

"Complete measurement"

Pol. photon: $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ horiz., vert.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\longleftrightarrow$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $\updownarrow$

$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$: $\swarrow\!\nearrow$ ($1/2$ ↔, $1/2$ ↕)

Could ask: Is photon ⤢-polarized or ⤡-polarized?

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$\Rightarrow$ Prob $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$: outcome ⤢, pms: $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Prob $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$: outcome ⤡, pms: $\begin{pmatrix} 1/\sqrt{2} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

Compl. meas (in comp. basis) $\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$ $\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

Given orthonormal ~~states~~ basis $|\phi_1\rangle, \dots, |\phi_n\rangle \in \mathbb{C}^n$,

complete meas. of $|\psi\rangle \in \mathbb{C}^n$

gives outcome i with prob $|\beta_i|^2$ and pms $|\phi_i\rangle$

if $|\psi\rangle = \beta_1 |\phi_1\rangle + \dots + \beta_n |\phi_n\rangle$

$|\psi\rangle = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$

Example:

Measure $|1\rangle$ in $|+\rangle, |-\rangle$ basis?

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $|-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}}_{\beta_0} \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{=|+\rangle} + \underbrace{\left(-\frac{1}{\sqrt{2}}\right)}_{\beta_1} \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}_{=|-\rangle}$

$\to \Pr[+] = |\beta_0|^2 = \frac{1}{2}$

$\Pr[-] = |\beta_1|^2 = 1/2$

More efficient calculation:

$\beta_i = \langle |\phi_i\rangle, |\psi\rangle \rangle$ $\langle \alpha, \gamma \rangle = \sum_i \overline{\alpha_i} \gamma_i$

$= \langle \phi_i | \psi \rangle$

Measure $|1\rangle$ $\left(= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ in $|+\rangle$ $\left(= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right)$, $|-\rangle$ basis.

Prob $[+] = |\langle + | 1 \rangle|^2 = |~~0 \cdot \frac{1}{\sqrt{2}}~~ + ~~1~~ \cdot \frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

Say 3 paths. $|\psi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ (up, middle, down)

Measure: Is photon in state $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ or $(\dots)$ or $(\dots)$?

"A" $|A\rangle$ "B"   "C"

Prob [outcome A] $= |\langle A | \psi \rangle|^2$

$= |\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + ~~\frac{1}{\sqrt{3}} \cdot 0~~|^2$

$= |\frac{2}{\sqrt{6}}|^2 = \frac{4}{6} = \frac{2}{3}$.   Pms: $|A\rangle$