Lecture March 3

- Quantum systems ($\mathbb{C}^n$)
- Q states ($\in \mathbb{C}^n$, length 1)
- Unitary ops ($U^\dagger U = I$)
- Measurement
  - ↳ Computational basis ($\psi = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}$, $\Pr[\text{outcome } i] = |\alpha_i|^2$, pms: $|i\rangle$)
  - ↳ Complete meas (given meas basis $|\phi_i\rangle$, $\Pr[\text{outcome } i] = |\langle \phi_i | \psi \rangle|^2$, pms: $|\phi_i\rangle$)

Missing:
- Projective measurement
- Composition of systems

Projective measurement

Q system: $\mathbb{C}^3$ (eg. photon left/middle/right)

Q state: $\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} = |\psi\rangle$

Measurement:
Is state a) on the left
b) not on the left

Step 1: Vector spaces for each outcome

$V_a = \text{span}\{|\text{left}\rangle\} = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}\alpha\\0\\0\end{pmatrix} : \alpha \in \mathbb{C}\right\}$

$V_b = \text{span}\{|\text{middle}\rangle, |\text{right}\rangle\}$
$= \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$
$= \left\{\begin{pmatrix}0\\\alpha\\\beta\end{pmatrix} : \alpha, \beta \in \mathbb{C}\right\}$ $\qquad \begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

Step 2: Write $|\psi\rangle$ as:

$$\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} = |\psi\rangle = \underset{\in V_a}{|\psi_a\rangle} + \underset{\in V_b}{|\psi_b\rangle} = \begin{pmatrix}1/\sqrt{3}\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

Step 3: $\Pr[\text{outcome } a] = \| |\psi_a\rangle \|^2 = \left(\frac{1}{\sqrt{3}}\right)^2 + 0^2 + 0^2 = \frac{1}{3}$

$\Pr[\text{outcome } b] = \| |\psi_b\rangle \|^2 = 0^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{2}{3}$

Step 4: Pms for a: $\frac{|\psi_a\rangle}{\| |\psi_a\rangle\|} = \frac{\begin{pmatrix}1/\sqrt{3}\\0\\0\end{pmatrix}}{1/\sqrt{3}} = \begin{pmatrix}1\\0\\0\end{pmatrix} = |\text{left}\rangle$

Pms for b: $\frac{|\psi_b\rangle}{\| |\psi_b\rangle\|} = \frac{\begin{pmatrix}0\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}}{\sqrt{2/3}} = \begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{2}}|\text{middle}\rangle + \frac{1}{\sqrt{2}}|\text{right}\rangle$

A projective measurement on $\mathbb{C}^n$ consists of subspaces $V_1, \dots, V_k \subseteq \mathbb{C}^n$
s.t.: V_i orthog. to V_j ($i \neq j$)
$\sum V_i = \mathbb{C}^n$ (aka: $\text{span}(V_1 \cup \dots \cup V_k)$)
(has outcomes $1, \dots, k$)

Given $|\psi\rangle$, when measuring $|\psi\rangle$, get outcome i
with prob. $\| |\psi_i\rangle \|^2$ and pms $|\psi_i\rangle / \| |\psi_i\rangle \|$
where $|\psi\rangle = \underset{\in V_1}{|\psi_1\rangle} + \dots + \underset{\in V_k}{|\psi_k\rangle}$

Example: Complete meas: $|\phi_1\rangle, \dots, |\phi_n\rangle \in \mathbb{C}^n$

$V_1 = \text{span}\{|\phi_1\rangle\}$ .... $V_n = \text{span}\{|\phi_n\rangle\}$

All V_i orthog: ✓ (because $|\phi_i\rangle$ orthog)
$\sum V_i = \mathbb{C}^n$: ✓ (because $|\phi_i\rangle$ basis)
⇒ this is proj. meas.

Given $|\psi\rangle$. $\Pr[\text{outcome } i] = \begin{cases} |\langle \phi_i | \psi \rangle|^2 & \text{(compl. meas)} \\ \| |\psi_i\rangle \|^2 & \text{(proj. meas)} \end{cases}$

$\| |\psi_i\rangle \|^2 = |\alpha_i|^2 = |\langle \psi | \phi_i \rangle|^2$

$|\psi\rangle = |\psi_1\rangle + \dots + |\psi_n\rangle$, with $|\psi_1\rangle = \alpha_1 |\phi_1\rangle$, ..., $|\psi_n\rangle = \alpha_n |\phi_n\rangle$

$\alpha_i = \langle \psi | \phi_i \rangle$

Pms: $|\phi_i\rangle$ (compl. meas)

$\frac{|\psi_i\rangle}{\| |\psi_i\rangle \|} = \frac{\alpha_i |\phi_i\rangle}{|\alpha_i|} = \frac{\alpha_i}{|\alpha_i|} |\phi_i\rangle$ (proj. meas)

Different view (same thing):
For V_i, let P_i be (orthog) proj. onto V_i

Proj. meas:
Consists of $P_1, \dots, P_k$ where
- Each P_i is orthog proj. ($P^2 = P$, $P^\dagger = P$)
- $P_i P_j = 0$ ($i \neq j$)
- $\sum_i P_i = I$

$\Pr[\text{outcome } i] = \| P_i |\psi\rangle \|^2$
Pms: $\frac{P_i |\psi\rangle}{\| P_i |\psi\rangle \|}$

(crossed out box: $|\psi\rangle = |\psi_1\rangle + \dots + |\psi_k\rangle$, $|\psi_1\rangle \in V_1$, $|\psi_k\rangle \in V_k$; ⇒ $|\psi_1\rangle = P_1 |\psi\rangle$ etc.)

$|\psi\rangle = \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$

a): $|\text{left}\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\qquad P_a = \begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix}$ $\qquad \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \to \begin{pmatrix}\alpha\\0\\0\end{pmatrix}$

b) $\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\qquad P_b = \begin{pmatrix}0&&\\&1&\\&&1\end{pmatrix}$ $\qquad \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \to \begin{pmatrix}0\\\beta\\\gamma\end{pmatrix}$

$\Pr[\text{outcome } b] = \| P_b |\psi\rangle \|^2 = \left\| \begin{pmatrix}0\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} \right\|^2 = \frac{2}{3}$

Composing systems:

Given: Q system A with n class. poss. ($\mathcal{H}_A = \mathbb{C}^n$) (→ R, G, B)
— " — B — " — m — " — ($\mathcal{H}_B = \mathbb{C}^m$) (→ up, down)

What is: AB, $\mathcal{H}_{AB} = ?$

AB has $n \cdot m$ class poss. (R up, R down, G up, G down, B up, B down)

$\Rightarrow \mathcal{H}_{AB} = \mathbb{C}^{n \cdot m} =: \mathcal{H}_A \otimes \mathcal{H}_B$

Eg: $\mathcal{H}_1, \dots, \mathcal{H}_n = \mathbb{C}^2$
$\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n = \mathbb{C}^{2 \cdot 2 \cdots 2} = \mathbb{C}^{2^n}$ $\qquad \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_{10} = \mathbb{C}^{2^{10}}$

Given $|\psi_A\rangle \in \mathcal{H}_A = \mathbb{C}^n$, $|\psi_B\rangle \in \mathcal{H}_B \in \mathbb{C}^m$

Combined state: $|\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

Eg. $|\psi_A\rangle = \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}\begin{matrix}R\\G\\B\end{matrix}$ $\qquad |\psi_B\rangle = \begin{pmatrix}x\\y\end{pmatrix}\begin{matrix}\text{up}\\\text{down}\end{matrix}$

$|\psi_A\rangle \otimes |\psi_B\rangle = \begin{pmatrix}\alpha \cdot x\\ \alpha \cdot y\\ \beta \cdot x\\ \beta \cdot y\\ \gamma \cdot x\\ \gamma \cdot y\end{pmatrix}\begin{matrix}\text{R up}\\\text{R down}\\\text{G up}\\\text{G down}\\\text{B up}\\\text{B down}\end{matrix}$

General: $|\psi_A\rangle = \begin{pmatrix}\alpha_1\\\vdots\\\alpha_n\end{pmatrix}$ $\qquad |\psi_B\rangle = \begin{pmatrix}\beta_1\\\vdots\\\beta_m\end{pmatrix}$

$$|\psi_A\rangle \otimes |\psi_B\rangle = \begin{pmatrix}\alpha_1\beta_1\\\alpha_1\beta_2\\\vdots\\\alpha_1\beta_m\\\alpha_2\beta_1\\\alpha_2\beta_2\\\vdots\\\alpha_n\beta_m\end{pmatrix} = \begin{pmatrix}\alpha_1|\psi_B\rangle\\\alpha_2|\psi_B\rangle\\\vdots\\\alpha_n|\psi_B\rangle\end{pmatrix}$$

Remaining questions: Combining Unitaries, meas.

If we apply U to $\mathcal{H}_A$, V to $\mathcal{H}_B$,
this applies $U \otimes V$ to $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$U \otimes V = \begin{pmatrix}u_{11}V & u_{12}V & \cdots & u_{1n}V\\ u_{21}V & & & \vdots\\ \vdots & & & \\ \vdots & \cdots & & u_{nn}V\end{pmatrix}$$

E.g:
$X = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $n \times n$
$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$ $m \times m$

$$X \otimes H = \begin{pmatrix}0 \cdot H & 1 \cdot H\\ 1 \cdot H & 0 \cdot H\end{pmatrix} = \begin{pmatrix}0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0&0\end{pmatrix} \quad nm \times nm$$

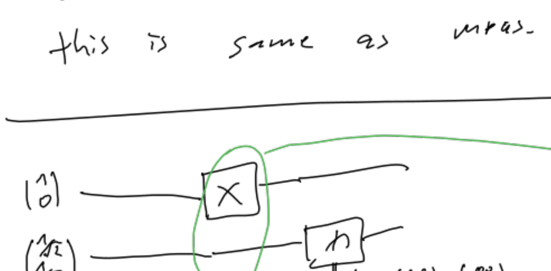
Combining proj. measurements:

$M_1 = \{P_1 \dots P_k\}$ $\qquad M_2 = \{Q_1 \dots Q_\ell\}$

$M_1 \otimes M_2 = \{P_1 \otimes Q_1, P_1 \otimes Q_2, \dots P_1 \otimes Q_\ell, P_2 \otimes Q_1, \dots, P_k \otimes Q_\ell\}$

(Here $P_i \otimes Q_j$ corresponds to outcome (i, j))

Systems A, B, measure A using $M = \{P_1 \dots P_k\}$
this is same as meas. AB using $M \otimes I = \{P_1 \otimes I, \dots, P_k \otimes I\}$



$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, second wire $\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$; measurement $\to \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$; $\Pr[\text{outcome } 0]$

$X \otimes I = \begin{pmatrix}0I & 1I\\1I & 0I\end{pmatrix} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ $\qquad \begin{pmatrix}0&1\\1&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$

Meas: $I \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}$ (outcome 0) $= \begin{pmatrix}1&&&\\&0&&\\&&1&\\&&&0\end{pmatrix}$ $\qquad I \otimes \begin{pmatrix}0&0\\0&1\end{pmatrix}$ (outcome 1)

$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix} = |\psi\rangle$

$(X \otimes I)|\psi\rangle = \begin{pmatrix}0\\0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

$\Pr[\text{outcome } 0] = \left\| \left(I \otimes \begin{pmatrix}1&0\\0&0\end{pmatrix}\right)\begin{pmatrix}0\\0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} \right\|^2 = \left\| \begin{pmatrix}0\\0\\1/\sqrt{2}\\0\end{pmatrix} \right\|^2 = \frac{1}{2}$

Pms (for outcome 0): $\begin{pmatrix}0\\0\\1/\sqrt{2}\\0\end{pmatrix} \Big/ \frac{1}{\sqrt{2}} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$