Practice March 23

Density op.

basis states $|b_1\rangle \dots |b_n\rangle$

pure state $|\psi\rangle\langle\psi|$

mixed states $\sum_i p_i |\psi_i\rangle\langle\psi_i|$

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$\rho_0 = |+\rangle\langle+|$

$\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

Performing a unitary U: $\rho \to U\rho U^\dagger$

Measuring: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$$|\psi_i\rangle \to U|\psi_i\rangle$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad \rho' = \sum_i (U|\psi_i\rangle)(U|\psi_i\rangle)^\dagger = \sum_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\left(\sum_i p_i|\psi_i\rangle\langle\psi_i|\right)U^\dagger = U\rho U^\dagger$$

$P_1 \dots P_n \qquad P_1 \rho P_1^\dagger$

$$P_r[\text{outcome } j] = \operatorname{tr} P_j \rho P_j^\dagger$$

$|\psi\rangle \qquad |\psi\rangle\langle\psi|$

$|\psi\rangle = \alpha_1 b_1 + \dots \alpha_n b_n$

$|\psi\rangle\langle\psi|_{ij} = \alpha_i \alpha_i^* = |\alpha_i|^2$

$\rho = \rho^\dagger$

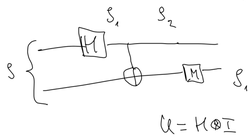
$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$\operatorname{tr}\rho = \operatorname{tr}\sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i \underbrace{\operatorname{tr}|\psi_i\rangle\langle\psi_i|}_{} = \sum_i p_i = 1$

proj. $P = P^\dagger \quad PP = P$

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$\rho^\dagger = \left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)^\dagger = \sum_i p_i (|\psi_i\rangle\langle\psi_i|)^\dagger = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho$



$\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|10\rangle\langle 10|$

$\rho_1 = U\rho U^\dagger = (H\otimes I)\left(\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|10\rangle\langle 10|\right)(H\otimes I) =$

$U = H\otimes I$

$= \frac{1}{2}(H\otimes I)|00\rangle\langle 00|(H\otimes I) +$

$\frac{1}{2}(H\otimes I)|10\rangle\langle 10|(H\otimes I) =$

$= \frac{1}{2}(|+\rangle\otimes|0\rangle)(\langle +|\otimes\langle 0|) +$

$\frac{1}{2}(|-\rangle\otimes|0\rangle)(\langle -|\otimes\langle 0|) =$

$= \frac{1}{2}|+\rangle\langle +|\otimes|0\rangle\langle 0| + \frac{1}{2}|-\rangle\langle -|\otimes|0\rangle\langle 0|$

$C\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$C\left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|10\rangle\right) = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$

$\rho_2 = C\rho_1 C^\dagger = \frac{1}{2}C(|+\rangle\otimes|0\rangle)(\langle +|\otimes\langle 0|)C^\dagger + \frac{1}{2}C(|-\rangle\otimes|0\rangle)(\langle -|\otimes\langle 0|)C^\dagger =$

$U|\psi\rangle = |\varphi\rangle$

$\langle\psi|U^\dagger = (U|\psi\rangle)^\dagger = (|\varphi\rangle)^\dagger = \langle\varphi|$

$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)\left(\frac{1}{\sqrt{2}}\langle 00| + \frac{1}{\sqrt{2}}\langle 11|\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle\right)\left(\frac{1}{\sqrt{2}}\langle 00| - \frac{1}{\sqrt{2}}\langle 11|\right) =$

$= \frac{1}{2}\left(\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|00\rangle\langle 11| + \frac{1}{2}|11\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| + \frac{1}{2}|00\rangle\langle 00| - \frac{1}{2}|00\rangle\langle 11| - \frac{1}{2}|11\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|\right) =$

$= \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$

$|\varphi_1\rangle = |00\rangle \qquad \rho' = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$

$|\varphi_2\rangle = |11\rangle$

$P_1 = I\otimes|0\rangle\langle 0|$

$P_2 = I\otimes|1\rangle\langle 1|$

$P_r[\text{outcome } 1] = \operatorname{tr} P_1 \rho P_1^\dagger = \operatorname{tr} P_1\rho$

$\operatorname{tr}(AB)C = \operatorname{tr} CAB$

$\operatorname{tr} AB = \operatorname{tr} BA$

$\operatorname{tr} ABC \overset{?}{=} \operatorname{tr} ACB$

$|00\rangle = |0\rangle\otimes|0\rangle$

$|00\rangle\langle 00| = |0\rangle\langle 0|\otimes|0\rangle\langle 0|$

$\operatorname{tr} P_1^\dagger P_1 \rho = \operatorname{tr} P_1 P_1 \rho = \operatorname{tr} P_1 \rho$

$I\otimes|0\rangle\langle 0|\left(\frac{1}{2}|0\rangle\langle 0|\otimes|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\otimes|1\rangle\langle 1|\right) =$

$= \frac{1}{2}|0\rangle\langle 0|\otimes|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\otimes|0\rangle\langle 0|\langle 1|\rangle\langle 1| =$

$= \frac{1}{2}|0\rangle\langle 0|\otimes|0\rangle\langle 0| = \frac{1}{2}|00\rangle\langle 00|$

PMS: $\frac{\frac{1}{2}|00\rangle\langle 00|}{\frac{1}{2}} = |00\rangle\langle 00|$

$\operatorname{tr} \underline{\quad} = \frac{1}{2}$

$|1\rangle\langle 1|$

$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}(0\ 0\ 1\ 0) = \begin{pmatrix}0&&&\\&0&&\\&&1&\\&&&0\end{pmatrix}$

$\begin{pmatrix}\frac{1}{2}&&&\\&0&&\\&&0&\\&&&0\end{pmatrix} = \frac{1}{2}|00\rangle\langle 00|$

Observable $O = \sum_m m\cdot P_m$

$O = 1\cdot|0\rangle\langle 0| - 1\cdot|1\rangle\langle 1| = \begin{pmatrix}1&\\&-1\end{pmatrix} = Z$

$Z = \begin{pmatrix}1&\\&-1\end{pmatrix}$

$|0\rangle \quad m=1 \quad P_1 = |0\rangle\langle 0|$

$|1\rangle \quad m=-1 \quad P_{-1} = |1\rangle\langle 1|$

$\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1\ 1) \qquad \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1\ -1)$

$X = \begin{pmatrix}0&1\\1&0\end{pmatrix}$

$O = 1\cdot|+\rangle\langle +| - 1|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1&1\\1&-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&2\\2&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix} = X$

$P_1 = |+\rangle\langle +|$

$P_{-1} = |-\rangle\langle -|$

$X|+\rangle = |+\rangle$

$X|-\rangle = -|-\rangle$

$X\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle = -1\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = -1|-\rangle$

$E(\text{measurement on } \rho) = \sum_m m\cdot P_r[\text{outcome } m] = \sum_m m\cdot\operatorname{tr} P_m\rho P_m^\dagger = \sum_m m\cdot\operatorname{tr} P_m\rho = \operatorname{tr}\sum_m m P_m \rho = \operatorname{tr} O\rho$

$E(X \text{ on } |0\rangle\langle 0|) = \operatorname{tr}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \operatorname{tr}\begin{pmatrix}0&0\\1&0\end{pmatrix} = 0 = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1)$

$E(X \text{ on } |+\rangle\langle +|) = \operatorname{tr}\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix} = \operatorname{tr}\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} = 1$

$\rho = \frac{3}{4}|+\rangle\langle +| + \frac{1}{4}|0\rangle\langle 0|$

$\operatorname{tr} X\rho = \operatorname{tr}\left(\frac{3}{4}X|+\rangle\langle +| + \frac{1}{4}X|0\rangle\langle 0|\right) = \frac{3}{4}\underbrace{\operatorname{tr} X|+\rangle\langle +|}_{=1} + \frac{1}{4}\underbrace{\operatorname{tr} X|0\rangle\langle 0|}_{=0} = \frac{3}{4}$

$A_i = \begin{cases}\text{unitary}\\ \text{measurement}\\ \text{extend}\end{cases}$

$x \in \{0,1\} \qquad \begin{matrix}0 \text{ w/ } p_0\\ 1 \text{ w/ } p_1\end{matrix}$

$A_1 = \text{unitary } U, \quad A_2 = \text{unitary } V$

$E = \{|0\rangle@p_0, |1\rangle@p_1\}$

$UE = \{U|0\rangle@p_0, U|1\rangle@p_1\}$

$(U\otimes I)(I\otimes V) = U\otimes V = (I\otimes V)(U\otimes I)$

$\rho_1 = (I\otimes V)(U\otimes I)\rho(U^\dagger\otimes I)(I\otimes V^\dagger) = (U\otimes V)\rho(U^\dagger\otimes V^\dagger)$

$\|$

$\rho_2 = \underline{(U\otimes I)(I\otimes V)}\rho\underline{(I\otimes V^\dagger)(U^\dagger\otimes I)} = (U\otimes V)\rho(U^\dagger\otimes V^\dagger)$

$|\varphi\rangle$ — $\boxed{M}$ — ρ