Partial trace

Motivation

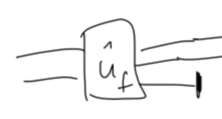
Have: $f : \{0,1\}^n \to \{0,1\}^m$

Want: $U_f : |x, y\rangle \to |x, y \oplus f(x)\rangle$

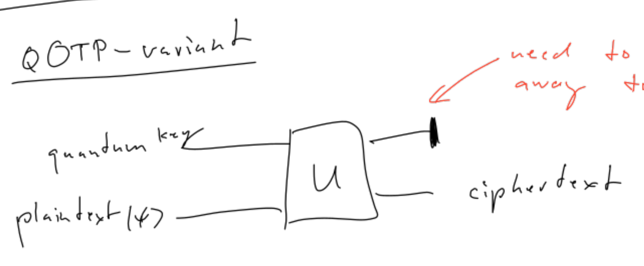
Try 1: $\hat{U}_f : |x, y\rangle \to |x, y \oplus f(x), \text{more stuff}\rangle$

Try 2: $\hat{U}_f : |x, y\rangle \to |x, y \oplus f(x), 0\rangle$

$\to$ want to throw away 0-bits

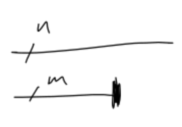


QOTP-variant



Want to check: Is the final density op $\rho = \frac{1}{2}I$?

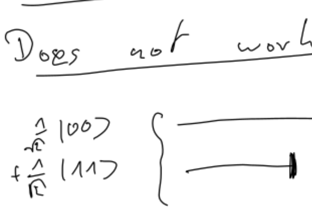
How can we mathematically describe



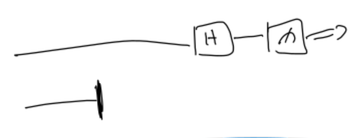
?

Do we want a function : $\mathbb{C}^{nm} \to \mathbb{C}^n$?

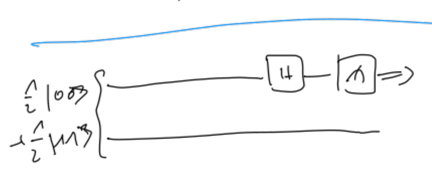
Does not work:



$|\psi\rangle$ what is here? $|0\rangle$ ↓ no, $|1\rangle$ ↓ no, $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$ ↓ no



if $|\psi\rangle = |+\rangle$, outcome is 0



$\Rightarrow$ same outcome 0 (wrong)

we get random bit!

This contradiction works for any final state

$\Rightarrow$ state after "throwing away" cannot be written as a pure state!

Can we write it as function: density op $\to$ density op?

(density op $\in \mathbb{C}^{nm \times nm}$ $\to$ density op $\in \mathbb{C}^{n \times n}$)

Yes!

is modeled by tr_B ("partial trace")

$tr_B(\sigma \otimes \tau) := \sigma \cdot tr\,\tau$ (linear)

For general ρ: $tr_B\,\rho = tr_B \sum \sigma_i \otimes \tau_i$
$= \sum tr_B\, \sigma_i \otimes \tau_i$
$= \sum \sigma_i \cdot tr\,\tau_i$

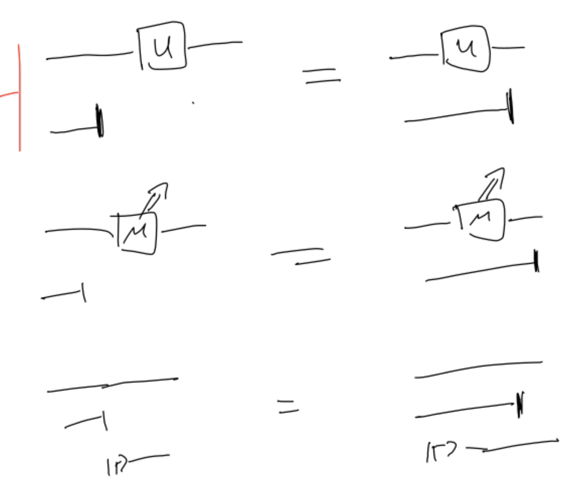
($\rho = \sum_i \sigma_i \otimes \tau_i$)

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 5 & 3 & 6 \\ 4 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & \\ & \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ & \end{pmatrix} + \begin{pmatrix} & \\ 3 & \end{pmatrix} + \dots$$

$2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Question: Is tr_B really "throwing away"?

Show:



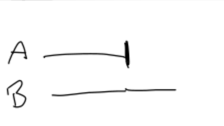
Then —| is modeled correctly!

$U(tr_B\,\rho)U^+ \overset{!}{=} tr_B\,(U \otimes I)\rho(U \otimes I)^+$

Proof: Say $\rho = \sum \sigma_i \otimes \tau_i$

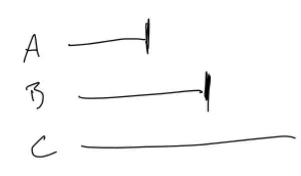
$U(tr_B\,\rho)U^+ = U(\sum \sigma_i \cdot tr\,\tau_i)U^+$
$= \sum tr\,\tau_i \cdot U\sigma_i U^+$

$tr_B(U \otimes I)\rho(U \otimes I)^+ = tr_B \sum (U \otimes I)(\sigma_i \otimes \tau_i)(U^+ \otimes I)$
$= tr_B \sum U\sigma_i U^+ \otimes \tau_i$
$= \sum U\sigma_i U^+ \cdot tr\,\tau_i$ (equal)



$tr_A\, \sigma \otimes \tau = \tau \cdot tr\,\sigma$

$tr_B\, \sigma \otimes \tau \otimes \gamma = \sigma \otimes \gamma \cdot tr\,\tau$

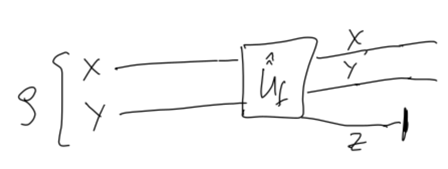


$tr_B(tr_A\,\rho) = tr_{AB}\,\rho$

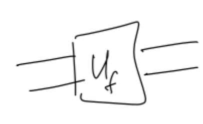
$tr_B(tr_A(\sigma \otimes \tau \otimes \gamma)) = tr_B\,\tau \otimes \gamma \cdot tr\,\sigma = \gamma \cdot tr\,\tau \cdot tr\,\sigma$
$= \gamma \cdot tr(\sigma \otimes \tau)$
$= tr_{AB}(\sigma \otimes \tau) \otimes \gamma$

$\hat{U}_f\, |x, y\rangle \to |x, y \oplus f(x), 0\rangle$

$\overset{\text{make}}{\leadsto}$ $U_f : |x, y\rangle \to |x, y \oplus f(x)\rangle$

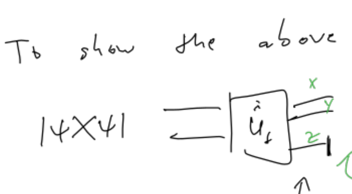


$\overset{?}{=}$



$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$

To show the above: show



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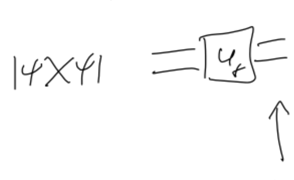


$\hat{U}_f |\psi\rangle\langle\psi| \hat{U}_f^+$
$=$
$(|\phi\rangle \otimes |0\rangle)(|\phi\rangle \otimes |0\rangle)^+$
$=$
$|\phi\rangle\langle\phi| \otimes |0\rangle\langle 0|$

$tr_Z(|\phi\rangle\langle\phi| \otimes |0\rangle\langle 0|)$
$= |\phi\rangle\langle\phi| \cdot \underbrace{tr\,|0\rangle\langle 0|}_{=1}$
$= |\phi\rangle\langle\phi|$

$\underbrace{U_f |\psi\rangle}_{= |\phi\rangle}\langle\psi| U_f^+ = |\phi\rangle\langle\phi|$

$\hat{U}_f |\psi\rangle = \hat{U}_f \sum_{xy} \alpha_{xy} |xy\rangle$
$= \sum_{xy} \alpha_{xy} |x, y \oplus f(x), 0\rangle$
$= \underbrace{\left(\sum \alpha_{xy} |x, y \oplus f(x)\rangle\right)}_{= U_f|\psi\rangle = |\phi\rangle} \otimes |0\rangle$
$= |\phi\rangle \otimes |0\rangle$

Quantum operation (superoperator, CPTPM "completely positive trace-preserving map")

"Anything that can happen to a density op"

$\mathcal{E} : \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m}$

Def 1: $\mathcal{E}$ is a q-op iff $\exists E_k \in \mathbb{C}^{m \times n}$: with $\sum E_k^+ E_k = I$ and
$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^+$$

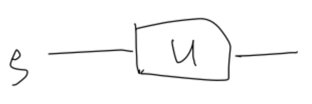
Def 2: $\mathcal{E}$ is q-op iff $\exists$ q-reg. Z, R, unitary U s.t. $\mathcal{E}(\rho) = tr_R\, U(\rho \otimes |0\rangle\langle 0|_Z)U^+$

Def 3: $\mathcal{E}$ is q-op iff
- $\mathcal{E}$ is linear
- $\mathcal{E}$ is compl. pos
- $\mathcal{E}$ is trace preserving

($\mathcal{E}$ pos $\Leftrightarrow$ If ρ pos then $\mathcal{E}(\rho)$ pos. $\mathcal{E}$ compl. pos $\Leftrightarrow$ $\mathcal{E} \otimes id$ pos)

(If $tr\,\rho = 1$ then $tr\,\mathcal{E}(\rho) = 1$)

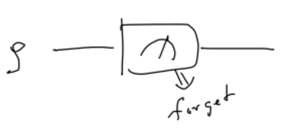
Examples:



$U\rho U^+$ $\quad \mathcal{E}(\rho) = U\rho U^+$

$\}\, tr_B\,\rho$ $\quad \mathcal{E}(\rho) = tr_B(\rho)$ $\quad$ $tr_B = id \otimes tr$

$tr\,\rho$ $\quad \mathbb{C}^1 = \mathbb{C}$ $\quad \mathcal{E}(\rho) = tr\,\rho$



$\sum \frac{|i\rangle\langle i|\rho|i\rangle\langle i|}{~~tr\,\text{—"—}~~} \cdot \text{~~tr —"—~~}$ $\quad \mathcal{E}(\rho) = \sum |i\rangle\langle i|\rho|i\rangle\langle i|$