$$\mathrm{tr}_B\, \sigma \otimes \tilde{\sigma} = \sigma \cdot \mathrm{tr}\,\tilde{\sigma}$$

$\uparrow A \quad \uparrow B$

$\mathrm{tr}_B\, \rho\,?$ $\quad |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|) = \langle\psi|$

$|\psi\rangle\langle\psi| = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + \underline{|11\rangle\langle 00|} + |11\rangle\langle 11|\right) =$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

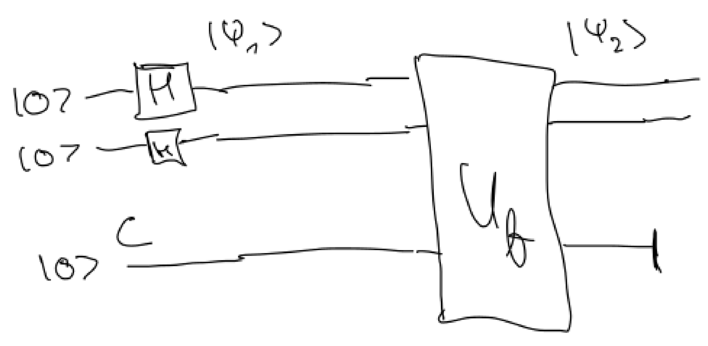
$|0\rangle\langle 1| \otimes |0\rangle\langle 1|$

$$\frac{1}{2}\left(\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ \cdots \right)$$

$|11\rangle = |1\rangle \otimes |1\rangle$

$|11\rangle\langle 00| = (|1\rangle \otimes |1\rangle)(\langle 0| \otimes \langle 0|) = |1\rangle\langle 0| \otimes |1\rangle\langle 0|$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0| \otimes |0\rangle\langle 0|$



$U_f|00, y\rangle \to |00, \bar{y}\rangle$ $\quad y = 0$
$U_f|01, y\rangle \to |01, y\rangle$
$U_f|10, y\rangle \to |10, y\rangle$ $\quad y = 1$
$U_f|11, y\rangle \to |11, \bar{y}\rangle$

001
010
100
111
000
011
101
110

$|i\rangle\langle i|$

$|\psi_1\rangle = (H \otimes H \otimes I)|000\rangle = |+\rangle \otimes |+\rangle \otimes |0\rangle = \sum_{x \in \{0,1\}^2} \frac{1}{2}|x, 0\rangle = \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |110\rangle)$

$\mathrm{tr}_C\, |00\rangle\langle 01| \otimes |1\rangle\langle 0| = |00\rangle\langle 01| \cdot \mathrm{tr}\,|1\rangle\langle 0|$ $\quad 0$

$|\psi_2\rangle = U_f|\psi_1\rangle = \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle)$

$|00\rangle\langle 00| \otimes |1\rangle\langle 1| \qquad |00\rangle\langle 01| \otimes |1\rangle\langle 0|$

$\rho = |\psi_2\rangle\langle\psi_2| = \frac{1}{4}\big(|001\rangle\langle 001| + \underline{|001\rangle\langle 010|} + \underline{|001\rangle\langle 100|} + |001\rangle\langle 111| +$
$+ \underline{|010\rangle\langle 001|} + |010\rangle\langle 010| + |010\rangle\langle 100| + \underline{|010\rangle\langle 111|} +$
$+ \underline{|100\rangle\langle 001|} + |100\rangle\langle 010| + |100\rangle\langle 100| + \underline{|100\rangle\langle 111|}$
$+ \ldots \big)$

$\rho = \frac{1}{4}\sum_{x, \tilde{x} \in \{0,1\}} |x\rangle\langle\tilde{x}|$

$f(00) = 1$
$f(01) = 0$
$f(10) = 0$
$f(11) = 1$

$U_f(x, y) = |x, y \oplus f(x)\rangle$

$\mathrm{tr}_C\, \rho = \frac{1}{4}\big(|00\rangle\langle 00| + |00\rangle\langle 11| + |01\rangle\langle 01| + |01\rangle\langle 10| +$
$|10\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 00| + |11\rangle\langle 11|\big) =$

$$\mathrm{tr}_C\, \rho = \frac{1}{4}\left(\begin{array}{l} \overset{f(\cdot)=1}{(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)} + \\ \overset{f(\cdot)=0}{(|10\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 01| + |01\rangle\langle 10|)} \end{array} \right) =$$

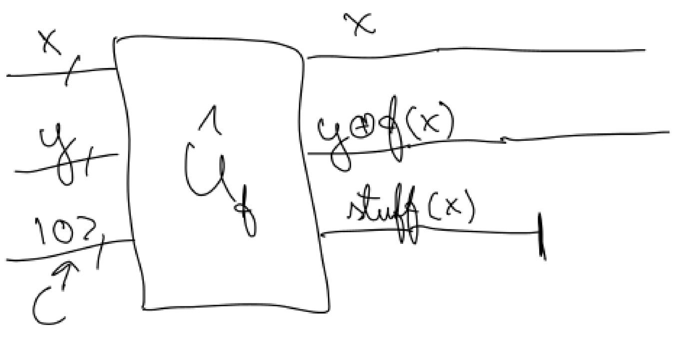
$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$$= \frac{1}{4}\left(\sum_{\substack{x, \tilde{x}: \\ f(x)=f(\tilde{x})=1}} |x, \tilde{x}| + \sum_{\substack{x, \tilde{x} \\ f(x)=f(\tilde{x})=0}} |x, \tilde{x}| \right) = \frac{1}{4}\sum_{\substack{x, \tilde{x}, z: \\ f(x)=f(\tilde{x})=z}} |x\rangle\langle\tilde{x}|$$

$z = \{0, 1\}$

$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$



$\hat{U}_f|x, y, z\rangle \to |x, y \oplus f(x), z \oplus \mathrm{stuff}(x)\rangle$

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ $\qquad |\psi_i\rangle = \sum_{x,y} \alpha_{xy}|x, y, 0\rangle$

$|x\rangle\langle\tilde{x}| \otimes |y\rangle\langle\tilde{y}| \otimes |0\rangle\langle 0|$

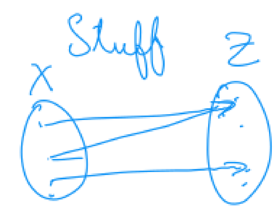
$|\psi_i\rangle\langle\psi_i| = \sum_{\substack{x,y \\ \tilde{x}\tilde{y}}} \alpha_{xy}\alpha^*_{\tilde{x}\tilde{y}} |x, y, 0\rangle\langle\tilde{x}, \tilde{y}, 0|$

$\rho = \sum_i p_i \sum_{\substack{x,y \\ \tilde{x}\tilde{y}}} \alpha_{xy_i}\alpha^*_{\tilde{x}\tilde{y}_i} |x, y, 0\rangle\langle\tilde{x}, \tilde{y}, 0|$

$\begin{cases} 0 & \text{if } \mathrm{stuff}(x) \neq \mathrm{stuff}(\tilde{x}) \\ 1 & \text{if } \mathrm{stuff}(x) = \mathrm{stuff}(\tilde{x}) \end{cases}$

$\rho = \sum_i p_i \sum_{\substack{x,y \\ \tilde{x}\tilde{y}}} \alpha_{xy_i}\alpha^*_{\tilde{x}\tilde{y}_i} |x, y \oplus f(x), \mathrm{stuff}(x)\rangle\langle\tilde{x}, \tilde{y} \oplus f(\tilde{x}), \mathrm{stuff}(\tilde{x})|$

$|x\rangle\langle\tilde{x}| \otimes |y \oplus f(x)\rangle\langle\tilde{y} \oplus f(\tilde{x})| \otimes |\mathrm{stuff}(x)\rangle\langle\mathrm{stuff}(\tilde{x})|$



$$\mathrm{tr}_C\, \rho = \sum_i p_i \sum_{\substack{x, y, \\ \tilde{x}, \tilde{y}, \\ z \in Z \\ \mathrm{stuff}(x) = \mathrm{stuff}(\tilde{x})}} \alpha_{xy_i}\alpha^*_{\tilde{x}\tilde{y}_i} |x, y \oplus f(x)\rangle\langle\tilde{x}, \tilde{y} \oplus f(\tilde{x})|)$$

(Partial) Trace is quantum op.

$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \qquad \sum_k E_k^\dagger E_k = I$

$\mathrm{tr}\, \rho = \mathrm{tr} \sum_{ab} \rho_{ab}|a\rangle\langle b|$

$|i\rangle\langle i|$ $\quad 4 \times 4$

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad |1\rangle\langle 0| \otimes |0\rangle\langle 1|$$

$\mathrm{tr}_A\, \sigma \otimes \tilde{\sigma} = \mathrm{tr}\,\sigma \cdot \tilde{\sigma} = \sum_i (\langle i| \otimes I)\rho(|i\rangle \otimes I) =$

$\mathrm{tr}\,\rho = \sum_i \langle i|\rho|i\rangle = \sum_{i\,a\,b} \langle i|a\rangle\langle b|i\rangle \rho_{ab}$ $\quad |1\rangle\langle 0| \otimes |1\rangle\langle 0|$

$= \sum_i (\langle i| \otimes I)(\sigma \otimes \tilde{\sigma})(|i\rangle \otimes I) = \sum_i \langle i|\sigma|i\rangle \otimes \tilde{\sigma}$

$E_i = (\langle i| \otimes I) \qquad \sum_i E_i^\dagger E_i = \sum_i (|i\rangle \otimes I)(\langle i| \otimes I) = \sum_i |i\rangle\langle i| \otimes I = I \otimes I = I$

Replace op

$\mathcal{E}(\rho) = |\psi\rangle\langle\psi|$

$\rho = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|$

$\mathcal{E}(\rho) = \mathcal{E}(p_0|0\rangle\langle 0|) + \mathcal{E}(p_1|1\rangle\langle 1|) = p_0|\psi\rangle\langle\psi| + p_1|\psi\rangle\langle\psi|$

$\mathcal{E}(\rho) = |\psi\rangle\langle\psi| \cdot \mathrm{tr}\,\rho = |\psi\rangle\langle\psi| \cdot \sum_i \langle i|\rho|i\rangle =$

$= \sum_i |\psi\rangle\langle i|\rho|i\rangle\langle\psi| = \sum_i (|\psi\rangle\langle i|)\rho(|i\rangle\langle\psi|) \leftarrow E_i^\dagger$

$\uparrow E_i \qquad (|\psi\rangle\langle i|)^\dagger = (\langle i|)^\dagger(|\psi\rangle)^\dagger = |i\rangle\langle\psi|$

$\sum_i E_i^\dagger E_i = \sum_i |i\rangle\underbrace{\langle\psi|\psi\rangle}_{=1}\langle i| = \sum_i |i\rangle\langle i| = I$

$\mathcal{E}(\rho) = \sum_k (|\psi\rangle\langle k|)\rho(|k\rangle\langle\psi|) = E_k \rho E_k^\dagger$