Physical view of QM

Example: A single particle in space

In classical physics:

Position: $\mathbb{R}^3$ (coordinates)

$x: \underset{\text{time}}{\mathbb{R}} \to \underset{\text{pos}}{\mathbb{R}^3}$ $\quad x(t) \hat{=}$ pos. at time t

$p: \underset{\text{time}}{\mathbb{R}} \to \underset{\text{momentum}}{\mathbb{R}^3}$

$$\boxed{\frac{\partial x(t)}{dt} = \frac{1}{m} p(t)}$$

In quantum physics

For given moment in time:

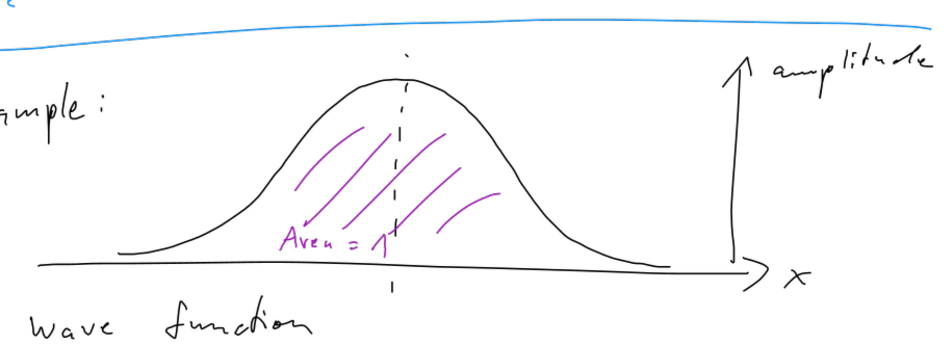
$\psi: \underset{\text{pos}}{\mathbb{R}^{\not 3}} \to \underset{\text{amplitude}}{\mathbb{C}}$

$\psi: \underset{\text{time}}{\mathbb{R}} \times \underset{\text{pos}}{\mathbb{R}^{\not 3}} \to \mathbb{C}$ $\quad \boxed{\forall t. \int |\psi(t,x)|^2 dx = 1}$

~~$\forall t. \sum_x |\psi(t,x)|^2 = 1$~~

Simplify: 1-d space

Example:



$\psi \hat{=}$ wave function

Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{dt} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)$$

$i\hbar$ ← some constant. Here: $\hbar = 1$

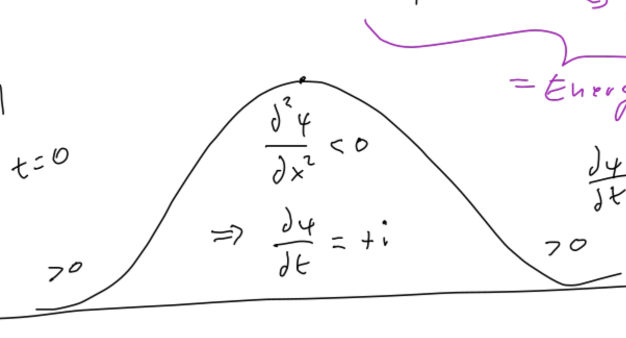
$\frac{\hbar^2}{2m}$: $=1$

$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2}$ → kinetic energy; $V(x,t)$: potential; together = Energy

$$i\hbar \frac{\partial \psi}{dt} = \hat{H}(t)\, \psi(x,t)$$

$\hat{H}(t)$: operator computing energy

At $t=0$



Schr. eq.

$$i \frac{\partial \psi(x,t)}{dt} = -\frac{\partial \psi(x,t)}{\partial x^2} + V(x,t)$$

Simpler case: V does not dep. on t

Time-indep Schr. eq: Find $E \in \mathbb{R}$, $\psi: \underset{\text{pos}}{\mathbb{R}} \to \mathbb{C}$:

$$-\frac{\partial^2 \psi_0(x)}{\partial x^2} + V(x) = E\, \psi_0(x)$$

① Why does solving TI. SE solve the SE?

② How do we solve it?

Say ψ_0, E are a solution to TISE.

Let $\psi(x,t) := \psi_0(x) \cdot \underbrace{e^{-iEt}}_{1 \cdot 1 = 1}$

E.g. $\psi_0 = x^2$, $E = 1$, $\psi = x^2 e^{-it}$

Then ψ is sol. to SE!

~~$i \frac{\partial \psi}{dt} = i \psi_0(x) \frac{d e^{-iEt}}{dt} = i \psi_0 e^{-iEt} (-iE)$~~ $= -\frac{\partial^2 \psi}{\partial x^2} + V$

$=$ ~~e^{-iEt}~~ $\frac{\partial \psi_0}{\partial x^2} + V$

~~Hagen~~

In general: Any sol. to SE is lin. comb. of sol. $\psi_0 \cdot e^{-iEt}$ for ψ_0, E being TISE sol.

Given wave function $\psi_{start}(x)$ at $t=0$

Find solutions ψ_E, E to TISE

write $\psi_{start} = \sum \alpha_E \psi_E = \sum \alpha_E \psi_E e^{-iE \cdot 0}$ (α_E: find)

$\Rightarrow \psi(x,t) = \sum \alpha_E \psi_E e^{-iEt}$ is solution to SE

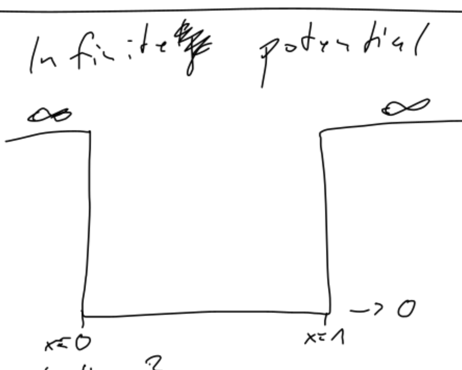
$\psi(x,0) = \psi_{start}$

Important facts:

- Any SE solution (with V time-indep) is linear comb. of TISE sols.
- ψ_0, E of TISE corresponds to a ~~state~~ particle with energy E

Example: Infinite potential well

$V =$



TISE solution?

$\psi_0 \in L^2(\mathbb{R})$

$\psi_0(x) = 0 \quad x < 0$ or $x > 1$

Inside $[0,1]$:

$$-\frac{d^2\psi_0}{dx^2} \cancel{+ V} = E \psi_0$$

Solutions to this:

$\psi_0 = A \sin(kx) + B\cos(kx) \quad E \geq 0$

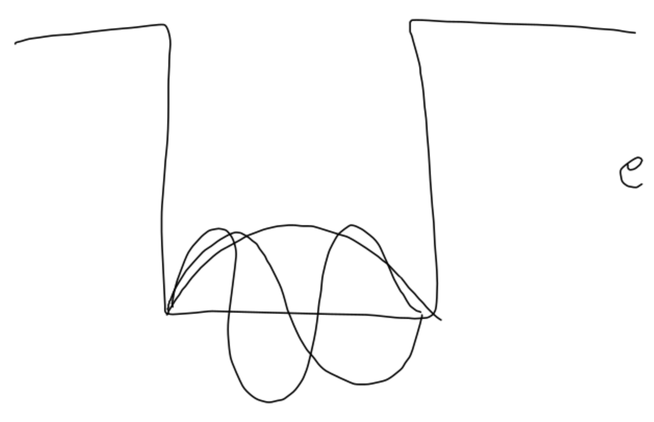
with $k = \sqrt{E}$

$\psi_0(0) = 0 \Rightarrow B = 0$

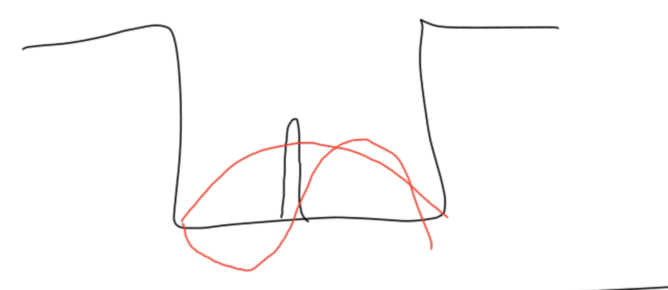
$\psi_0(1) = 0 \Rightarrow A \sin(k) = 0 \Rightarrow k$ multiple of π

$E = k^2 \Rightarrow E = n^2\pi^2$

$\Rightarrow \psi_0 = \sin(n\pi x)$ (or scalar multiple)



$e^{-in^2\pi^2 t}$



How does this connect to our formalism?

In the well, have every state is lin. comb. of

$\psi_n = \sin(n\pi x)$

These ψ_n span a Hilbert space $L^2([0,1])$

$\Rightarrow$ Every function is $\sum \alpha_n \psi_n \hat{=} (\alpha_1, \alpha_2, \alpha_3, \dots)$ (with $\sum |\alpha_i|^2 < \infty$)

Moreover: $\int \overline{\psi_n} \psi_m \, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

$\Rightarrow \psi_n$ are orthonormal basis if $\langle \psi, \phi \rangle = \int \overline{\psi} \phi \, dx$

Abstractly: Replace ψ_n by $|n\rangle$

Then every wave func. is $\text{span}\{|n\rangle\}_n$

$V(x) = \|x - \text{nucleus}\|^{\text{something}}$