Quantum Cryptography (spring 2021)	Dominique Unruh

Exercise Sheet 01

Out: 2021-02-21	Due: 2021-03-01

Read me: You will need 50% of all homeworks to qualify for the exam. (That is, if you get at least 50%, your final grade will be the exam grade. And if you do not get 50%, you do not pass the course.)

You may hand in your solutions in person or by email. If you submit by email, either scan a handwritten solution or typeset your solution readably. I do not consider ASCII formulas readable. For nicely typeset solutions, you can get up to three extra points for the effort.

When submitting, indicate your name and your matriculation number. On your first submission, please also indicate a password, this password will be needed for accessing the solutions and your points online.

You may work in teams to solve the problems. If you do, everyone has to formulate their own solution! (No copy&paste.)

Each problem has a table with some data to be filled by you. The prefilled field "knowlets" refers to the knowlets that this problem is based on (see the lecture notes). If you think the problem depends on a knowlet not indicated there, please add it. In the field "time", please indicate how long it took for you to solve this problem. In the field "difficulty", please indicate the perceived difficulty level as a grade from A–E. (E.g., B means that B-level student would probably be able to solve this fully.) The data you enter here does not affect your grade, it is used for statistics for future semesters. (If the exercise sheet itself is not part of your submission, please just provide the data somewhere in your submission.)

1 Working with quantum states

(a)

Knowlets:	QState	ProblemID: QState
Time:		
Difficulty:		

Which of the following are valid quantum states:

$$\begin{pmatrix}0\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}, \quad \frac{1}{\sqrt{3}}\begin{pmatrix}1\\2\end{pmatrix}, \quad \begin{pmatrix}\sqrt{\frac{2}{3}}\\ \frac{i}{\sqrt{3}}\end{pmatrix}$$

(b)

Knowlets:	CBMeas	ProblemID: CBMeas
Time:		
Difficulty:		

For each of the *valid* quantum states from Problem 1 (a), answer the following: You perform a measurement (i.e., you ask "whether the state is a classical 0 or a classical 1). What is the probability of answer 0 (i.e., yes), what is the probability of answer 1 (i.e., no)? What is the state after the measurement in each of those cases?

(c)

Knowlets:	ComplMeas ProblemID: MeasSum1
Time:	
Difficulty:	

Let a quantum state $\psi \in \mathbb{C}^2$ and an (orthonormal) measurement basis $\phi_{yes}, \phi_{no} \in \mathbb{C}^2$ be given. Measure ψ in that measurement basis. Let P_{yes} be the probability of outcome yes, and P_{no} the probability of outcome no. Show that $P_{\text{yes}} + P_{\text{no}} = 1$.

(d)

Knowlets:	UniTrafo ProblemID: UTInv
Time:	
Difficulty:	

Show that by applying a unitary transformation to a quantum state, no information is ever lost. More exactly, assume that a unitary transformation U is applied to a given quantum state Ψ, resulting in a state Φ. Then show that there is another unitary transformation V (not depending on Ψ or Φ) such that applying V to Φ gives Ψ again.

(e)

Knowlets:	QState, Rota, ComplMeas ProblemID: RotaPolPhoton
Time:	
Difficulty:	

Assume that a photon is in the state $\Psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. (Here the α component corresponds to the vertical part and the β component to the horizontal part.) Let R be a rotation of angle $\theta = \frac{\pi}{3}$. Let F denote a polarisation filter that lets only vertically polarised light through. Assume that the photon Ψ is first sent through R and then through F. It turns out that in this setting, the photon is absorbed by F with probability 1.

Given these informations, what do you know about α? (I.e., what are the possible values of α?)

(f)

Knowlets:	ComplMeas ProblemID: RepeatMeas
Time:	
Difficulty:	

What is wrong with the following approach:

Alice has a qubit $|\Psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. She wants to initialise the qubit to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. She knows that when measuring Ψ, with probability $\frac{1}{2}$ she get the measurement outcome 0 and the qubit will be in state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus she repeatedly measures the qubit in the computational basis until she gets the outcome 0. Since the probability is $\frac{1}{2}$ each time, the expected number of measurements until she gets her $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$-initialised qubit is 2.

(g)

Knowlets:	UniTrafo ProblemID: UniTrafo
Time:	
Difficulty:	

Which of the following are valid (unitary) transformations:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}.$$

(h)

Knowlets:	QState ProblemID: 5Paths
Time:	
Difficulty:	

Consider a system in which a single photon may be sent through 5 different paths. The photon may be polarised in any direction. Give a Hilbert space for describing the state of this photon and give a natural basis for expressing this state. How do you write that the photon is 45°-polarised and on path 3?

(i)

Knowlets:	QState ProblemID: 5PathsMulti
Time:	
Difficulty:	

Consider a system in which each of 5 paths may contain a photon (or not), and each of these photons may be polarised in any direction. Give a Hilbert space for describing the state of these photons and give a natural basis for expressing this state. How do you write that there is a photon on path 3 that is 45°-polarised and no photons on the other paths?