Quantum Cryptography (spring 2021)	Dominique Unruh
Exercise Sheet 03	
Out: 2021-03-08	Due: 2021-03-16

1 Composed systems

(a)

Knowlets:	QState, ProjMeas — ProblemID: MeasCat
Time:	
Difficulty:	

Consider the following state on n qubits: $\frac{1}{\sqrt{2}}|0\dots0\rangle + \frac{1}{\sqrt{2}}|1\dots1\rangle \in \mathbb{C}^{2^n}$.[1] Someone measures the last qubit (i.e., whether it is 0 or 1). What happens to the state?[2]

(b)

Knowlets:	ProjMeas — ProblemID: ProjMeasPr1
Time:	
Difficulty:	

Show that in a projective measurement with outcomes $i \in I$, it holds that $\sum_{i\in I} \Pr[\text{outcome } i \text{ occurs}] = 1$. (I.e., some outcome will always occur.)

Note: Recall that $\|x\|^2$ for any vector x is $x^\dagger x$. And that $P^\dagger P = P$ for orthogonal projectors P. And that $(xy)^\dagger = y^\dagger x^\dagger$. Then take the formula for the measurement probability and just simplify.

(c)

Knowlets:	ProjMeas — ProblemID: 5PathsMeas
Time:	
Difficulty:	

In the situation of Homework 1, Problem **??** (**??**),[3] we measure whether there is a photon on path 3. Formulate this mathematically (i.e., as a projective measurement).

Note: You only need to formulate the measurement. You are not required to apply it (i.e., to compute probabilities and post-measurement states).

[1] We call this a *cat state* because of its similarity to Schrödinger's cat: A cat which is dead can be seen as consisting of n dead particles ($|\text{dead},\dots,\text{dead}\rangle$), and a living cat can be seen as consisting of n living particles ($|\text{alive},\dots,\text{alive}\rangle$). (This is of course a simplification!).

[2] This can be seen as an explanation what happens if we try to implement Schrödinger's cat: Even with a very high quality box, information about at least one atom of the cat will leak to the outside (i.e., it is measured whether the atom is "alive"). This has then an effect on the state of the whole cat.

[3] Reminder: "Consider a system in which a single photon may be sent through 5 different paths. The photon may be polarised in any direction. Give a Hilbert space for describing the state of this photon and give a natural basis for expressing this state. How do you write that the photon is 45°-polarised and on path 3?"

(d)

Knowlets:	ProjMeas ProblemID: 5PathsMultiMeas
Time:	
Difficulty:	

In the situation of Homework 1, Problem **??** (**??**),[4] we measure whether there is a photon on path 3. Formulate this mathematically (i.e., as a projective measurement).

Note: You only need to formulate the measurement. You are not required to apply it (i.e., to compute probabilities and post-measurement states).

2 Quantum circuits

(a)

Knowlets:	PauliX, Hada, UniTrafo ProblemID: CircXHXH
Time:	
Difficulty:	

What is the state after this quantum circuit?

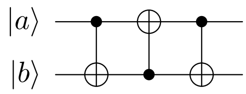
$$\begin{pmatrix}0\\1\end{pmatrix} \;-\boxed{X}-\boxed{H}-\boxed{X}-\boxed{H}-$$

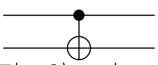
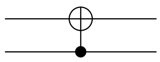
Note that $X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ is the bit flip, and $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$ is the Hadamard transform.

(b)

Knowlets:	UniTrafo, ComposQState, CNOT ProblemID: TripleCNOT
Time:	
Difficulty:	

What state comes out of the following circuit (for $a, b \in \{0, 1\}$)?



Here  denotes the controlled NOT, i.e., the operation defined by $\text{CNOT}|a, b\rangle = |a, a \oplus b\rangle$. (And  analogously denotes the operation mapping $|a, b\rangle$ to $|a \oplus b, b\rangle$.)

What useful (and simple) function does the above circuit perform?

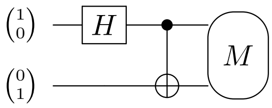
Note: Recall that $|a\rangle$ for some bit a simply stands for one of the computational basis vectors. E.g., $|a\rangle = |0\rangle = \binom{1}{0}$ if $a = 0$. And similarly, $|a, b\rangle$ stands for one of the four basis vectors of a 2-qubit system. E.g., $|a, b\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ if $a = 0$, $b = 1$.

[4]Reminder: "Consider a system in which each of 5 paths may contain a photon (or not), and each of these photons may be polarised in any direction. Give a Hilbert space for describing the state of these photons and give a natural basis for expressing this state. How do you write that there is a photon on path 3 that is 45°-polarised and no photons on the other paths?"

(c)

Knowlets:	ComposQState, ComposUni, Hada, CNOT, ComplMeas ProblemID: CircHCNOTM
Time:	
Difficulty:	

What are the possible outcomes of the measurement M? With which probabilities do they occur?



Here $-\!(M)$ is the complete measurement in the computational basis on the first and the second qubit.