Quantum Cryptography (spring 2021)	Dominique Unruh
Exercise Sheet 04	
Out: 2021-03-15	Due: 2021-03-23

1 Deutsch-Jozsa Algorithm

Assume that $f : \{0,1\}^n \to \{0,1\}$ is a function that satisfies one of the following two properties:

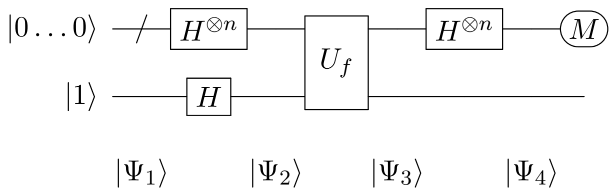
- f is constant (i.e., $f(x) = f(y)$ for all $x, y \in \{0,1\}^n$), or
- f is balanced (i.e., $|\{x : f(x) = 0\}| = |\{x : f(x) = 1\}| = 2^{n-1}$).

That is, we have the promise that f is constant or balanced, but we do not know which of the two holds.

Let U_f be the unitary transformation on $\mathbb{C}^{2^{n+1}}$ defined by

$$U_f|x, y\rangle = |x, y \oplus f(x)\rangle \qquad (x \in \{0,1\}^n, y \in \{0,1\}).$$

Consider the following circuit:



where M is a complete measurement in the computational basis.

The $|\Psi_i\rangle$ denote the intermediate states after the individual steps of the algorithm. E.g., $|\Psi_1\rangle = |0\dots01\rangle$.

(a)

Knowlets:	ComposUni	ProblemID: DeutschJozsaPsi2
Time:		
Difficulty:		

What is $|\Psi_2\rangle$?

(b)

Knowlets:	UniTrafo	ProblemID: DeutschJozsaPsi3
Time:		
Difficulty:		

Show that

$$|\Psi_3\rangle = \sum_{x \in \{0,1\}^n} 2^{-n/2-1/2}|x, f(x)\rangle - 2^{-n/2-1/2}|x, \overline{f(x)}\rangle.$$

(Here $\overline{f(x)} := 1 - f(x)$.)

(c)

Knowlets:	ComposQState ProblemID: DeutschJozsaPsi3T
Time:	
Difficulty:	

Show that

$$|\Psi_3\rangle = \Big(2^{-n/2} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}|x\rangle\Big) \otimes |-\rangle$$

Here $|-\rangle$ is short for $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.

(d)

Knowlets:	Tensor ProblemID: HadaN
Time:	
Difficulty:	

(Bonus problem) Show that $H^{\otimes n}|x\rangle = 2^{-n/2}\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle$ where $x \cdot z := \sum_{i=1}^n x_i z_i$.

(e)

Knowlets:	ComposUni ProblemID: DeutschJozsaPsi4
Time:	
Difficulty:	

What is $|\Psi_4\rangle$?

(f)

Knowlets:	ComplMeas, ComposQState ProblemID: DeutschJozsaProb
Time:	
Difficulty:	

Show that the probability P of measuring $0\dots0$ in the measurement is $\big(2^{-n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\big)^2$.

(g)

Knowlets:	ProblemID: DeutschJozsaConst
Time:	
Difficulty:	

Compute the probability P of measuring $0\dots0$ in the case that f is constant.

(h)

Knowlets:	ProblemID: DeutschJozsaBala
Time:	
Difficulty:	

Compute the probability P of measuring $0\dots0$ in the case that f is balanced.

2 Quantum State Probability Distributions and Density Operators

(a)

Knowlets:	QDistr, Density ProblemID: 3QSPD
Time:	
Difficulty:	

Consider the following quantum state probability distributions:

$$E_1 = \{|0\rangle @ \tfrac{1}{2},\ |+\rangle @ \tfrac{1}{2}\},$$
$$E_2 = \{|0\rangle @ \tfrac{1}{4},\ |1\rangle @ \tfrac{3}{4}\},$$
$$E_3 = \{|0\rangle @ \tfrac{1}{4},\ |1\rangle @ \tfrac{1}{4},\ |+\rangle @ \tfrac{1}{4}, |-\rangle @ \tfrac{1}{4}\}.$$

Compute the corresponding density operators ρ_1, ρ_2, ρ_3 as explicitly given matrices. (Note: $|+\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle := \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.)

(b)

Knowlets:	QDistr, QDistrU, Density — ProblemID: URandom
Time:	
Difficulty:	

Consider the following process: First, a random value $x \in \{0,1\}^n$ is chosen. Then an n-bit quantum register is prepared to have the value $|\Psi\rangle := |x\rangle$. Then a unitary transformation U is applied to Ψ. What is the density operator corresponding to the resulting quantum state probability distribution?

Hint: As the first step, consider the case that U is the identity.

(c)

Knowlets:	QDistr, QDistrM, Density — ProblemID: MeasureForget
Time:	
Difficulty:	

Let a measurement M consisting of projectors $P_1, \dots, P_n$ be given. Let a quantum state $|\Psi\rangle$ be given. Assume that $|\Psi\rangle$ is measured using M but the measurement outcome **is not recorded** (i.e., it is forgotten, erased). What is the quantum state probability distribution describing the state of the system after this experiment? What is the corresponding density operator?

Note: The formula in the lecture was for the case where the measurement outcome is **not** forgotten.

(d)

Knowlets:	QDistrM, QDistrM, Density — ProblemID: MeasureForgetD
Time:	
Difficulty:	

(Bonus problem) Assume a quantum system is in the state described by a density operator ρ. We apply a measurement M consisting of projectors $P_1, \dots, P_n$ to the system and forget the outcome. What is the density operator describing the resulting state of the system?