1 Bomb tester, improved

Consider a beam splitter that is parametrised by an angle $\theta$. This beam splitter performs the following operation $B_\theta$:

$\begin{align*}
B_\theta(\text{1}) &= \cos(\theta)\text{1} + \sin(\theta)\text{0} \\
B_\theta(\text{0}) &= -\sin(\theta)\text{1} + \cos(\theta)\text{0}
\end{align*}$

Here $\text{1}$ is a photon that is on the upper path, and $\text{0}$ a photon that is on the lower path.

(In other words, instead of reflecting half of the incoming light as did the beam splitter in the lecture, this beam splitter lets $(\sin \theta)^2$ of the light through and reflects $(\cos \theta)^2$.) Note that $B_{\pi \over 4}$ and $B_{-\pi \over 4}$ are the beam splitters described in the lecture.

Now consider the following setup: Let $n \in \mathbb{N}$. Fix $\theta := \pi \over 2n$. Take a photon and send it through the up-input of the beam splitter $B_\theta$ (i.e., the photon enters the beam splitter in state $\text{1}$). Then the photon exits the beam splitter in a superposition $\Psi_1$ between $\text{1}$ and $\text{0}$. Put the box with the bomb in the down-path. After passing (or not passing) the box, the photon is in a superposition $\Phi_1$ between $\text{1}$ and $\text{0}$ (which depends on whether there was a bomb in the box or not).

Now take the photon and send it into the beam splitter again (without destroying the superposition $\Phi_1$). The photon leaves the beam splitter in a superposition $\Psi_2$. Put the box in the down-path. The photon is in state $\Phi_2$. Etc.

After $n$ iterations, measure $\Phi_n$.

This can be done with the experimental setup described in Figure 1 where the mirrors (a) and (b) need to be switched away at the right moment to let the light go into or come out of the experiment at the right iterations.

For notational convenience, define $\Gamma_\alpha := \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$.

Assume that no bomb is in the box. Show that then $\Psi_j = \Phi_j = \Gamma_j \Phi$ for $j = 1, \ldots, n$. What is $\Gamma_n$? What is the probability of measuring $\text{0}$ after the experiment (i.e., for measuring $\Phi_n$ as $\text{0}$)?
For the following questions, assume that there is a bomb in the box. What is the value of $\Psi_1$? What is the probability that the bomb explodes when $\Psi_1$ passes through the box? What is the state $\Phi_1$ of the photon after the box was in its path (under the condition that the bomb does not explode)?

Show that the probability that the bomb does not explode in any of the $n$ iterations (i.e., that the state $\Psi_i$ will be measured as being in the up-path for each $i$) is $(\cos \theta)^{2n}$.

Assuming that the bomb does not explode, what is the state coming out of the experiment? With what probability do we measure $\Phi_n$ as $\left| \frac{1}{\sqrt{2}} \right>$?
(e) **Knowlets:** Bomb  
**ProblemID:** BombImproveTable  
**Time:**  
**Difficulty:**

Fill out the following table (in terms of $n$):

<table>
<thead>
<tr>
<th>Event</th>
<th>Probability if bomb</th>
<th>Probability if no bomb</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bomb explodes</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Photon is in up-path</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Photon is in down-path</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

For interpreting these results, note that for $n \to \infty$, we have that $(\cos \frac{\pi}{2n})^{2n} \to 1$.

## 2 Independent operations

(a) **Knowlets:** ComposUni, Tensor  
**ProblemID:** IndepU  
**Time:**  
**Difficulty:**

Show that the following two circuits perform the same unitary operation.

\[
\begin{array}{cc}
& U \\
/ & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cc}
& U \\
/ & \\
\end{array}
\]

By this we mean that in the first case, first $U$ is applied to the first system while nothing is done to the second, and then $V$ is applied to the second system while nothing is done to the first. In the second case, both operations are applied simultaneously.

(Note that this implies that on independent subsystems, it does not matter whether we first operate on the first and then the second, or vice versa.)

(b) **Knowlets:** Tensor, ComposMeas  
**ProblemID:** IndepM  
**Time:**  
**Difficulty:**

**Bonus question** Assume that the measurement $M_1$ is given by projectors $P_1, \ldots, P_n$ and that the measurement $M_2$ is given by projectors $Q_1, \ldots, Q_m$. Show that the following two circuits have the same effect. I.e., prove that for each $i, j$, the probability of getting the outcomes $i, j$ is the same in both circuits, and the state after performing the measurements is the same.

\[
\begin{array}{cc}
& M_1 \\
/ & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cc}
& M_1 \\
/ & \\
\end{array}
\]

\[
\begin{array}{cc}
& M_2 \\
/ & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cc}
& M_2 \\
/ & \\
\end{array}
\]
Explain (shortly) why Problem 2(a) and Problem 2(b) imply that one cannot use quantum mechanics to transfer information faster than light. (I.e., the only way to transfer information is to actually send something.)