Quantum Cryptography (spring 2022)	Dominique Unruh

Exercise Sheet 02

Out: 2022-03-02 Due: 2022-03-10

1 Bomb tester, improved

Consider a beam splitter that is parametrised by an angle θ. This beam splitter performs the following operation B_θ:

$$B_\theta \binom{1}{0} = \cos\theta \binom{1}{0} + \sin\theta \binom{0}{1}$$
$$B_\theta \binom{0}{1} = -\sin\theta \binom{1}{0} + \cos\theta \binom{0}{1}$$

Here $\binom{1}{0}$ is a photon that is on the upper path, and $\binom{0}{1}$ a photon that is on the lower path.

(In other words, instead of reflecting half of the incoming light as did the beam splitter in the lecture, this beam splitter lets $(\sin\theta)^2$ of the light through and reflects $(\cos\theta)^2$.) Note that $B_{\frac{\pi}{4}}$ and $B_{-\frac{\pi}{4}}$ are the beam splitters described in the lecture.

Now consider the following setup: Let $n \in \mathbb{N}$. Fix $\theta := \frac{\pi}{2n}$. Take a photon and send it through the up-input of the beam splitter B_θ (i.e., the photon enters the beam splitter in state $\binom{1}{0}$).

Then the photon exits the beam splitter in a superposition Ψ_1 between $\binom{1}{0}$ and $\binom{0}{1}$. Put the box with the bomb in the down-path. After passing (or not passing) the box, the photon is in a superposition Φ_1 between $\binom{1}{0}$ and $\binom{0}{1}$ (which depends on whether there was a bomb in the box or not).

Now take the photon and send it into the beam splitter again (without destroying the superposition Φ_1). The photon leaves the beam splitter in a superposition Ψ_2. Put the box in the down-path. The photon is in state Φ_2. Etc.

After n iterations, measure Φ_n.

This can be done with the experimental setup described in Figure 1 where the mirrors (a) and (b) need to be switched away at the right moment to let the light go into or come out of the experiment at the right iterations.

For notational convenience, define $\Gamma_\alpha := \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$.

(a)

Knowlets:	Bomb ProblemID: BombImproveNo
Time:	
Difficulty:	

Assume that no bomb is in the box. Show that then $\Psi_j = \Phi_j = \Gamma_{j\theta}$ for $j = 1, \ldots, n$. What is $\Gamma_{n\theta}$? What is the probability of measuring $\binom{0}{1}$ after the experiment (i.e., for measuring Φ_n as $\binom{0}{1}$)?

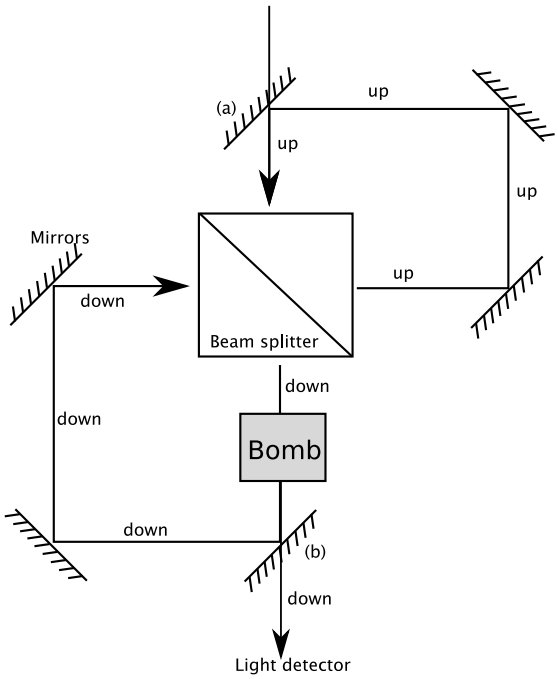


Figure 1: Bomb tester

(b)

Knowlets:	Bomb ProblemID: BombImproveOne
Time:	
Difficulty:	

For the following questions, assume that there is a bomb in the box. What is the value of Ψ_1? What is the probability that the bomb explodes when Ψ_1 passes through the box? What is the state Φ_1 of the photon after the box was in its path (under the condition that the bomb does not explode)?

(c)

Knowlets:	Bomb ProblemID: BombImprove3
Time:	
Difficulty:	

Show that the probability that the bomb does not explode in any of the n iterations (i.e., that the state Ψ_i will be measured as being in the up-path for each i) is $(\cos\theta)^{2n}$.

(d)

Knowlets:	Bomb ProblemID: BombImprove4
Time:	
Difficulty:	

Assuming that the bomb does not explode, what is the state coming out of the experiment? With what probability do we measure Φ_n as $\binom{1}{0}$?

(e)

Knowlets:	Bomb ProblemID: BombImproveTable
Time:	
Difficulty:	

Fill out the following table (in terms of n):

Event	**Probability if bomb**	**Probability if no bomb**
Bomb explodes		0
Photon is in up-path		
Photon is in down-path		

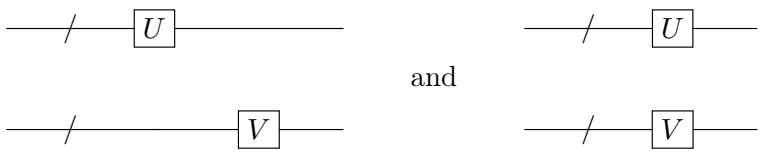
For interpreting these results, note that for $n \to \infty$, we have that $(\cos \frac{\pi}{2n})^{2n} \to 1$.

2 Independent operations

(a)

Knowlets:	ComposUni, Tensor ProblemID: IndepU
Time:	
Difficulty:	

Show that the following two circuits perform the same unitary operation.



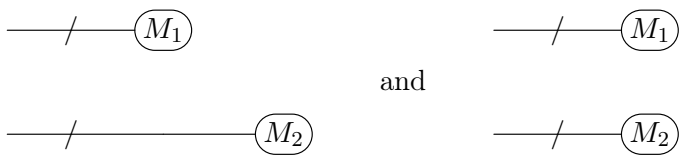
By this we mean that in the first case, first U is applied to the first system while nothing is done to the second, and then V is applied to the second system while nothing is done to the first. In the second case, both operations are applied simultaneously.

(Note that this implies that on independent subsystems, it does not matter whether we first operate on the first and then the second, or vice versa.)

(b)

Knowlets:	Tensor, ComposMeas ProblemID: IndepM
Time:	
Difficulty:	

(Bonus question) Assume that the measurement M_1 is given by projectors $P_1, \ldots, P_n$ and that the measurement M_2 is given by projectors $Q_1, \ldots, Q_m$. Show that the following two circuits have the same effect. I.e., prove that for each i, j, the probability of getting the outcomes i, j is the same in both circuits, and the state after performing the measurements is the same.



(c)

Knowlets:	ProblemID: NoFTL
Time:	
Difficulty:	

Explain (shortly) why Problem 2 (a) and Problem 2 (b) imply that one cannot use quantum mechanics to transfer information faster than light. (I.e., the only way to transfer information is to actually send something.)