Quantum Cryptography (spring 2023)	Dominique Unruh

Exercise Sheet 01

Out: 2023-02-13 Due: 2023-02-20

1 Working with quantum states

(a)

Knowlets:	QState ProblemID: QState
Time:	
Difficulty:	

Which of the following are valid quantum states:

$$\begin{pmatrix}0\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}, \quad \frac{1}{\sqrt{3}}\begin{pmatrix}1\\2\end{pmatrix}, \quad \begin{pmatrix}\sqrt{\frac{2}{3}}\\ \frac{i}{\sqrt{3}}\end{pmatrix}$$

(b)

Knowlets:	CBMeas ProblemID: CBMeas
Time:	
Difficulty:	

For each of the *valid* quantum states from Problem 1 (a), answer the following: You perform a measurement (i.e., you ask "whether the state is a classical 0 or a classical 1). What is the probability of answer 0 (i.e., yes), what is the probability of answer 1 (i.e., no)? What is the state after the measurement in each of those cases?

(c)

Knowlets:	ComplMeas ProblemID: ComplMeas
Time:	
Difficulty:	

For each of the *valid* quantum states from Problem 1 (a), answer the following: You perform a measurement in basis $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$ (we call the corresponding outcomes "+" and "−"). What is the probability of answer +, what is the probability of answer −? What is the state after the measurement in each of those cases?

(d)

Knowlets:	ComplMeas ProblemID: MeasSum1
Time:	
Difficulty:	

Let a quantum state $\psi \in \mathbb{C}^2$ and an (orthonormal) measurement basis $\phi_{yes}, \phi_{no} \in \mathbb{C}^2$ be given. Measure ψ in that measurement basis. Let P_{yes} be the probability of outcome yes, and P_{no} the probability of outcome no. Show that $P_{\text{yes}} + P_{\text{no}} = 1$.

(e)

Knowlets:	UniTrafo ProblemID: UTInv
Time:	
Difficulty:	

Show that by applying a unitary transformation to a quantum state, no information is ever lost. More exactly, assume that a unitary transformation U is applied to a given quantum state Ψ, resulting in a state Φ. Then show that there is another unitary transformation V (not depending on Ψ or Φ) such that applying V to Φ gives Ψ again.

(f)

Knowlets:	ComplMeas ProblemID: RepeatMeas
Time:	
Difficulty:	

What is wrong with the following approach:

Alice has a qubit $|\Psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$. They want to initialise the qubit to $\begin{pmatrix}1\\0\end{pmatrix}$. They know that when measuring Ψ, with probability $\frac{1}{2}$ they get the measurement outcome 0 and the qubit will be in state $\begin{pmatrix}1\\0\end{pmatrix}$. Thus they repeatedly measure the qubit in the computational basis until they get the outcome 0. Since the probability is $\frac{1}{2}$ each time, the expected number of measurements until they get their $\begin{pmatrix}1\\0\end{pmatrix}$-initialised qubit is 2.

(g)

Knowlets:	UniTrafo ProblemID: UniTrafo
Time:	
Difficulty:	

Which of the following are valid (unitary) transformations:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}i & -1\\ i & 1\end{pmatrix}.$$

(h)

Knowlets:	QState ProblemID: 5Paths
Time:	
Difficulty:	

Consider a system in which a single photon may be sent through 5 different paths. The photon may be polarised in any direction. Give a Hilbert space for describing the state of this photon and give a natural basis for expressing this state. How do you write that the photon is 45°-polarised and on path 3?

(i)

Knowlets:	QState ProblemID: 5PathsMulti
Time:	
Difficulty:	

Consider a system in which each of 5 paths may contain a photon (or not), and each of these photons may be polarised in any direction. Give a Hilbert space for describing the state of these photons and give a natural basis for expressing this state. How do you write that there is a photon on path 3 that is 45°-polarised and no photons on the other paths?

2 Bomb tester, improved

Consider a beam splitter that is parametrised by an angle θ. This beam splitter performs the following operation B_θ:

$$B_\theta\tbinom{1}{0} = \cos\theta\tbinom{1}{0} + \sin\theta\tbinom{0}{1}$$
$$B_\theta\tbinom{0}{1} = -\sin\theta\tbinom{1}{0} + \cos\theta\tbinom{0}{1}$$

Here $\binom{1}{0}$ is a photon that is on the upper path, and $\binom{0}{1}$ a photon that is on the lower path.

(In other words, instead of reflecting half of the incoming light as did the beam splitter in the lecture, this beam splitter lets $(\sin\theta)^2$ of the light through and reflects $(\cos\theta)^2$.) Note that $B_{\frac{\pi}{4}}$ and $B_{-\frac{\pi}{4}}$ are the beam splitters described in the lecture.

Now consider the following setup: Let $n \in \mathbb{N}$. Fix $\theta := \frac{\pi}{2n}$. Take a photon and send it through the up-input of the beam splitter B_θ (i.e., the photon enters the beam splitter in state $\binom{1}{0}$).

Then the photon exits the beam splitter in a superposition Ψ_1 between $\binom{1}{0}$ and $\binom{0}{1}$. Put the box with the bomb in the down-path. After passing (or not passing) the box, the photon is in a superposition Φ_1 between $\binom{1}{0}$ and $\binom{0}{1}$ (which depends on whether there was a bomb in the box or not).

Now take the photon and send it into the beam splitter again (without destroying the superposition Φ_1). The photon leaves the beam splitter in a superposition Ψ_2. Put the box in the down-path. The photon is in state Φ_2. Etc.

After n iterations, measure Φ_n.

This can be done with the experimental setup described in Figure 1 where the mirrors (a) and (b) need to be switched away at the right moment to let the light go into or come out of the experiment at the right iterations.

For notational convenience, define $\Gamma_\alpha := \begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}$.

(a)

Knowlets:	Bomb — ProblemID: BombImproveNo
Time:	
Difficulty:	

Assume that no bomb is in the box. Show that then $\Psi_j = \Phi_j = \Gamma_{j\theta}$ for $j = 1, \dots, n$. What is $\Gamma_{n\theta}$? What is the probability of measuring $\binom{0}{1}$ after the experiment (i.e., for measuring Φ_n as $\binom{0}{1}$)?

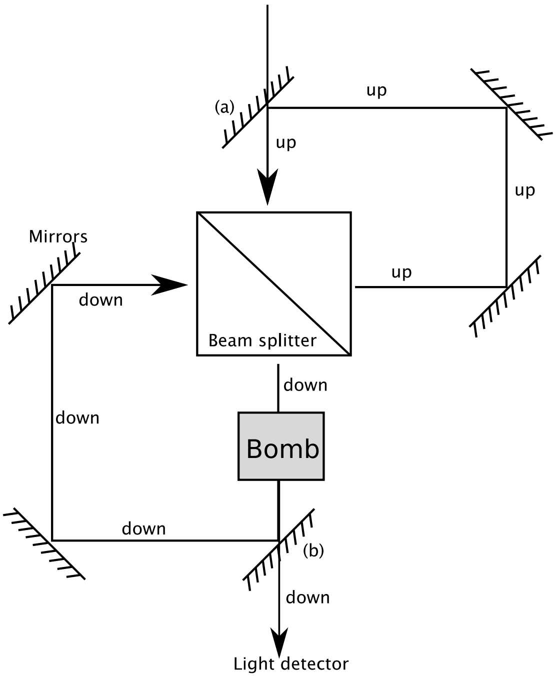

Figure 1: Bomb tester

(b)

Knowlets:	Bomb ProblemID: BombImproveOne
Time:	
Difficulty:	

For the following questions, assume that there is a bomb in the box. What is the value of Ψ_1? What is the probability that the bomb explodes when Ψ_1 passes through the box? What is the state Φ_1 of the photon after the box was in its path (under the condition that the bomb does not explode)?

(c)

Knowlets:	Bomb ProblemID: BombImprove3
Time:	
Difficulty:	

Show that the probability that the bomb does not explode in any of the n iterations (i.e., that the state Ψ_i will be measured as being in the up-path for each i) is $(\cos\theta)^{2n}$.

(d)

Knowlets:	Bomb ProblemID: BombImprove4
Time:	
Difficulty:	

Assuming that the bomb does not explode, what is the state coming out of the experiment? With what probability do we measure Φ_n as $\binom{1}{0}$?

(e)

Knowlets:	Bomb ProblemID: BombImproveTable
Time:	
Difficulty:	

Fill out the following table (in terms of n):

Event	**Probability if bomb**	**Probability if no bomb**
Bomb explodes		0
Photon is in up-path		
Photon is in down-path		

For interpreting these results, note that for $n \to \infty$, we have that $(\cos \frac{\pi}{2n})^{2n} \to 1$.