Quantum Cryptography (spring 2023)	Dominique Unruh
Exercise Sheet 03	
Out: 2023-02-27	Due: 2023-03-06

1 Quantum State Probability Distributions and Density Operators

(a)

Knowlets:	QDistr, Density — ProblemID: 3QSPD
Time:	
Difficulty:	

Consider the following quantum state probability distributions:

$$\begin{aligned} E_1 &= \{|0\rangle @ \tfrac{1}{2},\ |+\rangle @ \tfrac{1}{2}\}, \\ E_2 &= \{|0\rangle @ \tfrac{1}{4},\ |1\rangle @ \tfrac{3}{4}\}, \\ E_3 &= \{|0\rangle @ \tfrac{1}{4},\ |1\rangle @ \tfrac{1}{4},\ |+\rangle @ \tfrac{1}{4}, |-\rangle @ \tfrac{1}{4}\}. \end{aligned}$$

Compute the corresponding density operators ρ_1, ρ_2, ρ_3 as explicitly given matrices. (Note: $|+\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle := \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.)

(b)

Knowlets:	QDistr, QDistrU, Density — ProblemID: URandom
Time:	
Difficulty:	

Consider the following process: First, a random value $x \in \{0,1\}^n$ is chosen. Then an n-bit quantum register is prepared to have the value $|\Psi\rangle := |x\rangle$. Then a unitary transformation U is applied to Ψ. What is the density operator corresponding to the resulting quantum state probability distribution?

Hint: As the first step, consider the case that U is the identity.

(c)

Knowlets:	QDistr, QDistrM, Density — ProblemID: MeasureForget
Time:	
Difficulty:	

Let a measurement M consisting of projectors $P_1, \dots, P_n$ be given. Let a quantum state $|\Psi\rangle$ be given. Assume that $|\Psi\rangle$ is measured using M but the measurement outcome **is not recorded** (i.e., it is forgotten, erased). What is the quantum state probability distribution describing the state of the system after this experiment? What is the corresponding density operator?

Note: The formula in the lecture was for the case where the measurement outcome is **not** forgotten.

(d)	**Knowlets:**	QDistrM, QDistrM, Density ProblemID: MeasureForgetD
	Time:	
	Difficulty:	

Assume a quantum system is in the state described by a density operator ρ. We apply a measurement M consisting of projectors $P_1, \dots, P_n$ to the system and forget the outcome. What is the density operator describing the resulting state of the system?

(e)	**Knowlets:**	QDistr, Density, PhysInd, DensityPhysInd ProblemID: PhysIndBellIndep
	Time:	
	Difficulty:	

Consider the following experiments:

- Experiment A: A two-qubit system is initialised with probability $\frac{1}{2}$ to be in the state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ and with probability $\frac{1}{2}$ to be in the state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$.
- Experiment B: A uniformly random bit r is chosen, and then both qubits are individually prepared to be in the same state $|r\rangle$.

Note that in experiment A, we have entanglement: The state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ cannot be written in the form $|\Psi_1\rangle \otimes |\Psi_2\rangle$ (same for $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$). On the other hand, in experiment B, in each of the two cases $r = 0$ and $r = 1$, a state is prepared that is separable (of the form $|\Psi_1\rangle \otimes |\Psi_2\rangle$).

Show that the states produced in the two experiments are physically indistinguishable.

(f)	**Knowlets:**	QDistr, Density, PhysInd, DensityPhysInd ProblemID: GlobalPh
	Time:	
	Difficulty:	

In the lecture, we mentioned several times that a global phase, i.e., a factor $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ in front of a quantum state, is physically irrelevant.

Demonstrate this by showing that the two states $|\Psi\rangle$ and $\varphi|\Psi\rangle$ are physically indistinguishable.[1]

2 Physical indistinguishability – the opposite direction (bonus problem)

Knowlets:	QDistr, Density, ProjMeas, DensityM ProblemID: PhysIndReverse
Time:	
Difficulty:	

Let E_1 and E_2 be quantum state probability distributions with density matrices ρ_1 and ρ_2. Assume that $\rho_1 \neq \rho_2$. Prove that E_1 and E_2 are physically distinguishable by specifying a measurement $M = \{Q_{\text{yes}}, Q_{\text{no}}\}$ with the following property: When measuring

[1] More precisely, that the quantum state probability distributions $\{|\Psi\rangle@1\}$ and $\{\varphi|\Psi\rangle@1\}$ are physically indistinguishable.

E_1 and E_2 with M, we get the outcome yes with different probabilities P_1 and P_2 (where $P_i :=$ Pr[Outcome is yes when measuring ρ_i]).

Hint: Consider the matrix $\sigma := \rho_1 - \rho_2$. Show that σ is diagonalisable and that it therefore has an eigenvector $|\Psi\rangle$ with eigenvalue $\lambda \neq 0$. Set $Q_{\text{yes}} := |\Psi\rangle\langle\Psi|$. You may use without proof the fact that a density operator is always Hermitean and nonzero.