Andmete esitamine λ-arvutuses

- Baaskombinaatorid

$$\begin{array}{lcl} \mathbf{I} & \equiv & \lambda x.\ x \\ \mathbf{K} & \equiv & \lambda x\, y.\ x \\ \mathbf{S} & \equiv & \lambda f\ g\ x.\ f\ x\ (g\ x) \end{array}$$

- "Astendamine"

$$\begin{array}{lcl} E^0 E' & \equiv & E' \\ E^n E' & \equiv & \underbrace{E(E(\ldots(E}_{n\ \text{tükki}}\ E')\ldots)) \end{array}$$

- *NB!* $E^n(E\ E') \ \equiv\ E^{n+1}E' \ \equiv\ E\ (E^n E')$

Tõeväärtused

- Spetsifikatsioon

$$\begin{array}{lcl} \mathbf{not\ T} & = & \mathbf{F} \\ \mathbf{not\ F} & = & \mathbf{T} \end{array}$$

- Definitsioon

$$\begin{array}{lcl} \mathbf{T} & \equiv & \lambda xy.\ x \qquad (\equiv \quad \mathbf{K}) \\ \mathbf{F} & \equiv & \lambda xy.\ y \\ \mathbf{not} & \equiv & \lambda t.\ t\ \mathbf{F}\ \mathbf{T} \end{array}$$

- Näide:

$$\begin{array}{lcl} \mathbf{not\ T} & \equiv & (\lambda t.\ t\ \mathbf{F}\ \mathbf{T})\ \mathbf{T} \\ & \rightarrow & \mathbf{T}\ \mathbf{F}\ \mathbf{T} \\ & \equiv & (\lambda x.\ \lambda y.\ x)\ \mathbf{F}\ \mathbf{T} \\ & \rightarrow & (\lambda y.\ \mathbf{F})\ \mathbf{T} \\ & \rightarrow & \mathbf{F} \end{array}$$

Tingimuslause

- Spetsifikatsioon

$$\begin{aligned}\textbf{cond}\ \mathbf{T}\ E_1\ E_2 &= E_1\\ \textbf{cond}\ \mathbf{F}\ E_1\ E_2 &= E_2\end{aligned}$$

- Definitsioon

$$\textbf{cond} \equiv \lambda t\ x\ y.\ t\ x\ y$$

- Näide:

$$\begin{aligned}\textbf{cond}\ \mathbf{F}\ E_1\ E_2 &\equiv (\lambda t\ x\ y.\ t\ x\ y)\ \mathbf{F}\ E_1\ E_2\\ &\twoheadrightarrow \mathbf{F}\ E_1\ E_2\\ &\equiv (\lambda x\ y.\ y)\ E_1\ E_2\\ &\twoheadrightarrow E_2\end{aligned}$$

Paarid ja ennikud

- Paarid

$$\begin{aligned}\textbf{fst} &\equiv \lambda p.\ p\ \mathbf{T}\\ \textbf{snd} &\equiv \lambda p.\ p\ \mathbf{F}\\ (E_1, E_2) &\equiv \lambda f.\ f\ E_1\ E_2\end{aligned}$$

- Ennikud

$$\begin{aligned}(E_1, \ldots, E_n) &\equiv (E_1, (\ldots (E_{n-1}, E_n) \ldots))\\ E \overset{n}{\downarrow} 1 &\equiv \textbf{fst}\ E\\ E \overset{n}{\downarrow} 2 &\equiv \textbf{fst}\ (\textbf{snd}\ E)\\ &\ldots\\ E \overset{n}{\downarrow} i &\equiv \textbf{fst}\ (\textbf{snd}^{i-1} E)\\ &\ldots\\ E \overset{n}{\downarrow} n &\equiv \textbf{snd}^{n-1}\ E\end{aligned}$$

Naturaalarvud

- Standardnumbrid

$$
\begin{array}{lcll}
\ulcorner 0 \urcorner & \equiv & \lambda x.\, x & (\equiv \quad \mathbf{I}) \\
\ulcorner n{+}1 \urcorner & \equiv & (\mathbf{F}, \ulcorner n \urcorner) & \\
\mathbf{succ} & \equiv & \lambda n.\, (\mathbf{F}, n) & \\
\mathbf{pred} & \equiv & \lambda n.\, n\, \mathbf{F} & (\equiv \quad \mathbf{snd}) \\
\mathbf{iszero} & \equiv & \lambda n.\, n\, \mathbf{T} & (\equiv \quad \mathbf{fst})
\end{array}
$$

- Liitmine (?!)

$$\mathbf{add} \quad = \quad \lambda x\, y.\, \mathbf{cond}(\mathbf{iszero}\, x)\; y\; (\mathbf{add}(\mathbf{pred}\, x)(\mathbf{succ}\, y))$$

Naturaalarvud

- Church'i numbrid

$$
\begin{array}{lcl}
\underline{n} & \equiv & \lambda f\, x.\, f^n\, x \\
\mathbf{succ} & \equiv & \lambda n.\, \lambda f\, x.\, n\, f\, (f\, x) \\
\mathbf{iszero} & \equiv & \lambda n.\, n\, (\lambda x.\, \mathbf{F})\, \mathbf{T} \\
\mathbf{add} & \equiv & \lambda m\, n.\, \lambda f\, x.\, m\, f\, (n\, f\, x)
\end{array}
$$

- Näide

$$
\begin{array}{lcl}
\mathbf{add}\, \underline{2}\, \underline{1} & \equiv & (\lambda m\, n.\, \lambda f\, x.\, m\, f\, (n\, f\, x))\, \underline{2}\, \underline{1} \\
 & \twoheadrightarrow & \lambda f\, x.\, \underline{2}\, f\, (\underline{1}\, f\, x) \\
 & \twoheadrightarrow & \lambda f\, x.\, f\, (f\, (\underline{1}\, f\, x)) \\
 & \twoheadrightarrow & \lambda f\, x.\, f\, (f\, (f\, x)) \\
 & \equiv & \underline{3}
\end{array}
$$

- Korrutamine ja astendamine

$$
\begin{array}{lcl}
\mathbf{mul} & \equiv & \lambda m\, n.\, \lambda f\, x.\, m\, (n\, f)\, x \\
\mathbf{exp} & \equiv & \lambda m\, n.\, \lambda f\, x.\, n\, m\, f\, x
\end{array}
$$

Naturaalarvud

- Ühe lahutamine — abifunktsiooni spetsifikatsioon

$$\begin{array}{lcl} \textbf{prefn}\ f\ (\mathbf{T},x) & = & (\mathbf{F},x) \\ \textbf{prefn}\ f\ (\mathbf{F},x) & = & (\mathbf{F},f\ x) \\ (\textbf{prefn}\ f)^n\ (\mathbf{F},x) & = & (\mathbf{F},f^n\ x) \\ (\textbf{prefn}\ f)^n\ (\mathbf{T},x) & = & (\mathbf{F},f^{n-1}\ x) \end{array}$$

- Ühe lahutamine — definitsioon

$$\begin{array}{lcl} \textbf{prefn} & \equiv & \lambda f\ p.\ (\mathbf{F},(\textbf{cond}\ (\textbf{fst}\ p)\ (\textbf{snd}\ p)\ (f\ (\textbf{snd}\ p)))) \\ \textbf{pred} & \equiv & \lambda n.\ \lambda f\ x.\ \textbf{snd}\ (n\ (\textbf{prefn}\ f)\ (\mathbf{T},x)) \end{array}$$

- Näide

$$\begin{array}{lcl} \textbf{pred}\ \underline{n}\ f\ x & = & \textbf{snd}\ (\underline{n}\ (\textbf{prefn}\ f)\ (\mathbf{T},x)) \\ & = & \textbf{snd}\ ((\textbf{prefn}\ f)^n\ (\mathbf{T},x)) \\ & = & \textbf{snd}\ (\mathbf{F},f^{n-1}\ x) \\ & = & f^{n-1}\ x \end{array}$$

Listid

- Definitsioon

$$\begin{array}{lcll} \textbf{nil} & \equiv & \lambda z.\ z & (\equiv\ \mathbf{I}) \\ \textbf{cons} & \equiv & \lambda x\ y.\ (\mathbf{F},(x,y)) & \\ \textbf{null} & \equiv & \lambda z.\ z\ \mathbf{T} & (\equiv\ \textbf{fst}) \\ \textbf{hd} & \equiv & \lambda z.\ \textbf{fst}\ (\textbf{snd}\ z)) & \\ \textbf{tl} & \equiv & \lambda z.\ \textbf{snd}\ (\textbf{snd}\ z)) & \end{array}$$

- Näide

$$\begin{array}{lcl} \textbf{null nil} & \equiv & \textbf{fst}\ (\lambda z.\ z) \\ & \equiv & (\lambda p.\ p\ \mathbf{T})\ (\lambda z.\ z) \\ & \twoheadrightarrow & \mathbf{T} \end{array}$$

Püsipunktid

- *Püsipuktiteoreem:*

$$\forall F.\ \exists X.\ X = F\,X$$

- Termi M nimetatakse *püsipunkti kombinaatoriks* kui

$$\forall F.\ M\,F = F\,(M\,F)$$

- Curry "paradoksaalne" kombinaator

$$\mathbf{Y} \equiv \lambda f.(\lambda x.f(x\,x))\,(\lambda x.f(x\,x))$$

- "Tugev" püsipunkti kombinaator

$$\Theta \equiv (\lambda xy.\ y(xxy))\,(\lambda xy.\ y(xxy))$$

Püsipunktid

- *Lemma:* Olgu $G \equiv \lambda yf.f(yf) \qquad (\equiv \mathbf{SI})$

$$M \text{ on püsipunkti kombinaator} \quad \Longleftrightarrow \quad M = GM$$

- *Tõestus:*

($\Leftarrow$) Kui $M = GM$, siis:

$$\begin{array}{rcl} \forall F.\ M\,F & = & G\,M\,F \\ & \equiv & (\lambda yf.f(yf))\,M\,F \\ & = & F(M\,F) \end{array}$$

($\Rightarrow$) Kui M on püsipunkti kombinaator, siis:

$$\begin{array}{rcl} G\,M & = & \lambda f.f(Mf) \\ & = & \lambda f.Mf \\ & = & M \end{array}$$

Püsipunktid

- *Lemma:* Kõik jada

$$\begin{array}{rcl} \mathbf{Y}^0 & \equiv & \mathbf{Y} \\ \mathbf{Y}^{n+1} & \equiv & \mathbf{Y}^n\,G \end{array}$$

  elemendid on püsipunkti kombinaatorid.

- *Tõestus:*

$$\begin{array}{rcl} \mathbf{Y}^{n+1} & \equiv & \mathbf{Y}^n\,G \\ & = & G\,(\mathbf{Y}^n\,G) \\ & \equiv & G\,\mathbf{Y}^{n+1} \end{array}$$

- NB! $\mathbf{Y}^1 \twoheadrightarrow \Theta$

Rekursioon

- *Lemma:* Olgu $C \equiv C(f,\vec{x})$, siis
  - $\exists F.\ \forall\vec{N}.\ F\,\vec{N} = C(F,\vec{N})$
  - $\exists F.\ \forall\vec{N}.\ F\,\vec{N} \twoheadrightarrow C(F,\vec{N})$
- *Tõestus:* Võtame $F \equiv \Theta(\lambda f\vec{x}.C(f,\vec{x}))$
- Standardnumbrite liitmine

$$\begin{array}{rcl} \mathbf{add} & = & \lambda x\,y.\ \mathbf{cond}\,(\mathbf{iszero}\ x)\ y\ (\mathbf{add}(\mathbf{pred}\ x)(\mathbf{succ}\ y)) \\ \mathbf{add} & \equiv & \mathbf{Y}\,(\lambda f\ x\ y.\ \mathbf{cond}\,(\mathbf{iszero}\ x)\ y\ (f\,(\mathbf{pred}\ x)(\mathbf{succ}\ y))) \end{array}$$

Mitmekohalised funktsioonid

- "currymine"

$$\begin{array}{lcl}\mathbf{curry}_n & \equiv & \lambda f\ x_1\ \ldots\ x_n.\ f\ (x_1,\ldots,x_n)\\ \mathbf{uncurry}_n & \equiv & \lambda f\ p.\ f\ (p \overset{n}{\downarrow} 1)\ \ldots\ (p \overset{n}{\downarrow} n)\end{array}$$

- *NB!* $\mathbf{curry}_n(\mathbf{uncurry}_n\ N) = N$ $\mathbf{uncurry}_n(\mathbf{curry}_n\ M) = M$

- *Üldistatud* λ*-abstraktsioon*

$$\lambda(V_1,\ldots,V_n).\ E \quad \equiv \quad \mathbf{uncurry}_n\ (\lambda V_1\ \ldots\ V_n.\ E)$$

- *Üldistatud* β*-konversioon*

$$(\lambda(V_1,\ldots,V_n).\ E)(E_1,\ldots,E_n) \quad \to_\beta \quad E[E_1\ldots E_n/V_1\ldots V_n]$$

Rekursioon

- Vastastikune rekursioon

$$\begin{array}{lcl} f_1 & = & F_1\ f_1\ldots f_n\\ f_2 & = & F_2\ f_1\ldots f_n\\ & \ldots & \\ f_n & = & F_n\ f_1\ldots f_n\end{array}$$

- Definitsioon

$$\begin{array}{lcl} f & \equiv & \mathbf{Y}(\lambda(f_1,\ldots,f_n).\ (F_1 f_1\ldots f_n,\ \ldots,\ F_n f_1\ldots f_n))\\ f_1 & \equiv & f \overset{n}{\downarrow} 1\\ f_2 & \equiv & f \overset{n}{\downarrow} 2\\ & \ldots & \\ f_n & \equiv & f \overset{n}{\downarrow} n\end{array}$$

Arvutatavus

- *Church'i tees:* Kõik arvutatavad funktsioonid on esitatavad λ-arvutuses!
- *Primitiivrekursiivsed funktsioonid:*

(i) 0

(ii) $S(x) = x + 1$

(iii) $U_n^i(x_1, x_2, \ldots, x_n) = x_i$

(iv) $f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_r(x_1, \ldots, x_n))$

(v) $$\begin{aligned} f(0, x_2, \ldots, x_n) &= g(x_2, \ldots, x_n) \\ f(S(x_1), x_2, \ldots, x_n) &= h(f(x_1, x_2, \ldots, x_n), x_2, \ldots, x_n) \end{aligned}$$

Arvutatavus

- *Osaliselt rekursiivsed funktsioonid*

(vi) $f(x_1, \ldots, x_n) = \min \{ \ y \mid g(y, x_2, \ldots, x_n) = x_1 \ \}$

- *Minimiseerimine*

$$\begin{aligned} \mathbf{min}\, x\ f\ (x_1, \ldots, x_n) \quad = \quad & \mathbf{cond}\ (\mathbf{eq}\ (f(x, x_2, \ldots, x_n)\ x_1)\ \ x \\ & \qquad (\mathbf{min}\ (\mathbf{succ}\, x)\ f\ (x_1, \ldots, x_n)) \end{aligned}$$

- Definitsioon

$$\begin{aligned} \mathbf{min} \quad \equiv \quad \mathbf{Y}\ (\lambda m x f\, (x_1, \ldots, x_n).\ & \mathbf{cond}\ (\mathbf{eq}\ (f(x, x_2, \ldots, x_n))\ x_1)\ \ x \\ & \qquad (m(\mathbf{succ}\, x)\ f\ (x_1, \ldots, x_n))) \end{aligned}$$