Andmete esitamine λ-arvutuses

- Baaskombinaatorid

$$\begin{aligned}\mathbf{I} &\equiv \lambda x.\, x\\ \mathbf{K} &\equiv \lambda x\, y.\, x\\ \mathbf{S} &\equiv \lambda f\, g\, x.\, f\, x\, (g\, x)\end{aligned}$$

- “Astendamine”

$$\begin{aligned}E^0 E' &\equiv E'\\ E^n E' &\equiv \underbrace{E\,(E\,(\ldots(E}_{n \text{ tükki}}\; E')\ldots))\end{aligned}$$

- NB! $E^n(E\;E') \equiv E^{n+1}E' \equiv E\;(E^n E')$

Tõeväärtused

- Spetsifikatsioon

$$\begin{aligned} \textbf{not}\ \mathbf{T} &= \mathbf{F} \\ \textbf{not}\ \mathbf{F} &= \mathbf{T} \end{aligned}$$

- Definitsioon

$$\begin{aligned} \mathbf{T} &\equiv \lambda xy.\, x \qquad (\equiv \mathbf{K}) \\ \mathbf{F} &\equiv \lambda xy.\, y \\ \textbf{not} &\equiv \lambda t.\, t\ \mathbf{F}\ \mathbf{T} \end{aligned}$$

- Näide:

$$\begin{aligned} \textbf{not}\ \mathbf{T} &\equiv (\lambda t.\, t\ \mathbf{F}\ \mathbf{T})\ \mathbf{T} \\ &\rightarrow \mathbf{T}\ \mathbf{F}\ \mathbf{T} \\ &\equiv (\lambda x.\, \lambda y.\, x)\ \mathbf{F}\ \mathbf{T} \\ &\rightarrow (\lambda y.\, \mathbf{F})\ \mathbf{T} \\ &\rightarrow \mathbf{F} \end{aligned}$$

Tingimuslause

- Spetsifikatsioon

$$\begin{aligned} \mathbf{cond}\ \mathbf{T}\ E_1\ E_2 &= E_1 \\ \mathbf{cond}\ \mathbf{F}\ E_1\ E_2 &= E_2 \end{aligned}$$

- Definitsioon

$$\mathbf{cond} \equiv \lambda t\ x\ y.\ t\ x\ y$$

- Näide:

$$\begin{aligned} \mathbf{cond}\ \mathbf{F}\ E_1\ E_2 &\equiv (\lambda t\ x\ y.\ t\ x\ y)\ \mathbf{F}\ E_1\ E_2 \\ &\twoheadrightarrow \mathbf{F}\ E_1\ E_2 \\ &\equiv (\lambda x\ y.\ y)\ E_1\ E_2 \\ &\twoheadrightarrow E_2 \end{aligned}$$

Paarid ja ennikud

- Paarid

$$\begin{aligned} \mathbf{fst} &\equiv \lambda p.\ p\ \mathbf{T} \\ \mathbf{snd} &\equiv \lambda p.\ p\ \mathbf{F} \\ (E_1, E_2) &\equiv \lambda f.\ f\ E_1\ E_2 \end{aligned}$$

- Ennikud

$$\begin{aligned} (E_1, \ldots, E_n) &\equiv (E_1, (\ldots (E_{n-1}, E_n) \ldots)) \\ E \overset{n}{\downarrow} 1 &\equiv \mathbf{fst}\ E \\ E \overset{n}{\downarrow} 2 &\equiv \mathbf{fst}\ (\mathbf{snd}\ E) \\ &\ldots \\ E \overset{n}{\downarrow} i &\equiv \mathbf{fst}\ (\mathbf{snd}^{i-1} E) \\ &\ldots \\ E \overset{n}{\downarrow} n &\equiv \mathbf{snd}^{n-1}\ E \end{aligned}$$

Naturaalarvud

- Standardnumbrid

$$\begin{array}{llll} \ulcorner 0 \urcorner & \equiv & \lambda x.\, x & (\equiv \quad \mathbf{I}) \\ \ulcorner n{+}1 \urcorner & \equiv & (\mathbf{F}, \ulcorner n \urcorner) & \\ \mathbf{succ} & \equiv & \lambda n.\, (\mathbf{F}, n) & \\ \mathbf{pred} & \equiv & \lambda n.\, n\, \mathbf{F} & (\equiv \quad \mathbf{snd}) \\ \mathbf{iszero} & \equiv & \lambda n.\, n\, \mathbf{T} & (\equiv \quad \mathbf{fst}) \end{array}$$

- Liitmine (?!)

$$\mathbf{add} \quad = \quad \lambda x\, y.\, \mathbf{cond}(\mathbf{iszero}\; x)\; y\; (\mathbf{add}(\mathbf{pred}\; x)(\mathbf{succ}\; y))$$

Naturaalarvud

- Church'i numbrid

$$
\begin{array}{lcl}
\underline{n} & \equiv & \lambda f\,x.\ f^n\ x \\
\textbf{succ} & \equiv & \lambda n.\ \lambda f\,x.\ n\ f\ (f\ x) \\
\textbf{iszero} & \equiv & \lambda n.\ n\ (\lambda x.\ \textbf{F})\ \textbf{T} \\
\textbf{add} & \equiv & \lambda m\,n.\ \lambda f\,x.\ m\ f\ (n\ f\ x)
\end{array}
$$

- Näide

$$
\begin{array}{lcl}
\textbf{add}\ \underline{2}\ \underline{1} & \equiv & (\lambda m\,n.\ \lambda f\,x.\ m\ f\ (n\ f\ x))\ \underline{2}\ \underline{1} \\
& \twoheadrightarrow & \lambda f\,x.\ \underline{2}\ f\ (\underline{1}\ f\ x) \\
& \twoheadrightarrow & \lambda f\,x.\ f\ (f\ (\underline{1}\ f\ x)) \\
& \twoheadrightarrow & \lambda f\,x.\ f\ (f\ (f\ x)) \\
& \equiv & \underline{3}
\end{array}
$$

- Korrutamine ja astendamine

$$
\begin{array}{lcl}
\textbf{mul} & \equiv & \lambda m\,n.\ \lambda f\,x.\ m\ (n\ f)\ x \\
\textbf{exp} & \equiv & \lambda m\,n.\ \lambda f\,x.\ n\ m\ f\ x
\end{array}
$$

Naturaalarvud

- Ühe lahutamine — abifunktsiooni spetsifikatsioon

$$\begin{array}{lcl}
\mathbf{prefn}\ f\ (\mathbf{T},x) & = & (\mathbf{F},x) \\
\mathbf{prefn}\ f\ (\mathbf{F},x) & = & (\mathbf{F},f\ x) \\
(\mathbf{prefn}\ f)^n\ (\mathbf{F},x) & = & (\mathbf{F},f^n\ x) \\
(\mathbf{prefn}\ f)^n\ (\mathbf{T},x) & = & (\mathbf{F},f^{n-1}\ x)
\end{array}$$

- Ühe lahutamine — definitsioon

$$\begin{array}{lcl}
\mathbf{prefn} & \equiv & \lambda f\ p.\ (\mathbf{F},(\mathbf{cond}\ (\mathbf{fst}\ p)\ (\mathbf{snd}\ p)\ (f\ (\mathbf{snd}\ p)))) \\
\mathbf{pred} & \equiv & \lambda n.\ \lambda f\ x.\ \mathbf{snd}\ (n\ (\mathbf{prefn}\ f)\ (\mathbf{T},x))
\end{array}$$

- Näide

$$\begin{array}{lcl}
\mathbf{pred}\ \underline{n}\ f\ x & = & \mathbf{snd}\ (\underline{n}\ (\mathbf{prefn}\ f)\ (\mathbf{T},x)) \\
 & = & \mathbf{snd}\ ((\mathbf{prefn}\ f)^n\ (\mathbf{T},x)) \\
 & = & \mathbf{snd}\ (\mathbf{F},f^{n-1}\ x) \\
 & = & f^{n-1}\ x
\end{array}$$

Listid

- Definitsioon

$$
\begin{array}{lcll}
\textbf{nil} & \equiv & \lambda z.\ z & (\equiv \quad \textbf{I}) \\
\textbf{cons} & \equiv & \lambda x\ y.\ (\textbf{F}, (x, y)) & \\
\textbf{null} & \equiv & \lambda z.\ z\ \textbf{T} & (\equiv \quad \textbf{fst}) \\
\textbf{hd} & \equiv & \lambda z.\ \textbf{fst}\ (\textbf{snd}\ z)) & \\
\textbf{tl} & \equiv & \lambda z.\ \textbf{snd}\ (\textbf{snd}\ z)) &
\end{array}
$$

- Näide

$$
\begin{array}{lcl}
\textbf{null nil} & \equiv & \textbf{fst}\ (\lambda z.\ z) \\
 & \equiv & (\lambda p.\ p\ \textbf{T})\ (\lambda z.\ z) \\
 & \twoheadrightarrow & \textbf{T}
\end{array}
$$

Püsipunktid

- Püsipuktiteoreem:

$$\forall F.\ \exists X.\ X = F\,X$$

- Termi M nimetatakse püsipunkti kombinaatoriks kui

$$\forall F.\ M\,F = F\,(M\,F)$$

- Curry "paradoksaalne" kombinaator

$$\mathbf{Y} \equiv \lambda f.(\lambda x. f\,(x\,x))\,(\lambda x. f\,(x\,x))$$

- "Tugev" püsipunkti kombinaator

$$\Theta \equiv (\lambda xy.\ y(xxy))\,(\lambda xy.\ y(xxy))$$

Püsipunktid

- Lemma: Olgu $G \equiv \lambda y f. f(y f) \qquad (\equiv \mathbf{SI})$

$$M \text{ on püsipunkti kombinaator} \iff M = GM$$

- Tõestus:

$(\Leftarrow)$ Kui $M = GM$, siis:

$$\begin{aligned} \forall F.\, MF &= GMF \\ &\equiv (\lambda y f. f(y f)) MF \\ &= F(MF) \end{aligned}$$

$(\Rightarrow)$ Kui M on püsipunkti kombinaator, siis:

$$\begin{aligned} GM &= \lambda f. f(Mf) \\ &= \lambda f. Mf \\ &= M \end{aligned}$$

Püsipunktid

- Lemma: Kõik jada

$$
\begin{aligned}
\mathbf{Y}^0 &\equiv \mathbf{Y} \\
\mathbf{Y}^{n+1} &\equiv \mathbf{Y}^n\, G
\end{aligned}
$$

  elemendid on püsipunkti kombinaatorid.

- Tõestus:

$$
\begin{aligned}
\mathbf{Y}^{n+1} &\equiv \mathbf{Y}^n\, G \\
&= G\,(\mathbf{Y}^n\, G) \\
&\equiv G\,\mathbf{Y}^{n+1}
\end{aligned}
$$

- NB! $\mathbf{Y}^1 \twoheadrightarrow \Theta$

Rekursioon

- Lemma: Olgu $C \equiv C(f, \vec{x})$, siis
  - $\exists F.\ \forall \vec{N}.\ F\,\vec{N} = C(F, \vec{N})$
  - $\exists F.\ \forall \vec{N}.\ F\,\vec{N} \twoheadrightarrow C(F, \vec{N})$
- Tõestus: Võtame $F \equiv \Theta(\lambda f \vec{x}.C(f, \vec{x}))$
- Standardnumbrite liitmine

$$\begin{array}{lcl} \textbf{add} & = & \lambda x\, y.\ \textbf{cond}\,(\textbf{iszero}\ x)\ y\ (\textbf{add}(\textbf{pred}\ x)(\textbf{succ}\ y)) \\ \textbf{add} & \equiv & \textbf{Y}\,(\lambda f\ x\, y.\ \textbf{cond}\,(\textbf{iszero}\ x)\ y\ (f\,(\textbf{pred}\ x)(\textbf{succ}\ y))) \end{array}$$

Mitmekohalised funktsioonid

- “currymine”

$$\begin{aligned}\mathbf{curry}_n &\equiv \lambda f\ x_1\ \ldots\ x_n.\ f\ (x_1,\ldots,x_n)\\ \mathbf{uncurry}_n &\equiv \lambda f\ p.\ f\ (p \overset{n}{\downarrow} 1)\ \ldots\ (p \overset{n}{\downarrow} n)\end{aligned}$$

- NB!

$\mathbf{curry}_n(\mathbf{uncurry}_n\ N) = N$ $\qquad$ $\mathbf{uncurry}_n(\mathbf{curry}_n\ M) = M$

- Üldistatud λ-abstraktsioon

$$\lambda(V_1,\ldots,V_n).\ E \quad\equiv\quad \mathbf{uncurry}_n\ (\lambda V_1\ \ldots\ V_n.\ E)$$

- Üldistatud β-konversioon

$$(\lambda(V_1,\ldots,V_n).\ E)(E_1,\ldots,E_n) \quad\to_\beta\quad E[E_1\ldots E_n/V_1\ldots V_n]$$

Rekursioon

- Vastastikune rekursioon

$$
\begin{array}{lcl}
f_1 & = & F_1\ f_1 \ldots f_n \\
f_2 & = & F_2\ f_1 \ldots f_n \\
 & \ldots & \\
f_n & = & F_n\ f_1 \ldots f_n
\end{array}
$$

- Definitsioon

$$
\begin{array}{lcl}
f & \equiv & \mathbf{Y}(\lambda(f_1, \ldots, f_n).\ (F_1\, f_1 \ldots f_n,\ \ldots,\ F_n\, f_1 \ldots f_n)) \\
f_1 & \equiv & f \overset{n}{\downarrow} 1 \\
f_2 & \equiv & f \overset{n}{\downarrow} 2 \\
 & \ldots & \\
f_n & \equiv & f \overset{n}{\downarrow} n
\end{array}
$$

Arvutatavus

- Church'i tees: Kõik arvutatavad funktsioonid on esitatavad λ-arvutuses!
- Primitiivrekursiivsed funktsioonid:

(i) 0

(ii) $S(x) = x + 1$

(iii) $U_n^i(x_1, x_2, \ldots, x_n) = x_i$

(iv) $f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_r(x_1, \ldots, x_n))$

(v) $$f(0, x_2, \ldots, x_n) = g(x_2, \ldots, x_n)$$
$$f(S(x_1), x_2, \ldots, x_n) = h(f(x_1, x_2, \ldots, x_n), x_2, \ldots, x_n)$$

Arvutatavus

- Osaliselt rekursiivsed funktsioonid

(vi) $f(x_1, \ldots, x_n) = \min \{ y \mid g(y, x_2, \ldots, x_n) = x_1 \}$

- Minimiseerimine

$$\begin{aligned} \mathbf{min}\, x\ f\ (x_1, \ldots, x_n) \quad &= \quad \mathbf{cond}\ (\mathbf{eq}\ (f(x, x_2, \ldots, x_n)\ x_1)\ \ x \\ &\qquad\qquad (\mathbf{min}\ (\mathbf{succ}\ x)\ \ f\ (x_1, \ldots, x_n)) \end{aligned}$$

- Definitsioon

$$\begin{aligned} \mathbf{min} \quad &\equiv \quad \mathbf{Y}\ (\lambda m\, x\, f\, (x_1, \ldots, x_n).\ \mathbf{cond}\ (\mathbf{eq}\ (f(x, x_2, \ldots, x_n))\ x_1)\ \ x \\ &\qquad\qquad (m(\mathbf{succ}\ x)\ \ f\ (x_1, \ldots, x_n))) \end{aligned}$$